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Koster, M.; Boonen, T.

Publication date
2014

Document Version
Submitted manuscript

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November 19, 2014

Abstract

Undertaking joint projects in practice involves a lot of uncertainty, especially when it comes to the final costs. This paper addresses the problem of sharing realized costs by the participants, subject to their individual liquidity constraints. If all cost levels can be accounted for, and if the individual agent's liquidity is at least the fair share of the average risk-adjusted costs, then it is possible to find an egalitarian solution to this problem. We characterize this rule by means of properties that are defined for solutions in the context of rationing problems. The rule we propose is reminiscent to the constrained equal award rule in rationing models. First, there is a vector of transfers allocated to the agents if there are no costs of the project. The constrained equal award rule is applied to the variable component, i.e., when the liquidities are corrected for the transfers.

Keywords: risk sharing, fairness, egalitarian rule, rationing, constrained equal award rule, uncertainty

1 Introduction

In this paper, we characterize an egalitarian rule for rationing problems, where the amount to divide is stochastic. Special examples are bankruptcy problems (see, e.g., Thomson, 2003) where the estate is stochastic, and risk sharing under liquidity constraints.

A rationing problem describes the situation in which we allocate a given amount among a group of agents when the available amount is not enough to satisfy all their claims. Rationing problems are solved by means of rules that satisfy some desirable properties and determine, for each specific problem, an allocation in which no agent gets more than she claims. The total available amount is stochastic, if a project is evaluated before possible rationing. Stochastic rationing problems are also discussed by Habis and Herings (2013). They assume the values of the estate and claims to be stochastic, and focus on the weak-sequential core to formalize and study stable cooperation. We differ

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from Habis and Herings (2013) by characterizing egalitarian solutions. In our setting, the risky project is equally beneficial for all parties, but the egalitarian costs are not always affordable by all agents. Other authors that discuss a stochastic environment in cooperative decision making are, e.g., Suijs et al. (1999) and Timmer et al. (2004).

The standard formulation of our problem is as an ordered pair (C, L), where C is a stochastic cost, and $L \in \mathbb{R}^N$ is the vector of liquidities of the agents in N. A rule, or solution, to such a problem is a device which shows an allocation of each possible realization of the cost C. Different from Habis and Herings (2013) we seek to allocate cost in a way that can be interpreted as *a priori* egalitarian, in the sense that the risk due to the way of allocation is as equal spread amongst the agents – if possible; this requires that it makes sense to compare their attitudes towards risk. Agents are possibly risk-averse in our setting. We assume that agents optimize a trade-off between risk and return (see, e.g., Giorgi and Post, 2008), where risk is measured by a coherent risk measure (Artzner et al., 1999) and return equals the expected value. This setting corresponds with dual utility maximizing agents (Yaari, 1989). We only focus on comonotonic risk sharing profiles which implies that the allocated costs for every agent weakly increases if the total costs C increase, i.e., no agent will benefit from a higher cost. In non-trivial cases, where not all liquidities are high enough to be able to bear a fair share of $1/|N|$ of the realized cost, then high cost levels will demand more from agents with high liquidities. Then, if the allocated costs are relatively high when the total costs are high, these agents require a large compensation if the total costs are low. We will show that we can choose such *transfers* at low levels and high levels such that agents are assured to be set out to the same level of risk. Given these transfers the incremental costs are shared using ideas laid down in the constrained egalitarian rule for rationing. We will characterize our egalitarian rule following Yeh (2008), by properties that have a natural counterpart in the (deterministic) rationing context.

The characterization consists of two parts. First, we characterize the transfers; a side-payment that is paid or received irrespective of the outcome of the stochastic cost C. Marginal changes in the stochastic cost C are then determined via the natural cost counterpart of the constrained equal award rule in the rationing context. We use an alternative (marginal) formulation of properties consistency and composition up. Consistency is a general idea, that tells us how a solution behaves over problems with different agent sets. Especially, we may reduce a given problem by an agent, if at the moment of departure the transfers are already paid/received, and only the contingent cost allocation will be deducted from the problem. The property of composition up requires that a problem is solved consistently along with increases of cost. Given an increase of cost, new cost shares can be allocated from the information of the earlier cost shares alone.

This paper is set out as follows. In Section 2, we specify the model. In Section 3, we characterize a risk sharing rule. Finally, in Section 4, we explain how the rule can be used in the situation where egalitarian solutions do not exist, i.e., where some of the agents have too low liquidities to take a fair share of the risk.

2 Model

In this section, we specify the model. In Subsection 2.1 we formalize the global rationing model and the model for risk sharing under liquidity constraints , followed by Subsection 2.2 on the notion of solution. In Subsection 2.3, we define the preference relations of the agents. In Subsection 2.4, we specify the concept of egalitarian solutions.

2.1 Rationing and risk sharing under liquidity constraints

Throughout the paper the focus will be on a finite population of agents $N := \{1, 2, \ldots, n\}$. A *rationing problem* for N is a profile $(t, q) \in \mathbb{R}_+ \times \mathbb{R}_+^N$ such that $\sum_{i \in N} q_i \geq t$, where

- $t \geq 0$ is the amount of a divisible good to be allocated to a group of agents N;
- $q_i \geq 0$ is a justified claim of units of the good for agent $i \in N$.

The space of all rationing problems for the group of agents N is denoted R^N. So, the essence of such rationing problems is that if the agents cannot all get the claim amount, then the allocated amount to every agent has to be rationed.

In this paper, agents aim to share the cost of a risky project. The cost of the project, denoted by C, is a random variable on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the state space, $\mathcal{F}$ is the σ-algebra on Ω and $\mathbb{P}$ is the probability measure in the class of probability measures on $(\Omega, \mathcal{F})$, denoted by $\mathcal{P}(\Omega, \mathcal{F})$. The class of all such random cost variables is denoted L^∞. A *risk sharing problem* is an ordered pair $(C, L) \in L^\infty \times \mathbb{R}_{++}^N$ where C is the random cost that has to be allocated to the agents in N, so that the allocation for an agent i is consistent with its ability to pay, i.e. its liquidity $L_i > 0$. Moreover, we will assume that the range of C is given by $[0, C^*]$, where $C^* > 0$. Moreover, let L be such that it is *admissible*, i.e., $\sum_{i \in N} L_i \geq C^*$; admissibility implies that whatever the realization of the project will turn out to be, the collective of agents can afford it. Below it will be convenient to think of L as an ordered vector, i.e.,

$$L_1 \leq L_2 \leq \cdots \leq L_n.$$

We denote the class of all such risk sharing problems by P^N.

2.2 Solutions

A rationing solution is a mapping $r : R^N \to \mathbb{R}^N$ such that for each $(t, q) \in R^N$ we have

$$0 \leq r(t, q) \leq q \text{ and } \sum_{i \in N} r_i(t, q) = t.$$

Examples of rationing solutions are respectively the proportional solution r^p defined by

$$r^p(t, q) = \frac{q}{\sum_{i \in N} q_i} t,$$

and the *constrained equal award rule* $r^{CEA}(t,q) = \min\{\lambda, q_i\}$ where λ solves

$$\sum_{i\in N} \min\{\lambda, q_i\} = t.$$

For a discussion of rationing problems and other solutions, we refer to, e.g., Thomson (2003) and Moulin (2002).
A risk sharing rule is a mapping $\varphi : \cup P^N \to \cup \mathbb{R}^N$ such that for $(C,L) \in P^N$ we have

$$\sum_{i\in N} \varphi_i(C,L) = C, \tag{1}$$

$$\varphi_i(C,L) \le L_i, \text{ for all } i \in N, \tag{2}$$

where (1) requires that the cost C is allocated, and (2) requires agents to be able to pay the costs. Our subsequent analysis of risk sharing solutions is driven by the idea that each rationing rule defines a non-negative risk sharing rule, only with input (C,L) instead of (t,q). Take any rationing solution r, then under for any admissible L we may define a risk sharing solution φ^r through $\varphi^r(C,L) := r(C,L)$. For example, the proportional risk sharing rule φ^p is defined by $\varphi^p(C,L) := r^p(C,L)$, and the *constrained equal cost rule* is defined through the constrained equal award rule for rationing problems by $\varphi^{EC}(C,L) = r^{CEA}(C,L)$.
For each realization of C the set of admissible cost allocations is easily described. Define $X(C,L) := \left\{x \in \mathbb{R}^N : \sum_{i\in N} x_i = C, x \le L\right\}$. Then the extreme points of $X(C,L)$ are given by the vectors $m^1,\ldots,m^n$ defined by

$$m^i_j := \begin{cases} L_j & \text{if } j \ne i, \\ C - \sum_{k\in N\setminus i} L_k & \text{if } j = i. \end{cases}$$

and $X(C,L) = co\left\{m^1, m^2, \ldots, m^n\right\}$, i.e., the convex hull of these vectors. In particular, this shows that the challenge here is not finding solutions but to find solutions which have further special properties.
Remark Note that our notion of a risk sharing solution is not equivalent to a rationing solution because we do not require non-negativity of cost shares. In particular, $\varphi(0,L)$ is not necessarily the zero vector.

2.3 Preferences

In order to be able to focus on solutions and their degree of egalitarianism we need intercomparison of the preferences of the agents at the proposed allocations. Here we will assume that agents are endowed with the same preferences over outcomes.[1] Within the rationing model it makes sense to take as utility function $u_i(r_i(t,q)) = r_i(t,q)$ or any monotonic transformation of the allotted amount of good. However, due to the

[1] See, e.g., Koster (2002) for ideas of egalitarianism under asymmetric status quo, for transferable utility games.

stochastics, the model of risk sharing is much richer and therefore the preferences are more involved. We choose to focus on preferences that can characterized by a *risk measure*, i.e., a function $\rho : L^\infty \to \mathbb{R}$ that maps bounded stochastic variables into real numbers.

We take the agents to be risk-reward investors as in De Giorgi and Post (2008) and Asimit et al. (2013). Agents value a stochastic payment via a trade-off between a return (expected value) and a risk measure ρ^*, i.e., the utility of a bounded, stochastic cost $\zeta_i \in L^\infty$ is given by

$$U(\zeta_i) = -\mathbb{E}_F[\zeta_i] - \alpha \cdot \rho^*(\zeta_i), \tag{3}$$

with $\alpha \geq 0$. If $\alpha = 0$, all agents are risk-neutral and aim to minimize the expected value of ζ_i. A common characterization of risk measures is *coherence*. Artzner et al. (1999) characterize coherent risk measures as the risk measures satisfying the following four properties:

- *Sub-additivity*: for all $Y, Z \in L^\infty$, we have $\rho(Y + Z) \leq \rho(Y) + \rho(Z)$.
- *Monotonicity*: for all $Y, Z \in L^\infty$ such that $Y \geq Z$ almost surely, we have $\rho(Y) \geq \rho(Z)$.
- *Positive homogeneity*: for every $Y \in L^\infty$ and every $c \in \mathbb{R}_+$, we have $\rho(cY) = c\rho(Y)$.
- *Translation invariance*: for every $Y \in L^\infty$ and every $c \in \mathbb{R}$, we have $\rho(Y + c) = \rho(Y) + c$.

The relevance of these properties is widely discussed by Artzner et al. (1999). Particularly, Sub-additivity of a risk measure implies that the risk weakly decreases if risks are pooled. It also implies that there is no incentive for a firm to split its risk into pieces and evaluate them separately.

We also define the following property:

- *Comonotonic additivity*: for all $Y, Z \in L^\infty$ that are comonotone, we have

$$\rho(Y + Z) = \rho(Y) + \rho(Z).$$

In the sequel of this paper, we assume that the risk measure ρ^* is coherent and satisfies comonotonic additivity. The preference relation U in (3) is equivalent to maximizing

$$\begin{aligned}\widetilde{U}(\zeta_i) &= \frac{1}{1+\alpha} V(\zeta_i) \\ &= -\gamma \cdot \mathbb{E}_F[\zeta_i] - (1-\gamma) \cdot \rho^*(\zeta_i) \\ &:= -\rho(\zeta_i),\end{aligned}$$

with $\gamma = \frac{1}{1+\alpha} \in [0, 1]$. Hence, maximizing the utility U is equivalent with minimizing a given risk measure ρ, i.e., the objective of an agent is to minimize V, denoted by

$$V(\zeta_i) = \rho(\zeta_i), \text{ for all } \zeta_i \in L^\infty. \tag{4}$$

One can show that if ρ^* is coherent and comonotonic additive, ρ is coherent and comonotonic additive as well. Hence, we focus on a preference relation that is given by only a risk measure ρ. Moreover, if $\alpha > 0$ and ρ^* is strictly preserving second order stochastic dominance[2], ρ is strictly preserving second order stochastic dominance as well. Our setting allows for dual utility (Yaari, 1987) as well as weakly risk-averse agents. Artzner et al. (1999) show that a risk measure ρ is coherent if and only if there exists a set of probability measures $Q \subset \mathcal{P}(\Omega, \mathcal{F})$ such that

$$\rho(X) = \sup \{E_{\mathbb{Q}}[X] : \mathbb{Q} \in Q\}, \qquad \text{for all } X \in L^\infty. \tag{5}$$

The set Q need not be unique. Moreover, Delbaen (2000) shows that for every coherent risk measure ρ satisfying comonotonic additivity, there is a submodular function $\upsilon^\rho : \mathcal{F} \to \mathrm{I\!R}_+$ with $\upsilon^\rho(\emptyset) = 0$ and $\upsilon^\rho(\Omega) = 1$ such that the following set Q is generating ρ:[3,4]

$$Q = \{\mathbb{Q} \in \mathcal{P}(\Omega, \mathcal{F}) : \mathbb{Q}(A) \geq \upsilon^\rho(A) \text{ for all } A \in \mathcal{F}\}. \tag{6}$$

From (6) and submodularity of υ^ρ, we directly get the following proposition.

Proposition 1 *If ρ is coherent and comonotonic additive, it holds that for any risk sharing problem $(C, L) \in P^N$,*

$$\rho(\zeta_i) = \mathbb{E}_{F_{\mathbb{Q}}^{\zeta_i}}[\zeta_i], \text{ with } F_{\mathbb{Q}}^{\zeta_i}(x) = \upsilon^\rho(A(x)), x \in \mathrm{I\!R}, \tag{7}$$

where υ^ρ is as in (6) *and $A_i(x) \in \mathcal{F}$ is such that $\zeta_i(a) \geq \zeta_i(x) \Leftrightarrow a \in A_i(x)$.*

Example 2 In this example, we explain a broad class of risk measures satisfying the abovementioned properties. Distortion risk measures ρ are characterized as the risk measures that are coherent, comonotonic additive and satisfying the following property (see Wang et al., 1997):

- *Conditional state independence*: $\rho(Y)$ depends on the risk $Y \in L^\infty$ only via its distribution.

Wang (1995) defines a distortion risk measure by

$$\rho(Y) = \int_0^\infty g^\rho(1 - F_Y(x))\, dx + \int_{-\infty}^0 (g^\rho(1 - F_Y(x)) - 1)\, dx, \text{ for all } Y \in L^\infty, \tag{8}$$

[2]Random variables that can be ordered using second order stochastic dominance should generate risk measures that retain that ordering. Here risk X is second order stochastic dominating risk Y if

$$\int_{-\infty}^x F_X(t)dt \leq \int_{-\infty}^x F_Y(t)dt,$$

for all $x \in \mathrm{I\!R}$, with a strict inequality for some x (see, e.g., Rothschild and Stiglitz, 1970).

[3]This result is deduced by Delbaen (2000) from earlier results of Denneberg (1994) and Schmeidler (1986).

[4]A function $\upsilon : \mathcal{F} \to \mathrm{I\!R}$ is submodular if $\upsilon(S \cup T) + \upsilon(S \cap T) \geq \upsilon(S) + \upsilon(T)$ for all $S, T \in \mathcal{F}$.

for a continuous, concave and increasing distortion function $g^\rho : [0,1] \to [0,1]$ with $g^\rho(0) = 0$ and $g^\rho(1) = 1$, where F_Y is the cumulative density function (CDF) of risk Y. Here, convergence of the integrals is guaranteed by boundedness of the risk Y. For distortion risk measures, we can write (7) with

$$F_{\mathbb{Q}}^{\varphi_i(C,L)}(x) = 1 - g^\rho(\mathbb{P}(\varphi_i(C,L) \geq \varphi_i(x,L)))$$

for all $x \in \mathbb{R}$. Wirch and Hardy (2001) show that distortion risk measures satisfy strictly preserving second order stochastic dominance if and only if the function g^ρ is strictly concave. $\bigtriangledown$

2.4 V-egalitarian solutions

In this subsection, we define V-egalitarian solutions. Here, V is a preference relation as in (4). For the stochastic cost $C \in L^\infty$, a solution $\varphi_i(C,L), i \in N$ is called V-*egalitarian* if it satisfies the following three properties:

- *Comonotonic*: $\varphi_i, i \in N$ are non-decreasing as function of cost share;
- *Egalitarian*:
$$V(\varphi_i(C,L)) = V(\varphi_j(C,L)) \text{ for all } i,j \in N;$$
- *Pareto optimal*: there does not exist another solution $\widetilde{\varphi}_i(C,L), i \in N$ such that $V(\widetilde{\varphi}_i(C,L)) \leq V(\varphi_i(C,L))$ for all $i \in N$ with a strict inequality for at least one $i \in N$.

Comonotonicity is a natural property to impose. If the aggregate cost C increases, no agent should benefit by paying less, ceteris paribus.

From (4) and Proposition 1 we get that if $\varphi_i, i \in N$ is Pareto optimal and comonotonic, then

$$V(\varphi_i(C,L)) = E_{F_{\mathbb{Q}}^{\varphi_i(C,L)}}[\varphi_i(C,L)] = E_{F_{\mathbb{Q}}^{C}}[\varphi_i(C,L)], i \in N.$$

So, if the ordering of states is known (almost surely), we can state the value function as expectation under a given alternative probability measure $\mathbb{Q}$. Therefore, any V-egalitarian solution satisfies

$$V(\varphi_i(C,L)) = \frac{1}{n}V(C), i \in N. \tag{9}$$

In the next section we will consider combinations of risk sharing problems (C,L) and preferences V such that each agent can take a fair share of the total risk, i.e., $L_i \geq \frac{1}{n}V(C)$ for all $i \in N$. Formally,

- Risk sharing problem $(C,L) \in P^N$ is V-*sufficient* if $L_i \geq \frac{1}{n}V(C), i \in N$.

We can show that both φ^p as φ^{EC} do not generate V-egalitarian solutions unless $L_1 = \cdots = L_n$ resp. $L_1 \geq \frac{1}{n}C^*$ under the assumption of an ordered profile L. Moreover, we can show that admissibility and V-sufficiency are sufficient and necessary conditions to have an egalitarian solution that meets the liquidity constraints.[5]

Existence of a Pareto optimal comonotonic risk redistribution is shown by Landsberger and Meilijson (1994), who show that any allocation of an aggregate risk is dominated for second order stochastic dominance by a comonotonic allocation. Carlier et al. (2012) extend this result by showing that every strictly second order dominance preserving preference relation is such that for every non-comonotonic risk redistribution, there exists a comonotonic risk redistribution that Pareto dominates it. So, from this we obtain the following proposition.

Proposition 3 *A solution $\varphi_i(C, L), i \in N$ is Pareto optimal if $\varphi_i(C, L), i \in N$ is comonotonic. Moreover, if $V = \rho$ is strictly preserving second order stochastic dominance, every Pareto optimal solution $\varphi_i(C, L), i \in N$ is comonotonic.*

From Proposition 3, we get that if a risk redistribution is comonotonic and egalitarian, it is V-egalitarian. Moreover, if $V = \rho$ is strictly preserving second order stochastic dominance, any solution satisfying (9) is V-egalitarian.

3 Properties and characterization

Suppose V represents the utility function of an agent in N and take $(C, L) \in P^N$. In this section, we characterize risk sharing rules on the class of V-sufficient problems that allow for the following structure:

$$\varphi(C, L) = t + g(C, L - t), \tag{10}$$

where t is a vector of transfers such that $\sum_{i \in N} t_i = 0$ and g is a risk sharing rule as well. In particular we will characterize a rule and a unique vector of transfers such that $\varphi(C, L) = t + \varphi^{CE}(C, L - t)$ is V-egalitarian. We will now illustrate the central risk sharing rule in this paper.

Example 4 In this example, we provide the intuition for the risk sharing rule φ that we have in mind, without technical details. Let $V(C) = \mathbb{E}[C]$, $N = \{1, 2, 3\}$ and, moreover, let the risk sharing problem be given by $C \sim Un(0, 10)$ and $L = (2, 3, 5)$. So, $V(C) = 5$. It is easily verified that the problem (C, L) satisfies admissibility and V-sufficiency. Then, we derive a unique vector of transfers $t \approx (0.51, -0.13, -0.38)$ such that we obtain a V-egalitarian solution φ of type (10) which is displayed in Figure 1. The rule is linear in the sense that marginal contributions due to cost increase are equally shared amongst the agents whose liquidities are not fully met. Below we will point out – using standard continuity arguments – how the *cut-off* points γ_i and transfers at 0 cost may be determined from the distribution of C. $\triangledown$

[5] To see that admissibility and V-sufficiency are necessary conditions is trivial. It the sequel of this paper, we show that admissibility and V-sufficiency are sufficient as well.

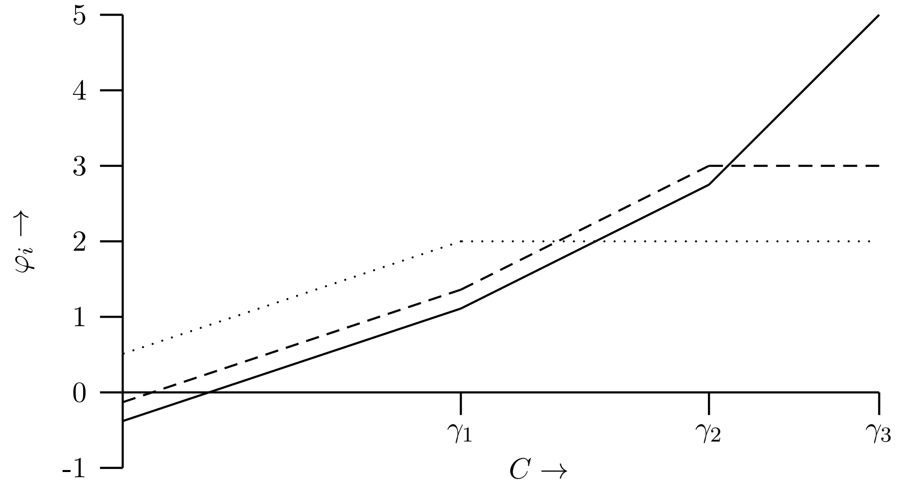

Figure 1: Graphical illustration of the V-egalitarian risk sharing rule $\varphi(C, L) = t + \varphi^{EC}(C, L - t)$ corresponding to Example 4. The dotted line represents φ_1, the dashed line φ_2 and the solid line φ_3. Here, $\gamma \approx (4.47, 7.75, 10)$ are as defined in the proof of Lemma 8.

The rest of the paper is devoted to a precise definition of the rule, and some of its properties. This will accumulate in the main theorem which provides a first characterization of the egalitarian rule. But first we will need some preparations.
We define the function h as

$$h(L) = L - \varphi(0, L), \tag{11}$$

for all admissible liquidity profiles L. After the transfers $\varphi(0, L)$ are allocated to the agents, the function h corrects the liquidities for the change in wealth.

In this section, we impose the following condition on φ:

Bijection (B) The mapping $L \mapsto L - \varphi(0, L)$ defines a bijection.

In particular, with bijection h, there exists a mapping g, as introduced in (10), such that:

$$g(C, L) = \varphi\left(C, h^{-1}(L)\right) - \varphi\left(0, h^{-1}(L)\right), \tag{12}$$

for all $(C, L) \in P^N$. Similar to (C, L), we have that $(C, h(L))$ is a risk sharing problem since the liquidity profile $h(L) \geq 0$ is admissible, which follows from

$$\sum_{i \in N} h(L) = \sum_{i \in N} (L_i - \varphi_i(0, L)) = \sum_{i \in N} L_i \geq C^*. \tag{13}$$

In this section, we focus on the dynamics of the vector $\varphi(0, L)$ and the function g separately.

Hence, g is a risk sharing rule that defines to every risk sharing problem $(C, h(L))$ a solution. If we fill in $C = 0$ in (12), we get $g(0, h(L)) = 0$. for all $(C, L) \in P^N$. So, the risk sharing rule g allocates only the C-contingent costs. If $t = \varphi(0, L)$, we get

$$\varphi(C, L) = t + g(C, L - t),$$

with

$$t \in \left\{ a \in \mathbb{R}^N : \sum_{i \in N} a_i = 0, a \leq L \right\} := \mathcal{T}.$$

A vector in $\mathcal{T}$ is referred to as vector of *transfers*; it is the allocation that takes place regardless the allocation of the remaining variable component C. The transfers $t = \varphi(0, L)$ may depend on the probability distribution of C.

Just like mimicking rationing rules we may also mimic properties known for rationing rules. Young (1988) defines *composition up* for rationing problems. A rationing rule r satisfies composition up if for all $(t, q) \in R^N$ and $0 \leq \bar{t} < t$, we have

$$r(t, q) = r(\bar{t}, q) + r(t - \bar{t}, q - r(\bar{t}, q)).$$

If we apply this property directly to a V-egalitarian risk sharing rule φ, we get by substituting $C = 0$ that $\varphi(0, L) = 0$. Instead we consider the solution of the residual problem without the transfers, as these are already allocated when we aim to share the C-contingent costs. This leads to the following property.

Obligation Composition Up (OCU): for all $(C, L) \in P^N$ and $0 \leq \bar{C} \leq C$, we have

$$g(C, h(L)) = g\left(\bar{C}, h(L)\right) + g\left(C - \bar{C}, h(L) - g\left(\bar{C}, h(L)\right)\right).$$

The idea is that one may allocate costs and apply the same allocation principles to the remaining and reduced problem where the amount to be allocated is the cost left over, and the remaining liquidities are the original ones minus the pre-paid amounts.

For the rationing model, Moreno-Ternero and Villar (2004) introduce *secured lower bound*[6] which requires that for all $(t, q) \in R^N$ and $i \in N$ that

$$r_i(t, q) \geq \frac{1}{n} \min\{q_i, t\}.$$

We desire a different but related version of this property as in the present format we get for $C = 0$ a nonnegative solution, which then allocates 0 to all. Instead we consider the minimal contribution of an agent in addition to his payment of transfer at $C = 0$.

Obligation Lower Bound (OLB): for all $(C, L) \in P^N$ and $i \in N$, we have

$$g_i(C, h(L)) \geq \frac{1}{n} \min\{h_i(L), C\}.$$

[6] This property weakens the *lower bound* by Moulin (2002) which reads $r_i(t, q) \geq \min\left\{q_i, \frac{1}{n}t\right\}$.

For all $(C,L) \in P^N$ and $i \in N$, the property OLB writes

$$\varphi_i(C,L) - \varphi_i(0,L) \geq \frac{1}{n} \min \{L_i - \varphi_i(0,L), C\}.$$

Notice that for solutions with $\varphi(0,L) = 0$ this property is in fact the *secured lower bound.*

We resolve all asymmetries in the problem at hand by a vector of transfers $t = \varphi(0,L)$, then apart from this $t \in \mathcal{T}$ there are no further asymmetries to deal with, so that a standard solution in the remainder resides. We show this in the next lemma when $n = 2$.

Lemma 5 *Suppose φ is a rule with the properties OCU and OLB. Then for $(C,L) \in P^N$ with $n = 2$ we have*

$$\varphi(C,L) = \varphi(0,L) + \varphi^{EC}(C,h(L)).$$

Proof It follows from OCU of φ that

$$g(C,h(L)) = g\left(\bar{C},h(L)\right) + g\left(C - \bar{C}, h(L) - g\left(\bar{C},h(L)\right)\right). \tag{14}$$

Note that (14) does depend on L via $h(L)$ only.

Young (1988) defines composition up only for $\bar{C} < C$. Note that $\bar{C} = C$ is not important since we get from (14) that $g(0,h(L) - g(C,h(L))) = 0$, which holds for rationing problems as well. Combining this with (14) implies that the rule g satisfies composition up.
Since φ satisfies OLB, it follows that

$$g_i(C,h(L)) \geq \frac{1}{n} \min \{h_i(L), C\}.$$

This implies that the rule g satisfies secured lower bound.

Since OLB implies that $g(C,h(L))$ is non-negative for all $(C,L) \in P^N$, we have that $\hat{g}$ defines rationing solutions for any $(C,h(L))$, where $(C,L) \in P^N$. Then the result of Yeh (2008) applies here and we find that $g \equiv \varphi^{EC}$. Hence, from this and (12) we get

$$\begin{aligned}\varphi(C,L) &= \varphi(0,L) + g(C,h(L)) \\ &= \varphi(0,L) + \varphi^{EC}(C,h(L)).\end{aligned}$$

This concludes the proof. □

A well-known property within the field of rationing rules is *consistency.* Consistency is the idea that there may be interdependencies between allocation problems with different populations of agents. Here we will interpret the idea as follows. Suppose that before realization of the costs we let an agent i go with her signature on a contract that she will stick to the original allocation, whatever the realization of the costs will be. Then the rest of the agents face the *reduced cost allocation problem* where only the remaining costs need to be shared. We will call a rule φ *consistent* if it is consistent as in the rationing interpretation, i.e., if for all $(C, L) \in P^N$ and $i \in N$, it holds that

$$\varphi_{N\backslash i}(C, L) = \varphi\left(C - \varphi_i(C, L), L_{N\backslash i}\right).$$

As indicated before, the possibility of having transfers at cost 0 is of great interest for us. Then we will have to do with a more general idea of consistency. We call a rule φ Zero-Consistent if for all problems $(C, L) \in P^N$ and $i \in N$ leaving, we first resolve the allocation of transfers at $C = 0$, before we continue with allocating the remaining C-contingent component.

Zero-Consistency (ZCONS): for all $(C, L) \in P^N$ and $i \in N$, it holds that

$$\varphi_{N\backslash i}(C, L) = \varphi\left(C - g_i(C, h(L)), L_{N\backslash i}\right). \tag{15}$$

Lemma 6 *If φ is Zero-Consistent, it holds that the risk sharing rule g is consistent.*

Proof If φ is Zero-Consistent, we get from substituting $C = 0$ in (15) that $\varphi_{N\backslash i}(0, L) = \varphi(0, L_{N\backslash i})$, and, hence, $h_{N\backslash i}(L) = h(L_{N\backslash i})$. From the fact that φ is Zero-Consistent, it follows that

$$\begin{aligned}
g_{N\backslash i}(C, h(L)) &= \varphi_{N\backslash i}(C, L) - \varphi_{N\backslash i}(0, L) \\
&= \varphi\left(C - g_i(C, L), h(L_{N\backslash i})\right) - \varphi_{N\backslash i}(0, L) \\
&= \varphi(0, L_{N\backslash i}) + g\left(C - g_i(C, h(L)), h(L_{N\backslash i})\right) - \varphi_{N\backslash i}(0, L) \\
&= g\left(C - g_i(C, h(L)), h(L_{N\backslash i})\right) \\
&= g\left(C - g_i(C, h(L)), h_{N\backslash i}(L)\right).
\end{aligned}$$

Hence, for all y on the image of h, we have $g_{N\backslash i}(C, y) = g\left(C - g_i(C, y), y_{N\backslash i}\right)$. The image of h is all admissible L. Hence, g is consistent for y. □

The following theorem follows from Lemma 5, Lemma 6 and Yeh (2008).

Theorem 7 *Let φ be a Zero-Consistent rule with the properties OCU and OLB. Then, for all $(C, L) \in P^N$ we have*

$$\varphi(C, L) = \varphi(0, L) + \varphi^{EC}(C, h(L)).$$

Lemma 8 *For any $(C, L) \in P^N$ such that $L_1 > \frac{1}{n}V(C)$, there exists a unique profile of transfers $t \in \mathcal{T}$ such that*

$$V\left(t_i + \varphi_i^{EC}(C, L - t)\right) = \tfrac{1}{n}V(C) \text{ for all } i \in N. \tag{16}$$

Proof Recall that the vector L is ordered such that $L_1 \leq L_2 \leq \cdots \leq L_n$. We will define numbers $\{t_i\}_{i \in N}$ and $\{\gamma_i\}_{i \in N \cup \{0\}}$, with $\gamma_0 = 0$, as follows.

Step 1: the function $s \mapsto V\left(\left(s + \frac{1}{n}C\right) \wedge L_1\right)$ is strictly increasing on $[-C^*, L_1 - \frac{1}{n}\operatorname{ess\,inf}(C)]$, where $\operatorname{ess\,inf}(C) \in [0, C^*]$ is the essential infimum of all realizations of C

with positive density. We apply the intermediate value theorem to this continuous and strictly increasing function:

$$V\left(\left(-C^* + \frac{1}{n}C\right) \wedge L_1\right) \leq V\left(0 \wedge L_1\right) = V\left(0\right) = 0 \leq \frac{1}{n}V\left(C\right),$$

$$\begin{aligned} V\left(\left(L_1 - \frac{1}{n}\operatorname{ess\,inf}(C) + \frac{1}{n}C\right) \wedge L_1\right) &\geq V\left(\left(L_1 + 0\right) \wedge L_1\right) \\ &= V\left(L_1\right) = L_1 \geq \frac{1}{n}V\left(C\right), \end{aligned}$$

where the last inequality follows from V-sufficiency of L. So, there is a unique value of t_1 such that $V\left(\left(t_1 + \frac{1}{n}C\right) \wedge L_1\right) = \frac{1}{n}V\left(C\right)$. Step 1 is completed by putting $\gamma_1 = n\left(L_1 - t_1\right) \wedge C^*$.

Now suppose that we have defined $t_1, \ldots, t_k$, and $\gamma_0, \gamma_1, \ldots, \gamma_k \in [0, C^*]$ in step 1 until step k. Then in step $k+1$ we will define t_{k+1}, γ_{k+1}.

Step $k+1$: define the function $f_{k+1} : [0, C^*] \to \mathbb{R}_+$ by

$$f_{k+1}\left(C\right) = \sum_{j=1}^{k} \frac{\left(\gamma_j \wedge C - \gamma_{j-1}\right)_+}{n-j+1} + \frac{\left(C - \gamma_k\right)_+}{n-k}.$$

Notice that $0 \leq f_{k+1}\left(C\right) \leq C \leq C^*$. Consider the continuous function h_{k+1} defined by

$$h_{k+1}\left(s\right) := V\left(\left(s - \sum_{j=1}^{k} \frac{t_j}{n-j} + f_{k+1}\left(C\right)\right) \wedge L_{k+1}\right).$$

The function h_{k+1} is strictly increasing on the interval

$$\left[-C^* + \sum_{j=1}^{k} \frac{t_j}{n-j}, L_{k+1} - \frac{1}{n}\operatorname{ess\,inf}(C) + \sum_{j=1}^{k} \frac{t_j}{n-j}\right].$$

Use the intermediate value theorem to see that there is a unique number t_{k+1} such that $h_{k+1}\left(t_{k+1}\right) = \frac{1}{n}V\left(C\right)$:

$$\begin{aligned} h_{k+1}\left(-C^* + \sum_{j=1}^{k} \frac{t_j}{n-j}\right) &= V\left(\left(-C^* + f_{k+1}\left(C\right)\right) \wedge L_{k+1}\right) \\ &\leq V\left(0 \wedge L_{k+1}\right) \\ &= V\left(0\right) \\ &= 0 \\ &\leq \frac{1}{n}V\left(C\right), \text{ and} \end{aligned}$$

$$h_{k+1}\left(L_{k+1} - \frac{1}{n}\operatorname{ess\,inf}(C) + \sum_{j=1}^{k} \frac{t_j}{n-j}\right) = V\left(\left(L_{k+1} - \tfrac{1}{n}\operatorname{ess\,inf}(C) + f_{k+1}\left(C\right)\right) \wedge L_{k+1}\right)$$

$$
\begin{aligned}
&\geq V\left(\left(L_{k+1}+0\right) \wedge L_{k+1}\right) \\
&= V\left(L_{k+1}\right) \\
&= L_{k+1} \\
&\geq \frac{1}{n} V\left(C\right),
\end{aligned}
$$

where the last inequality follows from V-sufficiency of L. Now step $k+1$ is complete by putting

$$\gamma_{k+1}=\left(\sum_{i=1}^{k} L_i+(n-k)\left(L_{k+1}-t_{k+1}\right)\right) \wedge C^*.$$

Iterate the steps until step n, and complete the definition of $\{t_i\}_{i \in N}$ and cost levels $\{\gamma_i\}_{i \in N \cup\{0\}}$.

Then define numbers $t_1^{EC}, t_2^{EC}, \ldots, t_n^{EC}$ by

$$\left\{\begin{array}{l}
t_1^{EC}=t_1, \\
t_i^{EC}:=t_i-\sum_{j=1}^{i-1} \frac{t_j}{n-j} \quad \text{for } i \geq 2.
\end{array}\right.$$

Notice that $t_i^{EC} \in\left[-C^*, L_i\right]$ for all $i \in N$ and, moreover,

$$
\begin{aligned}
h_i(t_i^{EC}) &= V\left(\left(t_i^{EC}+f_i\left(C\right)\right) \wedge L_i\right) \\
&= \frac{1}{n} V(C).
\end{aligned} \tag{17}
$$

Now take $q<n$. Then for $C \in\left(\gamma_q, \gamma_{q+1}\right]$ we have

$$\left(t_i^{EC}+f_i\left(C\right)\right) \wedge L_i-t_i^{EC}=\left\{\begin{array}{ll}
L_i-t_i^{EC}=\sum_{j=1}^{i} \frac{\gamma_j-\gamma_{j-1}}{n-j+1} & \text{if } i \leq q, \\
\sum_{j=1}^{q} \frac{\gamma_j-\gamma_{j-1}}{n-j+1}+\frac{C-\gamma_q}{n-q} & \text{if } k>q.
\end{array}\right.$$

By construction we have

$$
\begin{aligned}
\sum_{i \in N}\left(\left(t_i^{EC}+f_i\left(C\right)\right) \wedge L_i-t_i^{EC}\right) &= \sum_{i \leq q} \sum_{j=1}^{i} \frac{\gamma_j-\gamma_{j-1}}{n-j+1}+\sum_{i>q}\left(\sum_{j=1}^{q} \frac{\gamma_j-\gamma_{j-1}}{n-j+1}+\frac{C-\gamma_q}{n-q}\right) \\
&= \sum_{i \leq q} \sum_{j=1}^{i} \frac{\gamma_j-\gamma_{j-1}}{n-j+1}+(n-q)\left(\sum_{j=1}^{q} \frac{\gamma_j-\gamma_{j-1}}{n-j+1}+\frac{C-\gamma_q}{n-q}\right) \\
&= \sum_{j \leq q}\left(\gamma_j-\gamma_{j-1}\right)+\left(C-\gamma_q\right) \\
&= C.
\end{aligned} \tag{18}
$$

Then, we get

$$V\left(C\right)=\sum_{i \in N} \frac{1}{n} V\left(C\right)$$

$$
\begin{aligned}
&= \sum_{i \in N} V\left(\left(t_i^{EC} + f_i(C)\right) \wedge L_i\right) && (19)\\
&= \sum_{i \in N} V\left(t_i^{EC} + \left(\left(t_i^{EC} + f_i(C)\right) \wedge L_i - t_i^{EC}\right)\right) \\
&= \sum_{i \in N} t_i^{EC} + \sum_{i \in N} V\left(\left(t_i^{EC} + f_i(C)\right) \wedge L_i - t_i^{EC}\right) && (20)\\
&= \sum_{i \in N} t_i^{EC} + V\left(\sum_{i \in N}\left(\left(t_i^{EC} + f_i(C)\right) \wedge L_i - t_i^{EC}\right)\right) && (21)\\
&= \sum_{i \in N} t_i^{EC} + V(C), && (22)
\end{aligned}
$$

where (19) follows from (17), (20) follows from Translation invariance of $V = \rho$, (21) follows from comonotonic additivity of $V = \rho$ and (22) follows from (18). Hence $\sum_{i \in N} t_i^{EC} = 0$ and, so, $t^{EC} \in \mathcal{T}$. Consequently, the liquidities $L - t^{EC}$ are admissible due to (13). We claim that $L - t^{EC}$ is ordered just like L. First we will show that $t_1^{EC} \geq t_2^{EC} \geq \cdots \geq t_n^{EC}$. To see this recall that by construction

$$
V\left(\left(t_i^{EC} + f_i(C)\right) \wedge L_i\right) = V\left(\left(t_j^{EC} + f_j(C)\right) \wedge L_j\right) \text{ for all } i, j \in N,
$$

and moreover if $i \leq j$ then $L_i \leq L_j$ and $f_j(C) \geq f_i(C)$. So the transfers t_j^{EC} decrease with j. But since $L_i \leq L_j \Leftrightarrow i \leq j$ we have $L_i - t_i^{EC} \leq L_j - t_j^{EC} \Leftrightarrow i \leq j$. Due to this ranking it is not hard to see that by construction we have

$$
\left(t_i^{EC} + f_i(C)\right) \wedge L_i = t_i^{EC} + \varphi_i^{EC}\left(C, L - t^{EC}\right),
$$

for all $i \in N$. This concludes the proof. $\square$

Note that if $(C, L) \in P^N$ is such that $L_1 = \cdots = L_j = \frac{1}{n}V(C)$ for $j \in N$, we can set $t_i = L_i$ for all $i \in \{1, \ldots, j\}$; all other choices of transfers leads to either the same risk (for events with positive density), or to solutions that are not V-egalitarian.

The following result is the main theorem of this paper.

Theorem 9 *There is a unique V-egalitarian rule φ that has the properties B, ZCONS, OCU and OLB. This rule is given by*

$$
\varphi(C, L) = t + \varphi^{EC}(C, L - t), \tag{23}
$$

for all $(C, L) \in P^N$, where $t \in \mathcal{T}$ is a unique vector of transfers.

It is easy to see that the risk sharing rule in (23) is comonotonic and, due to Proposition 3, Pareto optimal. Moreover, the properties OCU and OLB are easily verified. It is egalitarian by construction (see Lemma 8). Therefore, Theorem 9 follows from Theorem 7, Lemma 8, and Yeh (2008).

In the proof of Lemma 8, we showed that $t_1 \geq \cdots \geq t_n$. Moreover, $t_i = t_j$ if and only if $L_i = L_j$ or $\varphi_i(C, L) < L_i$ and $\varphi_j(C, L) < L_j$ for all $C < C^*$, where φ is as in (23). If liquidities are not tight, we allocate the costs equally for all C. This is shown in the following corollary.

Corollary 10 *If $L_1 \geq \frac{1}{n}C^*$, then the unique V-egalitarian rule that is consistent and has the properties OCU and OLB is given by*

$$\varphi\left(C, L\right) = \frac{1}{n}C.$$

The following proposition shows the function h is a bijection. This implies that for the rule in (23), we have that g in (12) is well-defined, and the properties OCU, OLB and ZCONS are valid for this rule.

Proposition 11 *For the unique vector of transfers t in* (16), *we have that the mapping $h : L \to L - \varphi(0, L)$ is a bijection on all admissible liquidity profiles in $\mathbb{R}^N$.*

Proof Note that we assumed without loss of generality $L_1 \leq L_2 \leq \cdots \leq L_n$. Then by construction we have $\varphi_1\left(0, L\right) \geq \varphi_2\left(0, L\right) \geq \cdots \geq \varphi_n\left(0, L\right)$ so that

$$L_i - \varphi_i\left(0, L\right) \leq L_j - \varphi_j\left(0, L\right) \Leftrightarrow i \leq j.$$

First, we show that the function h is injective. Suppose that for the liquidity profiles L, L^* we have

$$L - \varphi\left(0, L\right) = L^* - \varphi\left(0, L^*\right). \tag{24}$$

Then, we have

$$L_i^* - \varphi_i\left(0, L^*\right) \leq L_j^* - \varphi_j\left(0, L^*\right) \Leftrightarrow i \leq j.$$

We write $t_k = \varphi_k(0, L)$ and $t_k^* = \varphi_k(0, L^*)$ for all $k = 1, \ldots, n$. We keep $L^* - t^*$ constant, and we will show that $L = L^*$. Start with agent 1; we have

$$\begin{aligned} V\left(\left(t_1^* + \tfrac{1}{n}C\right) \wedge L_1^*\right) &= V\left(\left(t_1 + \tfrac{1}{n}C\right) \wedge L_1\right) \\ &= V\left(\left(L_1 - \left(L_1^* - t_1^*\right) + \tfrac{1}{n}C\right) \wedge L_1\right). \end{aligned} \tag{25}$$

Notice that the righthand side of this equality is strictly increasing in L_1. This means that for given (L_1^*, t_1^*) there can be at most one L_1 such that (25) holds. So, we have $L_1 = L_1^*$. Now we may proceed with induction. Suppose that for the first k agents we have $L_i = L_i^*$, then we have

$$\begin{aligned} & V\left(\left(t_{k+1}^* + \frac{1}{n-k-1}\left(C - \sum_{i=1}^{k} \varphi_k(C, L^*)\right)\right) \wedge L_{k+1}^*\right) \\ &\quad = V\left(\left(t_{k+1} + \frac{1}{n-k-1}\left(C - \sum_{i=1}^{k} \varphi_k(C, L)\right)\right) \wedge L_{k+1}\right) \\ &\quad = V\left(\left(L_{k+1} - \left(L_{k+1}^* - t_{k+1}^*\right) + \frac{1}{n-k-1}\left(C - \sum_{i=1}^{k} \varphi_k(C, L^*)\right)\right) \wedge L_{k+1}\right). \end{aligned}$$

Notice again that the righthand side of this equality is strictly increasing in L_{k+1}, so that $L_{k+1} = L_{k+1}^*$, and, hence, $L = L^*$ Hence, the function h is injective.

Next, we show that the function h is surjective. Let L^* be an admissible liquidity profile. Assume without loss of generality that $L_1^* \leq L_2^* \leq \cdots \leq L_n^*$. We aim to find profiles (L,t) such that $L - t = L^*$ and $V((t + \varphi^{EC}(C, L-t)) = \frac{1}{n}V(N)$. Then, we get

$$\tfrac{1}{n}V(C) = V\left(\left(t_1 + \tfrac{1}{n}C\right) \wedge L_1\right) = V\left(\left(L_1 - L_1^* + \tfrac{1}{n}C\right) \wedge L_1\right). \tag{26}$$

If $L_1 = 0$, the righthand side of (26) becomes negative which is smaller than $\frac{1}{n}V(C)$. If L_1 goes to goes to infinity, the righthand side of (26) goes to infinity. Moreover, the righthand side of (26) is strictly increasing and continuous in L_1. Hence, there exists a L_1 such that (26) holds. Suppose that for the first k agents we have a (L_i, t_i), with $L_i - t_i = L_i^*$, that yield a V-egalitarian rule for them. Then, we have

$$\begin{aligned}\frac{1}{n}V(C) &= V\left(\left(t_{k+1} + \frac{1}{n-k-1}\left(C - \sum_{i=1}^{k}\varphi_k(C, L_k)\right)\right) \wedge L_{k+1}\right)\\ &= V\left(\left(L_{k+1} - L_{k+1}^* + \frac{1}{n-k-1}\left(C - \sum_{i=1}^{k}\varphi_k(C, L_k)\right)\right) \wedge L_{k+1}\right).\end{aligned} \tag{27}$$

If $L_{k+1} = 0$, the righthand side of (27) is negative which is smaller than $\frac{1}{n}V(C)$. If L_{k+1} goes to goes to infinity, the righthand side of (27) goes to infinity. Again, the righthand side of (27) is strictly increasing and continuous in L_{k+1}. Hence, there exists a L_{k+1} such that (27) holds. Since (L,t) yields a V-egalitarian solution, we get from (22) that $\sum_{i\in N}(L_i - t_i) = \sum_{i\in N} L_i^*$, and, so, $t \in \mathcal{T}$ and L is admissible. Hence, the function h is surjective and, so, bijective. This concludes the proof. □

Independence of composition up, secured lower bound, and consistency is shown by Yeh (2008) for rationing problems. Showing independence of OCU and OLB and ZCONS is analogue to Yeh (2008), as the setting in this paper captures the class of rationing problems for a given cost C. Hence, the properties OCU, OLB, and ZCONS are logically independent. It follows from Theorem 7 that those three properties do not necessarily lead to a V-egalitarian solution. It is easy to see that V-egalitarianism does not imply any of the properties OCU, OLB and ZCONS.

4 What if the liquidity profile is not V-sufficient?

In this section, we discuss the case where the liquidity profile L is admissible but not V-sufficient. Then, it holds that

$$L_1 < \frac{1}{n}V(C).$$

It is easy to show that V-egalitarian solutions do not exist. As the project might be beneficial for all parties, and all parties are needed to support the project, we consider a solution. We want the risk sharing rule to be a egalitarian as possible. For instance, we can lexicographic minimize the vector $V(\varphi_i(C))$ for $i \in N$. Then, we erase Agent 1 that is not V-sufficient from the risk sharing problem. Agent 1 pays $t = L_1$ regardless of the realization of C. Consider the reduced risk sharing problem $\left(\tilde{C}, \tilde{L}\right) \in P^{N \backslash 1}$, where

$$\tilde{C} = C - L_1,$$
$$\tilde{L} = L_{N \backslash 1}.$$

Clearly, this problem is again admissible, but is it V-sufficient? If not, i.e., if $L_2 < \frac{1}{n-1} V(C - L_1)$, remove Agent 2 in the same way for the new problem and continue. If the reduced problem becomes V-sufficient, then apply (23) to the reduced risk sharing problem to obtain the risk sharing rule. Note that the cost $\tilde{C}$ may have negative realizations. So the idea is that where we are limited in our choices we propose to adopt the idea of egalitarianism under participation constraints as in Dutta and Ray (1989).

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