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Prym-Tjurin Constructions on Cubic Hypersurfaces

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Abstract. In this paper, we give a Prym-Tjurin construction for the cohomology and the Chow groups of a cubic hypersurface. On the space of lines meeting a given rational curve, there is the incidence correspondence. This correspondence induces an action on the primitive cohomology and the primitive Chow groups. We first show that this action satisfies a quadratic equation. Then the Abel-Jacobi mapping induces an isomorphism between the primitive cohomology of the cubic hypersurface and the Prym-Tjurin part of the above action. This also holds for Chow groups with rational coefficients. All the constructions are based on a natural relation among topological (resp. algebraic) cycles on X modulo homological (resp. rational) equivalence.

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1. Introduction

Algebraic cycles on a cubic hypersurface have been serving as an interesting but nontrivial example in the study of Chow groups. In [Sh], we obtained natural relations among 1-cycles and gave some applications of those relations. This paper is a continuation of [Sh] and generalization of the results to higher dimensional cycles on cubic hypersurfaces. Throughout the paper, we will work over the field $\mathbb{C}$ of complex numbers. In this article, all homology and cohomology groups are modulo torsion.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface of dimension $n \geq 3$. We first establish Theorem 2.2 which gives a natural relation among cycles on X. To get an idea of what such a relation is, we first fix a smooth curve $C \subset X$ of degree e. Let $T \subset X$ be a closed subvariety of dimension r. Let $T' \subset X$ be the subvariety swept out by lines on X that meet both C and T. When C and T

are in general position, then T' has expected dimension r. If $r \geq 2$, then the relation we get is

$$2e\,T + T' = a\,h^{n-r}, \quad \text{in } \mathrm{CH}_r(X)$$

for some integer $a > 0$, where h is the class of a hyperplane section of X. If $r = 1$, the relation we get is

$$2e\,T + T' + 2\deg(T)C = b\,h^{n-1}, \quad \text{in } \mathrm{CH}_1(X)$$

for some integer $b > 0$. The second case was first proved in [Sh].
Let F be the Fano scheme of lines on X. Let $p : P(X) \to F$ be the total family of lines on X and $q : P(X) \to X$ be the natural morphism. Then we have the natural cylinder homomorphism $\Psi = q_* p^*$ and its transpose, the Abel-Jacobi homomorphism, $\Phi = p_* q^*$. These are defined on the cohomology groups and the Chow groups. The following theorem is a more concise expression of the above relations; see Corollary 2.4.

THEOREM 1.1. *Let $\gamma \in \mathrm{H}_n(X, \mathbb{Z})$, $\mathfrak{a} \in \mathrm{CH}_1(X)$ and $\mathfrak{b} \in \mathrm{CH}_r(X)$. We use h to denote the class of a hyperplane in either the Chow group or the cohomology group of X. Then the following are true.*
(1) $2\deg(\mathfrak{a})\gamma + \Psi(\Phi(\gamma)\cdot\Phi([\mathfrak{a}])) = 3\deg(\gamma)\deg(\mathfrak{a})h^{\frac{n}{2}}$ *in* $\mathrm{H}_n(X, \mathbb{Z})$*, where the right hand side is* 0 *if* n *is odd.*
(2) If $n-2 \geq r \geq 2$*, then* $2\deg(\mathfrak{a})\mathfrak{b} + \Psi(\Phi(\mathfrak{a})\cdot\Phi(\mathfrak{b})) = 3\deg(\mathfrak{a})\deg(\mathfrak{b})h^{n-r}$ *in* $\mathrm{CH}_r(X)$.
(3) If $r = 1$*, then* $2\deg(\mathfrak{a})\mathfrak{b} + 2\deg(\mathfrak{b})\mathfrak{a} + \Psi(\Phi(\mathfrak{a})\cdot\Phi(\mathfrak{b})) = 3\deg(\mathfrak{a})\deg(\mathfrak{b})h^{n-1}$ *in* $\mathrm{CH}_1(X)$.

REMARK 1.2. If $n \geq 5$, then $\mathrm{CH}_1(X) \cong \mathbb{Z}$; see [Pa, Proposition 4.2]. In this case, $\mathfrak{a} \in \mathrm{CH}_1(X)$ is the same as $\deg(\mathfrak{a})[l]$, where $[l]$ is the class of a line. For fixed n, the groups $\mathrm{CH}_r(X)$ are expected to be trivial (isomorphic to $\mathbb{Z}$) when r is small or large enough. Hence, the above relations are interesting when r is in the middle range.

With these relations, we will realize the Hodge structure and Chow groups of X as Prym-Tjurin constructions.
A Prym variety is constructed from a curve together with an involution and they form a larger class of principally polarized abelian varieties (p.p.a.v.) than the Jacobian of curves. Mumford gives this a modern treatment in [Mu]. He also uses Prym varieties to characterize the intermediate Jacobian of a cubic threefold, see the appendix of [CG]. In [Tj], Tjurin generalizes this idea by replacing the involution with a correspondence that satisfies a quadratic equation. This gives what we now call Prym-Tjurin varieties. This was further developed and completed by S. Bloch and J.P. Murre, see [BM]. Welters has proved that all p.p.a.v.'s can be realized as Prym-Tjurin varieties, see [We]. Roughly speaking, this means that every principally polarized weight one Hodge structure can be realized via a Prym-Tjurin construction on some curve. We naturally ask whether similar constructions can be done for higher weight Hodge structures. The work of Lewis in [Lew] sheds some light on this question. Izadi gives a Prym construction for the cohomology of cubic hypersurfaces in [Iz].

Definition 1.3. Let Λ be an abelian group and $\sigma : \Lambda \to \Lambda$ an endomorphism. Assume that σ satisfies the quadratic equation $(\sigma - 1)(\sigma + q - 1) = 0$. Then the *Prym-Tjurin part* of Λ is defined as

$$P(\Lambda, \sigma) = \mathrm{Im}(\sigma - 1)$$

Remark 1.4. In many cases when Λ carries some extra structure such as a Hodge structure, the homomorphism σ is usually compatible with that extra structure and $P(\Lambda, \sigma)$ carries an induced such structure. One very interesting case is when $\Lambda = \mathrm{H}^*(Y, \mathbb{Z})$ together with the natural Hodge structure and σ is an action that is induced by some correspondence $\Gamma \in \mathrm{CH}^{\dim Y}(Y \times Y)$.

Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth cubic hypersurface as above. We start with a general rational curve $C \subset X$ of degree $e \geq 2$. Let S_C be a natural resolution (in fact the normalization) of the space of lines on X that meet C. Let $F = F(X)$ be the Fano scheme of lines on X, $p : P \to F$ the total family with a morphism $q : P \to X$. Then $S_C = q^{-1}(C)$. In particular, we have a morphism $q_0 : S_C \to C$. Consider the family $p_C : P_C \to S_C$ of lines parameterized by S_C with the morphism $q_C : P_C \to X$. We have the natural cylinder homomorphisms $\Psi_C = (q_C)_*(p_C)^* : \mathrm{H}^{n-2}(S_C, \mathbb{Z}) \to \mathrm{H}^n(X, \mathbb{Z})$ and $\Psi_C = (q_C)_*(p_C)^* : \mathrm{CH}_r(S_C) \to \mathrm{CH}_{r+1}(X)$. Similarly we have its transpose $\Phi_C = (p_C)_*(q_C)^* : \mathrm{H}^n(X, \mathbb{Z}) \to \mathrm{H}^{n-2}(S_C, \mathbb{Z})$ and $\Phi_C = (p_C)_*(q_C)^* : \mathrm{CH}_{r+1}(X) \to \mathrm{CH}_r(S_C)$. On S_C we have the incidence correspondence which induces an action σ on the cohomology groups and the Chow groups. Note that there is a natural morphism from S_C to the Grassmannian $G = G(2, n+2)$. The Plücker embedding of G induces a ample class g on S_C. On S_C, we have another divisor class $g' = q_0^*(pt)$ for some $pt \in C$. Let $\mathbb{Q}[g, g'] \subset \mathrm{H}^*(S_C, \mathbb{Q})$ be the subring generated by g and g'. Given a compact manifold Y, we can define the *intersection pairing* on $\mathrm{H}^*(Y)$ by

$$\langle \alpha, \beta \rangle = \int_Y \alpha \cup \beta.$$

By abuse of notation, when α and β are of complementary dimension, we also write $\alpha \cdot \beta$ for $\langle \alpha, \beta \rangle$. With this notion, we define the primitive cohomology (or cycle) classes on S_C to be those which are orthogonal to the classes in $\mathbb{Q}[g, g']$. The primitive cohomology is denoted by $\mathrm{H}^*(S_C, \mathbb{Z})^\circ$ and the primitive Chow group is denoted by $\mathrm{CH}_*(S_C)^\circ$. The subring $\mathbb{Q}[g, g']$ is invariant under the action of σ and hence σ acts on the primitive cohomology and also the primitive Chow groups. Our main result is the following

Theorem 1.5. *Let $C \subset X$ be a general rational curve of degree $e \geq 2$ as above. Let σ be the action of the incidence correspondence on either the primitive cohomology or the primitive Chow group of S_C. Then the following are true.*
(1) The action σ satisfies the following quadratic relation

$$(\sigma - 1)(\sigma + 2e - 1) = 0$$

(2) The map Φ_C induces an isomorphism of Hodge structures

$$\Phi_C : \mathrm{H}^n(X, \mathbb{Z})_{prim} \to P(\mathrm{H}^{n-2}(S_C, \mathbb{Z})^\circ, \sigma)(-1)$$

where the (-1) *on the right hand side means shifting of degree by* $(1,1)$*. The intersection forms are related by the following identity*

$$\Phi_C(\alpha)\cdot\Phi_C(\beta) = -2e\,\alpha\cdot\beta, \qquad \textit{for all } \alpha,\beta\in \mathrm{H}^n(X,\mathbb{Z})_{prim}.$$

(3) The map Φ_C *induces an isomorphism*

$$\Phi_C : \mathrm{A}_i(X)_{\mathbb{Q}} \to P(\mathrm{CH}_{i-1}(S_C)^{\circ}_{\mathbb{Q}}, \sigma)$$

This is proved in section 4 (Theorem 4.3). In [Iz], Izadi proved a variant of statement (2) for C being a line. In [Sh], we proved the above theorem for cubic threefolds where (3) holds true for integral coefficients. Besides the natural relations stated at the very beginning of this section, another ingredient to carry out the Prym-Tjurin construction is the surjectivity of Ψ_C on primitive cohomologies.

THEOREM 1.6. *The natural homomorphism*

$$\Psi_C : \mathrm{H}^{n-2}(S_C,\mathbb{Z})^{\circ} \to \mathrm{H}^n(X,\mathbb{Z})_{prim}$$

between the primitive cohomologies is surjective.

This is proved in Section 5 using the Clemens-Letizia method (see [Cl] and [Le]).

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NOTATION:

$X\subset \mathbb{P}^{n+1}$, smooth cubic hypersurface of dimension $n\geq 3$ with h the hyperplane class;

$G = \mathrm{G}(2, n+2)$, the Grassmannian of lines in $\mathbb{P}^{n+1}$;

$F = F(X) \subset G$, Fano scheme of lines on X, smooth of dimension $2n-4$; see [CG] and also [AK].

$l\subset X$, a line on X; $[l]\in F$ the corresponding point on F;

$P = P(X)$, the universal family of lines on X, namely we have the following diagram

$$\begin{tikzcd} P(X) \arrow[r, "q"] \arrow[d, "p"'] & X \\ F & \end{tikzcd}$$

$I\subset F\times F$, the incidence correspondence, i.e. the closure of $\{([l_1],[l_2]) : l_1 \neq l_2,\, l_1\cap l_2\neq \emptyset\}$; Note that I has codimension $n-2$ in $F\times F$;

$\Phi = p_*q^*$, homomorphism from either the cohomology groups or the Chow groups of X to those of F;

$\Psi = q_*p^*$, homomorphism from either the cohomology groups or the Chow groups of F to those of X;

g, the polarization on F that comes from the Plücker embedding, $g = \Phi(h^2)$;

$F_x\subset F$, the subscheme of lines passing through $x\in X$; it is a (2,3) complete

intersection in $\mathbb{P}(\mathcal{T}_{X,x})$ and smooth for general x.
$D_x \subset X$, the variety swept out by all lines through x; D_x is a cone over F_x;
$F_C \subset F$, the subscheme parameterizing lines meeting a curve $C \subset X$;
$S_C = q^{-1}(C) \subset P(X)$, note that there are natural morphism $i_C = p|_{S_C} : S_C \to F_C \subset F$ and $q_0 = q|_{S_C} : S_C \to C$;
$g|_{S_C} = (i_C)^* g$, by abuse of notation, we still use g to denote this class;
$g' = (q_0)^*[pt]$, where $[pt] \in C$ is a closed point;
$\mathbb{Q}[g, g'] \subset \mathrm{H}^*(S_C, \mathbb{Q})$, the subring generated by g and g';
$P_C = P(X)|_{S_C}$, to be more precise, we take the following fiber product

$$\begin{tikzcd} P_C \arrow[r, "j_C"] \arrow[d, "p_C"'] & P(X) \arrow[d, "p"] \\ S_C \arrow[r, "i_C"] & F \end{tikzcd}$$

$q_C = q \circ j_C$;
$\Phi_C = (p_C)_*(q_C)^*$, homomorphism from either the cohomology groups or the Chow groups of X to those of S_C;
$\Psi_C = (q_C)_*(p_C)^*$, homomorphism from either the cohomology groups or the Chow groups of S_C to those of X;
$D_C \subset X$, the divisor swept out by all the lines meeting C; D_C is linearly equivalent to $5\deg(C)h$; see Lemma 2.1;
Given a polarization H on Y, we use $\mathrm{H}^*(Y, \mathbb{Z})_{prim}$ to denote the primitive cohomology;
Let (Y, H) be a polarized variety, we define $\mathrm{A}_*(Y) \subset \mathrm{CH}_*(Y)$ to be the subgroup of degree 0 (with respect to H) cycles;
For a vector bundle $\mathscr{E}$ on Y, we use $\mathbb{P}(\mathscr{E})$ to denote the geometric projectivization which parameterizes all 1-dimensional linear subspaces of $\mathscr{E}$; more generally, if $1 \leq r_1 < \cdots < r_k < \operatorname{rk}\mathscr{E}$ is an increasing sequence of integers, we use $G(r_1, \ldots, r_k, \mathscr{E})$ to denote the relative flag variety of subspaces of $\mathscr{E}$ with corresponding ranks.

2. The fundamental relations

In this section we will establish a basic relation among algebraic/topological cycles on X. To do that, we need the following lemma which says that the space of lines on X meeting a given curve has the expected dimension.

Lemma 2.1. *(i)* $\dim F_x = n-3$ *for all but finitely many points* x_i*, called Eckardt points. For each* x_i*, we have* $\dim F_{x_i} = n-2$.
(ii) Let $C \subset X$ *be a smooth curve on* X*. Then* F_C *if of pure expected dimension* $n-2$.
(iii) The divisor D_C *on* X *is linearly equivalent to* $5\deg(C)h$.

Proof. In [CS, Lemma 2.1], it is shown that $\dim F_x = n-3$ for a general point $x \in X$. By [CS, Corollary 2.2], there are at most finitely many Eckardt points x_i; see [CS, Definition 2.3]. This proves (i). Statement (ii) follows from (i) directly. Statement (ii) further implies that D_C is a divisor. Since $\mathrm{Pic}(X) = \mathbb{Z}h$, the class of D_C has to be a multiple of h. Let $l \subset X$ be a general line, then the intersection number of l and D_C is equal to the number of lines meeting both C and l. It is shown in [Sh, Lemma 3.10] that the above intersection number is $5\deg(C)$. $\square$

Theorem 2.2. *Let $C \subset X$ be a smooth curve of degree e. We use h to denote the class of a hyperplane section on X, viewed as an element either in the Chow group or the (co)homology group. Then the following are true.*
(1) Let γ be a topological cycle of real dimension n on $X(\mathbb{C})$. Then

$$2e[\gamma] + \Psi(\Phi([\gamma]) \cdot F_C) = 3e \deg(\gamma) h^{\frac{n}{2}}$$

in $\mathrm{H}_n(X)$, where $\deg(\gamma) = \gamma \cdot h^i$ if $n = 2i$ and $\deg(\gamma) = 0$ otherwise.
(2) Let γ be an algebraic cycle on X of dimension r with $2 \leq r \leq n-2$. Then

$$2e\gamma + \Psi(\Phi(\gamma) \cdot F_C) = 3e \deg(\gamma) h^{n-r},$$

in $\mathrm{CH}_r(X)$, where $\deg(\gamma) = \gamma \cdot h^r$.
(3) Let γ be an algebraic cycle of dimension 1 on X. Then

$$2e\gamma + \Psi(\Phi(\gamma) \cdot F_C) + 2\deg(\gamma) C = 3e \deg(\gamma) h^{n-1}$$

in $\mathrm{CH}_1(X)$.

Remark 2.3. Statement (1) holds for other dimensional topological cycles too. However, by the Lefschetz hyperplane theorem, it is only interesting when γ has dimension n.

Corollary 2.4. *Let $\gamma \in \mathrm{H}_n(X, \mathbb{Z})$, $\mathfrak{a} \in \mathrm{CH}_1(X)$ and $\mathfrak{b} \in \mathrm{CH}_r(X)$. Then the following are true*
(i) $2\deg(\mathfrak{a})\gamma + \Psi(\Phi(\gamma) \cdot \Phi([\mathfrak{a}])) = 3\deg(\mathfrak{a})\deg(\gamma) h^{\frac{n}{2}}$ in $\mathrm{H}_n(X, \mathbb{Z})$.
(ii) If $n - 2 \geq r \geq 2$, then $2\deg(\mathfrak{a})\mathfrak{b} + \Psi(\Phi(\mathfrak{a}) \cdot \Phi(\mathfrak{b})) = 3\deg(\mathfrak{a})\deg(\mathfrak{b}) h^{n-r}$ in $\mathrm{CH}_r(X)$.
(iii) If $r = 1$, then $2\deg(\mathfrak{a})\mathfrak{b} + 2\deg(\mathfrak{b})\mathfrak{a} + \Psi(\Phi(\mathfrak{a}) \cdot \Phi(\mathfrak{b})) = 3\deg(\mathfrak{a})\deg(\mathfrak{b}) h^{n-1}$ in $\mathrm{CH}_1(X)$.

Lemma 2.5. *Let γ_1 and γ_2 be two disjoint irreducible topological cycles on $\mathbb{P}^{n+1}$ of real dimensions r_1 and r_2. Let $L(\gamma_1, \gamma_2)$ be the set swept out by all complex lines meeting both γ_1 and γ_2. Assume that γ_1 and γ_2 are in general position and $L(\gamma_1, \gamma_2)$ has expected dimension. Then we have*

$$L(\gamma_1, \gamma_2) = \deg(\gamma_1)\deg(\gamma_2) h^{n - \frac{r_1 + r_2}{2}}$$

in $\mathrm{H}_{r_1+r_2+2}(\mathbb{P}^{n+1}, \mathbb{Z})$. Here $\deg(\gamma) = \gamma \cdot h^{\frac{r}{2}}$ if $r = \dim_{\mathbb{R}}(\gamma)$ is even and 0 otherwise. By convention $h^{k+\frac{1}{2}} = 0$ for all integer k.

Lemma 2.6. *Assume that γ_1 and γ_2 are two topological manifolds (resp. smooth projective varieties). Let $f_i : \gamma_i \to \mathbb{P}^{n+1}$, $i = 1, 2$, be two disjoint irreducible topological (resp. algebraic) cycles on $\mathbb{P}^{n+1}$. Let $\varphi : \gamma_1 \times \gamma_2 \to G(2, n+2)$ be the map (resp. morphism) sending a pair of points (x, y) to the unique complex line connecting them. Let $\mathscr{E}_2$ be the canonical rank 2 quotient bundle* *on $G(2,n+2)$ and $\mathscr{E}' = \varphi^*\mathscr{E}_2$ its pullback. Let $p_i : \gamma_1 \times \gamma_2 \to \gamma_i$, $i = 1,2$, be the two projections. Then*

$$c_1(\mathscr{E}') = p_1^*(f_1^* h) + p_2^*(f_2^* h)$$

in $\mathrm{H}^2(\gamma_1 \times \gamma_2, \mathbb{Q})$ (resp. $\mathrm{CH}^1(\gamma_1 \times \gamma_2)$), where h is the class of a hyperplane. In other words, $c_1(\mathscr{E}')$ is the pull-back of the class $\pi_1^* h + \pi_2^* h$ via the natural map $f_1 \times f_2 : \gamma_1 \times \gamma_2 \to \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$, where $\pi_i : \mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ are the projections.

Since the cohomology (resp. Chow ring) of $\mathbb{P}^{n+1}$ is generated by the class of a hyperplane, both of the above lemmas can be reduced to the case where both γ_1 and γ_2 are complex linear subspaces. The proofs are left to the reader.

Proof of Theorem 2.2. To prove (1), we first reduce to the case $\gamma = f_*[M]$, where $f : M \to X$ is a continuous map from an n-dimensional topological manifold M to X such that (i) C does not meet $f(M)$; (ii) f_*M intersects D_C transversally and $\Delta_0 := \{t \in M : f(t) \in D_C\}$ is of pure expected real dimension $\dim M - 2$. The reason why we can do the above reduction can be seen as follows. First, γ is always of the form of linear combinations of cycles of the form $f_*[M]$. If we can prove the statement (1) for each term in such a linear combination, then we prove (1) for γ. To get (i) and (ii), we note that C is a curve and D_C is a divisor and hence we can always move the cycle γ to a cycle γ' such that (i) and (ii) hold.
Now we assume the situation after reduction. For each pair of points $(t,x) \in M \times C$, there is a unique (complex projective) line passing through $f(t)$ and x. This gives a continuous map $i_0 : M \times C \to \mathrm{G}(2,n+2)$. We get the following diagram

(1)
$$\begin{tikzcd}
D_1 \cup D_2 \cup D \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d, "i_X"] \\
P \arrow[r, "i_0'"] \arrow[d, "\pi"'] & \mathrm{G}(1,2,n+2) \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{\pi}"'] & \mathbb{P}^{n+1} \\
M \times C \arrow[r, "i_0"] & \mathrm{G}(2,n+2) &
\end{tikzcd}$$

where all the squares are fiber products; D_1 and D_2 are sections of π corresponding to the points on $f(M)$ and C respectively; D is the set of the third points of the intersection of the lines with X. Let $\pi' : D \to M \times C$ be the restriction of π to D. Let $\Delta \subset M \times C$ be the closed subset of points (t,x) such that the line through $f(t)$ and x is contained in X. By definition, Δ_0 is the image of Δ under the projection to the first factor M and hence $\dim \Delta = \dim \Delta_0$. Then π' is one-to-one away from Δ while over Δ it is an S^2-bundle with trivial Euler class (this is because, by taking the point in which the line meets C, we have a section of this bundle). Consider the following diagram

$$
\begin{tikzcd}
\mathbb{P}^1 = S^2 \arrow[r] & E \arrow[r] \arrow[d] & D \arrow[d, "\pi'"] \\
& \Delta \arrow[r] & M \times C
\end{tikzcd}
\tag{2}
$$

For a general point $z = (t_0, x_0) \in \Delta$, we use E_z to denote the fiber of $E \to \Delta$ at the point z.

Claim 1: $E \cdot E_z = -1$.

Proof of Claim 1. Let $U_0 \subset C$ be a small 2-dimensional (meaning real dimension) disc centered at the point $x_0 \in C$. Let $U_z = \{t_0\} \times U_0 \subset M \times C$ be the corresponding small disc centered at z. By the assumption that $z \in \Delta$ is general, we see that U_z meets Δ transversally at the point z. Then we easily see that $\pi'^* U_z = \tilde{U}_z + E_z$, where $\tilde{U}_z = \pi'^{-1}(U_z \backslash \{z\}) \cup \{y_0 \in E_z\}$. The existence of the point y_0 can be seen as follows. Let $f_0 : C \dashrightarrow X$ be the rational map defined by sending $x \in C \subset X$ to the third point of the intersection of X with the line connecting $f(t_0)$ and x. Since C is a smooth curve, this rational map extends to a morphism $f_0 : C \to X$. Then one easily sees that $y_0 = f_0(x_0)$. This implies that $\tilde{U}_z \cdot E = 1$. The projection formula gives

$$(E_z + \tilde{U}_z) \cdot E = \pi'^* U_z \cdot E = U_z \cdot \pi_* E = U_z \cdot 0 = 0.$$

It follows that $E_z \cdot E = -1$. □

Let $g : D \to X$ and $g_i : D_i \to X$, $i = 1, 2$, be the natural maps. Let h be the (co)homology class of a hyperplane.

Lemma 2.7. *Let $d = \deg(f_*[M])$. The following are true.*
(i) $g_*[D] = ed\, h^{\frac{n}{2}-1}$ in $\mathrm{H}_{n+2}(X)$;
(ii) $g_*(h|_D) = ed\, h^{\frac{n}{2}}$ in $\mathrm{H}_n(X)$;
(iii) $h|_D + [E] = 2\pi'^* c_1(\mathscr{E})$, where $\mathscr{E} = i_0^* \mathscr{E}_2$ is the pull back of the canonical rank 2 quotient bundle $\mathscr{E}_2$ on $\mathrm{G}(2, n+2)$;
(iv) $c_1(\mathscr{E}) = p_1^*(h|_M) + p_2^*(h|_C)$, where $p_1 : M \times C \to M$ and $p_2 : M \times C \to C$ are the projections;
(v) $g_*(\pi'^* c_1(\mathscr{E})) = 2ed\, h^{\frac{n}{2}} - e\, f_*[M]$.

Proof of Lemma 2.7. For (i), we first note that the class of the image of $[D_1] + [D_2] + [D]$ on X is the restriction of $[P]$ viewed as a class on $\mathbb{P}^{n+1}$. As a class on $\mathbb{P}^{n+1}$, $[P] = ed\, h^{\frac{n}{2}-1}$ by Lemma 2.5. However, if we identify D_i, $i = 1, 2$, with $M \times C$, then the natural map $D_i \to X$ contacts either the factor M or the factor C. Hence the classes of $(g_1)_*[D_1]$ and $(g_2)_*[D_2]$ are all zero on X. This proves (i).
For (ii), we note that by the projection formula, we have $g_*(h|_D) = g_*[D] \cdot h$. Hence (ii) follows from (i). The identity in (iv) follows from Lemma 2.6.
To prove (iii), we first note that $h|_D + [E]$ restricts to zero on the fibers E_z of $E \to \Delta$ since $h \cdot E_z = 1$ and $E \cdot E_z = -1$ (Claim 1). By the Leray-Hirsch

theorem, this implies that $h|_D + [E] = \pi'^*\mathfrak{a}$ where $\mathfrak{a}$ is a homology class on $M \times C$. Applying π'_* to the above equation, we get $\mathfrak{a} = \pi'_*(h|_D)$. Note that on P, we have $D_1 + D_2 + D = 3h$, where, by abuse of notation, we still use h to denote the pullback of the class of a hyperplane to P. Hence

$$h \cdot D_1 + h \cdot D_2 + h \cdot D = 3h^2 \tag{3}$$

Since P is the projectivization of $\mathscr{E}$ and h is the first Chern class of the relative $\mathcal{O}(1)$-bundle, we have $h^2 - \pi^* c_1(\mathscr{E})h + \pi^* c_2(\mathscr{E}) = 0$. Applying π_* to (3), we get

$$\begin{aligned} \pi_*(h \cdot D) + \pi_*(h \cdot D_1) + \pi_*(h \cdot D) &= 3\pi_*(h^2) \\ &= 3\pi_*(\pi^* c_1(\mathscr{E}) \cdot h - \pi^* c_2(\mathscr{E})) \\ &= 3c_1(\mathscr{E}) \cdot \pi_* h = 3c_1(\mathscr{E}) \end{aligned}$$

We also easily get that $\pi_*(h \cdot D_1) = p_1^* f^* h$ and $\pi_*(h \cdot D_2) = p_2^*(h|_C)$. Combine this with (iv), we get

$$\mathfrak{a} = \pi'_*(h|_D) = \pi_*(h \cdot D) = 2c_1(\mathscr{E}).$$

To prove (v), we note that for any class $\mathfrak{a}$ on $M \times C$, if we pull back $\mathfrak{a}$ to $D_1 + D_2 + D$ and then push forward to X, what we get is the same class as we pull back $\mathfrak{a}$ to P, then push forward to $\mathbb{P}^{n+1}$ and then restrict to X. As a result, we always get a class coming from $\mathbb{P}^{n+1}$. When we take $\mathfrak{a} = c_1(\mathscr{E})$, we get that the class of $\pi^*\mathfrak{a}$ on $\mathbb{P}^{n+1}$ is equal to $2ed\, h^{\frac{n}{2}}$. This can be seen as follows (using notation from Lemma 2.5).

$$\begin{aligned} \tilde{\varphi}_*(i'_0)_* \pi^* c_1(\mathscr{E}) &= \tilde{\varphi}_*(i'_0)_* \pi^*(p_1^* f^* h + p_2^* h|_C) \\ &= L(f_* M \cdot h, C) + L(M, C \cdot h) \\ &= ed\, h^{n - \frac{(n-2)+2}{2}} + ed\, h^{n - \frac{n+0}{2}} \\ &= 2ed\, h^{\frac{n}{2}}. \end{aligned}$$

Hence we have the following equalities.

$$\begin{aligned} g_*(\pi'^* c_1(\mathscr{E})) &= 2ed\, h^{\frac{n}{2}} - (g_{1*}(\pi^* c_1(\mathscr{E})|_{D_1}) + g_{2*}(\pi^* c_1(\mathscr{E})|_{D_2})) \\ &= 2ed\, h^{\frac{n}{2}} - g_{1*}(\pi^* c_1(\mathscr{E})|_{D_1}) \\ &= 2ed\, h^{\frac{n}{2}} - e\, f_*[M] \end{aligned}$$

Here the second equality uses the fact that $g_2 : D_2 \to C \subset X$ contracts the factor M which has dimension $n \geq 3$. This finishes the proof of the lemma. $\square$

In the above lemma, we apply g_* to (iii) and then take (ii) and (v) into account.

$$\begin{aligned} g_*[E] &= 2g_*(\pi'^* c_1(\mathscr{E})) - g_*(h|_D) \\ &= 4ed\, h^{\frac{n}{2}} - 2e\, f_*[M] - ed\, h^{\frac{n}{2}} = 3ed\, h^{\frac{n}{2}} - 2e\, f_*[M] \end{aligned}$$

Note that $g_*[E] = \Psi(\Phi(f_*[M]) \cdot F_C)$. This way, we easily deduce the statement (1) of Theorem 2.2 for $\gamma = f_*[M]$ and the curve C.

Now we start to prove (2) and (3) of Theorem 2.2. The basic strategy is the same as above. The only difference is that we need to consider the more delicate rational equivalence rather than homological equivalence. By linearity of the equalities in (2) and (3) and the Chow moving lemma, we reduce to the case $\gamma = f_*M$ where $f : M \to X$ is a morphism from a smooth projective variety M to X such that (i) f is birational onto its image; (ii) $f(M)$ does not meet C and f_*M intersects D_C transversally. Let r be the dimension of M. As before we construct the natural morphism $i_0 : M \times C \to \mathrm{G}(2, n+2)$. Let $\mathscr{E}$ be the pull back of the canonical rank 2 quotient bundle to $M \times C$. In this way, we get the following diagram as before.

(4)
$$\begin{tikzcd}
D_1 \cup D_2 \cup D \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d] \\
P \arrow[r, "i_0'"] \arrow[d, "\pi"'] & \mathrm{G}(1,2,n+2) \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{\pi}"] & \mathbb{P}^{n+1} \\
M \times C \arrow[r, "i_0"] & \mathrm{G}(2,n+2) &
\end{tikzcd}$$

Here all the squares are fiber products; D_1 and D_2 are sections of π and contracted to $f(M)$ and C via $g_1 = \tilde{\varphi}\circ i_0'|_{D_1} : D_1 \to X$ and $g_2 = \tilde{\varphi}\circ i_0'|_{D_2} : D_2 \to X$ respectively; D corresponds to the third point of the intersection of the lines with X. Let $\pi' : D \to M \times C$ be the restriction of π to D. Then π' is a birational morphism. Let $\Delta \subset M \times C$ be the subvariety that consists of points (t, x) such that the line through $f(t)$ and x is contained in X. By the assumption that f_*M intersects D_C transversally, we conclude that Δ is generically smooth. Let Δ^{sing} be the singular locus of Δ. Let $E = (\pi')^{-1}\Delta \subset D$ and we have $E \cong \mathbb{P}(\mathscr{E}|_\Delta)$.

LEMMA 2.8. *Away from Δ^{sing}, π' is the blow-up along Δ.*

Proof. Let $\mathscr{E}_2$ be the canonical rank 2 quotient bundle on $G(2, n+2)$ and $\mathscr{E} = i_0^*\mathscr{E}_2$. The point $i_0(t, x) \in G(2, n+2)$ represents the line connecting $f(t)$ and x, which is also naturally $\mathbb{P}(\mathscr{E}^\vee_{(t,x)})$. Hence we have canonical homomorphisms $\mathcal{L}_{f(t)} \hookrightarrow \mathscr{E}^\vee_{(t,x)}$ and $\mathcal{L}_x \hookrightarrow \mathscr{E}^\vee_{(t,x)}$, where $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^{n+1}}(-1)$ is the tautological line bundle on $\mathbb{P}^{n+1}$. And since $f(t) \neq x$, we have the natural identification

$$\mathscr{E}^\vee_{(t,x)} = \mathcal{L}_{f(t)} \oplus \mathcal{L}_x.$$

As $(t, x) \in M \times C$ varies, this identification glues and gives a canonical isomorphism

$$\mathscr{E}^\vee \cong p_1^* f^* \mathcal{L} \oplus p_2^*(\mathcal{L}|_C).$$

Assume that $\phi(X_0, \ldots, X_{n+1}) \in \mathrm{H}^0(\mathbb{P}^{n+1}, \mathcal{O}(3))$ is the homogeneous polynomial defining X. Since we have the canonical identification $\tilde{\pi}_*\tilde{\varphi}^*\mathcal{O}(3) = \mathrm{Sym}^3(\mathscr{E}_2)$, the section ϕ induces a global section $\tilde{\phi}$ of $\mathrm{Sym}^3(\mathscr{E}_2)$ and $F \subset G(2, n+2)$ is defined by $\tilde{\phi} = 0$; see [AK]. Let $\phi' = i_0^*\tilde{\phi}$ be the induced global section of $\mathrm{Sym}^3 \mathscr{E}$. It follows from the expression of $\mathscr{E}^\vee$ that

$$\mathscr{E} \cong p_1^* f^* \mathcal{O}_X(1) \oplus p_2^* \mathcal{O}_C(1)$$

and hence

$$\mathrm{Sym}^3\mathscr{E} = p_1^*f^*\mathcal{O}_X(3) \oplus \mathscr{E} \otimes p_1^*f^*\mathcal{O}_X(1) \otimes p_2^*\mathcal{O}_C(1) \oplus p_2^*\mathcal{O}_C(3).$$

Accordingly, we can write $\phi' = \phi'_{3,0} + \phi'_{\mathrm{mid}} + \phi'_{0,3}$. Since $f(M) \subset X$ and ϕ vanishes on X, we conclude that $\phi'_{3,0} = 0$. Similarly, we have $\phi'_{0,3} = 0$. It follows that

$$\phi' = \phi'_{\mathrm{mid}} \in \mathrm{H}^0(M \times C, \mathscr{E} \otimes p_1^*f^*\mathcal{O}_X(1) \otimes p_2^*\mathcal{O}_C(1)).$$

The vanishing $\phi' = 0$ at a point (t, x) exactly means that the line connecting $f(t)$ and x is contained in X. Hence $\Delta \subset M \times C$ is defined by $\phi' = 0$. In particular, $\Delta \subset M \times C$ is a local complete intersection and

$$\mathscr{N}_{\Delta/M\times C} \cong \mathscr{E} \otimes p_1^*f^*\mathcal{O}_X(1) \otimes p_2^*\mathcal{O}_C(1)|_\Delta.$$

The section ϕ' can be viewed as a homomorphism

$$\phi' : p_1^*f^*\mathcal{O}_X(-1) \otimes p_2^*\mathcal{O}_C(-1) \longrightarrow \mathscr{E}.$$

This gives a section $s : M \times C\backslash\Delta \to P$ of π such that the image of s is the open subset $D\backslash E$ of D. Let $\sigma : \widetilde{M \times C} \to M \times C$ be the blow-up of $M \times C$ along Δ and $E_0 = \sigma^{-1}\Delta$. As a consequence of the above description of $\mathscr{N}_{\Delta/M\times C}$, we see that ϕ' extends to $\widetilde{M \times C}\backslash\sigma^{-1}\Delta^{sing}$ and gives a rank one subbundle

$$\tilde{\phi}' : \mathscr{L} \hookrightarrow \sigma^*\mathscr{E}$$

where $\mathscr{L} = \sigma^*(p_1^*f^*\mathcal{O}_X(-1) \otimes p_2^*\mathcal{O}_C(-1)) \otimes \mathcal{O}(E_0)$ is a line bundle on $\widetilde{M \times C}\backslash\sigma^{-1}\Delta^{sing}$. This further gives a morphism

$$\tilde{s} : \widetilde{M \times C}\backslash\sigma^{-1}\Delta^{sing} \hookrightarrow P$$

such that $\sigma = \pi \circ \tilde{s}$. Note that $\tilde{s}$ extends s. Hence the image of $\tilde{s}$ is contained in D since that of s is. Since $\tilde{E} \to \Delta$ and $E \to \Delta$ are all $\mathbb{P}^1$-bundles, we conclude that The image of $\tilde{s}$ is exactly $D\backslash(\pi')^{-1}\Delta^{sing}$. This implies that

$$\tilde{s} : \widetilde{M \times C}\backslash\sigma^{-1}\Delta^{sing} \longrightarrow D\backslash(\pi')^{-1}\Delta^{sing}$$

is an isomorphism, since they are the same locally closed subvariety of P. □

Back to the proof of the theorem. We know from the above lemma that the possible singularities of D can only appear over Δ^{sing}. Take a resolution of singularities $r' : \tilde{D} \to D$. Let E_i be the exceptional divisors of r'. We use E' to denote the strict transform of E in $\tilde{D}$. We still use h to denote the class of a hyperplane in $\mathbb{P}^{n+1}$. Let $g : \tilde{D} \to X$ be the natural morphism.

Lemma 2.9. *Let $d = \deg(f_*[M])$, then the following are true.*
(i) $g_*\tilde{D} = ed\, h^{n-r-1}$ in $\mathrm{CH}_{r+1}(X)$;
(ii) $g_*(h|_{\tilde{D}}) = ed\, h^{n-r}$;
(iii) $h|_{\tilde{D}} + E' + \sum a_iE_i = 2(\pi' \circ r')^*c_1(\mathscr{E})$, for some $a_i \in \mathbb{Z}$;
(iv) $c_1(\mathscr{E}) = p_1^*(f^*h) + p_2^*(h|_C)$;
(v) If $\dim M \geq 2$, then $g_*(r'^*\pi'^*c_1(\mathscr{E})) = -e\, f_*M + 2ed\, h^{n-r}$ in $\mathrm{CH}_r(X)$;
(vi) If M is a curve, then $g_*(r'^*\pi'^*c_1(\mathscr{E})) = ed\, h^{n-1} - e\, f_*M - dC$ in $\mathrm{CH}_1(X)$.

Proof of Lemma 2.9. The proof is similar to that of Lemma 2.7. We note that the push forward of $D_1 + D_2 + D$ to X is a class coming from $\mathbb{P}^{n+1}$ because it is the restriction of the class of the image of P in $\mathbb{P}^{n+1}$. By Lemma 2.5, the class of P in $\mathbb{P}^{n+1}$ is equal to $ed\, h^{n-r-1}$. Since D_1 and D_2 are contracted to smaller dimensions via g_1 and g_2 respectively, we get (i).
Conclusion (ii) follows from the projection formula as before. (iv) follows from Lemma 2.6.
For (iii), note that on $D - \pi'^{-1}(\Delta^{sing})$, the class of $h + E$ comes from a class $\mathfrak{a}$ on $M \times C - \Delta^{sing}$ via pull back by π'. Since the codimension of Δ^{sing} in $M \times C$ is at least 3, we know that the divisor class group of $M \times C - \Delta^{sing}$ is the same as that of $M \times C$. Hence we can view $\mathfrak{a}$ as a divisor class on $M \times C$. The equality

$$h + E = \pi'^*\mathfrak{a}, \quad \text{on } D \backslash \pi'^{-1}(\Delta^{sing})$$

gives an equation

$$r'^*(h + E) + \sum a_i E_i = r'^*\pi'^*\mathfrak{a}$$

for some $a_i \in \mathbb{Z}$. By a very similar argument as before we know that the class $\mathfrak{a}$ is $2c_1(\mathscr{E})$.
To prove (v) and (vi), we do the following explicit calculation.

$$\begin{aligned}
g_* r'^* \pi'^* c_1(\mathscr{E}) &= g_* r'^* \pi'^* (p_1^* f^* h + p_2^* h|_C), \qquad \text{by (iv)} \\
&= \tilde{g}_* \pi^* (p_1^* f^* h + p_2^* h|_C)|_{D \cup D_1 \cup D_2} - g_{1*} \pi^* (p_1^* f^* h + p_2^* h|_C)|_{D_1} \\
&\quad - g_{2*} \pi^* (p_1^* f^* h + p_2^* h|_C), \qquad \tilde{g} : D \cup D_1 \cup D_2 \to X \\
&= \tilde{\varphi}_* (i'_0)_* \pi^* (p_1^* f^* h + p_2^* h|_C)|_X - g_{1*}(p_2^* h|_C) - g_{2*}(p_1^* f^* h) \\
&\qquad \text{(here we identify } D_1 \text{ and } D_2 \text{ with } M \times C) \\
&= L(f_* M \cdot h, C)|_X + L(f_* M, C \cdot h)|_X - e\, f_* M - g_{2*}(p_1^* f^* h) \\
&= 2ed\, h^{n-r} - e\, f_* M - g_{2*}(p_1^* f^* h)
\end{aligned}$$

Note that $g_{2*}(p_1^* f^* h)$ is supported on the curve C. If $\dim M \geq 2$, then $g_{2*}(p_1^* f^* h) = 0$; if $\dim M = 1$, then $g_{2*}(p_1^* f^* h) = \deg(f_* M) C$. Hence (v) and (vi) follow from the above computation. □

Now we come back the the proof of the theorem. In the above lemma, we first apply g_* to (iii) and note that all the E_i's map to zero. We also easily see that $g_* E' = \Psi(\Phi(f_* M) \cdot F_C)$. Combine all these with (v) (or (vi) in the case of 1-cycles), we get the conclusion (2) and (3) for $\gamma = f_* M$. By linearity, the conclusions hold for any given γ. □

3. The action of the incidence correspondence

Let $C \subset X$ be a smooth rational curve of degree e on X. We also assume that C is general, meaning that it comes from a non-empty open subset of the corresponding component of the Hilbert scheme. Let $S_C = q^{-1} C$ be the inverse image of C under the morphism $q : P(X) \to X$. The points on S_C can be described as

$$S_C = \{([l], x) \in F \times C : x \in l\}.$$

Let $q_0 = q|_{S_C} : S_C \to C$.

Lemma 3.1. *If C is general, then S_C is smooth of dimension $n-2$.*

We will prove this lemma later. In [Sh, Lemma 3.4], we show that for a general rational curve C of degree $e \geq 2$, there exist $n_e = \frac{5e(e-3)}{2} + 6$ secant lines of C, i.e. lines meeting C twice. Let $L_{C,i}$, $i = 1, \ldots, n_e$, be all the secant lines of C and $[L_{C,i}] \in F$ the corresponding points on the variety of lines. Above each point $[L_{C,i}]$, we have a pair of points $\{([L_{C,i}], y_i), ([L_{C,i}], z_i)\}$ on S_C, where y_i and z_i are the two points in which C intersects $L_{C,i}$. Then Lemma 3.1 implies that $[L_{C,i}]$'s are the only singular points of F_C and $p|_{S_C} : S_C \to F_C$ is the normalization and also a desingularization since S_C is smooth.
From now on, we make the assumption that $e \geq 2$, unless otherwise stated.

Definition 3.2. We say that a correspondence $\Gamma \subset Y \times Y$ is generically defined by $y \mapsto \sum y_i$ if Γ is the closure of the graph of this multi-valued map.

For a general point $[l] \in F_C$, let $x = C \cap l$ be the intersection point. By $[l] \mapsto ([l], x) \in S_C$, we view $[l]$ as a point on S_C.

Lemma 3.3. *There exist $5e-5$ lines $l_1, l_2, \ldots, l_{5e-5}$ meeting both l and C in points different from x.*

By abuse of language, these lines are called *secant lines* of the pair (l, C); see [Sh, Definition 3.1]. Each point $[l_i]$ can again be viewed as a point $([l_i], x_i)$ on S_C, where $x_i = l_i \cap C$. A line meeting two disjoint curves $C_1, C_2 \subset X$ will be called an *incidence line* of C_1 and C_2 (note that they are also called secant lines in [Sh]).

Proof of Lemma 3.3. Note that if $\tilde{l}$ is a general line, the number of incidence lines of $\tilde{l}$ and C is $5e$; this follows from a degree computation using (3) of Theorem 2.2 or [Sh, Lemma 3.10]. When $\tilde{l}$ specializes to l, five of these incidence lines of $\tilde{l}$ and C will specialize to five lines passing through x and those five lines are not counted as the secant lines of the pair (l, C). We will describe this specialization in more details later. Hence the number of secant lines of the pair (l, C) is $5e-5$. □

Definition 3.4. Let the incidence correspondence $I_C \subset S_C \times S_C$ be generically defined by

$$I_C : ([l], x) \mapsto \sum_{i=1}^{5e-5} ([l_i], x_i), \tag{5}$$

where l is a line meeting C and l_i are the secant lines of the pair (l, C). Let σ denote the action of I_C on either the cohomology groups or the Chow groups of S_C. On F, we have the incidence correspondence $I = \{([l_1], [l_2]) \in F \times F : l_1 \cap l_2 \neq \emptyset\} \subset F \times F$. This induces a correspondence

$$I'_C = (i_C \times i_C)^* I \in \mathrm{CH}_{n-2}(S_C \times S_C),$$

where $i_C = p|_{S_C} : S_C \to F$ is the natural morphism.

REMARK 3.5. Note that $(i_C \times i_C)^{-1}I$ has a component, namely $S_C \times_C S_C$, which has dimension larger than expected. The correspondence $I_C \subset (i_C \times i_C)^{-1}I$ is a component of expected dimension. It turns out later that one key ingredient to understand the action of σ is the difference between I_C and I'_C.

Note that on F, we have a natural polarization g given by the Plücker embedding of $\mathrm{G}(2, n+2)$. It can also be written as $g = \Phi(h^2)$. We fix $g|_{S_C} = (i_C)^* g$ as the polarization of S_C. By abuse of notation, we still use g to denote its restriction to S_C. Recall that S_C admits a natural morphism $q_0 = q|_{S_C} : S_C \to C$. This gives an extra class $g' = q_0^*[pt]$ on S_C. Note that g' is never ample. The following is the main result of this section.

THEOREM 3.6. *Let $C \subset X$ be a general rational curve of degree $e \geq 2$ and σ the action of the incidence correspondence as above. Then the following are true.*
(1) Let $\mathfrak{a}$ be a topological cycle of odd dimension on S_C. Then
$$\sigma(\mathfrak{a}) = \Phi_C(\Psi_C(\mathfrak{a})) + \mathfrak{a}$$
in $\mathrm{H}^*(X, \mathbb{Z})$.
(2) Let $\mathfrak{a}$ be a topological cycle of dimension $2m$ or an algebraic cycle of dimension m on S_C, then in either the cohomology $\mathrm{H}^{2n-2m-4}(X, \mathbb{Z})$ or Chow group $\mathrm{CH}_m(X)$, we have
$$\sigma(\mathfrak{a}) = \Phi_C(\Psi_C(\mathfrak{a})) + \mathfrak{a} + \mathrm{const.}$$
where the constant only depends on the intersection numbers $\mathfrak{a} \cdot g^m$ and $\mathfrak{a} \cdot g' g^{m-1}$; the constant is zero if $\mathfrak{a} \cdot g^m = 0$ and $\mathfrak{a} \cdot g' g^{m-1} = 0$.

Before proving the above theorem, we give some technical constructions related to the geometry of S_C.

3.1. THE NORMAL BUNDLE OF A LINE MEETING C. Recall from [CG, §6] that for any line l on X, we have either
$$\mathscr{N}_{l/X} \cong \mathcal{O}(1)^{n-3} \oplus \mathcal{O}^2,$$
in which case l is said to be of *first type*, or
$$\mathscr{N}_{l/X} \cong \mathcal{O}(1)^{n-2} \oplus \mathcal{O}(-1),$$
in which case l is said to be of *second type*. A general line is of first type.

DEFINITION 3.7. Let $\mathscr{G}$ be a vector bundle on $\mathbb{P}^1$. Then we have a decomposition $\mathscr{G} = \bigoplus_{i=1}^{\mathrm{rk}\,\mathscr{G}} \mathcal{O}(a_i)$, $a_i \in \mathbb{Z}$. We define the *positive part* of $\mathscr{G}$ to be
$$\mathrm{Pos}(\mathscr{G}) = \bigoplus_{a_i > 0} \mathcal{O}(a_i).$$
Similarly, we define the *nonnegative part* of $\mathscr{G}$ to be
$$\mathrm{NN}(\mathscr{G}) = \bigoplus_{a_i \geq 0} \mathcal{O}(a_i).$$

LEMMA 3.8. *Assume that $C \subset X$ is general and l is a line meeting C in a point x. Then l meets C transversally at the point x.*

Proof. If l is tangent to C at the point x, then l is a secant line of C. We call a secant line of C *simple* if it meets C in two distinct points. Then we only need to show that the secant lines of a general curve C are simple. Note that having simple secant lines is an open condition and we only need to find a degenaration of C which has all secant lines being simple. First we note that the concept of a secant line can be naturally extended to most nodal curves; see [Sh]. For example, let $C_1 = L_1 \cup L_2 \cup L_3$ be a chain of three distinct lines with $L_1 \cap L_2 = x$ and $L_2 \cap L_3 = y$. Assume that the plane spanned by L_1 and L_2 (resp. L_2 and L_3) is not contained in X. Let l_{12} (resp. l_{23}) be the residue line, or equivalently the secant line, of $L_1 \cup L_2$ (resp. $L_2 \cup L_3$). If we further assume that the pair (L_1, L_3) has finitely many incidence lines $\{E_0 = L_2, E_1, E_2, E_3, E_4\}$. Then the set of secant lines of C_1 is given by

$$\{l_{12}, l_{23}, E_1, E_2, E_3, E_4\}.$$

Let $e \geq 2$ be the degree of C. We construct a chain $C' = \cup_i l_i$, $i = 1, \ldots, e$, of distinct lines such that: (i) l_i and l_{i+1} meet in a point y_i, $i = 1, \ldots, e-1$; (ii) there are finitely many distinct secant lines of C', which are different from the lines in $\{l_i : i = 1, \ldots, e\}$; (iii) all the lines l_i are of first type. Such a curve C' can be constructed inductively. The condition (ii) says that the curve C' has only simple secant lines. The condition (iii) implies that C' can be deformed to a smooth degree e rational curve on X. Since the space of degree e rational curves on X are irreducible by [CS], we see that C' is a specialization of C. Hence the lemma is proved. $\square$

DEFINITION 3.9. Let $([l], x) \in S_C$ be a point where l is a line on X and x is a point in which l meets C. Since C is general, l meets C transversally at the point x. We define $\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \hookrightarrow \mathscr{N}_{l/X}$ to be the subsheaf of sections s such that $s(x)$ is in the direction of $\mathcal{T}_{C,x}$. Or equivalently, $\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle$ fits into the following short exact sequence

$$0 \longrightarrow \mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \longrightarrow \mathscr{N}_{l/X} \longrightarrow \frac{\mathcal{T}_{X,x}}{\mathcal{T}_{l,x} \oplus \mathcal{T}_{C,x}} \longrightarrow 0$$

The tangent space of S_C at a point $([l], x)$ is canonically identified as

$$\mathcal{T}_{S_C,([l],x)} = \mathrm{H}^0(l, \mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle)$$

See [Ko, §II.1]. It follows from the short exact sequence in Definition 3.9 that

$$\chi(\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle) = \chi(\mathscr{N}_{l/X}) - (n-2) = n-2 = \dim S_C$$

where we use the fact that $\dim S_C = n-2$ which is a consequence of Lemma 2.1. Hence $\dim \mathcal{T}_{S_C,([l],x)} = \dim S_C$ if and only if $\mathrm{H}^1(l, \mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle) = 0$. Equivalently, we have the following

LEMMA 3.10. *The variety S_C is smooth at $([l], x)$ if and only if $h^1(\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle) = 0$.*

PROPOSITION 3.11. *The splitting of $\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle$ has the following possibilities.*

(i) If l is of first type and the image of $\mathcal{T}_{C,x}$ in $\mathscr{N}_{l/X,x}$ is not contained in $\mathrm{Pos}(\mathscr{N}_{l/X})_x$*, then*

$$\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \cong \mathcal{O}^{n-2}\oplus \mathcal{O}(-1).$$

(ii) If l is of first type and the image of $\mathcal{T}_{C,x}$ in $\mathscr{N}_{l/X,x}$ is contained in $\mathrm{Pos}(\mathscr{N}_{l/X})_x$*, then*

$$\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \cong \mathcal{O}(1)\oplus \mathcal{O}^{n-4}\oplus \mathcal{O}(-1)^2.$$

(iii) If l is of second type and the image of $\mathcal{T}_{C,x}$ in $\mathscr{N}_{l/X,x}$ is not contained in $\mathrm{Pos}(\mathscr{N}_{l/X})_x$*, then*

$$\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \cong \mathcal{O}^{n-2}\oplus \mathcal{O}(-1).$$

(iv) If l is of second type and the image of $\mathcal{T}_{C,x}$ in $\mathscr{N}_{l/X,x}$ is contained in $\mathrm{Pos}(\mathscr{N}_{l/X})_x$*, then*

$$\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \cong \mathcal{O}(1)\oplus \mathcal{O}^{n-3}\oplus \mathcal{O}(-2).$$

Proof. Let $r_+ = \mathrm{rk}(\mathrm{Pos}(\mathscr{N}_{l/X}))$ and $a = \dim_{\mathbb{C}} \mathrm{Im}\{\mathrm{Pos}(\mathscr{N}_{l/X}) \to \frac{\mathcal{T}_{X,x}}{\mathcal{T}_{l,x}\oplus\mathcal{T}_{C,x}}\}$. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{Q} \arrow[r] & \mathbb{C}^{n-2-a} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \arrow[r] \arrow[u] & \mathscr{N}_{l/X} \arrow[r] \arrow[u] & \frac{\mathcal{T}_{X,x}}{\mathcal{T}_{l,x}\oplus\mathcal{T}_{C,x}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}(1)^{r_+-a}\oplus\mathcal{O}^a \arrow[r] \arrow[u] & \mathrm{Pos}(\mathscr{N}_{l/X}) \arrow[r] \arrow[u] & \mathbb{C}^a \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where all columns and rows are short exact. Note that $\mathscr{Q} = \mathcal{O}^2$ if l is of first type and $\mathscr{Q} = \mathcal{O}(-1)$ if l is of second type.

If l is of first type and $\mathcal{T}_{C,x}$ is not contained in $\mathrm{Pos}(\mathscr{N}_{l/X})$, then $\mathrm{Pos}(\mathscr{N}_{l/X})_x \to \frac{\mathcal{T}_{X,x}}{\mathcal{T}_{l,x}\oplus\mathcal{T}_{C,x}}$ is injective and hence $a = n-3$. Since $\mathscr{Q} \cong \mathcal{O}^2$, the top row of the above diagram gives $\mathscr{G} \cong \mathcal{O}\oplus\mathcal{O}(-1)$. It follows from the first column that $\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \cong \mathcal{O}^{n-2}\oplus\mathcal{O}(-1)$ since $r_+ = n-3$.

If l is of first type and $\mathcal{T}_{C,x}$ is contained in $\mathrm{Pos}(\mathscr{N}_{l/X})$, then $\mathrm{Pos}(\mathscr{N}_{l/X})_x \to \frac{\mathcal{T}_{X,x}}{\mathcal{T}_{l,x}\oplus\mathcal{T}_{C,x}}$ has a 1-dimensional kernel and hence $a = n-4$. Since $\mathscr{Q} \cong \mathcal{O}^2$, the top row of the above diagram gives $\mathscr{G} \cong \mathcal{O}(-1)^2$. It follows from the first column that $\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \cong \mathcal{O}(1)\oplus\mathcal{O}^{n-4}\oplus\mathcal{O}(-1)^2$ since $r_+ = n-3$.

If l is of second type and $\mathcal{T}_{C,x}$ is not contained in $\mathrm{Pos}(\mathscr{N}_{l/X})$, then $\mathrm{Pos}(\mathscr{N}_{l/X})_x \to \frac{\mathcal{T}_{X,x}}{\mathcal{T}_{l,x}\oplus\mathcal{T}_{C,x}}$ is injective and hence $a = n-2$. Since $\mathscr{Q} \cong \mathcal{O}(-1)$, the top row of the above diagram gives $\mathscr{G} \cong \mathcal{O}(-1)$. It follows from the first column that $\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \cong \mathcal{O}^{n-2}\oplus\mathcal{O}(-1)$ since $r_+ = n-2$.

If l is of second type and $\mathcal{T}_{C,x}$ is contained in $\mathrm{Pos}(\mathscr{N}_{l/X})$, then $\mathrm{Pos}(\mathscr{N}_{l/X})_x \to \frac{\mathcal{T}_{X,x}}{\mathcal{T}_{l,x}\oplus\mathcal{T}_{C,x}}$ has a 1-dimensional kernel and and hence $a = n-3$. Since $\mathscr{Q} \cong \mathcal{O}(-1)$, the top row of the above diagram gives $\mathscr{G} \cong \mathcal{O}(-2)$. It follows from the first column that $\mathscr{N}_{l/X}\langle \mathcal{T}_{C,x}\rangle \cong \mathcal{O}(1)\oplus\mathcal{O}^{n-4}\oplus\mathcal{O}(-2)$ since $r_+ = n-2$. $\square$

Corollary 3.12. *S_C is singular at $([l], x)$ if and only if l is of second type and the image of $\mathcal{T}_{C,x}$ in $\mathscr{N}_{l/X,x}$ is contained in* $\mathrm{Pos}(\mathscr{N}_{l/X})_x$.

Definition 3.13. The curve C has a *bad direction* at a point $x \in C$ if there exists a line l of second type through x such that the image of $\mathcal{T}_{C,x}$ in $\mathscr{N}_{l/X,x}$ is contained in $\mathrm{Pos}(\mathscr{N}_{l/X})_x$. Otherwise, we say that C has a *good direction* at x.

Lemma 3.14. *If C is general, then C has good directions everywhere.*

Proof. Given a line $l \subset X$ of second type and a point $x \in l$, the positive part of $\mathscr{N}_{l/X}$ together with l itself determines an $n-1$ dimensional linear subspace $\mathcal{P}_{l,x}$ of $\mathcal{T}_{X,x}$. Let $F_0 \subset F$ be the closed subscheme of lines of second type on X. By [CG, Corollary 7.6], we know that $\dim F_0 \leq n-2$. Let $D_2 \subset X$ be the locus swept by all lines of second type. Then $\dim D_2 \leq n-1$. Let C be a general rational curve on X. Then C can meet D_2 in at most finitely many points x_i, $i = 1, \dots, m$ and through each point x_i, there is a unique line l_i of second type. The condition that C has a good direction at x_i is equivalent to $\mathcal{T}_{C,x_i} \notin \mathcal{P}_{l_i,x_i}$. We first note that it is proved in [Iz, Lemma 1.4] that a general line has good directions everywhere. It is proved in [CS] that the moduli space of degree e rational curves on X is irreducible. Hence to prove the lemma, we only need to construct one such curve C whose tangent directions are all good. This can be obtained by smoothing a chain of lines as follows. Take a chain of lines $L = \cup_{j=1}^{e} L_j$ such that (i) each line L_j is of first type and has good directions everywhere and (ii) none of the nodes of L is on a line of second type. Since all the components L_j are of first type, the chain L is smoothable; see [Ko, Chapter II, Theorem 7.9]. Then a general smoothing of L has good directions everywhere since having a bad direction is a closed condition. $\square$

Lemma 3.15. *Let $l \subset X$ be a line of first type and $x \in l$ be a point, then F_x is smooth at $[l]$.*

Proof. Deformation theory gives us

$$\mathcal{T}_{F_x,[l]} = \mathrm{H}^0(l, \mathscr{N}_{l/X} \otimes \mathcal{O}_l(-x)) = \mathrm{H}^0(l, \mathcal{O}^{n-3} \oplus \mathcal{O}(-1)^2) \cong \mathbb{C}^{n-3}.$$

Hence $\dim \mathcal{T}_{F_x,[l]} = n-3 = \dim F_x$. It follows that F_x is smooth at $[l]$. $\square$

Proof of Lemma 3.1. We first deal with the case when C is a general line. In this case, we see from [Iz, Lemma 1.4] that F_C is smooth away from the point $[C]$ and hence so is $S_C - p^{-1}[C]$. Thus the singularities of S_C can only appear on $\tilde{C} = p^{-1}[C] \cong C$. Recall that we have the natural morphism $q_0 = q|_{S_C} : S_C \to C$ and $\tilde{C}$ is a section of q_0. If $q_0^{-1}y$ is smooth at the point $\tilde{y} = \tilde{C} \cap q_0^{-1}y$ for some $y \in C$, then S_C is smooth at $\tilde{y}$. Note that if we identify $q_0^{-1}y$ with F_y, then $\tilde{y} = [C] \in F_y$. It follows that singularities of S_C can only appear at the points $\tilde{y} \in \tilde{C}$ such that F_y is singular at the point $[C]$, where $y = q_0(\tilde{y}) \in C$. But this can never happen if C is of first type by Lemma 3.15.
Assume that $e \geq 2$. By Corollary 3.12, we only need to show that when C is general, it does not have a bad direction, which is established in Lemma 3.14. $\square$

3.2. FIRST ORDER DEFORMATIONS OF C.

DEFINITION 3.16. Let $C \subset X$ be a general smooth rational curve of degree $e \geq 2$. Then C is free since it passes through a general point of X; see [Ko, Chapter II, Theorem 3.11]. In other words, $\mathscr{N}_{C/X}$ is globally generated. Let $v \in \mathrm{H}^0(C, \mathscr{N}_{C/X})$ be a general section of the normal bundle. Then v is nowhere vanishing since $\mathscr{N}_{C/X}$ is globally generated with rank $n-1 \geq 2$. We define the rank 2 subbundle $\mathcal{T}_v \subset \mathcal{T}_X|_C$, associated to v, to be such that

$$\mathcal{T}_{v,x} = \{\tau \in \mathcal{T}_{X,x} : \bar{\tau} \in \mathbb{C}v \subset \mathscr{N}_{C/X,x} = \frac{\mathcal{T}_{X,x}}{\mathcal{T}_{C,x}}\}$$

for all $x \in C$. Let

$$\Sigma_v = \{([l], x) \in S_C : \mathcal{T}_{l,x} \subset \mathcal{T}_{v,x}\}.$$

REMARK 3.17. Note that we have natural embeddings

$$S_C \xhookrightarrow{j_1} \mathbb{P}(\mathcal{T}_X|_C) \xhookleftarrow{j_2} \mathbb{P}(\mathcal{T}_v),$$

where j_1 is defined by $([l], x) \mapsto \{\mathcal{T}_{l,x} \subset \mathcal{T}_{X,x}\}$ and j_2 is induced by the natural subbundle structure $\mathcal{T}_v \subset \mathcal{T}_X|_C$. We know that $\dim S_C = n-2$, $\dim \mathbb{P}(\mathcal{T}_X|_C) = n$ and $\dim \mathbb{P}(\mathcal{T}_v) = 2$. Furthermore, the set Σ_v is precisely the intersection of $\mathbb{P}(\mathcal{T}_v)$ and S_C.

LEMMA 3.18. *If v is general, then S_C intersect $\mathbb{P}(\mathcal{T}_v)$ transversally in the set $\Sigma_v = \{([L_i], x_i)\}_{i=1}^{r_0}$, where $x_i \neq x_j$ for $i \neq j$. In particular, Σ_v is finite.*

Proof. By [CG, Lemma 6.5], we see that S_C is a fiberwise (2,3) complete intersection of $\mathbb{P}(\mathcal{T}_X|_C)$. In other words, for each point $x \in C$, there exist

$$\phi_{2,x} : \mathrm{Sym}^2(\mathcal{T}_{X,x}) \to \mathbb{C} \quad \text{and} \quad \phi_{3,x} : \mathrm{Sym}^3(\mathcal{T}_{X,x}) \to \mathbb{C}$$

such that $q_0^{-1}x = F_x \subset \mathbb{P}(\mathcal{T}_{X,x})$ is defined by $\phi_{2,x} = 0$ and $\phi_{3,x} = 0$. The tangent direction $\mathcal{T}_{C,x}$ defines a point $\tilde{x} = [\mathcal{T}_{C,x}] \in \mathbb{P}(\mathcal{T}_{v,x}) \subset \mathbb{P}(\mathcal{T}_{X,x})$. Lemma 3.8 implies that the point $\tilde{x}$ is not contained in F_x. Note that $\mathbb{P}(\mathcal{T}_{v,x})$ can be viewed as a line in $\mathbb{P}(\mathcal{T}_{X,x})$ passing through the point $\tilde{x}$. Since $\mathscr{N}_{C/X}$ is globally generated, we see that for a fixed $x \in C$ the line $\mathbb{P}(\mathcal{T}_{v,x})$ is a general line passing through $\tilde{x}$. Now since $F_x \subset \mathbb{P}(\mathcal{T}_{X,x})$ has codimension 2, the line does not meet F_x. This shows that when v is general, there are at most finitely many points $x_i \in C$ such that F_{x_i} meets $\mathbb{P}(\mathcal{T}_{v,x_i})$. For any of such point x_i, since $\tilde{x}_i \notin F_x$, the line $\mathbb{P}(\mathcal{T}_{v,x_i})$ meets F_{x_i} in at most finitely many points. This proves that S_C intersects $\mathbb{P}(\mathcal{T}_v)$ transversally. We still need to rule out the situation $x_i = x_j = y$.

Let $\tilde{X} \to X$ be the blow-up of X along the curve C with $E \cong \mathbb{P}(\mathscr{N}_{C/X}) \subset \tilde{X}$ being the exceptional divisor. Consider the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"'] & \tilde{X} \arrow[d] \\ C \arrow[r] & X \end{tikzcd}$$

For a point $x \in C$ and a line L passing through x, Lemma 3.8 implies that $\mathcal{T}_{L,x}$ defines a 1-dimensional linear space in $\mathscr{N}_{C/X,x}$ and hence gives a point $z \in E_x = \pi^{-1}x$. When (L,x) varies, the point z traces out a subvariety $Z \subset E$. Note that z is simply the intersection point of the strict transform $\tilde{L}$ of L and the exceptional divisor E. This gives rise to a morphism $f_x : F_x \to E_x$. First we note that for general $x \in C$, the morphism f_x is generically 1-to-1. This can be seen as follows. When x is a general point, the tangent direction $\mathcal{T}_{C,x}$ defines a general point $\tilde{x} \in \mathbb{P}(\mathcal{T}_{X,x})$. Then for a general point $[L] \in F_x \subset \mathbb{P}(\mathcal{T}_{X,x})$, the line connecting $\tilde{x}$ and $[L]$ meets F_x in the single point $[L]$. Let $Z_1 \subset Z$ be the set of points having at least two preimages under f_x for some $x \in C$. The above discussion together with the fact $\dim F_x = n-3$ for all $x \in C$ (since C does not pass any Eckardt point) implies

$$\dim Z = n-2, \qquad \dim Z_1 \leq n-3.$$

A section $v \in \mathrm{H}^0(C, \mathscr{N}_{C/X})$ defines a section $C_v \subset E$. Since $\mathscr{N}_{C/X}$ is globally generated, the curves $\{C_v\}$ form a covering family of movable curves on E. Note that $\dim E = n-1$. It follows that for a general v, the curve C_v does not meet Z_1.

If $x_i = x_j = y$, then we have two distinct lines L_i and L_j passing through y such that $\mathcal{T}_{L_i,y}$, $\mathcal{T}_{L_j,y}$ and $v(y)$ give the same point $z \in \mathbb{P}(\mathscr{N}_{C/X,x})$. This means that C_v meets Z_1 at the pint z. But this does not happen for general v by the above discussion. □

Remark 3.19. The element $v \in \mathrm{H}^0(C, \mathscr{N}_{C/X})$ can be viewed as a first order deformation of C. Note that we have a canonical identification

$$\mathcal{T}_{\mathrm{Hilb}(X),[C]} = \mathrm{H}^0(C, \mathscr{N}_{C/X}).$$

Since $[C] \in \mathrm{Hilb}^e(X)$ is a smooth point, we can find a smooth pointed curve $\varphi : (T,0) \longrightarrow (\mathrm{Hilb(X)}, [C])$ such that $d\varphi(\mathcal{T}_{T,0}) = \mathbb{C}\, v$. The curve T parameterizes a 1-dimensional family of curves on X,

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & X \\ T & \end{tikzcd}$$

such that $\mathscr{C}_0 = C$. Let $\Sigma_t = \{L_{i,t}\}$, $t \neq 0$, be the set of all incidence lines of $\mathscr{C}_t$ and C (meaning lines meeting both $\mathscr{C}_t$ and C). We have proved in [Sh, Lemma 3.10] that the cardinality of Σ_t is equal to $5\deg(\mathscr{C}_t)\deg(C) = 5e^2$. If we take the limit as $t \to 0$ (see [Fu, §11.1]), then we see that Σ_t specializes to the set Σ_v. Hence we get $r_0 = |\Sigma_v| = 5e^2$. We will write

$$\Sigma_v = \{([L_i], x_i) : i = 1, 2, \ldots, 5e^2\}.$$

Lemma 3.20. *If C is general, then L_i is of first type and*

$$\mathscr{N}_{L_i/X}\langle \mathcal{T}_{C,x_i}\rangle \cong \mathcal{O}^{n-2} \oplus \mathcal{O}(-1), \quad 1 \leq i \leq 5e^2.$$

Furthermore, we have a decomposition

$$\mathrm{NN}(\mathscr{N}_{L_i/X}\langle \mathcal{T}_{C,x_i}\rangle)_{x_i} = \mathrm{Pos}(\mathscr{N}_{L_i/X})_{x_i} \oplus \mathcal{T}_{C,x_i}, \tag{6}$$

which is canonical upto a scalar multiplication on $\mathrm{Pos}(\mathscr{N}_{L_i/X})_{x_i}$.

Proof. When C and also v are general, all the lines L_i are of first type and $\mathcal{T}_{C,x_i}$ is not pointing to the positive part of $\mathscr{N}_{L_i/X}$. Hence (i) of Proposition 3.11 applies. Consider the following short exact sequence

$$0 \longrightarrow \mathrm{Pos}(\mathscr{N}_{L_i/X})_{x_i} \xrightarrow{\cdot t_i} \mathrm{NN}(\mathscr{N}_{L_i/X}\langle \mathcal{T}_{C,x_i}\rangle)_{x_i} \longrightarrow \overline{\mathcal{T}}_{C,x_i} \longrightarrow 0, \tag{7}$$

where t_i is a local uniformizer of C at the point x_i and $\overline{\mathcal{T}}_{C,x_i}$ is the image of $\mathcal{T}_{C,x_i}$ in $\mathscr{N}_{L_i/X,x_i}$. The natural homomorphism

$$\mathcal{T}_{C,x_i} \hookrightarrow \mathcal{T}_{X,x_i} \to \mathscr{N}_{L_i/X,x_i}$$

factors through $\mathrm{NN}(\mathscr{N}_{L_i/X}\langle \mathcal{T}_{C,x_i}\rangle)_{x_i} \longrightarrow \mathscr{N}_{L_i/X,x_i}$ and gives a homomorphism $\theta_i : \overline{\mathcal{T}}_{C,x_i} \to \mathscr{N}_{L_i/X}\langle \mathcal{T}_{C,x_i}\rangle)_{x_i}$. This homomorphism θ_i canonically splits the sequence (7). After identifying $\overline{\mathcal{T}}_{C,x_i}$ with $\mathcal{T}_{C,x_i}$, we get the decomposition. Note that the only choice made here is the local parameter t_i. A different choice of t_i induces a scalar multiplication on $\mathrm{Pos}(\mathscr{N}_{L_i/X})_{x_i}$. □

LEMMA 3.21. *Let $V \subset \mathcal{T}_{X,x}$ be a 3-dimensional vector space containing $\mathcal{T}_{v,x}$, for some $x \in C$. We view $\mathbb{P}(V)$ as the space of lines tangent to X at x in a direction of V. Then $F_x \cap \mathbb{P}(V)$ is a zero dimensional scheme of length 6 if $x \neq x_i$. If $x = x_i$ for some $1 \le i \le 5e^2$, then $F_x \cap \mathbb{P}(V)$ can have at most one component of dimension 1 and this component has to be a line (on $\mathbb{P}(V) = \mathbb{P}^2$) if it exists.*

Proof. If we show that $F_x \cap \mathbb{P}(V)$ is zero dimensional, then it of length 6 since it is a (2,3)-complete intersection in $\mathbb{P}(V) \cong \mathbb{P}^2$. Assume that $B \subset \mathbb{P}(V) \cap F_x$ is a curve. Then $B \cap \mathbb{P}(\mathcal{T}_{v,x})$ is nonempty, where $\mathbb{P}(\mathcal{T}_{v,x})$ is viewed as a line on $\mathbb{P}(V)$. Or in other words, $\mathbb{P}(\mathcal{T}_{v,x})$ contains a line L of X. This exactly means that $(L,x) \in \Sigma_v$. Hence such B does not exist if $x \neq x_i$. If $x = x_i$, we have already seen that $\mathbb{P}(\mathcal{T}_{v,x_i})$ contains the unique line L_i of X. Hence $B \subset \mathbb{P}(V)$ is of degree 1, namely a line, if it exists. This is because B can only meet the line $\mathbb{P}(\mathcal{T}_{v,x})$ in the point $[L_i]$. □

3.3. THE RATIONAL MAP ρ. Let $\mathscr{F}$ be the quotient of $\mathcal{T}_X|_C$ by $\mathcal{T}_v$. The natural quotient homomorphism $\mathcal{T}_X|_C \to \mathscr{F}$ induces a rational map $\beta : \mathbb{P}(\mathcal{T}_X|_C) \dashrightarrow \mathbb{P}(\mathscr{F})$. By definition, the indeterminacy locus of β is exactly $\mathbb{P}(\mathcal{T}_v)$. Let $\rho = \beta\circ j_1 : S_C \dashrightarrow \mathbb{P}(\mathscr{F})$, where $j_1 : S_C \hookrightarrow \mathbb{P}(T_X|_C)$ is the closed immersion induced by $([l],x) \mapsto [\mathcal{T}_{l,x} \subset \mathcal{T}_{X,x}]$; see Remark 3.17. Then the indeterminacy locus of ρ is the set Σ_v. This shows that we have a morphism

$$\rho : S_C - \Sigma_v \to \mathbb{P}(\mathscr{F}).$$

LEMMA 3.22. *Let $f : Y_1 \dashrightarrow Y_2$ be a rational map between two smooth projective varieties whose indeterminacy locus $\Sigma \subset Y_1$ is a finite set of closed points. Let $\Gamma_f \subset (Y_1 - \Sigma) \times Y_2$ be the graph of f and $\bar{\Gamma}_f \subset Y_1 \times Y_2$ the closure of Γ_f. We* *define the pull-back f^* and push-forward f_* (on either the cohomology groups or the Chow groups) as*

$$f^*\mathfrak{a} = p_{1,*}(\bar{\Gamma}_f \cdot p_2^*\mathfrak{a}), \qquad f_*\mathfrak{a} = p_{2,*}(\bar{\Gamma}_f \cdot p_1^*\mathfrak{a}),$$

where $p_i : Y_1 \times Y_2 \to Y_i$ is the projection to the corresponding factor. Then we have $f^*(\mathfrak{a}^k) = (f^*\mathfrak{a})^k$, for all $\mathfrak{a} \in \mathrm{Pic}(Y_2)$ and $k < \dim Y_1$.

Proof. Let $E = \Sigma \times Y_2 \cap \bar{\Gamma}_f$ be the exceptional set of the morphism $\bar{\Gamma}_f \to Y_1$. Let D and D_0 be two divisors on $Y_1 \times Y_2$ such that $D \cdot \bar{\Gamma}_f = D_0 \cdot \bar{\Gamma}_f + \gamma_1$ where γ_1 is a cycle supported on E. We conclude from the above that for all $1 \leq k \leq \dim Y_1$,

$$D^k \cdot \bar{\Gamma}_f = D_0^k \cdot \bar{\Gamma}_f + \gamma_k \tag{8}$$

where γ_k is a cycle supported on E. This can be proved by induction. In equation (8), we apply $p_{1,*}$ to both sides and note that $p_{1,*}\gamma_k = 0$, $1 \leq k \leq \dim Y_1 - 1$, since p_1 contracts E to points while $\dim \gamma_k \geq 1$. It follows that

$$p_{1,*}(D^k \cdot \bar{\Gamma}_f) = p_{1,*}(D_0^k \cdot \bar{\Gamma}_f), \quad 1 \leq k \leq \dim Y_1 - 1.$$

For any divisor D on $Y_1 \times Y_2$, we take $D_0 = p_1^* p_{1,*}(D \cdot \bar{\Gamma}_f)$. Since $p_1|_{\bar{\Gamma}_f} : \bar{\Gamma}_f \to Y_1$ is an isomorphism away from E, we see that $D \cdot \bar{\Gamma}_f = D_0 \cdot \bar{\Gamma}_f + \gamma_1$ for some cycle γ_1 supported on E. Hence we get

$$\begin{aligned} p_{1,*}(D^k \cdot \bar{\Gamma}_f) &= p_{1,*}(D_0^k \cdot \bar{\Gamma}_f) \\ &= p_{1,*}(p_1^*(D')^k \cdot \bar{\Gamma}_f), \qquad D' = p_{1,*}(D \cdot \bar{\Gamma}_f) \\ &= (D')^k \cdot p_{1,*}\bar{\Gamma}_f, \qquad \text{(the projection formula)} \\ &= (D')^k = (p_{1,*}(D \cdot \bar{\Gamma}_f))^k \end{aligned}$$

for all divisors D on $Y_1 \times Y_2$ and $1 \leq k \leq \dim Y_1 - 1$. In particular, if we take $D = p_2^*\mathfrak{a}$ for some $\mathfrak{a} \in \mathrm{Pic}(Y_2)$, then we get

$$f^*\mathfrak{a}^k = p_{1,*}(\bar{\Gamma}_f \cdot p_2^*\mathfrak{a}^k) = p_{1,*}(\bar{\Gamma}_f \cdot (p_2^*\mathfrak{a})^k) = (p_{1,*}(\bar{\Gamma}_f \cdot p_2^*\mathfrak{a}))^k = (f^*\mathfrak{a})^k,$$

where $1 \leq k \leq \dim Y_1 - 1$. The case $k = 0$ is automatic. $\square$

Let $\Gamma_\rho \subset (S_C - \Sigma_v) \times \mathbb{P}(\mathscr{F})$ be the graph of ρ and $\bar{\Gamma}_\rho \subset S_C \times \mathbb{P}(\mathscr{F})$ its closure. Note that for any 3-dimensional vector space $V \subset \mathcal{T}_{X,x}$ containing $\mathcal{T}_{v,x}$, the condition $(([l], x), V/\mathcal{T}_{v,x}) \in \Gamma_\rho$ implies that $l \subset \mathbb{P}(V)$. We define the closed subvariety $Z \subset S_C \times \mathbb{P}(\mathscr{F})$ as

$$Z = \{(([l], x), V/\mathcal{T}_{v,x}) \in S_C \times \mathbb{P}(\mathscr{F}) \mid l \subset \mathbb{P}(V)\}.$$

LEMMA 3.23. *$\bar{\Gamma}_\rho = Z$ in $\mathrm{CH}_{n-2}(S_C \times \mathbb{P}(\mathscr{F}))$.*

Proof. From Lemma 3.21, Z agrees with Γ_ρ away from Σ_v, namely

$$\Gamma_\rho = Z|_{(S_C - \Sigma_v) \times \mathbb{P}(\mathscr{F})}.$$

It follows that $\bar{\Gamma}_\rho \subset Z$. Hence we have

$$Z = \bar{\Gamma}_\rho + \sum_{i=1}^{5e^2} \{([L_i], x_i)\} \times \mathbb{P}(\mathscr{F}_{x_i}), \qquad \text{in } \mathrm{CH}_{n-2}(S_C \times \mathbb{P}(\mathscr{F})).$$

Since $\dim \mathbb{P}(\mathscr{F}_{x_i}) = n-3$, we see that $\{([L_i], x_i)\} \times \mathbb{P}(\mathscr{F}_{x_i}) = 0$ in $\mathrm{CH}_{n-2}(S_C \times \mathbb{P}(\mathscr{F}))$. Hence the lemma follows. $\square$

DEFINITION 3.24. Let $\Gamma_v \subset S_C \times S_C$ be the closure of $(S_C\backslash\Sigma_v) \times_{\mathbb{P}(\mathscr{F})} (S_C\backslash\Sigma_v)$.

REMARK 3.25. Let $([l], x) \in S_C$ be a general point. Then $\mathcal{T}_{l,x}$ and $\mathcal{T}_{v,x}$ span a linear 3-dimensional vector space $V \subset \mathcal{T}_{X,x}$ and $\rho([l], x) = [V/\mathcal{T}_{v,x}]$. Then by Lemma 3.21, there are 5 other lines l_i, $i = 1, \ldots, 5$, such that $\mathcal{T}_{l_i,x} \subset V$. Then Γ_v is the correspondence generically defined by

$$([l], x) \mapsto \sum_{i=0}^{5} ([l_i], x),$$

where $l_0 = l$.

LEMMA 3.26. *$\Gamma_v = \bar{\Gamma}_\rho^t \circ \bar{\Gamma}_\rho$ in $\mathrm{CH}_{n-2}(S_C \times S_C)$.*

Proof. By Lemma 3.23, we may replace $\bar{\Gamma}_\rho$ by Z. By the definition of composition of correspondences (see [Fu, §16.1]), we know that

$$\begin{aligned} Z^t \circ Z &= p_{13,*}(p_{12}^* Z \cdot p_{23}^* Z^t) \\ &= p_{13,*}(Z \times S_C \cap S_C \times Z^t) \end{aligned}$$

It follows that $Z^t \circ Z$ is represented by the cycle

$$\{(([l], x), ([l'], x)) \in S_C \times S_C : [l], [l'] \in F_x \cap \mathbb{P}(V) \text{ for some } [V/\mathcal{T}_{v,x}] \in \mathbb{P}(\mathscr{F}_x)\}$$

One directly checks that this cycle is Γ_v. $\square$

LEMMA 3.27. *Let Γ_v act either on the cohomology groups or on the Chow groups of S_C.*
(i) If $\mathfrak{a}$ is an odd dimensional topological cycle, then $(\Gamma_v)_*\mathfrak{a} = 0$;
(ii) If $\mathfrak{a}$ is zero dimensional, then $(\Gamma_v)_*\mathfrak{a}$ is a constant that only depends on the degree of $\mathfrak{a}$;
(iii) Let $\mathfrak{a} = [S_C]$, then $(\Gamma_v)_*\mathfrak{a} = 6\mathfrak{a}$;
(iv) If $\mathfrak{a}$ is a topological cycle of real codimension $2m$ or an algebraic cycle of codimension m, then $(\Gamma_v)_*\mathfrak{a}$ is a linear combination of g^m and $g'g^{m-1}$ which only depends on the intersection numbers $\mathfrak{a} \cdot g^{n-m-2}$ and $\mathfrak{a} \cdot g'g^{n-m-3}$.

Proof. By Lemma 3.26, we know that

$$(\Gamma_v)_* = (\bar{\Gamma}_\rho^t \circ \bar{\Gamma}_\rho)_* = (\bar{\Gamma}_\rho^t)_*(\bar{\Gamma}_\rho)_* = \rho^*\rho_*.$$

Let ξ be the relative $\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1)$ class on $\mathbb{P}(\mathscr{F})$. Then the cohomology of $\mathbb{P}(\mathscr{F})$ is naturally given by

$$\mathrm{H}^*(\mathbb{P}(\mathscr{F})) = \mathrm{H}^*(C)[\xi], \quad \xi^{n-2} + \pi^* c_1(\mathscr{F}) \cdot \xi^{n-3} = 0.$$

Where $\pi : \mathbb{P}(\mathscr{F}) \to C$ is the natural projection. Similarly, we have the description of the Chow ring as

$$\mathrm{CH}^*(\mathbb{P}(\mathscr{F})) = \mathrm{CH}^*(C)[\xi], \quad \xi^{n-2} + \pi^* c_1(\mathscr{F}) \cdot \xi^{n-3} = 0.$$

Consider the following diagram

$$\begin{tikzcd} S_C \arrow[r, "j_1"] \arrow[dr, dashed, "\rho"'] & \mathbb{P}(\mathcal{T}_X|_C) \arrow[r, "\alpha"] \arrow[d, dashed, "\beta"] & C \\ & \mathbb{P}(\mathscr{F}) \arrow[ur, "\pi"'] & \end{tikzcd}$$

The rational map β is defined by the natural homomorphism $\mathcal{T}_X|_C \to \mathscr{F}$. The locus where β is not defined is exactly $\mathbb{P}(\mathcal{T}_v)$. The pull back $\beta^*\xi$ restricts to a hyperplane class on each fiber of α. The variety $\mathbb{P}(\mathcal{T}_X|_C)$ parameterizes all lines in $\mathbb{P}^{n+1}$ which are tangent to X at some point $x \in C$ (remembering the point x). Let $\tilde{g} \subset \mathbb{P}(\mathcal{T}_X|_C)$ be the divisor corresponding to all the lines meeting a given linear $\mathbb{P}^{n-1} \subset \mathbb{P}^{n+1}$ in general position. Then we see that $i^*\tilde{g} = g|_{S_C}$ and this is denoted again by g for simplicity. One also sees that $\tilde{g}$ restricts to a hyperplane class of $\mathbb{P}(\mathcal{T}_{X,x}) = \alpha^{-1}(x)$. Thus $\tilde{g}$ and $\beta^*\xi$ differ by a class coming from C. After pulling back to S_C, we get

$$\rho^*\xi = g + rg'$$

for some integer r. By Lemma 3.22, we get the following key equality

$$\rho^*\xi^k = (\rho^*\xi)^k = (g + rg')^k \tag{9}$$

Let $\mathfrak{a} \in \mathrm{H}^{2m+1}(S_C)$ be a topological class of odd dimension, then $\rho_*\mathfrak{a}$ is an element in $\mathrm{H}^{2m+1}(\mathbb{P}(\mathscr{F}))$ which is zero. Hence $(\Gamma_v)_*(\mathfrak{a}) = \rho^*\rho_*\mathfrak{a} = 0$. Now let $\mathfrak{a}$ be an element of $\mathrm{H}^{2m}(S_C)$ (or $\mathrm{CH}^m(S_C)$), $0 < m < n-2$ (the cases of $m = 0, n-2$ are quite easy to deal with). Then $\rho_*\mathfrak{a}$ is an element in $\mathrm{H}^{2m}(\mathbb{P}(\mathscr{F}))$ (or $\mathrm{CH}^m(\mathbb{P}(\mathscr{F}))$). Then we have the following expression

$$\rho_*\mathfrak{a} = a\xi^m + b\pi^*[pt] \cdot \xi^{m-1},$$

for some $a, b \in \mathbb{Z}$. Apply ρ^* to the above identity and use Lemma 3.22, we get

$$(\Gamma_v)_*(\mathfrak{a}) = \rho^*\rho_*\mathfrak{a} = a(g + rg')^m + bg'(g + rg')^{m-1} = ag^m + (b + ma)g^{m-1}g'$$

Also, the numbers a and b can be determined in the following way

$$\begin{aligned} a &= (a\xi^m + b\pi^*[pt]\xi^{m-1}) \cdot \pi^*[pt]\xi^{n-m-3} \\ &= \rho_*\mathfrak{a} \cdot \pi^*[pt]\xi^{n-m-3} \\ &= \mathfrak{a} \cdot \rho^*(\pi^*[pt]\xi^{n-m-3}) \\ &= \mathfrak{a} \cdot g'(g + rg')^{n-m-3}, \qquad \text{(by Lemma 3.22)} \\ &= \mathfrak{a} \cdot g'g^{n-m-3} \end{aligned}$$

To get b, we consider

$$\begin{aligned} \rho_*\mathfrak{a} \cdot \xi^{n-m-2} &= a\xi^{n-2} + b \\ &= a(-\pi^*c_1(\mathscr{F}))\xi^{n-3} + b \\ &= -a \deg \mathscr{F} + b \end{aligned}$$

Hence we have

$$\begin{aligned} b &= a \deg \mathscr{F} + \mathfrak{a} \cdot \rho^* \xi^{n-m-2} \\ &= a \deg(\mathscr{F}) + \mathfrak{a} \cdot (g + rg')^{n-m-2} \end{aligned}$$

Thus b only depends on the intersection numbers $\mathfrak{a} \cdot g^{n-m-2}$ and $\mathfrak{a} \cdot g' g^{n-m-3}$. This finishes the proof. □

3.4. Proof of Theorem 3.6. Let $T \subset \mathrm{Hilb}^e(X)$ be a general smooth curve passing through the point $[C]$. Let $\{\mathscr{C}_t : t \in T\}$ be the corresponding 1-dimensional family of rational curves on X such that the fiber $\mathscr{C}_0 = C$ at the special point $0 = [C] \in T$. Let $S_t = S_{\mathscr{C}_t}$ and $i_t = i_{\mathscr{C}_t} : S_t \to F$. Let $I'_t \subset S_t \times S_C$, $t \neq 0$, be the natural incidence correspondence, namely

$$I'_t = \{(([l'], x'), ([l], x)) \in S_t \times S_C : l \cap l' \neq \emptyset\} = (i_t \times i_C)^* I \in \mathrm{CH}_{n-2}(S_t \times S_C).$$

Let $I_t \subset S_t \times S_C$ be the correspondence generically defined by

$$I_t : ([l'], x') \mapsto \sum_{i=1}^{5e} ([l'_i], x'_i), \tag{10}$$

where $([l]', x') \in S_t$ is a general point, l'_i the incidence lines of l' and C, $x'_i = l'_i \cap C$.

Lemma 3.28. *We have $I_t = I'_t$ in $\mathrm{CH}_{n-2}(S_t \times S_C)$.*

Proof. Let $\tilde{I}_t = (i_t \times i_C)^{-1} I$. By definition, I_t is equal to the cycle class of $\tilde{I}_t$. Consider the projection $p_1 : \tilde{I} \to S_t$. Let $([l'], x') \in S_t$ be any point. If l' does not meet C, then $p_1^{-1}([l'], x') = \sum_{i=1}^{5e} ([l_i], x_i)$ where l_i run through all the incidence lines of l' and C; if $l' \cap C = y$, then $p_1^{-1}([l'], x')$ is the union of $F_y = q_0^{-1} y$ and the set of all secant lines of the pair (l', C). It follows that

$$\tilde{I}_t = I'_t \cup (\bigcup_{j=1}^{5e^2} \{[L_{j,t}], x_{j,t})\} \times F_{y_{j,t}}),$$

where $\{L_{j,t}\}_{j=1}^{5e^2}$ is the set of incidence lines of $\mathscr{C}_t$ and C, $x_{j,t} = L_{j,t} \cap \mathscr{C}_t$ and $y_{j,t} = L_{j,t} \cap C$. Since $\dim F_{y_{j,t}} = n-3$, we see that the cycle class of $\tilde{I}_t$ is equal to that of I'_t. □

Back to the proof of Theorem 3.6. In equation (10), when t specializes to 0, l' specializes to a line l meeting C; there are 5 lines, say $l'_1, \ldots, l'_5$, among all the l'_i's, that specialize to five lines, E_i, $i = 1, 2, \ldots, 5$, passing through $x = l \cap C$; all the other lines of the l'_i's specialize to secant lines, l_i, of the pair (l, C). We have the following description of the E_i's.

Since T is a smooth curve in the Hilbert scheme of degree e rational curves on X, the tangent space $\mathcal{T}_{T,0}$ gives a one dimensional subspace of $\mathcal{T}_{\mathrm{Hilb}^e(X),[C]} = \mathrm{H}^0(C, \mathscr{N}_{C/X})$. Let $v \in \mathrm{H}^0(C, \mathscr{N}_{C/X})$ be a generator of $\mathcal{T}_{T,0}$. By choosing T general enough, we may assume that v, as a section of $\mathrm{H}^0(C, \mathscr{N}_{C/X})$, is

general. To this v, we can associate $\mathcal{T}_v \subset \mathcal{T}_X|_C$ with quotient $\mathscr{F}$, $\Sigma_v \subset S_C$ and $\rho : S_C \backslash \Sigma_v \to \mathbb{P}(\mathscr{F})$ as before.
We use the notation and results of [Fu, §11.1]. Let $I_0 = \lim_{t\to 0} I_t$. In the equation (10), we take the limit as $t \to 0$, then we get

$$I_0 : ([l], x) \mapsto \sum_{i=6}^{5e}([l_i'], x_i) + \sum_{i=1}^{5}([E_i], x), \tag{11}$$

where $\{l_i'\}_{i=6}^{5e}$ is the set of all secant lines of the pair (l, C) and $x_i = l_i' \cap C$. Note that the rule in (11) generically defines $I_C + \Gamma_v - \Delta_{S_C}$. We see that the difference of I_0 and $I_C + \Gamma_v - \Delta_{S_C}$ is supported on $\Sigma_v \times_C \mathbb{P}(\mathscr{F})$. Since $\dim \Sigma_v \times_C \mathbb{P}(\mathscr{F}) = n - 3$, we have

$$I_0 = I_C + \Gamma_v - \Delta_{S_C}, \quad \text{in } \mathrm{CH}_{n-2}(S_C \times S_C). \tag{12}$$

Note that $I_t = I|_{S_t \times S_C}$ for $t \neq 0$. By taking limits and applying [Fu, Proposition 11.1], we know that the class of I_0 is equal to I restricted to $S_C \times S_C$. Equivalently, we have $I_0 = I_C'$ in $\mathrm{CH}_{n-2}(S_C \times S_C)$. Thus we get the following key identity

$$I_C' = I_C + \Gamma_v - \Delta_{S_C}. \tag{13}$$

Since $\Psi_C = \Psi \circ (i_C)_*$, $\Phi_C = (i_C)^* \circ \Phi$ and $\Phi \circ \Psi = I_*$, we see that

$$\Phi_C \circ \Psi_C = (i_C)^* \circ I_* \circ (i_C)_* = ((i_C \times i_C)^* I)_* = (I_C')_*$$

Combine this with the equation (13), we obtain

$$\Phi_C \circ \Psi_C = \sigma - 1 + (\Gamma_v)_*$$

as actions on either the cohomology groups or the Chow groups of S_C. Then Theorem 3.6 follows easily from Lemma 3.27.

3.5. THE SUBALGEBRA $\mathbb{Q}[g, g']$. On S_C we have the polarization g and the class $g' = q_0^*[pt]$, where $q_0 = q|_{S_C} : S_C \to C$ is the natural morphism.

LEMMA 3.29. $\Psi_C(g' \cdot \Phi_C(h^m)) = 2h^{m+1}$.

Proof. First we note that $\Psi_C(g' \cdot \Phi_C(h^m)) = \Psi(F_x \cdot \Phi(h^m))$. Let $M \subset X$ be a general complete intersection of hyperplanes that represents h^m. Choose $x \in X$ to be a general point. Similar to the proof of Theorem 2.2, we consider the following diagram

$$\begin{tikzcd}
D_1 + D_2 + D \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d] \\
P \arrow[r] \arrow[d, "\pi"] & \mathrm{G}(1, 2, n+2) \arrow[r] \arrow[d] & \mathbb{P}^{n+1} \\
M \arrow[r, "\varphi"] & \mathrm{G}(2, n+2) &
\end{tikzcd}$$

where all the squares are fiber products; $\varphi(x')$ is the line through x and x', for all $x' \in M$; $D_1 \subset P$ is a section of π and the natural morphism $g_1 : D_1 \to$

X contracts D_1 to $x \in X$; D_2 is a section of π and the natural morphism $g_2 : D_2 \to X$ maps D_2 isomorphically to $M \subset X$; D corresponds to the third point in the intersection of the lines with X. Let $g : D \to X$ be the natural morphism. Let $\Delta \subset M$ be the intersection of M and D_x (recall that D_x is the variety swept out by lines through x). Then, by Bertini's theorem, Δ is smooth of codimension 2 in M. As in Lemma 2.8, the natural map $\pi' : D \to M$ is the blow-up along Δ. Let $E \subset D$ be the exceptional divisor. As before, we have

$$h|_D + E = 2(\pi|_D)^*(h|_M) \tag{14}$$

where $h|_D = g^*h$. One checks that the push-forward of $\pi^*(h|_M)|_{(D_1+D_2+D)}$ to X is equal to $3h^{m+1}$ and that

$$(g_2)_*(\pi^*(h|_M)|_{D_2}) = M \cdot h = h^{m+1}$$
$$(g_1)_*(\pi^*(h|_M)|_{D_1}) = 0$$

It follows that

$$g_*(\pi^*(h|_M)|_D) = 3h^{m+1} - h^{m+1} = 2h^{m+1}.$$

Note that the push-forward of $D + D_1 + D_2$ to X is $P|_X = 3h^m$. We also know that $(g_1)_*D_1 = 0$ and $(g_2)_*D_2 = M = h^m$. Hence we get $g_*D = 2h^m$. As a consequence, we have $g_*(h|_D) = g_*D \cdot h = 2h^{m+1}$. We apply g_* to the equality (14) and note that $g_*E = \Psi(F_x \cdot \Phi(h^m))$. Then we get

$$\Psi(F_x \cdot \Phi(h^m)) = 2g_*(\pi'^*h|_M) - g_*(g^*h) = 2(2h^{m+1}) - 2h^{m+1} = 2h^{m+1}.$$

Hence the lemma follows. $\square$

PROPOSITION 3.30. *The following statements hold in* $\mathrm{H}^*(S_C, \mathbb{Q})$.
(1) g^m *is a linear combination of* $\Phi_C(h^{m+1})$ *and* $g'\Phi_C(h^m)$; $g'g^{m-1}$ *is a multiple of* $g'\Phi_C(h^m)$. *The algebra* $\mathbb{Q}[g, g']$ *is generated by cycles of the form* $\Phi_C(h^{m+1})$ *with* $0 \le m \le n-2$ *and* $g'\Phi_C(h^m)$ *with* $1 \le m \le n-2$.
(2) The restriction of $\mathrm{H}^*(G)$ *to* $\mathrm{H}^*(S_C)$ *is equal to* $\mathbb{Q}[g, g'g] \subset \mathbb{Q}[g, g']$.
(3) The action of σ *preserves the algebra* $\mathbb{Q}[g, g']$.
(4) Under Ψ_C, *the image of* $\mathbb{Q}[g, g']$ *is* $\mathbb{Q}[h]_{0<\deg<n}$ *where* h *is the class of a hyperplane on* X.

Proof. Before proving the proposition, we review a bit of Schubert calculus. We refer to §14.7 of [Fu] for the details of the general theory. Schubert calculus on $G = \mathrm{G}(2, n+2)$ shows that $\mathrm{CH}^m(G) = \mathrm{H}^{2m}(G)$ is generated by Schubert varieties that are defined by flags $\mathbb{P}^a \subset \mathbb{P}^b \subset \mathbb{P}^{n+1}$ with $a < b \le n+1$ and $a + b = 2n + 1 - m$. A Schubert variety is the space of lines meeting $\mathbb{P}^a$ and contained in $\mathbb{P}^b$. In our case, most of these classes restrict to 0 on S_C unless $b = n$ or $b = n+1$. If $b = n+1$ and $a = n - m$, where $0 \le m \le n-1$, then the corresponding Schubert class restricts to $\Phi(h^{m+1})$ on F and hence restricts to $\Phi_C(h^{m+1})$ on S_C. If $b = n$ and $a = n - m + 1$,where $2 \le m \le n$, then the corresponding Schubert class restricts to $\deg(C)\, g' \cdot \Phi_C(h^m)$ on S_C. Here we note that $F_y \subset S_C$ represents g' on S_C, for any $y \in C$. As a cycle on F, the class of F_y comes from the restriction of some class from G. However, viewed as a class on S_C, F_y is not the restriction of any class from G. The degree of a nonzero homogeneous element of $\mathbb{Q}[g, g']$ means the degree of that element as a polynomial.

Since g^m comes from G, we know that it can be written as a linear combination of $\Phi_C(h^{m+1})$ and $g'\Phi_C(h^m)$. Hence we also see that $g'g^{m-1}$ is a multiple of $g'\Phi_C(h^m)$ since $(g')^2 = 0$. This proves (1).

The discussion of Schubert calculus above shows that the restriction of $\mathrm{H}^*(G)$ to S_C is generated by $\Phi_C(h^{m+1})$, $0 \le m \le n-2$ and $g'\Phi_C(h^m)$, $2 \le m \le n-2$. Combining this fact with (1), we see that the image of the restriction $\mathrm{H}^*(G) \to \mathrm{H}^*(S_C)$ consists of elements $f \in \mathbb{Q}[g, g']$ such that the coefficient of the term g' in f is zero. This proves that the image of $\mathrm{H}^*(G) \to \mathrm{H}^*(S_C)$ is $\mathbb{Q}[g, g'g]$.

To prove (3), we only need to consider the action of σ on $\Phi_C(h^{m+1})$ and $g'\Phi_C(h^m)$. By Proposition 3.6, the following equations hold modulo $\mathbb{Q}[g, g']$.

$$\begin{aligned}
&\sigma(\Phi_C(h^{m+1})) = \\
&\qquad = \Phi_C(h^{m+1}) + \Phi_C\Psi_C(\Phi_C(h^{m+1})) \\
&\qquad = \Phi_C \circ \Psi(\Phi(h^{m+1}) \cdot F_C) \\
&\qquad = \begin{cases} \Phi_C(3e\deg(h^{m+1})h^{m+1} - 2eh^{m+1}), & m < n-2, \\ \Phi_C(3e\deg(h^{m+1})h^{m+1} - 2eh^{m+1} - 2\deg(h^{n-1})C), & m = n-2 \end{cases} \\
&\qquad = \begin{cases} 0, & m < n-2 \\ -6\Phi_C(C), & m = n-2 \end{cases}
\end{aligned}$$

Here the first equation uses Proposition 3.6; the second equation uses the fact that $\Phi_C(h^{m+1}) \in \mathbb{Q}[g, g']$; the third equation uses Theorem 2.2; the last one uses the fact that the degree of h^m with respect to the polarization h is 3. In the case $m = n-2$, we have that the cohomology class of $\Phi_C(C)$ is $5e^2[pt] \in \mathrm{H}^{2n-4}(S_C, \mathbb{Q})$. Note that $\mathrm{H}^{2n-4}(S_C, \mathbb{Q}) = \mathbb{Q}[pt] = \mathbb{Q}[g, g']_{\deg = n-2}$. Hence we also have $\Phi_C(C) \in \mathbb{Q}[g, g']$. Similarly, the following equalities hold modulo $\mathbb{Q}[g, g']$.

$$\begin{aligned}
\sigma(g' \cdot \Phi_C(h^m)) &= \Phi_C \circ \Psi_C(g' \cdot \Phi_C(h^m)) + g' \cdot \Phi_C(h^m) \\
&= 2\Phi_C(h^{m+1}) + g'\Phi_C(h^m)
\end{aligned}$$

where we use Lemma 3.29.

To prove (4), we also consider the action of Ψ_C on $\Phi_C(h^{m+1})$ and $g'\Phi_C(h^m)$. By Theorem 2.2 and Lemma 3.29, we know that the images are multiples of h^{m+1}, $0 \le m \le n-2$ (we have this range since otherwise the cycles on S_C are automatically zero for dimension reasons). □

4. The quadratic relation and Prym-Tjurin construction

Let $C \subset X$ be a general smooth rational curve on X with $e = \deg(C) \ge 2$. Let $S_C = q^{-1}(C)$. As was shown in Lemma 3.1, S_C is smooth and it is the normalization of the variety of all lines meeting C. We will use the notation from the previous section.

DEFINITION 4.1. We define the *primitive cohomology*, $\mathrm{H}^*(S_C,\mathbb{Z})^\circ$, of S_C, to be the set of all elements $\alpha\in \mathrm{H}^*(S_C,\mathbb{Z})$ such that

$$\langle\alpha,\beta\rangle := \int_{S_C}\alpha\cup\beta = 0,\quad \forall\beta\in\mathbb{Q}[g,g'].$$

We define the *primitive Chow group*, $\mathrm{CH}^*(S_C)^\circ$, of S_C , to be the set of all elements $\alpha\in\mathrm{CH}^*(S_C)$ such that

$$\langle[\alpha],\beta\rangle := \int_{S_C}[\alpha]\cup\beta =_{num} 0,\quad \forall\beta\in\mathbb{Q}[g,g'],$$

where $[\alpha]$ means the cohomoloy class of α. An element $\alpha\in\mathrm{H}^*(S_C)^\circ$ is called a *primitive cohomology class*; an element $\alpha\in\mathrm{CH}^*(S_C)^\circ$ is called a *primitive cycle class.*

REMARK 4.2. Since the restriction of $\mathrm{H}^*(G,\mathbb{Q})$ to S_C is $\mathbb{Q}[g,gg']$, we deduce that $H^{n-2}(S_C,\mathbb{Z})^\circ$ $(n\neq 4)$ consists of elements α such that $\langle\alpha,\beta\rangle=0$ for all β coming from G. Note that the only difference between $\mathbb{Q}[g,g']$ and $\mathbb{Q}[g,gg']$ is the element g' since $(g')^2=0$; when $n\neq 4$, the equation $\langle\alpha,g'\rangle=0$ is automatic for dimension reasons since $g'\in\mathrm{H}^2(S_C)$ and $\alpha\in\mathrm{H}^{n-2}(S_C)$ are not of complementary dimension. If n is odd, then every cohomology class in $\mathrm{H}^{n-2}(S_C)$ is primitive by definition.
By abuse of notation, we will also use $\alpha\cdot\beta$ to denote $\langle\alpha,\beta\rangle$ for α and β of compementary dimensions.

By Proposition 3.30, the action σ induces an action, still denoted by σ, on the primitive cohomology and the primitive Chow groups of S_C. This is because σ is symmetric and preserves $\mathbb{Q}[g,g']$. Hence $\alpha\cdot\sigma(\beta)=\sigma(\alpha)\cdot\beta$. If α is a primitive class and $\beta\in\mathbb{Q}[g,g']$, then the above identity shows that $\sigma(\alpha)\cdot\beta=0$ and hence $\sigma(\alpha)$ is still primitive.

THEOREM 4.3. *Let $C\subset X$ be a general rational curve of degree $e\geq 2$ as above. Let σ be the action of the incidence correspondence on either $\mathrm{H}^*(S_C)^\circ$ or $\mathrm{CH}^*(S_C)^\circ$. Then the following are true.*
(1) On the primitive part of either the cohomology groups or the Chow groups, σ satisfies the following quadratic relation

$$(\sigma-1)(\sigma+2e-1)=0$$

(2) The map Φ_C induces an isomorphism of Hodge structures

$$\Phi_C:\mathrm{H}^n(X,\mathbb{Z})_{prim}\to P(\mathrm{H}^{n-2}(S_C,\mathbb{Z})^\circ,\sigma)(-1)$$

The intersection forms are related by the following identity

$$\Phi_C(\alpha)\cdot\Phi_C(\beta)=-2e\,\alpha\cdot\beta$$

(3) The map Φ_C induces an isomorphism

$$\Phi_C:\mathrm{A}_i(X)_{\mathbb{Q}}\to P(\mathrm{CH}_{i-1}(S_C)^\circ_{\mathbb{Q}},\sigma)$$

Proof. Since Γ_v is also symmetric, we know that it acts on the primitive cohomology and the primitive Chow groups. But by Lemma 3.27, the image of Γ_v is always non-primitive unless it is zero. Hence we get that $(\Gamma_v)_* = 0$ on primitive cohomology and Chow groups. Hence on the primitive part of the cohomology group and the Chow groups, we have

$$\sigma = \Phi_C \circ \Psi_C + 1.$$

The next fact that we need is

Lemma 4.4. *If $\mathfrak{a}$ is a primitive class in either the cohomology group or the Chow groups of S_C, then $\Psi_C(\mathfrak{a})$ has h-degree zero.*

The proof of the above lemma is easy. We note that $\Psi_C(\mathfrak{a})\cdot h^i = \mathfrak{a}\cdot \Phi_C(h^i) = 0$, since $\Phi_C(h^i) \in \mathbb{Q}[g, g']$; see (1) of Proposition 3.30. Now we can prove statement (1) of the theorem. Theorem 2.2 shows that, on the primitive cohomology and the primitive Chow groups, we have

$$\Psi_C(\Phi_C(\mathfrak{a})) + 2e\mathfrak{a} = 0.$$

From this we get

$$\begin{aligned}(\sigma-1)(\sigma+2e-1)(\mathfrak{a}) &= \Phi_C\circ\Psi_C(\Phi_C\circ\Psi_C + 2e)(\mathfrak{a})\\ &= \Phi_C(\Psi_C\circ\Phi_C(\Psi_C(\mathfrak{a}))) + 2e\Phi_C\circ\Psi_C(\mathfrak{a})\\ &= \Phi_C(-2e\Psi_C(\mathfrak{a})) + 2e\Phi_C\circ\Psi_C(\mathfrak{a}), \quad \text{(by Theorem 2.2)}\\ &= 0\end{aligned}$$

Now we prove (2). For simplicity, we write P for $P(\mathrm{H}^{n-2}(S_C,\mathbb{Z})^\circ, \sigma)$. Since $\Phi_C\circ\Psi_C = \sigma - 1$ and that Ψ_C is onto (Propositioon 4.5), we know that the image of Φ_C is exactly P. By Theorem 2.2, we know $\Psi_C\circ\Phi_C = -2e$. This implies that Φ_C is injective. Hence $\Phi_C : H^n(X,\mathbb{Z})_{prim} \to P$ is an isomorphism. Φ_C respects the Hodge structures and hence is an isomorphism of Hodge structures. The intersection forms are related by

$$\Phi_C(\alpha)\cdot\Phi_C(\beta) = \alpha\cdot\Psi_C(\Phi_C(\beta)) = \alpha\cdot(-2e\beta) = -2e\alpha\cdot\beta$$

Statement (3) can be proved exactly in the same way. □

Proposition 4.5. *(1) The homomorphism*

$$\Psi_C : \mathrm{H}^{n-2}(S_C,\mathbb{Z})^\circ \to \mathrm{H}^n(X,\mathbb{Z})_{prim}$$

on primitive cohomology is surjective.
(2) The homomorphism

$$\Psi_C : \mathrm{CH}_m(S_C,\mathbb{Q})^\circ \to \mathrm{A}_{m+1}(X,\mathbb{Q})$$

on primitive Chow groups is surjective.

Proof. Statement (1) follows from Theorem 5.1 (for $n \neq 4$) and Remark 5.10 (for $n = 4$). The proof of (2) is easy since we have $\mathbb{Q}$ coefficients. Let $\alpha \in \mathrm{CH}_{m+1}(X)_{\mathbb{Q}}$, take $\mathfrak{a} = -\frac{1}{2e}\Phi_C(\alpha)$. Then we have $\alpha = \Psi_C(\mathfrak{a})$. Now assume that α has h-degree 0. For any $\mathfrak{b} \in \mathbb{Q}[g, g']$, we have $\Phi_C(\alpha)\cdot\mathfrak{b} = \alpha\cdot\Psi_C(\mathfrak{b}) = 0$

since $\Psi_C(\mathfrak{b}) \in \mathbb{Q}[h]$. This means that $\Phi_C(\alpha)$ is a primitive cycle class. This gives the surjectivity of Ψ_C on primitive Chow groups. $\square$

Let $\pi : \mathscr{C} \to T$ be a family of curves together with a morphism $f : \mathscr{C} \to X$. Then we can define the Abel-Jacobi homomorphism $\Phi_T := \pi_* f^*$ and the cylinder homomorphism $\Psi_T := f_* \pi^*$ on the cohomology groups and the Chow groups. Let $C \subset X$ be a general smooth rational curve of degree $e \geq 2$ as before. Then there is a natural incidence correspondence $\Gamma_{T,C} \subset T \times S_C$ given by

$$t \mapsto \sum([l_i], x_i)$$

where l_i are the incidence lines of $\mathscr{C}_t$ and C (i.e. lines meeting both curves) and $x_i = l_i \cap C$. This correspondence induces

$$(\Gamma_{T,C})_* : \mathrm{CH}_r(T) \to \mathrm{CH}_r(S_C)$$

and

$$[\Gamma_{T,C}]^* : \mathrm{H}^{n-2}(S_C, \mathbb{Z}) \to \mathrm{H}^{n-2}(T, \mathbb{Z})$$

We can define the primitive cohomology groups and the primitive Chow groups of T as the classes that are orthogonal to $\Phi_T(\mathbb{Z}[h])$. Hence $\Gamma_{T,C}$ also induces homomorphisms between the primitive cohomology groups and the primitive Chow groups.

PROPOSITION 4.6. *Let $\mathscr{C} \to T$ be a family of curves on X and $C \subset X$ a general rational curve of degree e as above. All the homomorphisms are restricted to primitive classes. Then the following hold.*
(1) The homomorphism $[\Gamma_{T,C}]^* : \mathrm{H}^{n-2}(S_C, \mathbb{Z})^\circ \to \mathrm{H}^{n-2}(T, \mathbb{Z})$ factors as $\Phi'_T \circ (\sigma - 1)$, where $\Phi'_T : P(\mathrm{H}^{n-2}(S_C, \mathbb{Z})^\circ, \sigma) \to \mathrm{H}^{n-2}(T, \mathbb{Z})$ is a homomorphism such that the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{H}^{n-2}(S_C, \mathbb{Z}) \arrow[r, "\Psi_C"] \arrow[d, "\sigma-1"'] & \mathrm{H}^n(X, \mathbb{Z})_{prim} \arrow[dl, "\Phi_C"'] \arrow[d, "\Phi_T"] \\
P(\mathrm{H}^{n-2}(S_C, \mathbb{Z})^\circ, \sigma) \arrow[r, "\Phi'_T"] & \mathrm{H}^{n-2}(T, \mathbb{Z})
\end{tikzcd}$$

(2) This image of $(\Gamma_{T,C})_* : \mathrm{CH}_r(T, \mathbb{Q})^\circ \to \mathrm{CH}_r(S_C, \mathbb{Q})^\circ$ is contained in the subgroup $P(\mathrm{CH}_r(S_C, \mathbb{Q})^\circ, \sigma)$, where $\mathrm{CH}_r(T, \mathbb{Q})^\circ$ is the subgroup of primitive elements. In other words, the following diagram is commutative.

$$\begin{tikzcd}
 & \mathrm{A}_{r+1}(X, \mathbb{Q}) \arrow[dr, "\Phi_C"] & \\
\mathrm{CH}_r(T, \mathbb{Q})^\circ \arrow[ur, "\Psi_T"] \arrow[rr, "(\Gamma_{T,C})_*"] & & P(\mathrm{CH}_r(S_C, \mathbb{Q})^\circ, \sigma)
\end{tikzcd}$$

Proof. These are consequences of the identities $[\Gamma_{T,C}]^* = \Phi_T \circ \Psi_C$ and $(\Gamma_{T,C})_* = \Phi_C \circ \Psi_T$. $\square$

Let C_1 and C_2 be two general rational curves on X of degree e_1 and e_2 respectively. Then there is a natural incidence correspondence $\Gamma_{12} \subset S_{C_1} \times S_{C_2}$ defined by

$$([l], x) \mapsto \sum_{i=1}^{5e_2} ([l_i], x_i)$$

where $([l], x) \in S_{C_1}$, l_i are the incidence lines of l and C_2), $x_i = l_i \cap C_2$. Let $\gamma_{12} = (\Gamma_{12})_*$ be the induced homomorphism on either the cohomology groups or the Chow groups. Note that by definition, we have $\gamma_{12} = \Phi_{C_2} \circ \Psi_{C_1}$. Since both Φ_C and Ψ_C respect primitive classes, the above identity implies that γ_{12} takes primitive classes to primitive classes. We still use γ_{12} to denote the action on the primitive cohomology and the primitive Chow groups.

Proposition 4.7. *Let C_1, $C_2 \subset X$ be two general rational curves of degree at least 2 and γ_{12} be the homomorphism induced by incidence correspondence as above. Let σ_1 and σ_2 be the action of the self incidence correspondence on S_{C_1} and S_{C_2} respectively. Then the following are true.*
(1) The image of $\gamma_{12} : \mathrm{H}^{n-2}(S_{C_1}, \mathbb{Z})^\circ \to \mathrm{H}^{n-2}(S_{C_2}, \mathbb{Z})^\circ$ is always in the Prym-Tjurin part. Furthermore there is an isomorphism of Hodge structures

$$t_{12} : P(\mathrm{H}^{n-2}(S_{C_1}, \mathbb{Z})^\circ, \sigma_1) \to P(\mathrm{H}^{n-2}(S_{C_2}, \mathbb{Z})^\circ, \sigma_2)$$

such that $\Phi_{C_2} = t_{12} \circ \Phi_{C_1}$ and $\gamma_{12} = t_{12}(\sigma_1 - 1)$.
(2) The same conclusions as in (1) hold for primitive Chow groups with $\mathbb{Q}$-coefficient.

Proof. For simplicity, we write $\Lambda_i = \mathrm{H}^{n-2}(S_{C_i}, \mathbb{Z})^\circ$ for the primitive cohomology and $P_i = \mathrm{Im}(\sigma_i - 1)$ for the Prym-Tjurin part, $i = 1, 2$. Let $\Lambda = \mathrm{H}^n(X, \mathbb{Z})_{prim}$, $\Phi_i = \Phi_{C_i} : \Lambda \to P_i$. Then one easily checks that $t_{12} = \Phi_2 \circ \Phi_1^{-1}$ satisfies (1). The proof of (2) is similar. □

5. Surjectivity of Ψ_C on primitive cohomology

In this section we supply a proof of the surjectivity of Ψ_C on the primitive cohomologies. To do this, it is more convenient to consider homology instead of cohomology. We define

$$V_{n-2}(S_C, \mathbb{Z}) = \ker\{\mathrm{H}_{n-2}(S_C, \mathbb{Z}) \to \mathrm{H}_{n-2}(G, \mathbb{Z})\}, \tag{15}$$

and

$$V_n(X, \mathbb{Z}) = \ker\{\mathrm{H}_n(X, \mathbb{Z}) \to \mathrm{H}_n(\mathbb{P}^{n+1}, \mathbb{Z})\}. \tag{16}$$

Under the Poincaré duality $\mathrm{H}_{n-2}(S_C, \mathbb{Z}) \cong \mathrm{H}^{n-2}(S_C, \mathbb{Z})$, the subspace $V_{n-2}(S_C, \mathbb{Z})$ corresponds to $\mathrm{H}^{n-2}(S_C, \mathbb{Z})^\circ$ if $n \neq 4$. When $n = 4$, since the class g' is not from $G(2, 6)$, we see that $V_2(S, \mathbb{Z})$ is larger than $\mathrm{H}^2(S_C, \mathbb{Z})^\circ$. Actually, we have

$$\mathrm{H}^2(S_C, \mathbb{Z})^\circ = \{\sigma \in V_2(S_C, \mathbb{Z}) : g' \cdot \sigma = 0\}.$$

Similarly the Poincaré duality on X allows us to identify $V_n(X,\mathbb{Z})$ with $\mathrm{H}^n(X,\mathbb{Z})_{prim}$. Then the surjectivity of

$$\Psi_C : \mathrm{H}^{n-2}(S_C,\mathbb{Z})^\circ \to \mathrm{H}^n(X,\mathbb{Z})_{prim}$$

when $n \neq 4$, is equivalent to the following

Theorem 5.1. *The natural cylinder homomorphism*

$$\Psi_C : V_{n-2}(S_C,\mathbb{Z}) \to V_n(X,\mathbb{Z})$$

is surjective.

The idea of the proof is the Clemens-Letizia method, see [Cl] and [Le]. Our presentation closely follows that of [Shi, §2,3]. Let $\pi : V \to \Delta$ be a proper flat holomorphic map from a complex manifold V of dimension $m+1$ onto the unit disk Δ. This map is called a *degeneration* if π is smooth over the punctured disk $\Delta^* = \Delta - 0$ and $V_t := \pi^{-1}(t)$ is irreducible for $t \neq 0$. Let $\mathrm{Sing}(V_0)$ denote the singular locus of V_0.

Definition 5.2. ([Shi, Definition 1]) A degeneration $\pi : V \to \Delta$ is called *quadratic of codimension* r if $\mathrm{Sing}(V_0)$ is connected and, for every point $p \in \mathrm{Sing}(V_0)$, there exist local coordinates $(z_0,\ldots,z_m)$ of V around p such that $\pi = z_0^2 + \cdots + z_r^2$.

Proposition 5.3. *For any smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$, there exist a Lefschetz pencil $X_t \subset \mathbb{P}^{n+1}$ with $t \in B \cong \mathbb{P}^1$, such that*
(1) The total space, $\mathcal{F} = \coprod_{t\in B} F(X_t)$, of the associated Fano schemes of lines is smooth.
(2) $X_0 = X$ is the cubic hypersurface we start with and X_t is smooth unless $t \in \{t_1,\ldots,t_N\}$.
(3) For each degeneration point $t_j \in B$, $j = 1,\ldots,N$, the family $\beta : \mathcal{F} \to B$ is a quadratic degeneration of codimension $n-2$ at the point t_j.

This Proposition can be viewed as a special case of [Shi, Proposition 1]. Note that we have no isolated singularities in $F(X_t)$ since $F(X_t)$ is smooth as long as X_t is smooth. By the results of [CG, Theorem 7.8], we easily get a description of the fibers of β. If $t \in B - \{t_1,\ldots,t_N\}$ then $\mathcal{F}_t = F(X_t)$ is smooth of dimension $2n-4$. Let x_i be the ordinary double point of X_{t_i} and Γ_i be the lines on X_{t_i} that pass through x_i. We know that Γ_i is smooth of dimension $n-2$. The singular fiber $\mathcal{F}_{t_i}$ is irreducible with Γ_i being the singular locus which is an ordinary double subvariety. Let $z \in \Gamma_i$ be a point and $Y \subset \mathcal{F}$ be an $(n-1)$-dimensional complex submanifold is a small neighborhood of z. If Y meets Γ_i transversally in the point z, then z is a non-degenerate critical point of $\beta|_Y$. There is an associated vanishing cycle $\sigma_i \in \mathrm{H}_{n-2}(Y_{t_i+\epsilon},\mathbb{Z})$.
The next observation we make is that a cubic hypersurface with an isolated ordinary double point has enough rational curves in its smooth locus. Let $X \subset \mathbb{P}^{n+1}$ be a cubic hypersurface with an ordinary double point $x_0 \in X$. Then the projection from the point x_0 defines a morphism $\tau : \tilde{X} \to \mathbb{P}^n$, where

$\tilde{X} = \mathrm{Bl}_{x_0}(X)$ is the blow-up of X at the point x_0. Let $\sigma : \tilde{X} \to X$ be the blow-up morphism. Then it is known that τ is the blow-up of $\mathbb{P}^n$ along a smooth $(2,3)$-complete intersection $Z \subset \mathbb{P}^n$; see [CG, Lemma 6.5]. Let $Q \subset \mathbb{P}^n$ be the unique quadric hypersurface containing Z and $\tilde{Q} \subset \tilde{X}$ be the strict transform of Q. Then σ is an isomorphism on $\tilde{X}\backslash\tilde{Q}$ and it contracts $\tilde{Q}$ to the singular point x_0. Let $h \in \mathrm{Pic}(X)$ be the class of a hyperplane section and $H \in \mathrm{Pic}(\mathbb{P}^n)$ the class of $\mathcal{O}(1)$. Then on $\tilde{X}$, we have

$$\sigma^* h = 3\tau^* H - E.$$

Let $C \subset X\backslash\{x_0\}$ be a degree e rational curve and $\tilde{C} = \sigma^{-1}C \subset \tilde{X}$. Then this curve $\tilde{C}$ satisfies the following conditions

$$\tilde{C} \cdot (3\tau^* H - E) = e, \qquad \tilde{C} \cdot (2\tau^* H - E) = \tilde{C} \cdot \tilde{Q} = 0.$$

This implies that $\tilde{C} \cdot \tau^* H = e$ and $\tilde{C} \cdot E = 2e$. Hence $C' = \tau(\tilde{C}) \subset \mathbb{P}^n$ is a degree e rational curve which meets Z in $2e$ points and $\tilde{C}$ is the strict transform of C'. Conversely, if we start with a degree e rational curve $C' \subset \mathbb{P}^n$ which meets Z in $2e$ points. Let $\tilde{C}$ be the strict transform of C'. Then $C = \sigma(\tilde{C}) \subset X\backslash\{x_0\}$ is a rational curve of degree e.

Lemma 5.4. *Let $X \subset \mathbb{P}^{n+1}$ be a cubic hypersurface with an isolated ordinary double point $x_0 \in X$. Then there exists a free rational curve $C \subset X\backslash\{x_0\}$ of degree $e \geq 1$.*

Proof. We only need to show that there is a rational curve of degree e through a general point of $X\backslash\{x_0\}$ and contained in $X\backslash\{x_0\}$. We first do this for $e = 1$. In this case, we see from the above discussion that we only to show the following *Claim*: Let $y \in \mathbb{P}^n$ be a general point, then there is a line l' on $\mathbb{P}^n$ that meets Z in two points.

To prove the claim, we consider the projection, $pr_y : \mathbb{P}^n\backslash\{y\} \to \mathbb{P}^{n-1}$, from the point y. If there is no such line l', then $pr_y|_Z : Z \hookrightarrow \mathbb{P}^{n-1}$ is a closed immersion. But this would imply that Z is a degree 6 hypersurface in $\mathbb{P}^{n-1}$ and hence $-K_Z = (n-6)H$ where H is the class of a hyperplane section of Z. Since $Z \subset \mathbb{P}^n$ is a $(2,3)$-complete intersection, we also know that $-K_Z = (n+1-2-3)H = (n-4)H$, which is a contradiction.

For case of $e > 1$, we can take a chain of e lines on $X\backslash\{x_0\}$ and smooth to a degree e rational curve; see [Ko, §II.7]. Such a rational curve passes through a general point since a line does. □

Lemma 5.5. *Under the situation of Proposition 5.3, there exists a contractible analytic open neighborhood D of $0 \in B$ containing $\{t_1, \ldots, t_N\}$ such that*
(i) There is an analytic family of curves $\{C_t \subset X_t^{sm} : t \in D\}$ with $C_0 = C$, where X_t^{sm} is the smooth locus of X_t.
(ii) $\mathcal{S} = \cup_{t \in D} S_{C_t} \subset \mathcal{F}$ is a complex manifold and $\rho : \mathcal{S} \to D$ is smooth away from the t_i's;
(iii) As sub-manifolds of $\mathcal{F}$, $\mathcal{S}$ meets each Γ_i transversally at finitely many points $z_{i1}, \ldots, z_{ir}$.

Proof. Let $\mathcal{H}/B$ be the component of relative Hilbert scheme of rational curves on X_t's that contains C as an element. This is well-defined since C defines a smooth point in the relative Hilbert scheme. We claim that, after shrinking D, there exists a general analytic section $s : t \mapsto C_t \in \mathcal{H}$ satisfies $C_t \subset X_t^{sm}$. To prove this we first note that by Lemma 5.4, there is a free rational curve $C_i \in X_{t_i}^{sm}$ of the same degree e. Since C_i is free, it deforms to a curve C_t in nearby fibers. By the result of [CS, Theorem 1.1], the space of degree e rational curves on a smooth cubic hypersurface is irreducible. This implies that C_t is a point of $\mathcal{H}$ and hence so is C_i. So we can choose a family $s : t \mapsto C_t$ whose specializations does not meet the points $x_i \in X_{t_i}$. This proves (i). We also get (iii) easily by choosing s general enough. The smoothness of $\mathcal{S}$ follows from a deformation argument. First by shrinking D, we may assume that $\mathcal{S}_t$ is smooth for all $t \neq t_i$. By choosing C_i general, we may assume that S_{C_i} is smooth away from $\{z_{i1,\ldots,z_{ir}}\}$. We only need to show that $\mathcal{S}$ is smooth at the points $z = z_{ij}$. Let L be the line corresponds to z and $C_i = C_{t_i}$. Let $v \in \mathrm{H}^0(C_i, \mathscr{N}_{C_i/\mathbb{P}^{n+1}})$ be the section corresponding to $\frac{\partial}{\partial t}$ at the point C_i. Let $y = C_i \cap L$. Then $v(y) \in \mathscr{N}_{C_i/X,y} = \frac{\mathcal{T}_{X,y}}{\mathcal{T}_{C_i,y}}$ determines a 2-dimensional subspace $V_y \subset \mathcal{T}_{X,y}$. Then $\mathcal{T}_{\mathcal{S},z} \subset \mathcal{T}_{\mathcal{F},z}$ is naturally given by all sections $v' \in \mathcal{T}_{\mathcal{F},z} \subset \mathrm{H}^0(L, \mathscr{N}_{L/\mathbb{P}^{n+1}})$ with $v'(y) \in \bar{V}_y$, where $\bar{V}_y$ is the image of V_y in $\frac{\mathcal{T}_{X,y}}{\mathcal{T}_{L,y}} = \mathscr{N}_{L/\mathbb{P}^{n+1},y}$. When $v(y)$ is general, the above condition gives a codimension $n-2$ subspace of $\mathcal{T}_{\mathcal{F},z}$, i.e. $\dim \mathcal{S} = \dim \mathcal{T}_{\mathcal{S},z}$. This proves (ii). □

Remark 5.6. From the above proof, we see that the family $t \mapsto C_t$ can be made algebraic on some finite cover of $B = \mathbb{P}^1$. The points z_{ij} are exactly the critical points of ρ and all of them are non-degenerate.

Now we fix a small ϵ and let B_i be the closed ball of radius ϵ with center t_i. If ϵ is small enough, we have $B_i \subset D$. Pick a path l_i connecting 0 and $t_i + \epsilon$ such that $\cup_i l_i$ is star-shaped and contractible. Let $D_i \subset \mathbb{P}^{n+1}$ be a small open ball centered at the double point $x_i \in X_{t_i}$. Let $\tilde{D}_i \subset \mathcal{F}$ be the set of lines $L \in \mathcal{F}$ such that L meets D_i. Thus $\tilde{D}_i$ is a small open neighborhood of Γ_i. By construction,

$$\mathcal{S} \cap \tilde{D}_i = U_1 \cup \cdots \cup U_r$$

where U_j are disjoint open balls in $\mathcal{S}$. Let $p_j : P_j \to U_j$ be the family of lines parameterized by U_j. Hence we get the following commutative diagram

$$\begin{tikzcd} P_j \arrow[r, "q_j"] \arrow[d, "p_j"'] & Y \arrow[d, "\pi"] \\ U_j \arrow[r, "\rho_j"] & B \end{tikzcd}$$

where Y is the blow up of $\mathbb{P}^{n+1}$ along the base locus of the Lefschetz pencil and $\rho_j = \rho|_{U_j}$. We fix a general analytic section $s_j : U_j \to P_j$.

Lemma 5.7. *(*[Shi, Lemma 7]*) There exists local analytic coordinates $u_0, u_1, \ldots, u_n$ of $\mathbb{P}^{n+1}$ at the point x_j such that*

(i) The image of P_j in $\mathbb{P}^{n+1}$ is locally given by $u_0 + \sqrt{-1}u_1 = 0$;
(ii) The image $q_j \circ s_j(U_j)$ is given by $u_0 = u_1 = 0$;
(iii) $\pi = t_j + u_0^2 + u_1^2 + \cdots + u_n^2$.

Proof. We first show that q_j has injective tangent map. For any tangent vector $v \in \mathcal{T}_{U_j, z_j}$, it corresponds to a global section of $\mathscr{N}_{L_j/\mathbb{P}^{n+1}}$, where L_j is the line corresponding to z_j. By the choice of C_t, we know that C_{t_i} meets the divisor swept out by lines through x_i transversally. Then v does not vanish at the point $x_i \in L_j$. This means that the point $x_i \in L_j$ actually moves when we move L_j in U_j. Hence we get the injectivity of the tangent map of q_j. Then we can start with local coordinates on U_j and extend to P_j and then to $\mathbb{P}^{n+1}$. See the proof of [Shi, Lemma 7] for more details. □

Since we are doing local computations, by abuse of notation, we regard P_j as a submanifold of $\mathbb{P}^{n+1}$. Using the local coordinates obtained in Lemma 5.7, it is standard that a vanishing cycle associated to x_i is given by

$$\Sigma_{ij} = \{u_0^2 + \cdots + u_n^2 = \epsilon, \quad u_k \in \mathbb{R}\}$$

See [La] for more details. Note that the index j is used to remember that the local coordinates are chosen with respect to P_j as in Lemma 5.7. If we fix an orientation of Σ_{ij} and this gives an element $[\Sigma_{ij}] \in \mathrm{H}_n(X_{t_i+\epsilon}, \mathbb{Z})$. Also, by Lemma 5.7, we know that $\sigma_{ij} = (q_j \circ s_j)^{-1}\Sigma_{ij}$, with the induced orientation, gives a vanishing cycle for the critical point $z_{ij} \in \mathcal{S}$. Let

$$[\sigma_{ij}] \in \mathrm{H}_{n-2}(\mathcal{S}_{t_i+\epsilon}, \mathbb{Z})$$

be the corresponding homology class. We choose the orientation of Σ_{ij} such that $[\Sigma_{ij}] = [\Sigma_i]$ is a fixed class. Then σ_{ij} has an induced orientation.

Proposition 5.8. ([Shi, Proposition 2]) *For any $\mathfrak{a} \in \mathrm{H}_{n-2}(\mathcal{S}_{t_i+\epsilon}, \mathbb{Z})$, we have*

$$\mathfrak{a} \cdot ([\sigma_{i1}] + \cdots + [\sigma_{ir}]) = \Psi(\mathfrak{a}) \cdot [\Sigma_i]$$

where $\Psi = \Psi_{C_{t_i+\epsilon}}$.

Proof. The proof goes the same as that of [Shi, Proposition 2]; we sketch it here for completeness. We still use $\mathfrak{a}$ to denote a topological cycle that represents the class $\mathfrak{a}$. We use the notations above and set $\mathfrak{a}_j = \mathfrak{a} \cap U_j$. If $L \in \mathfrak{a} - \cup_j \mathfrak{a}_j$, then $L \cap \Sigma_i = 0$. We may assume that $\mathfrak{a}_j$ meets σ_{ij} transversally at μ points $a_1, \ldots, a_\mu \in \sigma_{ij}$. Then the intersection of $\cup_{L\in\mathfrak{a}_j} L$ and Σ_{ij} are exactly $s_j(a_1), \ldots, s_j(a_\mu)$. And by construction, this intersection is transversal. By chasing the orientations, we see that the contributions to the two sides of the identity are equal. □

Remark 5.9. Since $\cup_i l_i$ is contractible, $\mathrm{H}_n(X_{t_i+\epsilon}, \mathbb{Z})$ is naturally identified with $H_n(X, \mathbb{Z})$ via $(l_i)_*$. We can similarly identify $\mathcal{S}_{t_i+\epsilon}$ and $S_C = \mathcal{S}_0$. Hence we can view $[\sigma_{ij}]$ as elements in $V_{n-2}(S_C, \mathbb{Z})$ and $[\Sigma_i]$ as elements in $V_n(X, \mathbb{Z})$. Under this identification, the above proposition still holds true.

Now we are ready to prove the main result of this section. The proof follows that of [Shi, Proposition 4].

Proof. (of Theorem 5.1). By Lefschetz theory, $V_n(X,\mathbb{Z})$ is generated by vanishing cycles, see [La]. Since $\Psi = \Psi_C$ commutes with the specialization map, we know that $\Psi([\sigma_{i1}]) = \lambda[\Sigma_i]$ for some $\lambda \in \mathbb{Z}$. Now we see that

$$[\sigma_{i1}] \cdot ([\sigma_{i1}] + \cdots + [\sigma_{ir}]) = [\sigma_{i1}] \cdot [\sigma_{i1}] = \pm 2$$

See [La], p.40. By the above proposition, we get

$$\Psi([\sigma_{i1}]) \cdot [\Sigma_i] = \lambda[\Sigma_i] \cdot [\Sigma_i] = \pm 2$$

Comparing this with $[\Sigma_i] \cdot [\Sigma_i] = \pm 2$, we get $\lambda = \pm 1$. This shows that $[\Sigma_i]$ is in the image of Ψ. This proves the theorem since $[\Sigma_i]$ generates $V_n(X,\mathbb{Z})$. $\square$

Remark 5.10. To complete the picture, we still need to prove the surjectivity of

$$\Psi_C : \mathrm{H}^{n-2}(S_C,\mathbb{Z})^\circ \to \mathrm{H}^n(X,\mathbb{Z})_{prim}$$

for $n = 4$. Note that under Poincaré duality, $\mathrm{H}^2(S_C,\mathbb{Z})^\circ \subset \mathrm{H}^2(S_C,\mathbb{Z})$ consists of all classes in $V_2(S_C,\mathbb{Z})$ which is orthogonal to g'. Hence we only need to show that $\sigma_{ij} \cdot g' = 0$. But this is easy to see from construction. In fact, we pick a general point $x \in C_{t_i+\varepsilon}$, then g' is represented by F_x (all lines through x). Then by construction, all lines through x avoids σ_{ij} (this is essentially due to the fact that the surface swept out by lines in through x is $2h^2$; see Lemma 3.29). This means $\sigma_{ij} \cdot g' = 0$.

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