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Endogenous Fluctuations under Evolutionary Pressure in Cournot Competition*

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Abstract

An evolutionary game theoretic model of Cournot competition is investigated. Individuals choose from a finite set of different behavioural rules. Each rule specifies the quantity to be produced in the current period as a function of past quantities. Using more sophisticated rules may require extra information costs. Based upon realized payoffs, the fractions of the population choosing a certain behavioural rule are updated according to the replicator equation with noise. The long-run behaviour of the evolutionary system consisting of the population dynamics coupled with the quantity dynamics of the Cournot game may be complicated and endogenous fluctuations may arise. We consider a typical example where firms can choose between two rules: the rational rule and the best-reply rule. We show that, if the best-reply rule is unstable, a homoclinic tangency between the stable and unstable manifold of the equilibrium occurs as evolutionary pressure increases (that is, as the noise level decreases), implying bifurcation routes to complicated dynamics and strange attractors.

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1 Introduction

Firms are often large and complex organizations. Their major tasks are performed by different divisions within each of which information is generated and processed, decisions are made, results are evaluated, and procedures are changed. A possible way to take the complexity of a firm's organization into account (at least to some extent), is to model regular and predictable behavioural patterns of a firm by means of 'routines'. This behavioural approach to the firm is pioneered by Winter (1964), who uses the term 'routine' to denote characteristics of a firm that range from well-specified technical routines for producing output, through procedures for hiring and firing, or ordering new inventory, to policies regarding investment, or research and development, and business strategies about product diversification and advertising. Routines are a persistent feature of the firm and determine its possible behaviour. Furthermore, they are heritable in the sense that due to inertia tomorrow's firm generated from today's has many of the same characteristics, and they are selectable in the sense that firms with certain routines may do better than others, and if so, their relative importance in the industry increases over time due to imitation.

In this paper we refer to routines as behavioural rules. Though we assume that routines determine possible behaviour, actual behaviour is determined also by a firm's environment. As part of the environment of a firm consists of the other firms in the industry, it is particularly important to develop evolutionary models in which a firm's decisions include routine responses to the decisions of competitors (cf. Nelson and Winter (1982), p.408). In particular, for a firm's output decision the influence of competitors' behaviour is usually substantial.

The model to be studied is a model of Cournot competition. Every discrete-time period a large population of firms is matched in pairs randomly to play a symmetric Cournot duopoly game. To incorporate the procedural aspects of decision making mentioned above, we assume the firms to act according to behavioural rules. A behavioural rule specifies the quantity to be produced in the current period as a function of past observations. Rubinstein (1986) argues that behavioural rules are costly to operate. We therefore associate costs with every behavioural rule depending on the informational and computational requirements of implementing the rule. This seems a reasonable assumption since more complex behavioural rules are not only more likely to break down and more difficult to learn, but may also require more time to be implemented.

Each firm chooses a behavioural rule from a finite set of different rules, which we assume to be commonly known. When making a choice concerning the behavioural rules, a firm takes into account the past performance of the rules, i.e., the past realized profits net of the costs associated with the behavioural rules. We assume both past performance and costs to be publicly available. The firms' strategic behaviour implies that successful behavioural rules will continue to be used, while unsuccessful behavioural rules are dropped. The strategic behaviour therefore causes the distribution of the population fractions of firms over the given set of behavioural rules to change over time. We model these population dynamics by means of the replicator dynamics with noise (see e.g. Gale, Binmore and Samuelson (1995)). The population dynamics coupled with the quantity dynamics arising from the different behavioural rules result in a highly nonlinear evolutionary dynamical system, by which firms' behaviour patterns and market outcomes are simultaneously determined over time.

Our paper is also of interest from a methodological viewpoint, since it differs from standard evolutionary game theory in two important respects. Firstly, most applications of evolutionary game theory to economics deal with bimatrix games, that is, games where the different agents have a finite set of actions. Most games of economic interest, however, are games with a continuous action space (for example, games from the field of industrial organization, as the Cournot duopoly game studied here). The setup of our model allows us to study games with a continuous action space. Secondly, in contrast to most evolutionary game theory, we do not model agents as machines programmed to play a certain action, but agents are modelled with a certain ability to make inferences about their environment and to form beliefs about their optimal strategy. In particular, this setup allows for very naive, as well as very sophisticated or even rational players.

Our main result is that, as evolutionary pressure increases, or equivalently the noise level decreases, complicated endogenous quantity fluctuations around an unstable equilibrium arise, with firms switching between behavioural rules over time. This scenario arises in particular when there are sophisticated and inherently stable behavioural rules that are expensive and simple but inherently unstable behaviour rules that are much cheaper. In this paper we analyze a simple, but typical example where such a scenario leads to chaotic dynamics, namely the Cournot duopoly model with two rules, an expensive rational rule and a cheap best-reply rule. In this typical example we show that for a finite noise level, a *homoclinic bifurcation* between the stable and unstable manifolds of the equilibrium saddle point occurs. Existence of strange attractors under high evolutionary pressure is thus obtained as a direct application of recent results from homoclinic bifurcation theory; see Palis and Takens (1993) for an extensive mathematical treatment. The use of homoclinic bifurcation theory is rather new in economic theory and has been applied in a 2-dimensional version of the overlapping generations model with capital by de Vilder (1996), and more recently in a different 2-dimensional overlapping generations model by Yokoo (1999) and in the nonlinear cobweb model with heterogeneous beliefs by Goeree and Hommes (1999). In our model the simple economic mechanism of evolutionary pressure between simple, cheap destabilizing and sophisticated, but costly routines is the direct cause of homoclinic tangencies and their associated complicated endogenous fluctuations.

The above scenario is in fact similar to the *rational route to randomness* in the evolutionary cobweb dynamics detected by Brock and Hommes (1997), that is, a bifurcation route to strange attractors occurs when the intensity of choice parameter tends to infinity, or equivalently, when the noise level of the evolutionary fitness measure goes to zero. Although both the market equilibrium model (cobweb versus Cournot duopoly) and the evolutionary updating of fractions (discrete choice model versus replicator dynamics with noise) are different, surprisingly the routes to complexity seem to be very similar in both evolutionary systems.[1]

Another related model can be found in Gale and Rosenthal (1999). They study a model with two types of agents, experimenters, who randomly draw an action from some interval and test whether it performs better than their present action, and imitators, who adapt their action in the direction of the average action of the other agents. They find that the model is "stable in the large", in the sense that it converges to a compact neighbourhood of the unique and symmetric Nash equilibrium, but that it is "unstable in the small", that is, in this neighbourhood, there are endogenous fluctuations of agents' actions around the equilibrium levels.

Banerjee and Weibull (1995) look at the survival of nonrational agents in strategic environments modelled as finite symmetric two-player games in normal form. In their model most agents in the population act exactly like the agents in standard evolutionary game theory, i.e., they are programmed to play a fixed pure strategy. The remaining part of the population however consists of agents who are more sophisticated. These agents play a best reply against the current pure strategy distribution in the population. Banerjee and Weibull show that for a large class of games, all stable steady states of the replicator dynamics hold a positive share of programmed agents.

In the same strategic environment, Stahl (1993) considers a population of players who are drawn from a hierarchy of 'smartness' analogous to the levels of iterated rationalizability. He shows that nonrationalizable strategies die out, but if costs are associated with being smart and a manifest way to play the game emerges, then at least some fraction of the population will always consist of less smart players. By the emergence of a manifest way to play the game, Stahl (1993) means that there is at least one pure strategy that is always a best reply after some finite time. Survival of less smart players is also obtained in case the maximum level of smartness in the population is finite.

The paper is organized as follows. In section 2 we introduce the general evolutionary dynamical system and state some local instability results. Section 3 introduces our leading example, where firms can choose between two different behavioural rules, a rational rule and a best reply rule, and we perform a local bifurcation analysis for this case. Section 4 focuses on the global evolutionary dynamics and shows that a homoclinic bifurcation leading to strange attractors occurs. Section 5 concludes and some computations, technical details and proofs are contained in the Appendix.

[1] From a methodological viewpoint however, for the Cournot duopoly with replicator dynamics with noise the mathematical analysis is considerably simplified. Brock and Hommes (1997) had to carry out a complicated 'horseshoe' construction in order to show that homoclinic bifurcations between the stable and unstable manifolds of *periodic* saddle points occur.

2 The Model

Consider an infinite population of firms who are matched in pairs randomly each discrete-time period to play a symmetric Cournot duopoly game. However, instead of simultaneously choosing the supplied quantities directly, the firms act according to behavioural rules that exactly prescribe the quantity to be supplied. Note that even though the model is formulated for the duopoly case, it can easily be generalized to allow for any finite number of firms being randomly matched to play a Cournot oligopoly game. Before we turn to our evolutionary model let us first briefly review traditional Cournot duopoly analysis.

We consider a symmetric Cournot duopoly game, where x_i denotes the quantity supplied by firm i, $i = 1, 2$. In addition, let $P(x_i + x_j)$, $i \neq j$, denote the twice differentiable and nonincreasing inverse demand function and let $C(x_i)$ denote the twice differentiable nondecreasing cost function, which is the same for both firms. For firm i, $i = 1, 2$, the profit resulting from the above Cournot game is given by

$$\pi(x_i, x_j) = P(x_i + x_j)\, x_i - C(x_i).$$

We assume that the profit function of a firm is strictly concave in its own output x_i. Profit maximizing behaviour of firm i, $i = 1, 2$, taking the quantity supplied by the opponent j, $j \neq i$, as given, results in the well-known reaction function for firm i, which is given by

$$x_i = R_i(x_j) = \arg\max_{x_i} \left[P(x_i + x_j)\, x_i - C(x_i)\right].$$

Due to symmetry, firm i and j, $i \neq j$, have the same reaction function $R(\cdot)$. Furthermore, a symmetric Cournot-Nash equilibrium quantity x^* corresponds to the solution of

$$x^* = R(x^*).$$

Strict concavity of the profit functions and some boundary conditions ensures that such a Cournot-Nash equilibrium exists (see for example Novshek (1985)). For simplicity we assume that it is the unique symmetric Cournot-Nash equilibrium.[2]

Traditional Cournot analysis refers to a static environment. However, in a dynamic setting the reaction function introduced above can be used to study the so-called best-reply dynamics

$$x_{i,t+1} = R(x_{j,t}), i \neq j,$$

where $x_{i,t}$, $i = 1, 2$, denotes the quantity supplied by player i in period t. The Cournot-Nash equilibrium is stable (unstable) under the best-reply dynamics if $|R'(x^*)| < 1$

[2] Notice that the Cournot duopoly game may also have asymmetric Cournot-Nash equilibria, but they do not correspond to equilibria of the evolutionary game when there is a single population. For the linear-quadratic specification of the Cournot duopoly model specified below, there can indeed be asymmetric boundary equilibria, but they do not influence the dynamics of the evolutionary model.

$(|R'(x^*)| > 1)$.[3]

In part of this paper we focus on the following linear-quadratic specification of the Cournot duopoly game. The inverse demand and cost function are given by

$$P(x_1 + x_2) = a - b(x_1 + x_2) \text{ and } C(x_i) = cx_i - \frac{1}{2}dx_i^2, \quad i = 1, 2,$$

respectively. First, in order to have a strictly concave profit function we assume that $d < 2b$. Second, we require that at all times $x_i \leq \frac{c}{d}$ since the cost function is only upward sloping if this condition holds. The fraction $\frac{c}{d}$ can be interpreted as a capacity constraint. Furthermore, for a strictly positive price we require that $x_1 + x_2 \leq \frac{a}{b}$, a condition which is always satisfied for the examples in this paper. For the above specification of the inverse demand function and cost function the reaction function is given by

$$x_i = R(x_j) = \frac{a - c - bx_j}{2b - d}. \tag{1}$$

It can easily be calculated that in this case the Cournot-Nash equilibrium quantity, price, and profit are equal to $x^* = \frac{a-c}{3b-d}$, $p^* = P(2x^*) = \frac{a(b-d)+2bc}{3b-d}$, and $\pi^* = \pi(x^*, x^*) = \left(b - \frac{1}{2}d\right)\left(\frac{a-c}{3b-d}\right)^2$, respectively. Furthermore, it follows from (1) that the Cournot-Nash equilibrium in this linear-quadratic specification is stable (unstable) under the best-reply dynamics if $d < b$ ($d > b$).

We now turn to the description of our evolutionary model. Subsection 2.1 describes the set of behavioural rules firms can choose from and the quantity dynamics resulting from these rules; subsection 2.2 describes the evolutionary competition between these different rules. Finally, in subsection 2.3 some local instability results for the general evolutionary system are discussed.

2.1 Behavioural Rules

As mentioned before the firms act according to behavioural rules that exactly prescribe the quantity to be supplied in the Cournot game. In fact, the firms can choose between K different behavioural rules $H_1, H_2, \ldots, H_K$. With each behavioural rule H_i, $i = 1, \ldots, K$, we associate an information set I_i and corresponding information costs T_i. An example of such an information set is one that only contains information about average industry output in the previous period. Another, more complete, information set may contain additional information about the structure of the underlying model or information about the other behavioural rules. It seems obvious that obtaining the latter information set requires more resources than the former. Therefore the costs T_i associated to a larger information set are higher.

[3] The discussion of the possibility of instability of the best-reply dynamics was initiated by Theocharis (1959), who showed that for the linear specification of the Cournot oligopoly model (that is, for linear inverse demand function and constant marginal costs) and for more than 3 competitors the Cournot-Nash equilibrium is unstable.

Let $n_{i,t}$, $i = 1, \ldots, K$, denote the fraction of firms using rule H_i in period t and let $x_{i,t}$, $i = 1, \ldots, K$, denote the quantity supplied by a player using rule H_i in period t. Given a certain information set I_i the corresponding behavioural rule is given by

$$x_{i,t+1} = H_i \left(\mathbf{X}_t, \mathbf{N}_t \right), \tag{2}$$

where

$$\mathbf{X}_t = \begin{bmatrix} x_{1,t} & x_{1,t-1} & \cdots & x_{1,t-L} \\ x_{2,t} & x_{2,t-1} & & \vdots \\ \vdots & & \ddots & \\ x_{K,t} & \cdots & & x_{K,t-L} \end{bmatrix}$$

and

$$\mathbf{N}_t = \begin{bmatrix} n_{1,t} & n_{1,t-1} & \cdots & n_{1,t-L} \\ n_{2,t} & n_{2,t-1} & & \vdots \\ \vdots & & \ddots & \\ n_{K,t} & \cdots & & n_{K,t-L} \end{bmatrix}.$$

The above specification of $\mathbf{X}_t$ and $\mathbf{N}_t$ indicates that behavioural rules have a limited memory of $L + 1$ periods.[4] Note that, as the columns in $\mathbf{N}_t$ sum up to one, the last row of this matrix could be ignored. We include it, however, only for notational convenience. Together, the K heterogeneous behavioural rules constitute a $K-$dimensional dynamical system. We define the *quantity dynamics* of our model as this dynamical system (3) with constant population fractions ($N_t = N$, for all t), that is

$$\mathbf{x}_{t+1} = \mathbf{H} \left(\mathbf{X}_t, \mathbf{N} \right), \tag{3}$$

where $\mathbf{x}_t = (x_{1,t}, \ldots, x_{K,t})'$. To make sure that the quantity dynamics are well-behaved in terms of dynamic implications we impose the following regularity condition.

Assumption 1 The quantity dynamics (3) has a unique steady state, which is independent of $\mathbf{N}$. This steady state corresponds to the Cournot-Nash quantity x^*.

The latter assumption is a weak consistency assumption implying that the Cournot-Nash equilibrium x^* (or equivalently $\mathbf{X}^*$) is the unique equilibrium quantity of the quantity dynamics: given that all firms have been playing the Cournot-Nash equilibrium quantity for L periods, each behavioural rule specifies to play the Cournot-Nash equilibrium in the next period. Assumption 1 excludes multiplicity of of equilibria. We will see that even in this simple case with a unique equilibrium instability and endogenous fluctuations can arise. Assumption 1 also excludes the possibility that firms

[4] Notice that this specification allows for different memory lenghts for different behavioural rules. That is, if behavioural rule i has a memory of $L_i + 1$ periods, then L in the present notation should be interpreted as $\max_{i \in \{1,\ldots,K\}} L_i$.

unilaterally deviate from the Cournot-Nash equilibrium or the case where different rules lead to different types of equilibria, for example, the Cournot-Nash solution on the one hand and the kartel solution on the other hand. The emergence of endogenous fluctuations in the present model indicates that these fluctuations are generic features of more complicated evolutionary models.

2.2 Evolutionary Competition between Behavioural Rules

We will now describe how the fractions $n_{i,t}$ evolve over time. The choice of a behavioural rule is based upon its past performance. Let $\pi(x_{it}, x_{jt})$ denote the realized profit of a player supplying x_{it} when he is matched with a player supplying $x_{j,t}$ in period t. More specifically,

$$\pi_{ijt} \equiv \pi(x_{it}, x_{jt}) = P(x_{it} + x_{jt})\, x_{it} - C(x_{it}).$$

To keep the dynamics of the model analytically tractable, we only use the realized profit in the last period to determine the fitness measure of a behavioural rule. The probability of meeting a player supplying x_{jt} in period t is equal to n_{jt}. Hence, the net average profit of behavioural rule H_i in period t is

$$U_{i,t} = \sum_{j=1}^{K} n_{jt}\pi_{ijt} - T_i = \Pi^i(\mathbf{x}_t, \mathbf{n}_t) - T_i,$$

where $\mathbf{n}_t = (n_{1t}, \ldots, n_{Kt})'$ is the vector of population fractions, $\Pi^i(\mathbf{x}_t, \mathbf{n}_t)$ is the average profit of firms using behavioural rule H_i in period t and T_i are the associated information costs. We assume that the fitness measures U_{it} are publicly observable.[5]

The fitness measures U_{it} are used to determine the population fractions $n_{i,t+1}$. Population dynamics are often modelled by means of the replicator dynamics. As shown by Gale, Binmore, and Samuelson (1995), Binmore and Samuelson (1997), and Schlag (1998), using the replicator dynamics to represent the population dynamics can be motivated in the context of a learning, aspiration, or imitation story. Furthermore, the simplicity of the replicator dynamics lends it considerable appeal. In this paper we will mainly focus on the replicator dynamics with mutational noise (see e.g. Gale,

[5] The definition of the fitness measure can be generalized in a straightforward manner to include the performance of the behavioural rule over the last M periods. In case there is a delay in evaluating the behavioural rules, the fitness measure $U_{i,t}$ becomes

$$U_{i,t} = \sum_{m=0}^{M} w_{i,m} \sum_{j=1}^{K} n_{j,t-m}\pi_{i,j,t-m} - T(I_i) = \sum_{m=0}^{M} w_{i,m}\Pi^i(\mathbf{x}_{t-m}, \mathbf{n}_{t-m}) - T(I_i), \forall i,$$

where $\sum_{k=0}^{M} w_{i,k} = 1, \forall i$, and $w_{i,k} \geq 0, \forall i, k$. Including more memory in the fitness measure obviously increases the dimension of the evolutionary system, making it analytically intractable. Numerical simulations indicate that incorporating a delay yields similar results as working with the more basic definition of a fitness measure used in this paper.

Binmore and Samuelson (1995), Vega-Redondo (1996) and Young and Foster (1991)). The population dynamics then looks like

$$n_{i,t+1} = (1 - K\delta) \frac{n_{it} U_{it}}{\sum_{j=1}^{K} n_{jt} U_{jt}} + \delta, \qquad i = 1, \ldots, K, \tag{4}$$

where $\delta < \frac{1}{K}$ is assumed to be small. The parameter δ corresponds to the mutational noise level and its interpretation is as follows. Each period a fraction of $K\delta$ firms leaves the market and is replaced by new firms. These new firms randomly choose one of the K behavioural rules. Since this mutational noise is independently distributed across individuals, a law of large numbers argument implies that this noise corresponds to a *deterministic* perturbation of the replicator dynamics. In fact, it follows that each behavioural rule is chosen by a fraction $\frac{1}{K}$ of the new firms. It can be easily seen from (4) that the presence of the noise implies that all steady states have to be interior: the steady states in which only one behavioural rule is played are destroyed. In fact, the interior equilibria have the property that $\delta < n_i^* < 1 - (K-1)\delta$ for all $i = 1, \ldots, K$. By Assumption 1 we know that x^* is the unique equilibrium quantity of the dynamical system (3). Hence, in equilibrium profits (apart from information costs) Π^* from the duopoly game are equal to the Cournot-Nash equilibrium profits. Consequently, the equilibrium population fractions n_i^*, $i = 1, \ldots, K$, are determined by the following equations:

$$n_i^* = (1 - K\delta) \frac{n_i^* (\Pi^* - T_i)}{\sum_{j=1}^{K} n_j^* (\Pi^* - T_j)} + \delta, \tag{5}$$

for all i, $i = 1, \ldots, K$. We make the natural assumption that for each rule information costs are lower than equilibrium profits, $\Pi^* - T_i > 0$, for all i. Since (5) defines a continuous mapping from the compact set $[0,1]^K \subset \mathbb{R}^K$ into itself, Brouwer's fixed point theorem can be used to show existence of equilibrium population fractions n_i^*, $i = 1, \ldots, K$.

When $T_i = T$ for all i, $i = 1, \ldots, K$, it follows immediately from (5) that the equilibrium population fractions are unique and given by $n_i^* = \frac{1}{K}$, for all i, $i = 1, \ldots, K$. Uniqueness of the equilibrium population fractions in case the costs of the behavioural rules are different, follows from the next Lemma.

Lemma 1 *Assume $T_1 > \ldots > T_K \geq 0$. Then the equilibrium population fractions determined by (5) are unique. Furthermore, as δ approaches 0, n_K^* approaches 1 and n_i^* approaches 0 for all $i \neq K$.*

In order to indicate that the actual value of the equilibrium population fraction n_i^* depends on the parameter δ, we denote it as $n_i^*(\delta)$. From (5) it follows that for $\delta = 0$, there are K equilibria for the population dynamics, where in each equilibrium the whole population uses the same rule and by the above Lemma there is a unique equilibrium for $\delta > 0$. For δ small, in equilibrium the cheapest behavioural rule H_K is used by almost the whole population of firms, i.e., $n_K^*(\delta) \approx 1$.

2.3 Local Instability Results

The Cournot duopoly model with K heterogeneous behavioural rules introduced in the previous section, can be represented by a $2K$-dimensional dynamical system. We need the first K equations to keep track of the quantity dynamics as specified in (2), while the remaining K equations describe the population dynamics given by (4). In this subsection we present two results showing that the equilibrium in our evolutionary system can easily become locally unstable.

Let $(x^*, \mathbf{n}^*)$, with $\mathbf{n}^* = (n_1^*, \ldots, n_K^*)'$ denote the equilibrium of the dynamical system. Assumption 1 implies that x^* is the unique equilibrium quantity of the Cournot duopoly model with heterogeneous behavioural rules. Consider such a model with all population fractions being fixed at $\frac{1}{K}$, i.e., $x_{i,t+1} = H_i\left(\mathbf{X}_t, \bar{\mathbf{N}}\right)$ for all $i = 1, \ldots, K$, where $\bar{\mathbf{N}} \in M_{K\times(L+1)}\left(\left\{\frac{1}{K}\right\}\right)$. Thus, firms stay uniformly distributed over the set of behavioural rules. We make the following generic assumption:

Assumption 2 The Cournot-Nash equilibrium quantity x^* of the model with all population fractions being fixed at $\frac{1}{K}$ is hyperbolic, i.e., for all i, $i = 1, \ldots, K$, the linearization of $\mathbf{x}_{t+1} = \mathbf{H}\left(\mathbf{X}_t, \bar{\mathbf{N}}\right)$ at x^* has no eigenvalues on the unit circle, and at least one of its eigenvalues lies outside the unit circle.

The above assumption means that when all fractions of firms are fixed and uniformly distributed over the set of behavioural rules, the unique equilibrium quantity x^* of the resulting quantity dynamics is unstable.

The first instability result for our evolutionary system says that when all behavioural rules involve equal costs, the unstable equilibrium of the dynamical system with all population fractions being fixed at $\frac{1}{K}$ cannot be stabilized by switching of behavioural rules. Given that all behavioural rules involve equal costs, we can assume without loss of generality that $T_i = 0$ for all i, $i = 1, \ldots, K$.

Proposition 2 *Assume 1 and 2 and* $T_i = 0$ *for every behavioural rule* H_i, $i = 1, \ldots, K$. *Then the equilibrium* $(x^*, \mathbf{n}^*)$ *of the evolutionary dynamical system, given by (3) and (4), is locally unstable.*

A second instability result arises when there are different costs associated with the behavioural rules. Assume without loss of generality that $T_1 > \ldots > T_K \geq 0$. We replace Assumption 2 by Assumption 2'.

Assumption 2' When the whole population of firms uses the cheapest behavioural rule H_K, the Cournot-Nash quantity x^* is hyperbolic and locally unstable.

In an economic setting it is natural to take the cheapest behavioural rule to be some myopic or naive rule of thumb. Such a behavioural rule typically includes a time lag, does not incorporate any knowledge about the way the market works, and lacks information on the equilibrium quantities. Furthermore, a myopic rule is likely to include persistent forecast errors with respect to the behaviour of other market participants, or not to take this behaviour into account at all. It is not difficult to imagine that markets are unstable under such a behavioural rule.[6]

Consider the dynamical system given by (3) and (4) again. For a small noise level δ, at the equilibrium $(x^*, \mathbf{n})$, most firms will use the cheapest behavioural rule H_K. The second instability result says that local instability is inherent in situations where more sophisticated behavioural rules are more expensive.

Proposition 3 *Assume 1, 2', and $T_1 > \ldots > T_K \geq 0$. If the noise level δ is sufficiently small, then the equilibrium $(x^*, \mathbf{n}^*)$ of the evolutionary dynamical system, given by (3) and (4), is locally unstable.*

3 Best-Reply versus Rational Firms

In this section we consider a simple, but typical, example of the evolutionary Cournot model introduced in section 2 with $K = 2$ behavioural rules, namely, a best-reply behavioural rule and a so-called rational behavioural rule. Let n_t be the fraction of rational firms in period t and let x_t and y_t denote the quantities supplied by the best-reply firms and the rational firms in period t, respectively.

With respect to the best-reply firms we assume that they know average industry output in the previous period and play a best-reply against this average industry output. Best-reply firms thus in fact have naive expectations: they use average industry output in the previous period as a proxy for the expected output of the (unknown) competitor they have to play a Cournot duopoly game with in the current period. Since average industry output in period t is equal to $\overline{x}_t = n_t y_t + (1 - n_t) x_t$, the best-reply firms supply the quantity

$$x_{t+1} = R(\bar{x}_t) = R(n_t y_t + (1 - n_t) x_t) \tag{6}$$

in period $t + 1$. From the above expression it follows that the introduction of a certain amount of rational firms may stabilize the best-reply dynamics. In fact, we have $\frac{dR(\bar{x}_t)}{dx_t} = (1 - n_t) R'(\overline{x}_t)$. Consequently, if the fraction n_t is large enough the resulting best-reply dynamics may become stable, even though the original best-reply dynamics is not $(|R'(x^*)| > 1)$.

Rational firms reason much more sophisticated than best-reply firms. Rational firms are perfect foresight firms, knowing both future quantities x_{t+1} and y_{t+1} supplied

[6] One might argue that some of the cheapest rules are typically those rules that are very stable, for instance, the rule that specifies to produce the same amount in the current period as in the previous period. However, for this type of rules instability emerges since these rules lead to different equilibrium quantities (recall the discussion following Assumption 1) . For the subset of rules that have the Cournot-Nash equilibrium quantity as their unique equilibrium Assumption 2' seems to be more relevant, as argued in the text.

by both groups at time $t+1$ as well as the fractions n_{t+1} and $(1-n_{t+1})$ of firms playing according to the rational respectively best reply behavioural rule in period $t+1$. The rational strategy is implicitly defined as follows

$$y_{t+1} = R\left(n_{t+1}y_{t+1} + \left(1 - n_{t+1}\right)x_{t+1}\right). \tag{7}$$

The strategy of the rational firms is something like a Nash equilibrium in a game which is "contaminated" with a number of best-reply firms.[7] Because of the extreme computational and informational requirements of the rational behavioural rule it seems logical to associate costs $T > 0$ with using this rule. Since we assumed $R(\cdot)$ to be downward sloping the rational behavioural rule can be determined explicitly as

$$y_{t+1} = G\left(x_{t+1}, n_{t+1}\right).$$

Note that both rules discussed above are unbiased in the sense that they have the Cournot-Nash equilibrium quantity x^* as their unique equilibrium quantity. Furthermore, x^* is also the unique equilibrium quantity of the complete dynamics of the model with heterogeneous behavioural rules. In fact, this dynamical system is given by

$$\begin{aligned} x_{t+1} &= f\left(x_t, y_t, n_t\right) = R\left(n_t y_t + \left(1 - n_t\right)x_t\right) \\ y_{t+1} &= g\left(x_t, y_t, n_t\right) = G\left(f\left(x_t, y_t, n_t\right), h_\delta\left(x_t, y_t, n_t\right)\right) \\ n_{t+1} &= h_\delta\left(x_t, y_t, n_t\right), \end{aligned}$$

where the mapping $h_\delta\left(x_t, y_t, n_t\right)$ represents the replicator dynamics with noise level δ. With respect to the above dynamical system it should be noted that, using (6) and (7) we have

$$x_{t+1} = R\left(n_t y_t + \left(1 - n_t\right)x_t\right) = y_t.$$

Hence, the only difference between the rational strategy and the best-reply strategy is that the latter lags one period behind. Therefore, we can reduce the dimension of the system by one and rewrite the model in terms of x_t, as the 2-dimensional dynamical system

$$\begin{aligned} x_{t+1} &= R\left(n_t G\left(x_t, n_t\right) + \left(1 - n_t\right)x_t\right) = G\left(x_t, n_t\right) \\ n_{t+1} &= \left(1 - 2\delta\right)\frac{n_t\left(\Pi^r\left(x_t, n_t\right) - T\right)}{n_t\left(\Pi^r\left(x_t, n_t\right) - T\right) + \left(1 - n_t\right)\Pi^b\left(x_t, n_t\right)} + \delta, \end{aligned} \tag{8}$$

where Π^r and Π^b are realized profits; see below for an explicit example.

Next consider the local stability of the evolutionary model. In particular we focus on the question what happens when evolutionary pressure increases, that is, when the noise level δ approaches zero. The local stability properties of the equilibrium of the dynamical system (8) can be summarized as follows:

[7]Notice that if $n_{t+1} = 1$, that is, if every firm is rational, it follows that all firms supply the Cournot-Nash equilibrium quantity.

Proposition 4 *Consider the dynamical system (8). Let* $0 < \delta < \frac{1}{2}$, $T > 0$, *and* $R'(x^*) < -1$. *The equilibrium of (8) is* (x^*, n^*), *where* x^* *is the Cournot-Nash equilibrium quantity and*

$$n^* = \frac{1}{2T}\left(\delta\left(2\Pi^* - T\right) + T - \sqrt{\delta^2\left(2\Pi^* - T\right)^2 + \left(1 - 2\delta\right)T^2}\right).$$

Furthermore, the equilibrium (x^*, n^*) *is locally stable (unstable) for* $\delta > \delta^*$ $(\delta < \delta^*)$, *where*

$$\delta^* = \frac{\left(R'\left(x^*\right)^2 - 1\right)T}{2R'\left(x^*\right)\left(T\left(R'\left(x^*\right) + 1\right) - 2\left(\alpha + \Pi^*\right)\right)}.$$

At $\delta = \delta^*$ *a flip bifurcation occurs.*

Proposition 4 says that the equilibrium of (8) loses stability for low values of the noise level δ. The evolutionary dynamics becomes unstable when the eigenvalue λ_1 becomes smaller than -1, or equivalently, when

$$n^* < \frac{1}{2} + \frac{1}{2R'(x^*)}.$$

Note that if the best-reply dynamics are locally stable, i.e. if $-1 \leq R'(x^*) \leq 0$ and/or if there are no information costs, i.e. $T = 0$, the equilibrium is stable for all values of the noise level $\delta > 0$. However, when the best-reply dynamics are locally unstable, i.e. $R'(x^*) < -1$, and if information costs are strictly positive, i.e. $T > 0$, then as the noise level δ decreases, the equilibrium fraction n^* tends to 0, and the evolutionary dynamics becomes locally unstable. The economic intuition for this instability is simple. A low noise level means that there are hardly any firms who choose a behavioural rule at random. As a result, almost all firms will use the myopic behavioural rule whenever the dynamical system is close to the equilibrium, since the myopic rule generates higher net profit because there are no information costs associated with this rule. Since the equilibrium is unstable under this myopic behavioural rule quantities diverge from the Cournot-Nash equilibrium value.

4 Global Dynamics

According to Proposition 4, as evolutionary pressure increases, the equilibrium of the evolutionary Cournot duopoly model with the replicator dynamics with noise becomes locally unstable. In this section we investigate the global dynamics of the evolutionary Cournot model as evolutionary pressure increases. In order to investigate the global dynamics, we have to specify the firms reaction functions, and we focus on the linear-quadratic specification of the Cournot model. It will be convenient to write the model in terms of deviations X_t from the Cournot-Nash quantity x^*, that is, $X_t = x_t - x^*$. Straightforward calculations show that the model then becomes

$$
\begin{aligned}
X_{t+1} &= f(X_t, n_t) = -\frac{b(1-n_t)}{(2+n_t)b-d} X_t \qquad (9)\\
n_{t+1} &= g_\delta(X_t, n_t) = (1-2\delta)\frac{n_t\left(\Pi^r(X_t,n_t)-T\right)}{n_t\left(\Pi^r(X_t,n_t)-T\right)+(1-n_t)\Pi^b(X_t,n_t)}+\delta,
\end{aligned}
$$

with Π^b and Π^r the profits of the best reply and the rational firms respectively, given by

$$
\begin{aligned}
\Pi^b(X_t,n_t) &= \left(b-\frac{1}{2}d\right)\left(\frac{a-c}{3b-d}\right)^2 - b(1-n_t)\left(\frac{a-c}{3b-d}\right)\left(\frac{2b-d}{(2+n_t)b-d}\right)X_t \qquad (10)\\
&\quad -\left(b-\frac{1}{2}d\right)\left(\frac{(4-n_t)b-d}{(2+n_t)b-d}\right)X_t^2
\end{aligned}
$$

and

$$
\Pi^r(X_t,n_t) = \left(b-\frac{1}{2}d\right)\left(\frac{b(1-n_t)}{(2+n_t)b-d}X_t - \frac{a-c}{3b-d}\right)^2. \qquad (11)
$$

We shall refer to this dynamical system as $(X_{t+1}, n_{t+1}) = F_\delta(X_t, n_t)$, and investigate the global dynamical behaviour as evolutionary pressure increases, that is when the noise level δ decreases. In our global analysis we focus on a typical case, with all other parameters fixed at:

$$
a = 17,\ b = 1,\ c = 10,\ d = 1.1 \text{ and } T = 1. \qquad (12)
$$

We emphasize that for other nearby choices of the parameters essentially the same results as those presented below are obtained. With the parameters fixed as in (12), the equilibrium fraction is given by

$$
n^* = \frac{1}{2}\left(11\delta + 1 - \sqrt{(11\delta)^2 + (1-2\delta)}\right).
$$

From Proposition 4 it follows that the dynamical system undergoes the primary, period doubling bifurcation at $\delta^* = \frac{6859}{801020} \approx 0.008563$, in which a (locally) stable period 2 cycle is created as δ decreases.[8] As the noise level δ further decreases, a secondary bifurcation occurs in which the 2-cycle becomes unstable. In this particular example the period 2 cycle undergoes a Neimark-Sacker bifurcation at $\delta^{NS} \approx 0.00107$, in which

[8] The stability of a 2-cycle created in a period doubling bifurcation depends upon the third order coefficient of the normal form, which depends upon second and third order derivatives at the steady state; see e.g. Kuznetsov (1998, pp 119-123)). In our case, this normal form coefficient $c \approx 0.1253$, which is positive implying that the period doubling bifurcation is supercritical and the 2-cycle is indeed stable.

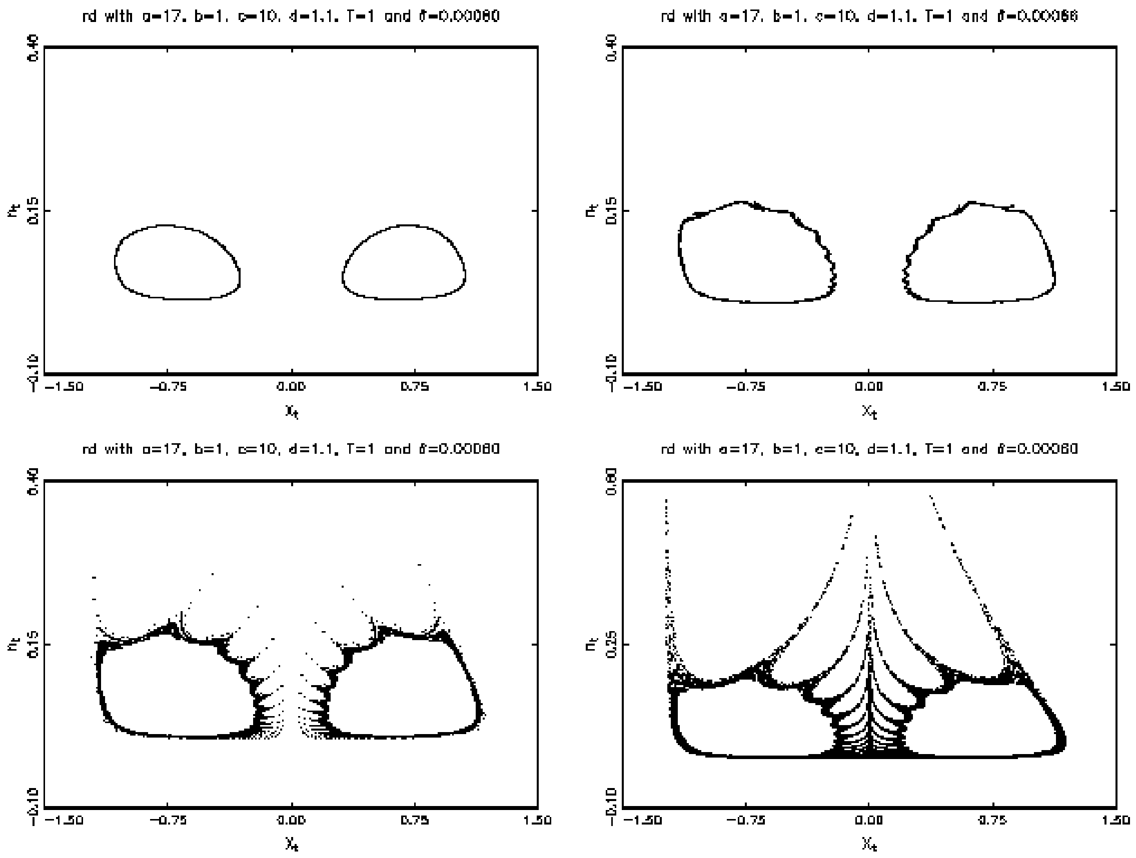

Figure 1: Attractors for the model with best-reply versus rational players and replicator dynamics with $a = 17$, $b = 1$, $c = 10$, $d = 1.1$, $T = 1$ and a) $\delta = 0.00080$, b) $\delta = 0.00066$, c) $\delta = 0.00064$ and d) $\delta = 0.00060$.

an attracting invariant set consisting of two 'circles', one around each point of the 2-cycle, is created.[9] Figure 1 shows some typical attractors of the evolutionary Cournot model, for decreasing values of $\delta < \delta^{NS}$, with (quasi-)periodic motion just after the Neimark-Sacker bifurcation and breaking of the invariant circles into a strange attractor as the parameter δ further decreases. Figure 2 shows the corresponding time series for $\delta = 0.00060$. The chaotic time series is characterized by periods of low volatility, with quantities close to the Cournot Nash equilibrium and periods of high volatility

[9] The Neimark-Sacker bifurcation with complex eigenvalues crossing the unit circle, is the discrete time analogue of the Hopf bifurcation. Whether the invariant 'circles' around the 2-cycle are attracting or repelling depends upon the coefficient of a cubic term in the normal form of the Neimark-Sacker bifurcation, which depends upon higher order derivatives at the 2-cycle; see e.g. Kuznetsov (1998, pp 125-137). In our case, this normal form coefficient $a \approx -9.1308$, which is negative implying that the Neimark-Sacker bifurcation is supercritical and the invariant circles are indeed locally attracting.

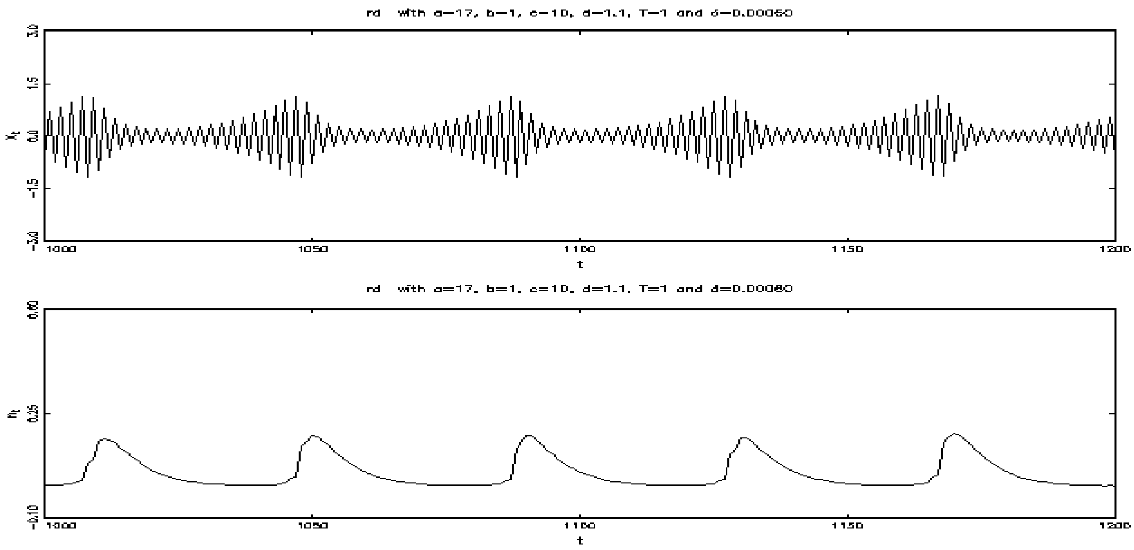

Figure 2: Time series of quantities and fraction of rational players for the model with best-reply versus rational players and replicator dynamics with $a = 17$, $b = 1$, $c = 10$, $d = 1.1$, $T = 1$ and $\delta = 0.00060$.

with quantities diverging from the Cournot-Nash equilibrium. If the dynamical system moves closer to the Cournot-Nash equilibrium quantity, the fraction of firms using the best-reply behavioural rule will become high. This causes the dynamical system to become unstable and drives the quantities away from the Cournot-Nash equilibrium quantity, making it profitable for the firms to use the rational behavioural rule. Consequently, the fraction of rational firms will increase and the dynamical system stabilizes again, i.e. quantities return close to the Cournot-Nash equilibrium.

4.1 Homoclinic Bifurcations

In fact, Figure 1 suggests the occurrence of chaos and strange attractors as evolutionary pressure increases. We will show that the occurrence of complicated evolutionary dynamics can be explained by a so-called homoclinic bifurcation between the stable and the unstable manifolds of the Cournot-Nash equilibrium saddle point. In order to be self-contained, we first briefly describe recent advances in the theory of homoclinic bifurcations, explaining the occurrence of strange attractors; see Palis and Takens (1993) for a comprehensive mathematical treatment, and de Vilder (1996) and Brock and Hommes (1997) for recent economic applications.

Consider a differentiable two-dimensional map $F_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, where $\alpha \in \mathbb{R}$ is a parameter. Let p be a saddle fixed point, that is, let the Jacobian of F_α evaluated at

p have two real eigenvalues λ_1 and λ_2, with $0 < |\lambda_2| < 1 < |\lambda_1|$.[10] The *local stable and unstable manifolds* of the fixed point p are defined as

$$\begin{aligned} W^s_{loc}(p) &= \left\{x \in U | F^t(x) \to p \text{ for } t \to +\infty\right\}, \\ W^u_{loc}(p) &= \left\{x \in U | F^t(x) \to p \text{ for } t \to -\infty\right\}, \end{aligned}$$

where U is some small neighborhood of p. These local manifolds are tangent to the corresponding stable and unstable eigenvectors of the linearized dynamical system at the fixed point. The *global stable and unstable manifolds* are then defined as

$$W^s(p) = \cup_{t \leq 0} F^t\left(W^s_{loc}(p)\right) \text{ and } W^u(p) = \cup_{t \geq 0} F^t\left(W^u_{loc}(p)\right).$$

Notice that these manifolds are invariant under F, that is, $F(W^s(p)) = W^s(p)$ and $F(W^u(p)) = W^u(p)$. If F is a linear system, these manifolds correspond to the stable and unstable eigenvectors. However, for a nonlinear mapping the stable and unstable manifolds can have a much more complicated structure. In particular, they can have intersections, that is, there may exist a point $q \neq p$ with $q \in W^s(p) \cap W^u(p)$. Such a point q is called a *point of homoclinic intersection.* Since the stable and unstable manifolds are invariant under F, we then must have that $F^t(q)$, $t = \pm 1, \pm 2, \ldots$, are also points of homoclinic intersection. The sequence of points $\{F^t(q)\}_{t=-\infty}^{t=\infty}$ then is called a *homoclinic orbit.* It was observed already by Poincaré that existence of such a homoclinic orbit implies a very complicated global geometric structure of the unstable and stable manifolds. Much later, Smale (1963) showed that existence of a homoclinic intersection implies that the map F has infinitely many *horseshoes,* that is, invariant *Cantor* sets Λ containing infinitely many periodic points as well as an uncountable set of non-periodic points, and that F exhibits *sensitive dependence on initial conditions* with respect to Λ. However, horseshoes are not attracting and the invariant sets Λ typically have Lebesgue measure zero, so that the complicated dynamics on Λ may only affect the short run behaviour, and for Lebesgue almost all initial states in the long run the orbits may still converge to a stable cycle or even a stable fixed point.

The map F_α, and therefore also the stable and unstable manifolds, depend upon the parameter α. A *homoclinic bifurcation* is said to occur at $\alpha = \alpha_0$ when for $\alpha > \alpha_0$ there is no intersection between the unstable manifold $W^u(p)$ and the stable manifold $W^s(p)$, for $\alpha = \alpha_0$ there is a point of homoclinic tangency between $W^s(p)$ and $W^u(p)$, where the stable and unstable manifolds are tangent, and for $\alpha < \alpha_0$ there is a point of transversal homoclinic intersection. The eigenvalues λ_1 and λ_2, with $0 < |\lambda_2| < 1 < |\lambda_1|$, at the fixed point turn out to be important for the global dynamics 'close' to a homoclinic bifurcation. A saddle point is called *dissipative* if the absolute value of the product of the eigenvalues is smaller than 1, i.e., $|\lambda_1 \lambda_2| < 1$. An important recent result in the theory of nonlinear dynamical systems is the following:

[10] The theory on homoclinic bifurcations can also be applied to a periodic saddle, but for the moment we focus on fixed points of F.

Theorem 5 *('Strange Attractor Theorem', Mora and Viana (1993); see also Palis and Takens (1993, pp. 139-143)) Let* $F_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ *be a two-dimensional map with parameter* α*, and let* p *be a dissipative (periodic) saddle point. If the map* F_α *exhibits a generic homoclinic bifurcation between the stable and the unstable manifolds of the (periodic) saddle point at* $\alpha = \alpha_0$*, then there exists a positive Lebesgue measure set* $\Omega \subset (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$ *, such that for all* $\alpha \in \Omega$ *the map* F_α *has a strange attractor.*

Roughly speaking the theorem states that close to a homoclinic bifurcation strange attractors arise with positive probability in the parameter space.[11] We have formulated the theorem for homoclinic bifurcations associated to a dissipative saddle fixed point, but the results also hold for homoclinic bifurcations associated to dissipative periodic saddle points. In addition to the existence of strange attractors, a number of other complicated phenomena occur due to homoclinic bifurcations, such as coexistence of infinitely many stable cycles (the so-called Newhouse phenomenon) and cascades of infinitely many period doubling and period halving bifurcations, but here we will focus on strange attractors.

4.2 A Homoclinic Bifurcation due to Evolutionary Pressure

The first step in the global analysis lies in identifying parts of the stable and unstable manifolds of our nonlinear evolutionary system (9). We know that for values of δ below the flip bifurcation value δ^* the equilibrium is a saddle-point since λ_1 is smaller than -1 and λ_2 lies between 0 and 1; the corresponding eigenvectors are given in Appendix 6.3. We first determine a part of the stable manifold of the equilibrium (x^*, n^*). The first picture in Figure 3 shows three components of this stable manifold.

The first component (component I) consists of the points with $X = 0$. Clearly, if the current quantity is at its Cournot-Nash equilibrium value, then according to both the best-reply rule and the rational rule the Cournot-Nash quantity has to be played in the next period, implying that $f(0, n) = 0$, for all n. Furthermore, for $X = 0$, the evolution of the fraction of rational firms is governed by

$$n_{t+1} = g_\delta(0, n_t) = (1 - 2\delta)\, n_t \frac{\Pi^* - T}{\Pi^* - n_t T} + \delta,$$

where Π^* are the profits made in equilibrium. $g_\delta(0, n)$ is an upward sloping function with a unique equilibrium, at which the slope lies between 0 and 1 (which, of course, is equal to the second eigenvalue λ_2). Therefore the fractions converge to the unique equilibrium n^*. The second component (II) consists of the points where all firms use the rational rule, i.e. where $n = 1$. If all firms are rational, everybody will

[11] A number of generic conditions concerning the homoclinic tangency have to be satisfied in order for the theorem to hold; see Palis and Takens (1993, p.35) for details. These conditions have been considerably simplified by Takens (1992), who provides conditions which are relatively easy to check for concrete examples.

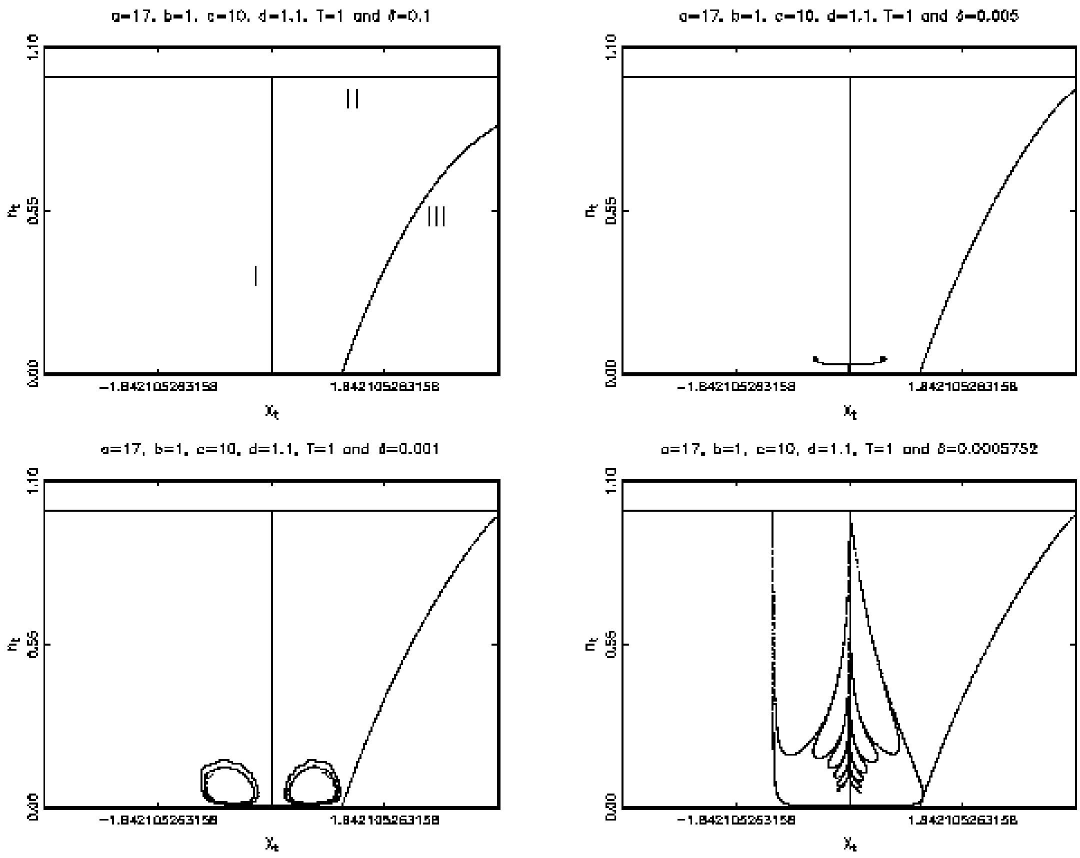

Figure 3: Stable and unstable manifolds for the evolutionary model of Cournot competition with replicator dynamics with $a = 17$, $b = 1$, $c = 10$, $d = 1.1$, $T = 1$. a) Three parts of the stable manifold for $\delta = 0.1$, b) Stable and unstable manifolds for $\delta = 0.005$, c) Stable and unstable manifolds for $\delta = 0.001$ and d) Stable and unstable manifolds for $\delta = 0.0005752$.

play the Cournot-Nash equilibrium in the next period. And indeed as can be seen from (9) the points with $n = 1$ are mapped onto the point $(0, 1-\delta)$, which lies in component I. Therefore all points of component II converge to the equilibrium by the same argument as above, so that component II must be part of the stable manifold. We can take this argument one step further by considering points (X, n) which are mapped onto component II, that is, points (X, n) satisfying

$$g_\delta (X, n) = 1.$$

Since $\Pi^r (X, n)$ and $\Pi^b (X, n)$ are quadratic in X we can compute two branches of solutions for this equation. The necessary computations are given in Appendix 6.4. One of these branches can be ignored, since $X < -x^*$ and therefore such points correspond to negative production; the other branch corresponds to component III

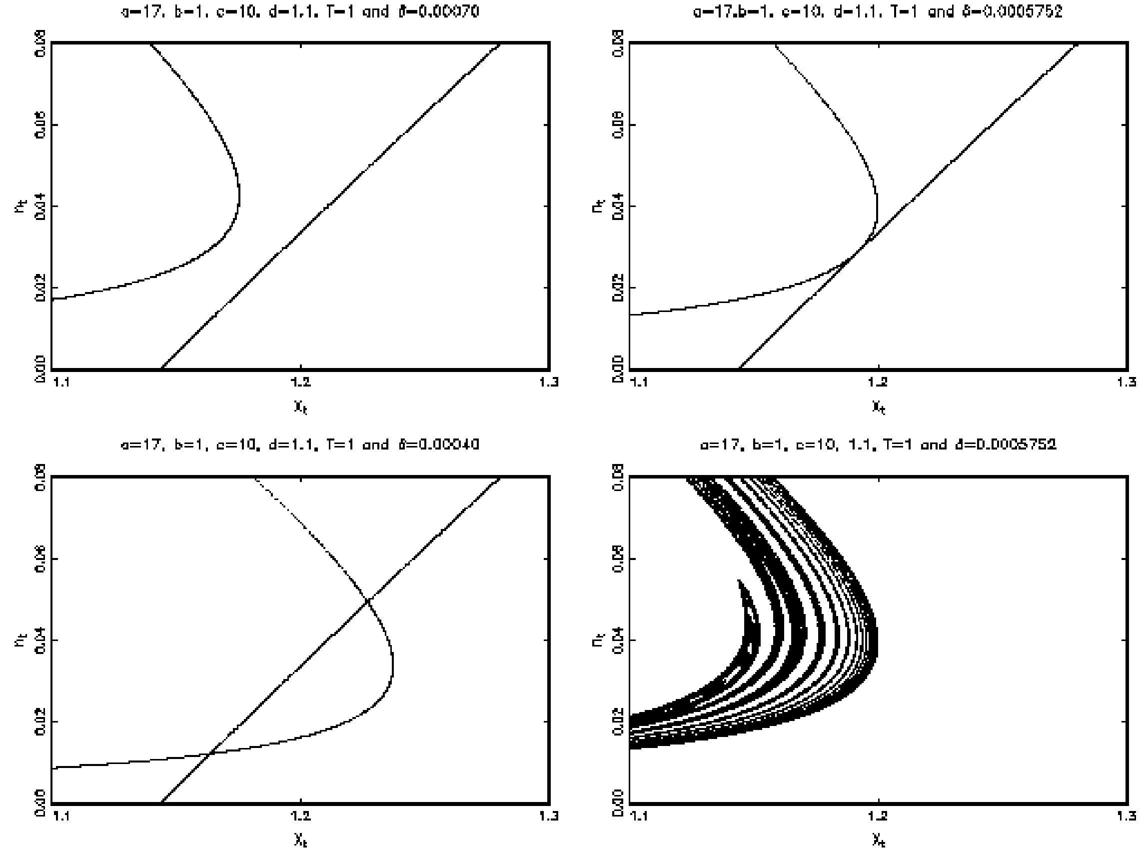

Figure 4: Homoclinic bifurcation: a) Stable and unstable manifolds for $\delta = 0.00070$, b) Stable and unstable manifolds for $\delta = 0.0005752$, c) Stable and unstable manifolds for $\delta = 0.00040$ and d) Part of the strange attractor for $\delta = 0.0005752$.

in Figure 3. The points in this third component are mapped onto points with $n = 1$, for which the argument given above can again be applied. Notice that this third component varies with the parameters of the model (in particular, it depends upon δ), whereas the first two components are independent of parameters.

We have been able to explicitly compute a part of the stable manifold. Unfortunately, one cannot compute the unstable manifold explicitly. However, since we know that the local unstable manifold lies tangent to the unstable eigenvector at the equilibrium we can approximate the local unstable manifold by a small piece of the unstable eigenvector. Figure 3 shows the stable and unstable manifolds for different values of δ. Since the unstable eigenvalue of the equilibrium $\lambda_1 < -1$, the unstable manifold has two branches. In Figure 3b ($\delta = 0.005$) we see that two branches of the unstable manifold of the equilibrium approach the stable period two orbit, whereas in Figure 3c ($\delta = 0.001$) the branches spiral to a set consisting of two invariant circles.

Figure 3d ($\delta = \delta^c \approx 0.005752$) shows a homoclinic tangency between the unstable manifold and component III of the stable manifold of the equilibrium. Figure 4 shows an enlargement of a small piece of the stable and unstable manifold. From the first three pictures it is clear that a homoclinic bifurcation occurs at $\delta^c \approx 0.0005752$. For $\delta = 0.00070$ there is no intersection between the stable and the unstable manifold, for $\delta = \delta^c$ there is a homoclinic tangency and for $\delta = 0.00040$ two points of transversal homoclinic intersection are shown. Figure 4 is thus numerical evidence of the occurrence of a homoclinic bifurcation as evolutionary pressure increases, that is when the parameter δ decreases and approaches 0. The fourth picture shows an enlargement of the strange attractor, with a complicated, Cantor-like structure, in this part of the state space. Furthermore, at $\delta = 0.0005752$, we have $|\lambda_1 \times \lambda_2| \approx 0.92235$, hence the equilibrium is dissipative at the homoclinic bifurcation and the strange attractor theorem from the previous section applies:

Theorem 6 *('Strange attractors under evolutionary pressure') Fix the other parameters as in (12). There exists a positive Lebesgue measure set of δ-values in the parameter interval $(\delta^c, \delta^c + \varepsilon)$ for which the map F_δ generating the evolutionary dynamics has a strange attractor.*

In fact, a sketch of the proof of this theorem has already been given by the geometric arguments discussed above; additional technical details that the strange attractor theorem indeed applies in this case are given in Appendix 6.6.[12] Recall that for slightly different choices of the other parameter values than those in (12) the theorem still applies, because the stable and unstable manifolds of the equilibrium depend continuously on the parameters.

Our results are related to the occurrence of complicated dynamics in the cobweb model with discrete choice dynamics in Brock and Hommes (1997). However, we would like to point out an important methodological simplification of the mechanism in the replicator dynamics with noise. Brock and Hommes (1997) have shown that a bifurcation route to strange attractors occurs when the intensity of choice parameter β tends to infinity, or equivalently, when observation noise to evolutionary fitness tends to zero. A nontrivial problem in their mathematical analysis was that, for finite β a homoclinic tangency between the stable and unstable manifolds of the equilibrium steady state can not arise, but that for $\beta = +\infty$ the stable and unstable manifolds in fact coincide. In order to circumvent these problems, they use a complicated horseshoe construction to show that the fourth iterate of their 2-D mapping has a full horseshoe, which implies that there must be homoclinic tangencies between the stable and unstable manifolds of *periodic* saddle points. In the replicator dynamics with noise, this complicated horseshoe construction is not necessary and the mathematical

[12] The homoclinic bifurcation occurs at parameter value δ^c, as δ decreases. In particular, it is shown in the appendix that strange attractors already arise close to, but before the homoclinic tangency, i.e. for $\delta > \delta^c$. After the homoclinic bifurcation, i.e. for $\delta < \delta^c$, fractions n_t of rational players may become larger than 1, and such trajectories have no well defined economic interpretation.

analysis is simplified considerably because a homoclinic tangency between the stable and unstable manifolds of the equilibrium steady state occurs for a small but positive δ-value.

5 Concluding Remarks

In this paper we have put forth an evolutionary model of Cournot competition. Our setup allows us to study games with a continuous action space and focuses on evolutionary competition between different modes of behaviour, that might reflect different degrees of sophistication of the firms. This approach can be applied to more general market games and in fact to any game with a continuous action space.[13] The evolutionary competition between behavioural rules coupled with the quantity dynamics arising from the interplay between these different behavioural rules leads to a highly nonlinear dynamical system, the behaviour of which critically depends upon the stability of the quantity dynamics. First, in case the cheapest behavioural rule is stable, the evolutionary process converges to a situation where most firms use this behavioural rule and produce quantities equal to the Cournot-Nash equilibrium quantity. Second, if the cheapest behavioural rule is unstable, complicated endogenous fluctuations may occur. In particular, when the noise level is low, high evolutionary pressure with respect to the choice of behavioural rules leads to highly irregular quantity dynamics converging to a strange attractor. Note that the nonlinearity causing this erratic behaviour arises from the evolutionary part of the model, because in our specification the reaction curves were linear. For the case of best-reply firms against rational firms we have shown that the complicated behaviour of the Cournot model with the replicator dynamics is due to a homoclinic tangency between the stable and unstable manifolds of the unstable equilibrium. Simulations suggest that this complicated behaviour also is a feature of the evolutionary Cournot model when we focus on other behavioural rules; see Droste and Tuinstra (1998) for additional examples. Existence of strange attractors thus seems to be a robust feature in evolutionary modelling.

We believe that the main contribution of this paper to the literature is conceptual. We have used the Cournot duopoly game because it seems to be well suited to get our main points across, which are: the importance for evolutionary game theory to consider games with a continuous action space, the focus on evolutionary competition between behavioural rules instead of evolutionary competition between actions and the observation that endogenous fluctuations are a generic feature of evolutionary models. At first sight the random matching interaction structure used in this paper

[13] Some evolutionary models of Cournot competition assume that there are only a finite number of different quantities (typically including the Cournot-Nash equilibrium quantity as well as the kartel solution) that can be produced by the firms. Notice that this is a special case of our model, where each behavioural rule specifies to play a constant quantity x_i in each period independent of the quantities specified by other rules, or the fractions of firm using these other rules.

might not seem to be the most accurate description of competition in a lot of markets. Here we want to offer two alternative interpretations for this interaction structure. Firstly, the random matching structure can be interpreted as taking place at the management level. Consider an economy with a number of different commodities, where each commodity is produced by two firms. There is a pool consisting of managers, which frequently go from one firm to the other and which use different management strategies or behavioural rules. The random matching structure therefore should be interpreted as a matching of different managers playing against each other, rather than a matching of different firms. Secondly, the model can be interpreted at a two-firm level, where the population fractions correspond to the probabilities with wich management chooses a certain strategy. The probability that a manager is replaced, or an alternative management strategy is chosen, again depends upon realized profits. Future research will focus on other interaction structures, such as local interaction, where the firms are located at different positions in the product space and only compete with their neighbours in this product space. It is interesting to see whether the dynamical features of the random matching model would be preserved in such a local interaction framework.

Other interesting extensions of the model are allowing for the possibility of multiple Cournot-Nash equilibria or allowing for behavioural rules that introduce the possibility of coordination on a 'non-Nash'-equilibrium, such as the joint profit maximum (the cartel solution) or the 'Walrasian' equilibrium (where quantities are chosen in such a way that marginal costs equal the price).

Finally, we want to refer to Cox and Walker (1998), who report on some laboratory experiments with a Cournot duopoly model with random matching and exactly the same linear-quadratic specification as in our model. They find that in general the subjects do not converge to the Cournot-Nash equilibrium if the best-reply dynamics are unstable. This seems to corroborate the model of this paper. Additional experiments might help us in judging the validity of the approach to Cournot duopoly games put forth in this paper.

6 Appendix

6.1 Proofs

Proof of Lemma 1. We want to show that there exists a unique solution to

$$n_i = (1 - K\delta)\frac{n_i U_i}{\sum_{j=1}^{K} n_j U_j} + \delta \tag{13}$$

where $n_i > 0$ for all i, $\sum_{i=1}^{n} n_i = 1$ and $U_i = \Pi^* - T_i$ is independent of the fractions. Equation (13) is a continuous mapping from the unit simplex into itself, so we know a fixed point exists. Now we have to show uniqueness of this fixed point. Equation

(13) can be written as

$$(n_i - \delta) \sum_{j=1}^{K} n_j U_j = (1 - K\delta)\, n_i U_i$$

Define $c \equiv \sum_{j=1}^{K} n_j U_j$ and $b_i \equiv (1 - K\delta)\, U_i$. Given the value of c the equilibrium population fractions are uniquely determined by

$$n_i = \frac{c\delta}{c - b_i}. \tag{14}$$

Notice that nonnegativity of the equilibrium fractions requires $c = \sum_{j=1}^{K} n_j U_j > (1 - K\delta)\, U_i = b_i$. Also notice that from (14) it follows that the fractions n_i are decreasing in c. Now suppose there are two distinct nonnegative solutions of (13), namely $\mathbf{n}$ and $\mathbf{m}$. It follows from (14) that $\sum_{j=1}^{K} n_j U_j = \sum_{j=1}^{K} m_j U_j$ implies that $\mathbf{n} = \mathbf{m}$. Now suppose that

$$c_n = \sum_{j=1}^{K} n_j U_j > \sum_{j=1}^{K} m_j U_j = c_m.$$

Given that $c_n > b_i$ and $c_m > b_i$ for all i (and therefore that all fractions n_i and m_i are positive), this implies that

$$n_i = \frac{c_n \delta}{c_n - b_i} < \frac{c_m \delta}{c_m - b_i} = m_i.$$

This leads to a contradiction since

$$1 = \sum_{i=1}^{n} n_i < \sum_{i=1}^{n} m_i = 1.$$

Hence there is a unique positive equilibrium $\mathbf{n}$.

Furthermore, multiplying n_i by U_i, aggregating over i and using (14) we see that c is a solution to

$$\sum_{i=1}^{K} \frac{U_i}{c - (1 - K\delta)\, U_i} = \frac{1}{\delta}.$$

As δ approaches 0, and given that $c > (1 - K\delta)\, U_i = b_i$, we find that $c = \sum_{j=1}^{K} n_j U_j$ has to approach $(1 - K\delta)\, U_K \approx U_K = \max_{i=1,\ldots,K} U_i$. This implies that n_K approaches 1 and all other fractions approach 0 as δ approaches 0. ■

Proof of Proposition 2. Let $M_{m\times n}(S)$ denote the set of all $m \times n$ matrices with entries from the set S. Notice that since $T_i = 0$ for all i ,$i = 1, \ldots, K$, the equilibrium population fractions are $n_i^* = \frac{1}{K}$ for all i, $i = 1, \ldots, K$. For all i, $i = 1, \ldots, K$, the linearization of $x_{i,t+1} = H_i(\mathbf{X}_t, \mathbf{N}_t)$ at the equilibrium $(x^*, \mathbf{n}^*)$ is given by

$$dx_{i,t+1} = \mathbf{e}^T \left[\left.\frac{\partial H_i}{\partial \mathbf{X}_t}\right|_{(x^*,\mathbf{n}^*)} \circ d\mathbf{X}_t \right] \mathbf{e} + \mathbf{e}^T \left[\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(x^*,\mathbf{n}^*)} \circ d\mathbf{N}_t \right] \mathbf{e} \tag{15}$$

where $dx_{i,t+1} = x_{i,t+1} - x^*$, $dn_{i,t} = n_{i,t} - n_i^*$, $\mathbf{e}^T \in M_{1\times K}(\{1\})$, $\mathbf{e} \in M_{(L+1)\times 1}(\{1\})$, and $\circ$ denotes the Hadamard or entrywise product. From Assumption 1 it follows that

$$\left(\left. \frac{\partial H_i}{\partial N_t} \right|_{(x^*,\mathbf{n}^*)} \right)_{g,l} = 0 \tag{16}$$

for all $i, g = 1, \ldots, K$ and $l = 1, \ldots, L+1$, which implies that the second term on the right-hand side of (15) cancels. We are therefore left with

$$dx_{i,t+1} = \mathbf{e}^T \left[\left. \frac{\partial H_i}{\partial \mathbf{X}_t} \right|_{(x^*,\mathbf{n}^*)} \circ d\mathbf{X}_t \right] \mathbf{e}$$

which is exactly the linearization of $x_{i,t+1} = H_i\left(\mathbf{X}_t, \bar{\mathbf{N}}\right)$ at x^*. Now, since by (16) the upper right block of the Jacobian of the linearized system contains only zeros, all eigenvalues of the linearization of the $x_{i,t+1} = H_i\left(\mathbf{X}_t, \bar{\mathbf{N}}\right)$ at x^* are eigenvalues of the linearization of the complete system at x^*. Hence, assumption 2 implies that the equilibrium $(x^*, \mathbf{n}^*)$ of the dynamical system, given by (3) and (4), is also unstable. ■

Proof of Proposition 3. Consider the dynamical system given by (3) and (4). According to the replicator dynamics (4), $n_{i,t+1} = f(\mathbf{x}_t, \mathbf{n}_t)$ for all $i = 1, \ldots, K$. Hence the dynamics can be represented by

$$x_{i,t+1} = H_i(\mathbf{X}_t, \mathbf{N}_t) = \tilde{H}_i(\mathbf{x}_t, \ldots, \mathbf{x}_{t-L}, \mathbf{n}_{t-L}), \tag{17}$$

where $\mathbf{x}_{t-l} = (x_{1,t-l}, \ldots, x_{K,t-l})'$ for all l, $l = 0, \ldots, L$, $\mathbf{n}_{t-L} = (n_{1,t-L}, \ldots, n_{K,t-L})'$. First, notice that including the population fraction $n_{K,t-L}$ in (17) is redundant since $\sum_{i=1}^{K} n_{i,t-L} = 1$. Second, notice that (17) is of dimension $(L+2)K - 1$.

Let $\mathbf{x}^* \in M_{K\times 1}(\{x^*\})$. By Assumption 1 we know that for all i, $i = 1, \ldots, K$, it holds that

$$\tilde{H}_i(\mathbf{x}^*, \ldots, \mathbf{x}^*, \mathbf{n}_{t-L}) = x^*$$

for any $\mathbf{n}_{t-L}$. Thus, the partial derivative

$$\frac{\partial \tilde{H}_i(\mathbf{x}^*, \ldots, \mathbf{x}^*, \mathbf{n}_{t-L})}{\partial n_{j,t-L}} = 0$$

for all $j = 1, \ldots, K-1$ and $i = 1, \ldots, K$. Let $\mathbf{n}^*(\delta) = (n_1^*(\delta), \ldots, n_K^*(\delta))'$ denote the equilibrium population fractions and consider the behavioural rules H_i, $i = 1, \ldots, K$, such that

$$x_{i,t+1} = H_i(\mathbf{X}_t, \mathbf{N}^*(\delta)) = \widehat{H}_i(\mathbf{x}_t, \ldots, \mathbf{x}_{t-L}), \tag{18}$$

with $\mathbf{N}^*(\delta) = [\mathbf{n}^*(\delta), \ldots, \mathbf{n}^*(\delta)]_{K\times(L+1)}$, describing the quantity dynamics when the population fractions of firms using behavioural rules $H_1, \ldots, H_K$ are fixed at the equilibrium population fractions of the dynamical system. Using (18) it can be shown that the Jacobian of (17), evaluated at the equilibrium has $K-1$ eigenvalues $\lambda_1, \ldots, \lambda_{K-1}$ such that $0 < \lambda_1, \ldots, \lambda_{K-1} < 1$, while the other $(L+1)\,K$ eigenvalues are equal to the eigenvalues of (18), evaluated at the equilibrium.

For δ small, the equilibrium population fraction using the cheapest behavioural rule H_K approaches one, i.e., $n_K^*(\delta) \approx 1$. Now consider the system describing the quantity dynamics when the whole population of firms uses behavioural rule H_K. By Assumption 2', the equilibrium quantity x^* of this system is unstable. Since, for δ small, the system given by (18) gets C^1-close to the system in which the whole population of firms uses behavioural rule H_K, we conclude that for δ sufficiently small, at the equilibrium, the system specified by (18) is unstable. This implies that the evolutionary dynamical system is locally unstable at the equilibrium. ∎

Proof of Proposition 4. Straightforward calculations show that (x^*, n^*) is the equilibrium of (8). Then differentiate $G(x,n) = R(nG(x,n) + (1-n)\,x)$ with respect to x and n and solve for $\partial G/\partial x$ and $\partial G/\partial n$. Evaluating these partial derivatives in equilibrium gives $\frac{\partial G}{\partial x} = \frac{(1-n^*)R'(x^*)}{1-n^*R'(x^*)}$ and $\frac{\partial G}{\partial x} = 0$. The Jacobian matrix evaluated at the equilibrium therefore becomes

$$\begin{pmatrix} \frac{(1-n^*)R'(x^*)}{1-n^*R'(x^*)} & 0 \\ \left.\frac{\partial h(x,G(x,n),n)}{\partial x}\right|_{(x^*,n^*)} & \left.\frac{\partial h(x,G(x,n),n)}{\partial n}\right|_{(x^*,n^*)} \end{pmatrix}$$

The eigenvalues of this Jacobian matrix are $\lambda_1 = \frac{(1-n^*)R'(x^*)}{1-n^*R'(x^*)} \leq 0$ and $\lambda_2 = \left.\frac{\partial h(x,G(x,n),n)}{\partial n}\right|_{(x^*,n^*)} = (1-2\delta)\,\frac{\Pi^*(\Pi^*-T)}{(\Pi^*-n^*T)^2}$. First, consider the eigenvalue λ_2. We have

$$0 < \lambda_2 = (1-2\delta)\,\frac{\Pi^*(\Pi^*-T)}{(\Pi^*-n^*T)^2} < \frac{\Pi^*(\Pi^*-T)}{(\Pi^*-n^*T)^2} < \frac{\Pi^*(\Pi^*-T)}{\left(\Pi^*-\frac{1}{2}T\right)^2} < 1.$$

Note that in the above derivation we use $\delta < n^* < \frac{1}{2}$. Second, because the eigenvalue λ_2 is always strictly between 0 and 1, the equilibrium (x^*, n^*) is locally stable (unstable) if

$$\lambda_1 = \frac{(1-n^*)\,R'(x^*)}{1-n^*R'(x^*)} > (<) -1.$$

Substituting the expression for n^* and solving for the noise level δ gives the condition as stated in Proposition 4. ∎

6.2 Derivation of Profit Functions

Consider the symmetric Cournot duopoly game with a linear demand function and quadratic cost functions, as specified in section 2. In that case, the profit of a firm supplying quantity x_1, when he is matched with a firm supplying x_2, is given by

$$\begin{aligned}\pi(x_1,x_2) &= P(x_1,x_2)\,x_1 - c(x_1) = (a - b(x_1+x_2))\,x_1 - cx_1 + \frac{d}{2}x_1^2 \\ &= (a-c-bx_2)\,x_1 + \left(\frac{1}{2}d - b\right)x_1^2.\end{aligned}$$

Now suppose the firms in the population can choose between the behavioural rules H_1 and H_2, which prescribe quantities x_1 and x_2, respectively. Let n denote the fraction of the population supplying x_1. Then, in the case of random matching between firms, the average profit of a firm supplying x_1 can be written as

$$\begin{aligned}\Pi^1(x_1,x_2,n) &= n\pi(x_1,x_1) + (1-n)\,\pi(x_1,x_2) \\ &= (a-c)\,x_1 + \left(\frac{1}{2}d - b\right)x_1^2 - b\,(nx_1 + (1-n)\,x_2)\,x_1.\end{aligned}$$

Rewritten in terms of deviations $X_1 = x_1 - x^*$ and $X_2 = x_2 - x^*$ from the Cournot-Nash equilibrium quantity x^*, the above expression becomes

$$\Pi^1(X_1,X_2,n) = \Pi^* + \left(\frac{1}{2}d - b\right)X_1^2 - b\,(nX_1 + (1-n)\,X_2)\left(X_1 + \frac{a-c}{3b-d}\right),$$

where $\Pi^* = \left(b - \frac{1}{2}d\right)\left(\frac{a-c}{3b-d}\right)^2$. Similarly, for the firm supplying x_2 we find

$$\Pi^2(X_1,X_2,n) = \Pi^* + \left(\frac{1}{2}d - b\right)X_2^2 - b\,(nX_1 + (1-n)\,X_2)\left(X_2 + \frac{a-c}{3b-d}\right).$$

6.3 The Jacobian matrix

In this appendix we derive the Jacobian matrix. The dynamical system $(X_{t+1}, n_{t+1})' = (f(X_t,n_t), g(X_t,n_t))'$ is defined by

$$\begin{aligned}f(X,n) &= -\frac{b\,(1-n)}{(2+n)\,b-d}X \\ g(X,n) &= (1-2\delta)\frac{n\,(\Pi^r(X,n)-T)}{n\,(\Pi^r(X,n)-T) + (1-n)\,\Pi^b(X,n)} + \delta,\end{aligned}$$

where $\Pi^b(X,n)$ and $\Pi^r(X,n)$ are given by (10) and (11), respectively. Straightforward computations show that the Jacobian matrix is

$$\begin{pmatrix} -\dfrac{b\,(1-n)}{(2+n)\,b-d} & \dfrac{b\,(3b-d)}{((2+n)\,b-d)^2}X \\ \kappa n\,(1-n)\left(\Pi^b\Pi^r_X - (\Pi^r - T)\,\Pi^b_X\right) & \kappa\left(\Pi^b(\Pi^r - T) + n\,(1-n)\left(\Pi^b\Pi^r_n - (\Pi^r - T)\,\Pi^b_n\right)\right)\end{pmatrix}$$

where $\kappa = \frac{1-2\delta}{\left(n(\Pi^r - T) + (1-n)\Pi^b\right)^2}$. From the profit functions it follows that if $X = 0$ (everybody plays the Cournot-Nash equilibrium quantity) we have

$$\Pi^b(0,n) = \Pi^r(0,n) \equiv \Pi,$$

furthermore, we have

$$\begin{aligned}
\Pi_X^r &= \frac{b(1-n)(2b-d)}{(2+n)b-d}\left(\frac{b(1-n)}{(2+n)b-d}X-\frac{a-c}{3b-d}\right),\\
\Pi_X^b &= -\frac{2b-d}{(2+n)b-d}\left(b(1-n)\left(\frac{a-c}{3b-d}\right)+((4-n)b-d)X\right),\\
\Pi_n^r &= -(2b-d)\frac{b(3b-d)}{((2+n)b-d)^2}X\left(\frac{b(1-n)}{(2+n)b-d}X-\frac{a-c}{3b-d}\right) \text{ and}\\
\Pi_n^b &= \frac{b(3b-d)}{((2+n)b-d)^2}X\left(\left(\frac{a-c}{3b-d}\right)+(2b-d)X\right).
\end{aligned}$$

At the Cournot-Nash equilibnrium these derivatives reduce to

$$\Pi_X^r(0,n)=\Pi_X^b(0,n)=\frac{b(1-n)(2b-d)}{(2+n)b-d}\frac{a-c}{3b-d} \text{ and } \Pi_n^r(0,n)=\Pi_n^b(0,n)=0.$$

Hence, the Jacobian matrix evaluated at the equilibrium is

$$\mathbf{J}^*=\begin{pmatrix} -\frac{b(1-n)}{(2+n)b-d} & 0\\ g_X & (1-2\delta)\frac{\Pi^*(\Pi^*-T)}{(\Pi^*-nT)^2}\end{pmatrix},$$

with eigenvalues

$$\lambda_1=-\frac{b(1-n)}{(2+n)b-d} \text{ and } \lambda_2=(1-2\delta)\frac{\Pi^*(\Pi^*-T)}{(\Pi^*-nT)^2}.$$

and corresponding eigenvectors

$$v_1=\begin{pmatrix}\lambda_1-\lambda_2\\ \frac{a-c}{3b-d}\frac{(2b-d)n(1-n)T}{\Pi^*(\Pi^*-T)}\lambda_1\lambda_2\end{pmatrix} \text{ and } v_2=\begin{pmatrix}0\\1\end{pmatrix}.$$

The product of the eigenvalues has absolute value

$$|\lambda_1\times\lambda_2|=(1-2\delta)\frac{b(1-n)}{(2+n)b-d}\frac{\Pi^*(\Pi^*-T)}{(\Pi^*-nT)^2}$$

which, for δ and n_δ^* close to 0, is approximately $\frac{b}{2b-d}\frac{\Pi^*-T}{\Pi^*}$.

6.4 Some Important Curves

In this section we derive some of the curves that play an important role in the analysis of our model. We start with the main computations necessary to determine L_δ^{III}, the third component of the stable manifold. Using the fact that the profit functions are quadratic in X we can derive the following solutions of $g_\delta(X,n) = 1$:

$$X_{1,2} = \frac{1}{2a_n}\left(-b_n \pm \sqrt{b_n^2 - 4a_n c_n}\right)$$

where

$$\begin{aligned} a_n &= \mu n \left(\frac{b(1-n)}{(2+n)b-d}\right)^2 - (1-n)\left(\frac{(4-n)b-d}{(2+n)b-d}\right), \\ b_n &= -2((\mu-1)n+1)\left(\frac{b(1-n)}{(2+n)b-d}\right)\left(\frac{a-c}{3b-d}\right) \text{ and} \\ c_n &= ((\mu-1)n+1)\left(\frac{a-c}{3b-d}\right)^2 - \frac{\mu n}{b-\frac{1}{2}d} \end{aligned}$$

and where $\mu = \frac{\delta}{1-\delta}$. These points solve

$$1 = (1-2\delta)\frac{n(\Pi^r(X,n)-1)}{n(\Pi^r(X,n)-1)+(1-n)\Pi^b(X,n)} + \delta.$$

Clearly for $n = 0$, there is no solution to this equation. However, it can be easily shown that $\lim_{n\to 0} X(n) = \frac{2b-d}{4b-d}\frac{a-c}{3b-d} \equiv X^f$, where $X(n)$ is the positive root and corresponds to L_δ^{III}.

Next we focus on the zero-profit curves L^b and L^r. Since $\Pi^b(X,n)$ and $\Pi^r(X,n)$ are quadratic functions in X the zero-profit functions can be easily solved for X, given n. The relevant solutions are given by (for illustration see Figure 5)

$$\begin{aligned} L^b &= \left\{(X,n)\,|X = \frac{(2+n)b-d}{(4-n)b-d}\frac{a-c}{3b-d}\right\} \text{ and} \\ L^r &= \left\{(X,n)\,|X = \frac{(2+n)b-d}{b(1-n)}\left(\frac{a-c}{3b-d} - \sqrt{\frac{2T}{2b-d}}\right)\right\}. \end{aligned}$$

Now consider the curve L^n, the curve consisting of the points at which the denominator of $g(X,n)$ vanishes. This occurs when

$$n\left(\Pi^r(X,n) - \Pi^b(X,n)\right) - nT + \Pi^b(X,n) = 0.$$

Again the right-hand side of this equation is quadratic in X and therefore can be solved. Now define

$$y = \frac{1}{(2+n)b-d}X \text{ and } z = \frac{a-c}{3b-d}.$$

Then

$$L^n = \{(X,n)\,|X = ((2+n)b-d)\,y_i\}$$

where y_i is the relevant solution of

$$\begin{aligned}
y_{1,2} &= \frac{1}{(1-n)\left((1-n)b^2-(3b-d)^2\right)}\left(zb(1-n)\pm\sqrt{D}\right) \text{ and} \\
D &= (1-n)\left((3b-d)^2 z^2+\left((1-n)b^2-(3b-d)^2\right)n\frac{2T}{2b-d}\right).
\end{aligned}$$

6.5 Diffeomorphisms

From the inverse function theorem it follows that a map has a local smooth inverse if the determinant is unequal to 0. So to determine where a local smooth inverse exists we want to find the set of points (X,n) at which the determinant vanishes. For our dynamical system $F(X,n)=(f(X,n),g(X,n))'$ we have (as computed in Appendix 6.3)

$$\begin{aligned}
\det J(X,n) &= -\frac{b(1-n)}{(2+n)b-d}\times(1-2\delta)\frac{\Pi^b(\Pi^r-T)+n(1-n)\left(\Pi^b\Pi^r_n-(\Pi^r-T)\Pi^b_n\right)}{\left(n(\Pi^r-T)+(1-n)\Pi^b\right)^2} \\
&\quad -\frac{b(3b-d)}{((2+n)b-d)^2}X\times(1-2\delta)\frac{n(1-n)\left[\Pi^b\Pi^r_X-(\Pi^r-T)\Pi^b_X\right]}{\left(n(\Pi^r-T)+(1-n)\Pi^b\right)^2}
\end{aligned}$$

Since all terms contain the term $(1-n)$, we know that the determinant vanishes at $n=1$. Now assume $n<1$ and $n(\Pi^r(X,n)-T)+(1-n)\Pi^b(X,n)\neq 0$ and divide $\det J(X,n)$ by the strictly positive number $(1-2\delta)(1-n)/\left(n(\Pi^r-T)+(1-n)\Pi^b\right)^2$. Then the determinant equals zero when

$$\begin{aligned}
D(X,n) &= -((2+n)b-d)*\left(\Pi^b(\Pi^r-T)+n(1-n)\left(\Pi^b\Pi^r_n-(\Pi^r-T)\Pi^b_n\right)\right) \\
&\quad -(3b-d)X*n\left[\Pi^b\Pi^r_X-(\Pi^r-T)\Pi^b_X\right]
\end{aligned}$$

equals zero. It can be easily checked that for all points with $\Pi^r(X,n)=T$ (that is, points in L^r) the determinant vanishes. We also computed another branch of points where the determinant vanishes numerically. This branch (which we call $L^{\det}$) starts in the focal point $\left(X^f,0\right)$.

6.6 Proof of Theorem 6

The theorem is an application of the 'strange attractor theorem' of Mora and Viana (1993); see also Palis and Takens (1993). However, before the theorem can be applied, three technical issues have to be resolved:

T1. The strange attractor theorem applies to (local) diffeomorphisms, whereas our map F_δ is non-invertible;

T2. The strange attractor theorem is valid when certain generic conditions are satisfied, which need to be checked in our particular case;

T3. It has to be checked that strange attractors already arise *before* the homoclinic bifurcation, i.e. for $\delta > \delta^c$, because for $\delta < \delta^c$ part of the unstable manifold is in the economically meaningless region where the fraction $n > 1$ (see also footnote 12).

Concerning T1 we will show that in the region where the homoclinic tangency occurs, our map F_δ is invertible, so that the strange attractor theorem applies. Concerning T2, we will show that the generic conditions given by Takens (1992) are satisfied, so that the strange attractor theorem can be applied. Finally, concerning T3 we will argue that the global geometric shape of the unstable and stable manifolds is equivalent to case B in Palis and Takens (1993, pp.192-194) for which horseshoes and homoclinic bifurcations of periodic cycles arise *before* the point of homoclinic bifurcations, i.e. for $\delta > \delta^c$, so that the strange attractor theorem applies.

In order to investigate in which region of the phase space the map F_δ is invertible, it is helpful to introduce the following three curves (Appendix 6.4 gives explicit expressions for these curves)

$$\begin{aligned} L^b &= \left\{(X,n)\,|\Pi^b(X,n) = 0\right\} \\ L^r &= \{(X,n)\,|\Pi^r(X,n) = T\} \\ L^n &= \left\{(X,n)\,|n\left(\Pi^r(X,n) - T\right) + (1-n)\,\Pi^b(X,n) = 0\right\}. \end{aligned}$$

These curves are shown in Figure 5. L^b and L^r are curves that give combinations of X and n at which, respectively, the profit for best-reply firms and the net profit for rational firms are equal to 0. The curve L^n gives the points (X,n) at which the numerator in $g(X,n)$ vanishes. Therefore the map (9) is not well-defined on L^n. Notice that these curves do not depend on the noise level δ. It can be easily seen that the curve L^r lies to the right of L^b. From its definition it follows that L^n must lie between L^b and L^r. L^b and L^n intersect the $n-$axis in the point $\left(X^f,0\right)$, where X^f is such that $\Pi^b\left(X^f,0\right) = 0$. In this point the map contains an element $0/0$, and such a point is called a *focal* point. Focal points can play an important role in the global dynamics of noninvertible dynamical systems, see for example Bischi, Gardini and Kopel (1999).

The map F_δ is non-invertible at points for which the determinant of the Jacobian matrix is equal to 0. The curve of points for which the determinant vanishes can not be determined analytically, but from appendix 6.4 we know that the determinant vanishes at the focal point $\left(X^f,0\right)$, for points with $n=1$ and for points on the curve L^r. Figure 5 shows the curves L^b, L^r and L^n and the curves L^r and $L^{\det}$ along which the determinant of F_δ vanishes. We have checked that the only intersection point of the curve where the determinant vanishes and the curve L^n is the focal point. Therefore, to the left of the curve L^n and strictly below the line $n=1$, the map F_δ is locally a diffeomorphism (except at the focal point).

Now we focus attention on the component of the stable manifold, that is mapped into points with $n=1$, that is, the component

$$L_\delta^{III} = \{(X,n)\,|g(X,n) = 1\}.$$

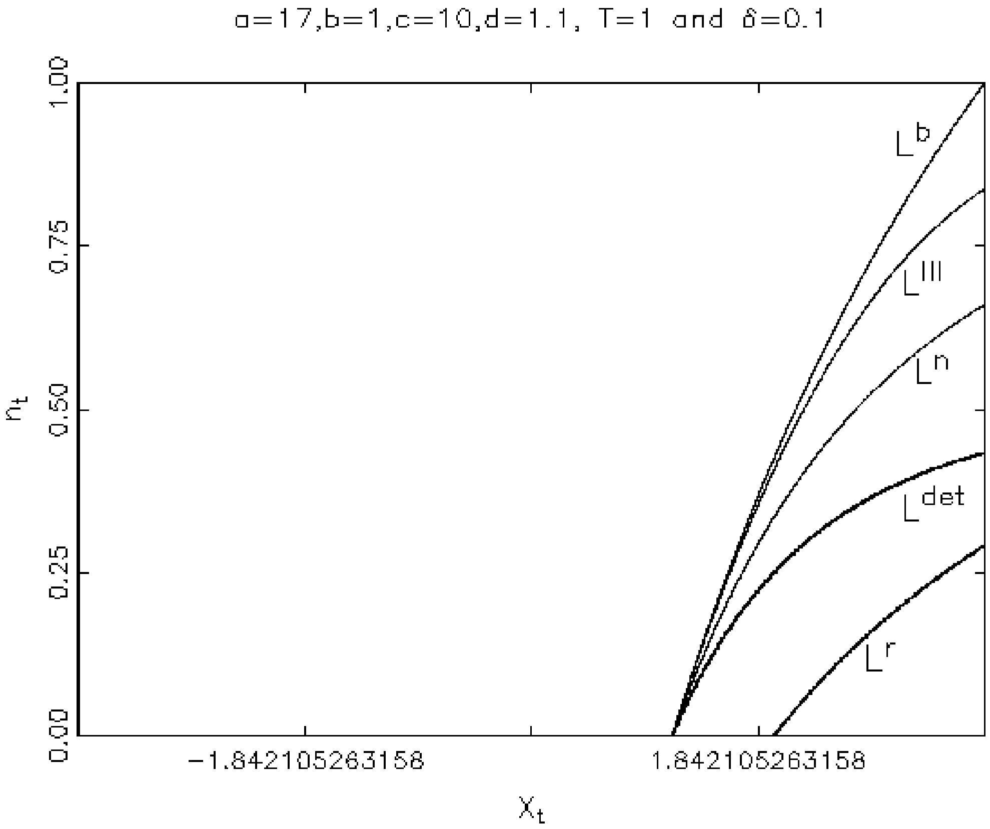

Figure 5: Some important curves: L^b and L^r are the curves along which net profits for the best-reply and rational rule are zero, respectively. L^n is the curve along which the denominator of $g(X,n)$ vanishes, $L^{\det}$ is the curve along which the determinant of the Jacobian matrix vanishes and L^{III} is the third component of the stable manifold of the equilibrium.

Consider

$$g(X,n) = (1-2\delta)\frac{n\left(\Pi^r(X,n)-T\right)}{n\left(\Pi^r(X,n)-T\right)+(1-n_t)\,\Pi^b(X,n)}+\delta = 1$$

and notice that for this equality to hold we must have $\Pi^b(X,n) < 0$, $\Pi^r(X,n) > 0$ and $n\left(\Pi^r(X,n)-T\right)+(1-n_t)\,\Pi^b(X,n) > 0$. In particular for $\delta = 0$, we have $L_0^{III} = L^b$. As δ increases profits associated to the best-reply rule have to decrease even further and it can be easily seen that as δ approaches $\frac{1}{2}$, L_δ^{III} approaches L^n. Clearly, for $0 < \delta < \frac{1}{2}$, L_δ^{III} lies between L^b and L^n and as δ increases it shifts from L^b to L^n. Consequently, at the homoclinic bifurcation, in a neighborhood of L_δ^{III}, the map F_δ is a diffeomorphism.

Next, let us check whether at the homoclinic bifurcation the generic conditions are satisfied. Takens (1992, pp.192-193) presents three alternative and simpler conditions, which if satisfied imply that the generic conditions of the strange attractor theorem are satisfied. Takens' three alternative conditions are: (a) F_δ should be a real analytic function; (b) the function $h(\beta) = -\ln|\lambda_1(\beta)|/\ln|\lambda_2(\beta)|$ is not constant, where $\lambda_i, i = 1, 2$, are the eigenvalues (depending upon the parameter) at the equilibrium; and (c) there has to be an 'inevitable homoclinic tangency', meaning that as the parameter changes the system moves from a situation without any homoclinic points to a situation with two (and hence infinitely many) transversal homoclinic points. Conditions (a) and (b) are clearly satisfied in our model; we will show that condition (c) is also satisfied, by making the geometric arguments in section 4.2 precise.

In order to show that an inevitable homoclinic tangency occurs, first consider the case $\delta = 0$, that is the replicator dynamics without noise. The system then becomes

$$\begin{aligned} X_{t+1} &= -\frac{b(1-n_t)}{(2+n_t)b-d}X_t \\ n_{t+1} &= \frac{n_t\left(\Pi^r(X_t,n_t)-T\right)}{n_t\left(\Pi^r(X_t,n_t)-T\right)+(1-n_t)\Pi^b(X_t,n_t)}. \end{aligned} \tag{19}$$

If $T > 0$ we have two equilibria: $(X^*, n^*) = (0, 0)$ and $(X^{**}, n^{**}) = (0, 1)$.[14] Hence, in an equilibrium the whole population is playing the same rule. Clearly the equilibrium where all firms play the rational behavioural rule is unstable, since a deviation to the best-reply strategy pays off since then information costs T do not have to be incurred.[15] On the other hand, the equilibrium where all firms use the best-reply behavioural rule is unstable when the best-reply dynamics are unstable. We are primarily interested in the equilibrium $(X^*, n^*) = (0, 0)$, since the other equilibrium disappears when $\delta > 0$. From (19) it follows that the stable manifold of this equilibrium is the $n-$axis, whereas the unstable manifold is the $X-$axis, except for the focal point $\left(X^f, 0\right)$ where the system is not defined. Using the fact that the unstable manifold varies continuously with δ, we will show that if δ is a small but positive number, there must be a point of homoclinic intersection.

For δ large, say close to but below the primary flip bifurcation value δ^*, the unstable manifold of the equilibrium tends to the stable 2-cycle, so that there can be no homoclinic intersection (for illustration, see Figure 3b). For δ small but positive we will show that a point of transversal homoclinic intersection exists.

Consider a box $OABCO$ with $O = (0, 0)$, $A = (0, n_0)$, $B = (X_0, n_0)$ and $C = (X_0, 0)$, where $0 < X_0 < X^f$ and $0 < n^* < n_0$, as in Figure 6. For δ sufficiently small, by the continuity argument made above, this box contains a piece of the unstable

[14] If $T = 0$ there is a continuum of equilibria: all states with $X = 0$ are equilibria, since profits for both rules are equal.

[15] Using terminology from evolutionary game theory, we say that this equilibrium is not *evolutionary stable,* since it can be *invaded* by the best-reply strategy, which, in the neighbourhood of this equilibrium does better against the rational strategy than the rational strategy does against itself.

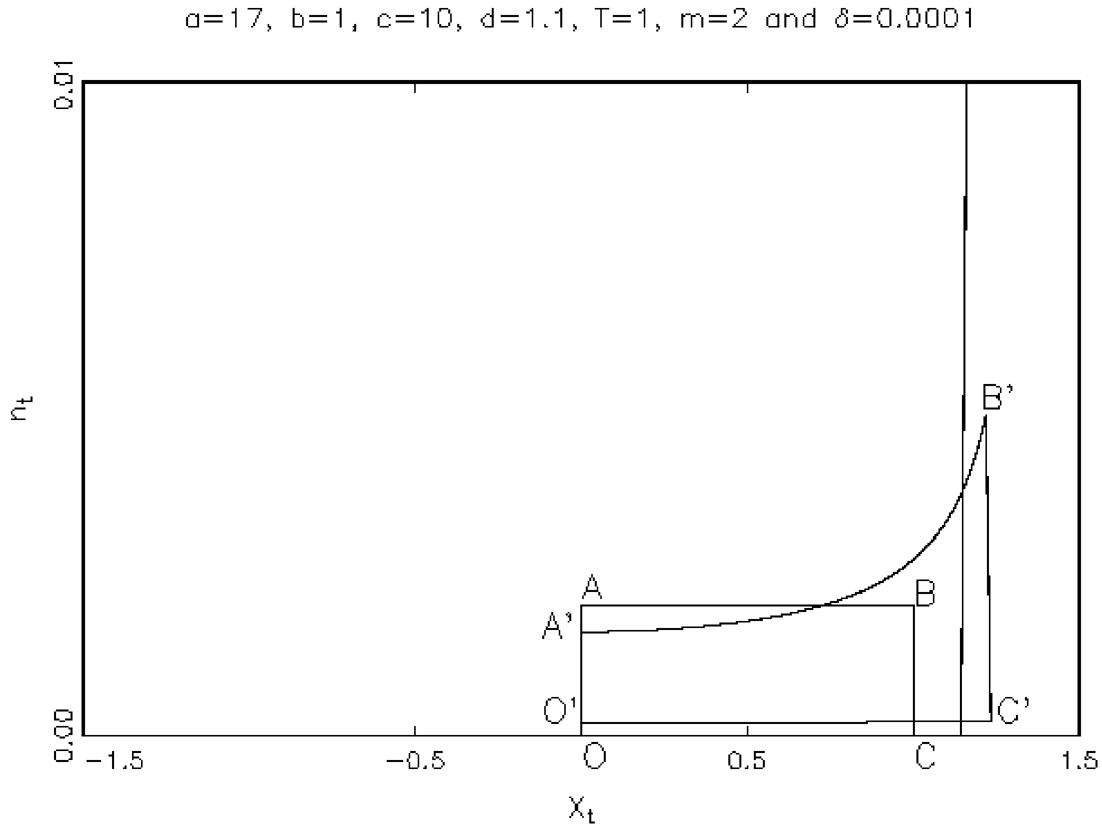

Figure 6: The second iterate of a suitable rectangular box intersects the stable manifold.

manifold leaving the box via the line segment BC. We claim that if the point C is sufficiently close (but at finite distance) to the focal point $(X^f, 0)$ and if the height of the box n_0 is sufficiently small, then the second iterate $O'A'B'C'O' = F_\delta^2\,(OABCO)$ of the box, intersects the stable manifold, as shown in Figure 6. This follows immediately from the fact that for the replicator dynamics without noise, i.e., for $\delta = 0$, if the point C is near the focal point $(X^f, 0)$, the second iterate $C' = F_\delta^2(C)$ lies to the right of the stable manifold L_δ^{III} and the continuity with respect to the parameter δ. We thus conclude that an inevitable homoclinic tangency occurs for some critical parameter value δ^c.

Finally, we have to show that strange attractors arise *before* the homoclinic bifurcation, i.e. for $\delta > \delta^c$. This depends upon the global geometric configuration of the stable and unstable manifolds of the equilibrium. Geometrically, our case is equivalent to case B in Palis and Takens (1993, pp. 192-194). The same construction as in Palis and Takens (1993, pp. 192-194) can therefore be applied to show that preliminary horseshoes and preliminary homoclinic bifurcations between stable and unstable manifolds of periodic points occur close to but before the homoclinic tangency between the stable and unstable manifolds of the equilibrium. Theorem 6 can therefore be obtained by applying the strange attractor theorem.

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