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Arjen Doelman; Philip Holmes

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Homoclinic explosions and implosions

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In this paper we study the interaction of local saddle node and Hopf bifurcations with global recurrent orbits. We consider a four-parameter family of three-dimensional vector fields which are small perturbations of an integrable system possessing a line of degenerate saddle points connected by a manifold of homoclinic loops. Our most striking finding is that homoclinic bifurcations occur in which a unique connecting orbit is replaced by a countable infinity of such orbits (an ‘explosion’) and in which a countable infinity collapses to a unique orbit (an ‘implosion’). We also discover a rich variety of heteroclinic connections among fixed points and periodic orbits. The family we study was motivated by the search for travelling ‘structures’ such as fronts and domain walls in the complex Ginzburg–Landau partial differential equation which models weakly nonlinear wave interactions near the onset of instability, and which leads to a special case of our more general unfolding.

1. Introduction

In this paper we study a three-dimensional dynamical system which provides an unfolding of a degenerate, integrable, ordinary differential equation possessing a manifold of centres and a manifold of saddle points connected by homoclinic orbits (see figure 1, below). The unfoldings possess up to four equilibria, three of which are always saddle points; while the fourth may be a saddle, a sink or a source. They also contain periodic orbits and homoclinic and heteroclinic orbits connecting these various limit sets to themselves and to each other. The interaction of simple local saddle-node and Hopf bifurcations with global bifurcations involving homo- and heteroclinic orbits is rich and, we believe, of considerable interest in its own right and we feel that the combination of local and perturbation analyses with geometrical and topological arguments that we have developed to solve this problem will be of general use. Nonetheless, we would probably not have chosen to study the system but for our interest in the three-dimensional vector field obtained when one seeks travelling wave solutions in the cubic Ginzburg–Landau (GL) partial differential equation with nearly real coefficients. This equation frequently appears in studying the nonlinear evolution of patterns at near-critical conditions (cf. Eckhaus 1992). In that case, as shown for example by Doelman (1993), the resulting ordinary differential equation (ODE) is

$$\left. \begin{aligned} \rho_\xi &= V, \\ V_\xi &= -\rho + \rho^3 + \frac{\Omega^2}{\rho^3} + \frac{\varepsilon}{\Delta} v V - \frac{a\varepsilon^2}{\Delta} \rho[(w-a) - \rho^2(b-a)] + \frac{a\varepsilon^2}{\Delta} v \frac{\Omega}{\rho}, \\ \Omega_\xi &= -\frac{\varepsilon}{\Delta} \rho^2[(w-a) - \rho^2(b-a)] + \frac{\varepsilon}{\Delta} v \Omega - \frac{a\varepsilon^2}{\Delta} v \rho V, \end{aligned} \right\} \quad (1.1)$$

with $\Delta = 1 + \varepsilon^2 a^2$. Here $\xi = x - vt$ is the coordinate in the uniformly translating frame, a and b are coefficients in the GL equation, v is the wave speed and w is the temporal frequency. Fixed points of (1.1) for $\rho \neq 0$ correspond to periodic (sinusoidal) wave trains, periodic solutions correspond to quasi-periodic (amplitude modulated) wave trains and homo- and heteroclinic orbits of (1.1) to domain walls and fronts. See van Saarloos & Hohenberg (1992), for a recent general review. The propagation and interaction of such waves is thought to play an important role in pattern formation in a variety of physical systems, but, apart from some numerical studies (see, for example, Keefe 1985; Rodriguez & Sirovich 1990) there is little evidence of complicated or chaotic wave trains. The present paper provides some rigorous results and can be seen in part as an extension of earlier work on the GL system (1.1) by Doelman (1993). Moreover, unlike most analytical studies of travelling structures in GL like equations, in which ODEs analogous to (1.1) are derived, our methods are more general. Instead of seeking particular solutions connecting specific equilibria, for example, we analyse in detail the geometrical structure of the *full* flow of a class of model problems, thereby obtaining information on whole families of travelling wave solutions.

As shown in Doelman (1993), the unfolding of the GL system for $\varepsilon \neq 0$ afforded by (1.1) is degenerate in a global sense: saddle node and homoclinic bifurcations coincide, at least to leading order in ε (also see §2, below). For this reason, and also to make our analysis more transparent, in place of (1.1) we treat a simpler but more general four-parameter model problem which we derive in §2. In doing so, we perform

unfolding and perturbation calculations which are also used in subsequent sections. Section 3 contains most of the ‘standard’ analysis of the resulting ordinary differential equations, both local and (semi-) global, and draws heavily on the calculations of §2. On the basis of this, we establish certain gross topological properties of the three-dimensional flow, including the relative positions of the perturbed invariant manifolds which split from the manifold of homoclinic orbits of the unperturbed problem. The finer topological consequences of these properties are elaborated in §4. We show that, on the global level, there are two cases to consider, referred to in §§3 and 4 as ‘outside’ and ‘inside’. These lead to utterly different solution structures. The main results of §4, theorems 4.3, 4.8 and 4.9, describe the sets of heteroclinic orbits connecting the two saddle points, originally part of a continuum, which survive on a slow manifold after perturbation, and the bifurcations in which they are created as parameters vary. In the ‘inside’ case infinitely many heteroclinic connections appear in an explosion which follows the creation of these equilibria; in the ‘outside’ case infinitely many connections disappear in an implosion, leaving a unique connection. In §5 we discuss the relationship of the periodic orbit and its stable and unstable manifolds to these connections. Finally in §6 we conclude and briefly discuss the implications of our results for the original GL problem and related travelling wave equations. Some of the more technical computational details are relegated to the appendices.

As mentioned above, the impulse for this paper came from our desire to prove existence and/or non-existence of ‘localized structures’ in evolution equations of GL type. But from the viewpoint of dynamical systems theory itself, we study the interaction of simple codimension one local and global bifurcations in a perturbed, three-dimensional, integrable system with parameters. We believe that our most striking results – the explosions and implosions of homo- and heteroclinic orbits – represent new phenomena which deserve to be studied in their own right. Furthermore, since the main ideas, methods and results of this paper are topological and do not rely on the specific form of the model ODE, it seems likely that these bifurcations should occur in a fairly general class of near-integrable systems with parameters.

Our general theme is to deduce fine global detail from relatively simple local and semi-global input, such as normal form and Melnikov calculations. Thus, after some tedious, but conceptually simple, analytical computation, the bulk of the paper in §§3, 4 and 5 is topological in nature. As such, it can be read as a self-contained example of how topological methods contribute to dynamical systems theory, as much as a study of a particular problem. We have included many illustrations, for most of our results were originally discovered and are essentially proved by the careful drawing of pictures. However, to keep the reader on her toes we have not included *all* the pictures we created ourselves. Perhaps the best way to really follow our arguments is to take a large pad of plain paper and several coloured pens. . .

2. The model problem

In this section we derive a three-dimensional model (2.14) which provides an unfolding of the vector field which appears in the study of travelling waves in the nearly real Ginzburg–Landau equation (see Doelman 1991). We wish to circumvent superfluous computations necessitated by the coordinate system inherited by the latter

‘GL system’, which has the following integrable system as basis:

$$\rho_\xi = V, \quad V_\xi = -\rho + \rho^3 + \Omega^2/\rho^3, \quad \Omega_\xi = 0.$$

This system degenerates at $\Omega = 0$ and has a saddle-centre bifurcation at $\Omega^2 = \frac{4}{27}$. To avoid these complications, we consider a model problem based on the simpler integrable system:

$$\dot{x} = y, \quad \dot{y} = x - x^2, \quad \dot{z} = 0. \quad (2.1)$$

Thus, our results apply to the GL equation only in the region $0 < \Omega^2 < \frac{4}{27}$. The phase portrait of (2.1) is sketched in figure 1, which shows the two manifolds (=lines) of degenerate equilibria, the two-dimensional homoclinic manifold to the line of saddles, $\mathcal{H}$ (see §3c), and the family of invariant cylinders filled with periodic orbits. This phase portrait is equivalent to the integrable GL phase portrait without the degenerations at $\Omega = 0$ and $\Omega^2 = \frac{4}{27}$. In the sequel we will study the fate of (2.1) under small perturbations. Specifically, we perturb it as follows:

$$\left. \begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^2 + \varepsilon(\alpha f(x, y, z; \lambda, \mu) + yg(x, y, z; \lambda, \mu)), \\ \dot{z} &= \varepsilon f(x, y, z; \lambda, \mu). \end{aligned} \right\} \quad (2.2)$$

The critical points of (2.2) lie on the lines of critical points of (2.1), due to the special structure of the perturbation. This is also the case in the GL system. There are ‘free’ parameters λ and μ , which play the roles of the wave speed v and the frequency w in the GL system. The functions f and g are modelled as

$$f(x, y, z; \lambda, \mu) = \lambda + (a_1 + \beta_1\lambda + \gamma_1\mu)x + (a_2 + \beta_2\lambda + \gamma_2\mu)y + 2\mu z + z^2, \quad (2.3)$$

$$g(x, y, z; \lambda, \mu) = (b_0 + \beta_0\lambda + \gamma_0\mu) + b_1x + b_2y + b_3z, \quad (2.4)$$

which may be thought of as leading terms in their Taylor series. In this section we will indicate how to determine suitable choices for the ‘unfolding’ parameters α , a_1 , a_2 ; $b_0, \dots, b_3$; $\beta_0, \beta_1, \beta_2$ and $\gamma_0, \gamma_1, \gamma_2$.

For $\varepsilon \neq 0$, the GL system exhibits a saddle-node bifurcation on the normally hyperbolic invariant curve given by $V = 0$ and the appropriate branch of solutions of $V_\xi = 0$; see Doelman (1993). In (2.2), this curve is simply $\{x = y = 0\}$ and consequently normal form theory (Guckenheimer & Holmes 1983) implies that f must contain the quadratic term $\pm z^2$ and the constant term λ . Without loss of generality we choose $+z^2$, and, to allow flexibility, include the linear term $2\mu z$, which we will later remove. The other terms in f and g necessary for a universal unfolding are not as easily determined, since we are concerned with global features such as periodic and homoclinic solutions. We study some such solutions in this section and in doing so determine which of the terms of (2.3)–(2.4), linear in x , y and z , may be removed without incurring loss of generality.

Note that the structure of f is such that the position of the (perturbed) saddle points of (2.2), $S^\pm$, can be controlled by λ and μ : they are given by $S^\pm = (0, 0, z_\pm(\lambda, \mu))$ with

$$z_\pm(\lambda, \mu) = -\mu \pm \sqrt{(\mu^2 - \lambda)}. \quad (2.5)$$

Thus, by varying λ and μ , one can have a saddle-node bifurcation (of perturbed saddles) on any plane $z = z_0$ in the phase space. Furthermore, the structure of f and g has been chosen as simple as possible; only the linear terms depend on λ and μ . We

Figure 1. The phase portrait of the unperturbed system (2.1).

will find that system (2.2) with the functions f and g as in (2.3) and (2.4) exhibits the same behaviour as the more complicated GL system (see (1.1) and Doelman (1993)), provided one chooses the unfolding parameters appropriately.

Our main tool in the derivation of the model problem is an approximation of a Poincaré map by which we can determine the existence – and ultimately the stability – of periodic solutions to (2.2) and thus to the model problem. This map will be used throughout the paper, so we develop its approximation here, relegating computational details to the appendices. Note that periodic orbits of the perturbed system must be $O(\varepsilon)$ close to some periodic orbit of the integrable system (2.1). These orbits may be parametrized via the integral k :

$$k = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3. \quad (2.6)$$

The centre points corresponds to $k = k_c = -\frac{1}{6}$, the saddles (and their homoclinic orbits) to $k_s = 0$; for $k \in (k_c, k_s)$ (2.1) has periodic solutions, while solutions with $k > k_s$ are unbounded. Thus, periodic orbits to (2.1) can be uniquely described by a pair (k, z) . We study the existence of periodic solutions by constructing and approximating the Poincaré map (as in Holmes (1986) and Doelman (1989, 1993)):

$$\mathcal{P}(k, z) = (k + \Delta K(k, z), z + \Delta Z(k, z)).$$

This map is only defined for k in an $O(\varepsilon)$ neighbourhood of (k_c, k_s) . The quantities $\Delta K(k_0, z_0)$ and $\Delta Z(k_0, z_0)$ measure the accumulated change in the slow variables k and z of a solution of (2.2) – with initial data on the cross section $\{y = 0, x \geq 1\}$ and x_0 such that $k = k_0$ (2.6) – at its next intersection with $\{y = 0, x \geq 1\}$. Thus,

$$\Delta Z(k_0, z_0) = \int_0^{T_\varepsilon(k_0, z_0)} \dot{z}(x_\varepsilon, y_\varepsilon, z_\varepsilon) dt,$$

where $(x_\varepsilon, y_\varepsilon, z_\varepsilon)$ is the solution of (2.2) with the above initial data and $T_\varepsilon(k_0, z_0)$ is the return time. Approximating $(x_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t))$ by $(x_0(t), y_0(t), z_0)$, the solution of (2.1) with the same initial data yields

$$\Delta Z(k_0, z_0) = \varepsilon \int_0^{T_0(k_0)} f(x_0(t), y_0(t), z_0) dt + O(\varepsilon^2),$$

where $T_0(k_0)$ is the period of the unperturbed periodic solution with ‘energy’ $k = k_0$. We define

$$G_k(x) = 2k + x^2 - \frac{2}{3}x^3 \quad (2.7)$$

and change variables to obtain

$$\Delta Z(k_0, z_0) = 2\varepsilon \int_{\hat{x}(k_0)}^{\tilde{x}(k_0)} \frac{f(x, 0, z_0)}{\sqrt{G_k(x)}} dx + O(\varepsilon^2),$$

using (2.6); here $0 < \hat{x}(k_0) < 1 < \tilde{x}(k_0) < \frac{3}{2}$ are the intersection points of the orbit corresponding to $k = k_0$ with $y = 0$. The substitution $y = 0$ (in f) is possible since the odd part of f – as a function of y – disappears in the above change of variables because the parts $y > 0$ and $y < 0$ of the integral cancel each other. Introducing

$$T_i(k) = \oint \frac{x^i}{\sqrt{G_k(x)}} dx \quad (2.8)$$

and using (2.3) finally yields

$$\Delta Z(k, z)/\varepsilon = (\lambda + 2\mu z + z^2)T_0(k) + (a_1 + \beta_1\lambda + \gamma_1\mu)T_1(k) + O(\varepsilon). \quad (2.9)$$

Analogously, one derives the expression for ΔK , using $\dot{k} = \varepsilon(\alpha y f + y^2 g)$. For computational convenience we henceforth set $\alpha = 0$, since the term ‘ $\varepsilon\alpha f$ ’ in (2.2) has no structural influence on the following analysis and its inclusion leads only to superfluous computations. In this case the slow manifolds of (2.2) are given explicitly by $\{x = y = 0\}$, the curve of saddle points in (2.1), and $\{x = 1, y = 0\}$, the centres in (2.1). Changing variables as above, we then find

$$\Delta K(k, z)/\varepsilon = [(b_0 + \beta_0\lambda + \gamma_0\mu) + b_3z]S_0(k) + b_1S_1(k) + O(\varepsilon), \quad (2.10)$$

where

$$S_i(k) = \oint x^i \sqrt{G_k(x)} dx \quad \text{for } i = 0, 1. \quad (2.11)$$

To derive the model problem, in Appendix A we impose certain conditions on (2.2), chosen to match behaviour to the flow of the GL system (1.1) as revealed in Doelman (1993). The most important is that for any pair (k_0, z_0) , $(k \in (k_c, k_s))$ there must be a unique pair $(\lambda_0(k_0, z_0), \mu_0(k_0, z_0))$ such that, for sufficiently small $\varepsilon \neq 0$, (2.2) has a periodic solution which is a continuation of the periodic solution of (2.1) corresponding to (k_0, z_0) . Equating (2.10) and (2.9) to 0, we deduce a first condition (A 2) on the coefficients of f and g . Next, we derive an explicit expression for the curve of global bifurcations

$$\mathcal{C}_{\text{glob}} = \left\{ (\lambda_*(z), \mu_*(z)) \right\} = \left\{ \left(\lim_{k \uparrow k_s} \lambda_0(k, z), \lim_{k \uparrow k_s} \mu_0(k, z) \right) \right\}, \quad (2.12)$$

see (A 6); $\mathcal{C}_{\text{glob}}$ defines a boundary in the (λ, μ) -plane of the region $\mathcal{P}$ in which (2.2) has periodic solutions. On this boundary the periodic orbit limits on an orbit homoclinic or heteroclinic to the saddle points $S^\pm(\lambda, \mu)$. A very special feature of the GL system is that $\mathcal{C}_{\text{glob}}$ coincides (up to $O(\varepsilon)$) with the curve $\mathcal{C}_{\text{SN}}$ on which the (perturbed) saddles merge. We require our model problem to reproduce this behaviour, but we also unfold it by including an extra parameter, b (see (2.14)): $\mathcal{C}_{\text{glob}}$ coincides with $\mathcal{C}_{\text{SN}}$ for $b = 0$, but $\mathcal{C}_{\text{glob}} \cap \mathcal{C}_{\text{SN}} = \emptyset$ for $b \neq 0$: cf. (A 9). Finally we impose a more technical condition (which yields (A 13)) and conclude that we may reduce (2.3) and (2.4) to

$$f = \lambda + a_1x + 2\mu z + z^2 \quad \text{and} \quad g = b_0 + \gamma_0\mu + \gamma_0z, \quad (2.13)$$

without losing relevant behaviour. System (2.2) can then be further simplified by a

shift in z :

$$\bar{z} = z + \mu,$$

which eliminates the parameter μ in (2.2). Therefore, the model problem which we will study may be written:

$$\dot{x} = y, \quad \dot{y} = x - x^2 + \varepsilon y(b + cz), \quad \dot{z} = \varepsilon(\nu + \rho x + z^2), \quad (2.14)$$

where we have dropped the overbar on $\bar{z}$ and renamed $\rho = a_1$, $b = b_0$, $c = \gamma_0$ and $\nu = \lambda - \mu^2$.

Remark 2.1. As we have noted, the general form of the perturbation functions f and g cannot be determined by normal form transformations as in the local bifurcation theory and the system (2.14) is certainly not a *universal* unfolding of the degenerate $\varepsilon = 0$ limit (2.1). It does, however, capture all the important aspects of the GL system which motivated the present study, and it unfolds certain ‘global’ degeneracies of that problem. In determining (2.14) we chose to satisfy a number of conditions on the global flow induced by (2.2) – see Appendix A for the details – and only to allow for one unfolding parameter b . While these choices were motivated and guided by our experience with the Ginzburg–Landau problem, and other, equally reasonable choices might be made, we do feel that they provide a balance between simplicity and generality. In particular we note that if other terms are retained in f , g , the simplifying shift $\bar{z} = z + \mu$ may not be possible. Section 6 contains a discussion of the implications of this shift in the GL context.

3. Studying the model problem by perturbation techniques

This section is divided in several subsections. In each of these subsections we study one or more fundamental aspects of model problem (2.14). Before doing this we observe a useful symmetry of our model equation: (2.14) is equivariant under

$$(x, y, z, t; \rho, \nu, b, c) \rightarrow (x, -y, -z, -t; \rho, \nu, -b, c).$$

This implies that we can restrict ourselves to $b \geq 0$ and need only consider unfoldings in the (ρ, ν) plane for $c < 0$ and $c > 0$. (The case $c = 0$ is degenerate and essentially trivial since the first two components of (2.14) are uncoupled from the third.)

(a) The local character of the fixed points

The critical points of (2.14) are given by: $S^\pm = (0, 0, \pm\sqrt{-\nu})$, the perturbed saddles, and $C^\pm = (1, 0, \pm\sqrt{-(\nu + \rho)})$, the perturbed centres. Therefore, there are two local bifurcation curves:

$$\mathcal{C}_{\text{SN}} = \{\nu = 0\} \quad \text{and} \quad \mathcal{C}_{\text{CN}} = \{\nu + \rho = 0\}, \quad (3.1)$$

each describing the coalescence of two critical points. The perturbed saddles

$$(0, 0, \pm\sqrt{-\nu})$$

have three eigenvalues: $\lambda_\pm^{\text{slow}}$ along the slow ($= O(\varepsilon)$) manifold; $\lambda_\pm^{\text{uu}}$, the fast ($= O(1)$) unstable direction; and $\lambda_\pm^{\text{ss}}$, the fast-stable direction. We find

$$\lambda_\pm^{\text{slow}} = \pm 2\varepsilon\sqrt{-\nu}, \quad \lambda_\pm^{\text{uu,ss}} = \pm 1 + \frac{1}{2}\varepsilon(b \pm c\sqrt{-\nu}) + O(\varepsilon^2), \quad (3.2)$$

with $c = \gamma_0 \neq 0$, see (A 2). Thus, $(0, 0, \sqrt{-\nu})$ is unstable and $(0, 0, -\sqrt{-\nu})$ is stable on the slow manifold $\{x = y = 0\}$ and they are always connected by a (slow) heteroclinic

orbit. We find, for $b \geq 0$:

$$-(b/c)^2 < \nu < 0 : \begin{cases} c < 0 : \lambda_+^{uu} > |\lambda_+^{ss}|, & \lambda_-^{uu} > |\lambda_-^{ss}|, \\ c > 0 : \lambda_+^{uu} > |\lambda_+^{ss}|, & \lambda_-^{uu} > |\lambda_-^{ss}|, \end{cases}$$

$$\nu < -(b/c)^2 : \begin{cases} c < 0 : \lambda_+^{uu} < |\lambda_+^{ss}|, & \lambda_-^{uu} > |\lambda_-^{ss}|, \\ c > 0 : \lambda_+^{uu} > |\lambda_+^{ss}|, & \lambda_-^{uu} < |\lambda_-^{ss}|. \end{cases}$$

Note that there are essentially four different local phase portraits, three of them are listed above and the fourth ' $\lambda_+^{uu} < |\lambda_+^{ss}|$, $\lambda_-^{uu} < |\lambda_-^{ss}|$ ' is equivalent to ' $\lambda_+^{uu} > |\lambda_+^{ss}|$, $\lambda_-^{uu} > |\lambda_-^{ss}|$ ' by the above symmetry; thus it can be obtained by taking $b < 0$.

A similar analysis carried out at the perturbed centres $C^\pm$ reveals that their eigenvalues are

$$\lambda_\pm^1 = \pm 2\varepsilon\sqrt{(-\nu - \rho)}, \quad \lambda_\pm^{2,3} = \frac{1}{2}\varepsilon(b \pm c\sqrt{(-\nu - \rho)}) \pm i\sqrt{(1 + O(\varepsilon^2))}, \quad (3.3)$$

from which we deduce the following, for $b \geq 0$:

$$-(b/c)^2 - \rho < \nu < -\rho : \begin{cases} c < 0 : C^+ \text{ source, } C^- \text{ saddle, } \dim W^s(C^-) = 1, \\ c > 0 : C^+ \text{ source, } C^- \text{ saddle, } \dim W^s(C^-) = 1, \end{cases}$$

$$\nu < -(b/c)^2 - \rho : \begin{cases} c < 0 : C^+ \text{ saddle, } \dim W^s(C^+) = 2, \\ \quad \quad \quad C^- \text{ saddle, } \dim W^s(C^-) = 1, \\ c > 0 : C^+ \text{ source, } C^- \text{ sink.} \end{cases}$$

Here we have denoted the stable manifolds of $C^\pm$ by $W^s(C^\pm)$. It can be checked that a Hopf bifurcation occurs for at C^+ for $c < 0$ (resp. at C^- for $c > 0$) on the set

$$\mathcal{C}_{\text{Hopf}} = \{\nu + (b/c)^2 = -\rho\}. \quad (3.4)$$

This formula gives an exact description of $\mathcal{C}_{\text{Hopf}}$, since the real part of $\lambda_\pm^{2,3}$ is given explicitly in (3.3) (i.e. not as a perturbation series in ε). Note that $C^\pm$ lie on the invariant line $\{x = 0, y = 1\}$ and that there is a heteroclinic orbit from C^+ to C^- along the segment of this line between the two fixed points for all $\nu < -\rho$.

(b) The existence and local character of the periodic solutions

It follows directly from (2.9) and (2.10) that, for the special case of (2.14), the components of the Poincaré map are

$$\left. \begin{aligned} \Delta Z(k, z)/\varepsilon &= (\nu + z^2)T_0(k) + \rho T_1(k) + O(\varepsilon), \\ \Delta K(k, z)/\varepsilon &= (b + cz)S_0(k) + O(\varepsilon). \end{aligned} \right\} \quad (3.5)$$

Note that $\Delta K = O(\varepsilon^2)$ in the plane $z = -b/c$. Therefore, the periodic solutions can exist only $O(\varepsilon)$ close to the $\{z = -b/c\}$ -plane (this corresponds in the coordinate system of (2.2) to $z = -(\mu + b/c)$, i.e. any plane, by adjusting μ). We define

$$\mathcal{T}(k) = T_1(k)/T_0(k) \quad (3.6)$$

and, using the computations of Appendix B (equation (B 1)), equate ΔZ to zero in the plane $\{z = -b/c\}$, implying that

$$\mathcal{T}(k, -b/c) = -[\nu + (b/c)^2]/\rho \in [0, 1] \quad (\rho \neq 0).$$

Figure 2. The four bifurcation curves $\mathcal{C}_{\text{SN}} = \{\nu = 0\}$, $\mathcal{C}_{\text{CN}} = \{\nu = -\rho\}$, $\mathcal{C}_{\text{glob}} = \{\nu = -(b/c)^2\}$ and $\mathcal{C}_{\text{Hopf}} = \{\nu = -\rho - (b/c)^2\}$. The region in which periodic solutions exist is shaded. No perturbed saddle points exist above $\mathcal{C}_{\text{SN}}$ ($\nu > 0$) and no perturbed centre points exist above $\mathcal{C}_{\text{CN}}$ ($\nu > -\rho$).

The limit $k \uparrow k_s$, for which $\mathcal{T} \downarrow 0$ approximately determines the curve $\mathcal{C}_{\text{glob}}$:

$$\mathcal{C}_{\text{glob}} = \{\nu + (b/c)^2 = 0 + O(\varepsilon)\}. \quad (3.7)$$

The curve $\mathcal{C}_{\text{Hopf}}$ can be determined analogously, by taking the limit $k \downarrow k_c$ (i.e. $\mathcal{T} \uparrow 1$). This yields (3.4) and is therefore in agreement with the local analysis at $C^\pm$ of §3*a*. Thus there are periodic solutions to the model problem if ν lies between $-(b/c)^2$ and $-\rho - (b/c)^2$.

The sign of b/c – and thus in essence the sign of c , since we can take $b \geq 0$ by the symmetry – plays an important role in the forthcoming analysis. Periodic solutions only exist $O(\varepsilon)$ near the $\{z = -b/c\}$ -plane, therefore, global bifurcations also take place $O(\varepsilon)$ near this plane. This determines which of the two perturbed saddles, $S^+ = (0, 0, \sqrt{-\nu})$ or $S^- = (0, 0, -\sqrt{-\nu})$, is involved in the global bifurcation:

$$b/c > 0 \quad \leftrightarrow \quad (0, 0, -\sqrt{-\nu}) \quad \text{and} \quad b/c < 0 \quad \leftrightarrow \quad (0, 0, \sqrt{-\nu}).$$

This observation shows how the GL case ($b = 0$) is unfolded by taking $b \neq 0$ in (2.14): the global bifurcation ‘jumps’ from the upper point $(0, 0, \sqrt{-\nu})$ to the lower $(0, 0, -\sqrt{-\nu})$ as b goes through 0. Perhaps more strikingly, for $b = 0$ $\mathcal{C}_{\text{Hopf}} = \mathcal{C}_{\text{CN}}$ *exactly*, while $\mathcal{C}_{\text{glob}}$ coincides with $\mathcal{C}_{\text{SN}}$ at leading order. We summarize the bifurcation behaviour discovered thus far in figure 2, which applies to all cases with $b, c \neq 0$.

We next study the local character of the periodic orbits by the Poincaré map defined in §2:

$$\mathcal{P} : \begin{pmatrix} k \\ z \end{pmatrix} \mapsto \begin{pmatrix} k \\ z \end{pmatrix} + \begin{pmatrix} \Delta K(k, z) \\ \Delta Z(k, z) \end{pmatrix}$$

(recall that both ΔK and ΔZ are $O(\varepsilon)$). Therefore, the matrix M which determines the local character of the periodic orbits – the fixed points of $\mathcal{P}$ – is given by

$$\begin{pmatrix} 1 + \frac{\partial}{\partial k} \Delta K|_{\Delta K = \Delta Z = 0} & \frac{\partial}{\partial z} \Delta K|_{\Delta K = \Delta Z = 0} \\ \frac{\partial}{\partial k} \Delta Z|_{\Delta K = \Delta Z = 0} & 1 + \frac{\partial}{\partial z} \Delta Z|_{\Delta K = \Delta Z = 0} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 1 + \varepsilon A_1 & \varepsilon B_1 \\ \varepsilon C_1 & 1 + \varepsilon D_1 \end{pmatrix},$$

at leading order. We write the eigenvalues of M as

$$\Lambda = 1 + \varepsilon \Lambda_1 + \text{h.o.t.}$$

and derive as eigenvalue equation for Λ_1

$$\Lambda_1^2 - \Lambda_1(\Lambda_1 + D_1) + (A_1 D_1 - B_1 C_1) = 0. \quad (3.8)$$

Note that one has to introduce the unknown higher order terms $\varepsilon^2 A_2$ and $\varepsilon^2 D_2$ on the diagonal of M in deriving (3.8). However, these terms are again cancelled in the end. It is quite easy to compute A_1 , B_1 and D_1 : since $\Delta K = 0$ at the fixed point, we have $b + cz = 0$ from (3.4) and thus

$$A_1 = \frac{\partial}{\partial k} \Delta K|_{\Delta K = \Delta Z = 0} = (b + cz) \frac{\partial}{\partial k} S_0|_{\Delta Z = 0} = 0, \quad (3.9)$$

and

$$B_1 = c S_0|_{\Delta Z = 0}, \quad D_1 = 2z T_0|_{\Delta Z = 0} = -2(b/c) T_0|_{\Delta Z = 0}. \quad (3.10)$$

We derive in Appendix B (equations (B 5) and (B 9)):

$$\frac{d}{dk} T_0 = -\frac{6k T_0 + T_1}{6k(6k + 1)} \quad \text{and} \quad \frac{d}{dk} T_1 = -\frac{T_0 - T_1}{6k + 1}. \quad (3.11)$$

Furthermore, since $\Delta Z = 0$, i.e. $\rho T_1 = -(\nu + (b/c)^2) T_0$, we can eliminate T_1 in the expression for C_1 :

$$C_1 = \rho \frac{X^2 - 12kX - 6k}{6k(6k + 1)} T_0|_{\Delta Z = 0} \quad \text{with} \quad X = \frac{\nu + (b/c)^2}{\rho}. \quad (3.12)$$

The discriminant of the polynomial $X^2 - 12kX - 6k$ is $24k(6k + 1)$ and so the polynomial is positive definite for $k \in (k_c, k_s)$. Using this, and recalling that $T_0 > 0$, $S_0 > 0$ and $k < 0$, we find that

$$\text{sgn}(B_1) = \text{sgn}(c), \quad \text{sgn}(C_1) = -\text{sgn}(\rho), \quad \text{sgn}(D_1) = -\text{sgn}(b/c).$$

Letting Λ_1^+ and Λ_1^- be the solutions of (3.8), we conclude by the above analysis that

$$\text{sgn}(\Lambda_1^+ + \Lambda_1^-) = -\text{sgn}(b/c) \quad \text{and} \quad \text{sgn}(\Lambda_1^+ \Lambda_1^-) = \text{sgn}(c\rho), \quad (3.13)$$

and, therefore:

- $\rho > 0$ and $c > 0$: the periodic solution is stable,
- $\rho > 0$ and $c < 0$: the periodic solution is of saddle type,
- $\rho < 0$ and $c > 0$: the periodic solution is of saddle type,
- $\rho < 0$ and $c < 0$: the periodic solution is unstable,

where we only considered $b > 0$. If the periodic solution is not of saddle type one has to evaluate the sign of $D_1^2 + 4B_1 C_1$ to determine whether the Poincaré map exhibits node or spiral behaviour. This requires a complicated analysis (in the case $B_1 C_1 < 0$), but since we do not need this information in the remainder of the paper, we do not pursue it further. However, note that when $b = 0$ the eigenvalue equation (3.8) reduces to $\Lambda_1^2 - B_1 C_1 = 0$, thus, linearly and at leading order, the periodic orbit is either a (degenerate) saddle ($c\rho < 0$) or a centre ($c\rho > 0$). See § 5 *a* below where we further consider this case.

(c) *The slow manifold $\{x = y = 0\}$ and its stable and unstable manifolds*

As we begin to concentrate more on global behaviour it is useful to first eliminate certain 'irrelevant' solutions from consideration. Recall that we are concerned primarily with solutions asymptotic as $t \rightarrow \pm\infty$ to fixed points on the slow manifold $\Gamma = \{x = y = 0\}$ – or to Γ itself – or to periodic orbits or other limit sets contained in $x \geq 0$, and that the parameters ν , ρ , b and c are supposed to be uniformly bounded, i.e. the parameters are $O(1)$ with respect to ε . The following lemma implies that we need only consider orbits for which $x(t) \geq 0$ for all t .

Lemma 3.1. *The sets $\mathcal{S}_+ = \{(x, y, z) : x < 0, y < 0\}$ and $\mathcal{S}_- = \{(x, y, z) : x < 0 < y\}$ are respectively positively and negatively invariant for the flow of (2.14).*

Proof. The assertion follows directly from examination of the first and second components of the vector field on the boundaries of $\mathcal{S}_\pm$, namely:

$$\dot{x}|_{x=0} = y \begin{cases} > 0 & \text{for } y > 0, \\ < 0 & \text{for } y < 0, \end{cases} \quad \text{and} \quad \dot{y}|_{y=0} = x - x^2 < 0 \quad \text{for } x < 0.$$

Together with the fact that $\Gamma = \{x = y = 0\}$ is invariant, this shows that solutions entering $\mathcal{S}_+$ can never leave $\mathcal{S}_+$ as t increases and solutions entering $\mathcal{S}_-$ as t decreases can never leave $\mathcal{S}_-$ as t decreases. ■

Standard Gronwall estimates imply that, for any $\tilde{\delta} > 0$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ any solution based at a point (x_0, y_0, z_0) with $x_0 > 0$ and

$$k_0 = \frac{1}{2}y_0^2 - \frac{1}{2}x_0^2 + \frac{1}{3}x_0^3 > \tilde{\delta} > k_s = 0,$$

will enter $\mathcal{S}_+$ (resp. $\mathcal{S}_-$) at some finite time $t > 0$ (resp. $t < 0$) and so, by lemma 3.1, can be henceforth discarded. Thus for any $\delta > 0$ and sufficiently small ε (depending on δ), solutions of interest belong to the strip $0 \leq x \leq \frac{3}{2} + \delta$. In this strip, from the third component of (2.14) we have

$$\dot{z} \geq \varepsilon(\nu - |\rho|(\frac{3}{2} + \delta(\varepsilon)) + z^2),$$

so that there is a constant $C = C(\nu, \rho, \delta(\varepsilon))$ such that $\dot{z} > 0$ for $|z| > C$.

This last observation prompts a modification to (2.14) which avoids difficulties associated with solutions which blow up in finite time, due to the dominant term $\dot{z} = \varepsilon z^2$ in the third component of (2.14). Note that $\dot{z} \neq o(1)$, i.e. $\dot{z} \not\rightarrow 0$ as $\varepsilon \rightarrow 0$, if z becomes larger than $O(\varepsilon^{-1/2})$. We conclude by the above considerations that solutions of interest satisfy $|z| < D$ for some $1 \ll D \ll \varepsilon^{-1/2}$. Thus, we introduce the C^∞ function $h(z)$:

$$h(z) = \begin{cases} 1 & |z| < D, \\ 0 & |z| > D + 1, \end{cases}$$

and replace the third component of (2.14) by

$$\dot{z} = \varepsilon[h(z)(\nu + \rho x + z^2) + (1 - h(z))].$$

Solutions in the domain of interest are unaffected by this modification and all solutions now exist for all t and so stable and unstable manifolds can be defined in the usual way. To this we now turn.

Let $W^s(\Gamma)$ and $W^u(\Gamma)$ denote those components of the stable and unstable manifolds of the slow manifold $\Gamma = \{x = y = 0\}$ of system (2.14) containing orbits which

approach Γ for $t \rightarrow +\infty$ (resp. $t \rightarrow -\infty$) with positive x -coordinate. Thus in the limit $\varepsilon \rightarrow 0$ both these two-dimensional manifolds coincide with the surface given by

$$\mathcal{H} = \{\tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 = 0 \quad (=k_s); x \geq 0\} \quad (3.14)$$

(see figure 1). For $\varepsilon \neq 0$ the stable and unstable manifolds $W^{s,u}(\Gamma)$ no longer coincide in a single surface like $\mathcal{H}$, but rather split and intersect only along orbits homoclinic to Γ . We analyse the form and intersections of $W^s(\Gamma)$ and $W^u(\Gamma)$ primarily by considering their intersections with the plane $\{y = 0\}$. These intersections can be numbered as follows: $\Gamma (= \{x = 0\})$ is the zeroth intersection of both $W^s(\Gamma)$ and $W^u(\Gamma)$ with $\{y = 0\}$ and the first intersections of $W^s(\Gamma)$ and $W^u(\Gamma)$ with $\{y = 0\}$ are $O(\varepsilon)$ close to the other component of $\mathcal{H} \cap \{y = 0\}$, namely the line $\{x = \frac{3}{2}\}$. More precisely, $(x_0, 0, z_0)$ belongs to the first intersection of $W^s(\Gamma)$ (resp. $W^u(\Gamma)$) with $\{y = 0\}$ if the ω - (resp. α -) limit set of the orbit $(x(t), y(t), z(t))$ based at $(x_0, 0, z_0)$ is contained in Γ and $y(t) \neq 0$ for any $t \in (0, \infty)$ (resp. $t \in (-\infty, 0)$). In the terminology of Wiggins (1992), this is a *primary intersection point*. At first we only consider these first intersections, later and in §4 the second, third, etc., intersections – if they exist – will be considered.

First we derive an expression for the distance as a function of z between these first intersections of $W^s(\Gamma)$ and $W^u(\Gamma)$ with $\{y = 0\}$. We denote these first intersections by $P^{-1}(\Gamma)$ (resp. $P(\Gamma)$). Note that P corresponds to ‘only half’ of the Poincaré map $\mathcal{P}$ defined in the previous section. We use the Melnikov method for systems with slowly varying coefficients, presented by Robinson (1988; cf. Wiggins 1988). This method measures the distance between $P(\Gamma)$ and $P^{-1}(\Gamma)$ in a $\{z = z_0\}$ plane. We define x_ε^u and x_ε^s as the intersections of $P(\Gamma)$ (resp. $P^{-1}(\Gamma)$) with $\{z = z_0\}$ on $\{y = 0\}$. Solutions $\gamma_\varepsilon^u(t) = (x_\varepsilon^u(t), y_\varepsilon^u(t), z_\varepsilon^u(t))$ in $W^u(\Gamma)$ and $\gamma_\varepsilon^s(t) = (x_\varepsilon^s(t), y_\varepsilon^s(t), z_\varepsilon^s(t))$ in $W^s(\Gamma)$ of (2.14) are determined by the initial condition $\gamma_\varepsilon^{u,s}(0) = (x_\varepsilon^{u,s}, 0, z_0)$ and $\gamma_0(t) = (x_0(t), y_0(t), z_0)$ is defined as the homoclinic solution to the unperturbed system (2.1) with $\gamma_0(0) = (\frac{3}{2}, 0, z_0)$. As is usual in Melnikov theory, we define the following time-dependent weighted distance function (Robinson 1983):

$$\Delta W(t, z_0) = \begin{pmatrix} y_0(t) \\ x_0(t) - x_0^2(t) \end{pmatrix} \wedge \begin{pmatrix} \frac{\partial}{\partial \varepsilon} [x_\varepsilon^u(t) - x_\varepsilon^s(t)]|_{\varepsilon=0} \\ \frac{\partial}{\partial \varepsilon} [y_\varepsilon^u(t) - y_\varepsilon^s(t)]|_{\varepsilon=0} \end{pmatrix},$$

where the wedge denotes the standard planar cross product. Note that

$$\Delta W(0, z_0) = \frac{3}{4} \frac{\partial}{\partial \varepsilon} [x_\varepsilon^u - x_\varepsilon^s]|_{\varepsilon=0}, \quad (3.15)$$

thus $\Delta W(0, z_0)$ measures the $O(\varepsilon)$ part of the distance between $P(\Gamma)$ and $P^{-1}(\Gamma)$ in the $\{z = z_0\}$ plane. Similar to the derivation of the Melnikov function for periodically driven systems (see for instance Guckenheimer & Holmes 1983) one shows (see Robinson 1983) that for the system (2.14)

$$\Delta W(0, z) = \int_{-\infty}^{\infty} \begin{pmatrix} y_0(t) \\ x_0(t) - x_0^2(t) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ y_0(t)[b + cz] \end{pmatrix} dt.$$

Note that the fact that the unperturbed system does not explicitly depend on the slow variable z simplifies the expression for $\Delta W(0, z)$ considerably (Robinson 1983).

Thus we find

$$\Delta W(0, z) = \int_{-\infty}^{\infty} (b + cz)y_0^2(t) dt = (b + cz)S_0(0) = \frac{\Delta K(0, z)}{\varepsilon} + O(\varepsilon) \quad (3.16)$$

(see (2.11), (2.7) and (3.5)). We can now interpret $\Delta W(0, z)$ in terms of k : $\Delta W(0, z)$ measures the difference between $k^u(z)$ and $k^s(z)$, where $k^{u,s}(z)$ are the values of integral k at the points $(x_\varepsilon^{u,s}, 0, z)$. This is confirmed by the observation that

$$x_\varepsilon^{u,s} = \frac{3}{2} + \varepsilon \tilde{x}_\varepsilon^{u,s} \longrightarrow k^{u,s} = \frac{3}{4}\varepsilon \tilde{x}_\varepsilon^{u,s} + O(\varepsilon^2), \quad (3.17)$$

which can be checked by straightforward computations. By the definition of ΔK in §2 and by combining the above observation with (3.14) we have, once again

$$\Delta W(0, z) = k^u(z) - k^s(z) = \Delta K(0, z)/\varepsilon + O(\varepsilon). \quad (3.18)$$

All in all, the combination of (3.15) and (3.16) gives a useful description of the distance between $P(\Gamma)$ and $P^{-1}(\Gamma)$ in the (k, z) -plane. We observe that this distance is a linear function of z and that $P(\Gamma)$ and $P^{-1}(\Gamma)$ intersect at

$$z = -b/c + O(\varepsilon),$$

i.e. the same plane (to $O(\varepsilon)$) as the one in which the periodic solution of (2.14) lies (see the previous section). Hence, provided $c \neq 0$ there is always an intersection of $W^u(\Gamma)$ and $W^s(\Gamma)$. Such an intersection corresponds to a solution $\gamma_{\text{int}}(t)$ of (2.14) with

$$\lim_{t \rightarrow \infty} d(\gamma_{\text{int}}(t), \Gamma) = \lim_{t \rightarrow -\infty} d(\gamma_{\text{int}}(t), \Gamma) = 0$$

($d(\cdot, \cdot)$ is the standard distance function). The solution $\gamma_{\text{int}}(t)$ is $O(\varepsilon)$ close – in some ‘central’ time interval $-\tau \leq t \leq \tau$ – to $\mathcal{H} \cap \{z = -b/c\} \cap \{x > \delta\}$, (see (3.13)) $\varepsilon \ll \delta \ll 1$, which is the part of the unperturbed homoclinic solution in the $z = -b/c$ plane bounded away from Γ .

The above analysis gives more information than just the distance between $P(\Gamma)$ and $P^{-1}(\Gamma)$: from (3.15) one can also determine the relative positions of $P(\Gamma)$ and $P^{-1}(\Gamma)$. Observe that the sign of c determines whether $k^u(z) - k^s(z)$ is decreasing or increasing as a function of z ((3.15) and (3.16)). This – *a priori* small – distinction yields two essentially different situations, as we shall see below and, in more detail, in §4.

When $c < 0$ we have

$$\frac{d}{dz} \Delta W(0, z) = cS_0(0) > 0,$$

implying that $x_\varepsilon^u - x_\varepsilon^s$ is positive for $z < -b/c$ and negative for $z > -b/c$ (see (3.14) and (3.15)). For $z < -b/c$ we say that $P(\Gamma)$ lies *outside* $P^{-1}(\Gamma)$. The converse occurs for $c > 0$: $P(\Gamma)$ lies *inside* $P^{-1}(\Gamma)$ for $z < -b/c$. See the upper panels of figures 3a, b. We now want to study the implications of these relative positions for the global structure of $W^u(\Gamma)$ and $W^s(\Gamma)$. As we shall see, this may be very complicated, involving countably many components in any cross section, and it is easiest first to understand the distinction between $c < 0$ and $c > 0$ in the case that $\nu > 0$ is large enough, not only that Γ contains no fixed points, but also that the z coordinate of $\gamma_{\text{int}}(t)$ increases monotonically with t ($\nu > \min\{0, -|\rho|(\frac{3}{2} + \delta)\}$ suffices, see above). Until near the end of this section we assume this to hold.

Let $\mathcal{C}^{u,s} = W^{u,s}(\Gamma) \cap \{z = z^*\}$ denote the curves in which the unstable and stable manifolds intersect the plane $\{z = z^* = -b/c + O(\varepsilon)\}$, on which $P(\Gamma)$ and $P^{-1}(\Gamma)$

intersect. Furthermore, we denote by $\mathcal{C}_{\text{loc}}^{\text{u,s}} \subset \mathcal{C}^{\text{u,s}}$ the closed segments bounded by the zeroth and primary intersection points $\{x = y = 0\}$ and $\{x = x_\varepsilon^{\text{u}} = x_\varepsilon^{\text{s}}, y = 0\}$ – by the definition of z^* . $\mathcal{C}_{\text{loc}}^{\text{u}} \cup \mathcal{C}_{\text{loc}}^{\text{s}}$ forms a closed loop $O(\varepsilon)$ near the level set $k = 0$ of the integral (2.6). Let $\mathcal{D}$ denote the region bounded by this loop. Consider now points contained in $\mathcal{C}^{\text{u}} \setminus \mathcal{C}_{\text{loc}}^{\text{u}}$ ‘beyond’ the first intersection point. The structure of the vector field (2.14) and the assumption above imply that such points belong to orbits which also contain points on $P(\Gamma)$ below $\{z = z^*\}$ and hence, for $c < 0$, outside $P^{-1}(\Gamma)$. Since $P(\Gamma)$ and $P^{-1}(\Gamma)$ intersect only at $z = z^*$ the orbits in question lie entirely outside $W^{\text{s}}(\Gamma)$ and so $\mathcal{C}^{\text{u}} \setminus \mathcal{C}_{\text{loc}}^{\text{u}}$ lies below $\mathcal{C}_{\text{loc}}^{\text{s}}$ with respect to y and so *outside* $\mathcal{D}$. For $c > 0$ the situation is reversed: $P(\Gamma)$ lies inside $P^{-1}(\Gamma)$ for $z < z^*$ and so $\mathcal{C}^{\text{u}} \setminus \mathcal{C}_{\text{loc}}^{\text{u}}$ lies above $\mathcal{C}_{\text{loc}}^{\text{s}}$ with respect to y and hence *inside* $\mathcal{D}$. See the lower panels of figures 3a, b.

Remark 3.2. Integrating the function k of (2.6) along the perturbed homoclinic orbit $\gamma_{\text{int}}(t) = (x_{\text{int}}(t), y_{\text{int}}(t), z_{\text{int}}(t))$ with initial condition $\gamma_{\text{int}}(0) = (x_\varepsilon^{\text{u}} = x_\varepsilon^{\text{s}}, 0, z^*)$ yields

$$k(t) = k(0) + \varepsilon \int_0^t (b + cz_{\text{int}}(t)) y_{\text{int}}^2(t) dt,$$

where $k(0) = (-\frac{1}{2}x^2 + \frac{1}{3}x^3)|_{x=x_\varepsilon^{\text{u}}=x_\varepsilon^{\text{s}}} = \frac{3}{4}\varepsilon x_\varepsilon^{\text{s,u}} + O(\varepsilon^2)$ (cf. (3.16)). Under our assumption that z increases monotonically we see that, for $t > 0$, $z > -b/c$ and so $k(t) < k(0)$ (resp. $k(t) > k(0)$) for $c < 0$ (resp. $c > 0$). Since $\lim_{t \rightarrow \infty} d(\gamma_{\text{int}}(t), \Gamma) = 0$ we have $\lim_{t \rightarrow \infty} k(t) = 0$ and thus $k(0) > 0$ for $c < 0$ and $k(0) < 0$ for $c > 0$. With similar considerations for $t < 0$, this implies that $\mathcal{C}^{\text{u,s}}$ lie entirely outside the interior of the level set $k = 0$ in case $c < 0$ and entirely inside it in case $c > 0$, in agreement with the ‘topological’ arguments above. We also show this on figures 3a, b.

As we have already indicated on the lower panel of figure 3a, for $c < 0$ the global structure of $\mathcal{C}^{\text{u}}$ and $\mathcal{C}^{\text{s}}$ on $\{z = z^*\}$ is rather simple: points on $\mathcal{C}^{\text{u}} \setminus \mathcal{C}_{\text{loc}}^{\text{u}}$ and $\mathcal{C}^{\text{s}} \setminus \mathcal{C}_{\text{loc}}^{\text{s}}$ near the first intersection lie outside $\mathcal{D}$ and hence, using the linearized local flow near Γ and estimating the behaviour of perturbed solutions by Gronwall’s lemma, we can deduce that $\mathcal{C}^{\text{u}} \setminus \mathcal{C}_{\text{loc}}^{\text{u}}$ and $\mathcal{C}^{\text{s}} \setminus \mathcal{C}_{\text{loc}}^{\text{s}}$ also contain components $O(\varepsilon)$ close to the unbounded components of the level set $k = 0$ (i.e. with $x < 0$). The first intersection point is the *only* element of $\mathcal{C}^{\text{u}} \cap \mathcal{C}^{\text{s}}$ other than the zeroth point $x = y = 0$.

It is now possible to deduce the structure of intersections of $W^{\text{u}}(\Gamma)$ and $W^{\text{s}}(\Gamma)$ with *any* section $\{z = \bar{z} = \text{const.}\}$ simply by transporting the structure on $\{z = z^*\}$ along solutions. As t (and thus z) increase, the orbit $\gamma_{\text{int}}(t)$ approaches Γ and a small disc in $W^{\text{u}}(\Gamma)$, intersecting $W^{\text{s}}(\Gamma)$ transversely, enters a neighbourhood of Γ . Appealing to the lambda lemma (see, for instance, Guckenheimer & Holmes 1983) one concludes that, as $z \rightarrow \infty$, one can find discs of $W^{\text{u}}(\Gamma)$ arbitrarily C^1 -close to any other disc in $W^{\text{u}}(\Gamma)$: loosely speaking, $W^{\text{u}}(\Gamma) \cap \{z = \bar{z}\}$ accumulates on itself as $\bar{z} \rightarrow \infty$. Similar remarks apply to $W^{\text{s}}(\Gamma)$ as t (and z) decrease. This leads to the structure shown in figure 4a, where we show intersections of $W^{\text{s}}(\Gamma)$ and $W^{\text{u}}(\Gamma)$ with sections $\{z = \text{const.}\}$ and $\{y = 0\}$. The number of ‘tongues’ of $W^{\text{u}}(\Gamma)$ and $W^{\text{s}}(\Gamma)$ appearing in a fixed interval $|z + b/c| \leq \text{const.}$ is determined by the ratio of the speeds at which a solution circulates inside $\mathcal{D}$ and climbs in the z direction. As the vertical speed decreases, so more tongues appear. This will be important in the next section when we study the global bifurcations occurring as ν decreases and the vertical speed drops to zero at some point(s).

Turning to the case $c > 0$ we first complete the picture of $\mathcal{C}^{\text{u}}$ and $\mathcal{C}^{\text{s}}$ on $\{z = z^*\}$, begun in figure 3b. Recall that $\mathcal{C}^{\text{u}} \setminus \mathcal{C}_{\text{loc}}^{\text{u}}$ and $\mathcal{C}^{\text{s}} \setminus \mathcal{C}_{\text{loc}}^{\text{s}}$ are trapped inside $\mathcal{D}$. We claim

Figure 3. The structure of $W^s(\Gamma)$ and $W^u(\Gamma)$ and their intersections C^s , C^u with the plane $\{z = z^* = -b/c + O(\varepsilon)\}$ for ν sufficiently large. (a) $c < 0$: outside case. (b) $c > 0$: inside case.

that this implies that C^u and C^u intersect in countably many points. First, we observe that on every section $\{z = \text{const.}\}$ the image of $C^u_{\text{loc}} \cup C^s_{\text{loc}}$ under the appropriate flow map encloses a region analogous to $\mathcal{D}$ and the resulting solid cylinder in the full space contains the components of $W^u(\Gamma)$ and $W^s(\Gamma)$ lying ‘beyond’ the first intersection $\gamma_{\text{int}}(t)$. Now each point on $P(\Gamma)$ with $z = \bar{z} < -b/c$ belongs to a solution which, as t (and z) increases eventually intersects $\{z = z^*\}$ at some point of $C^u \setminus C^u_{\text{loc}}$. As $\bar{z} \rightarrow -\infty$, the number of circuits the solution makes inside the solid cylinder before reaching $\{z = z^*\}$ grows without bound, due to the dominant $O(1)$ terms in (2.14). This implies that C^u winds infinitely many times clockwise inside $\mathcal{D}$. Similarly, considering orbits of points on $P^{-1}(\Gamma)$ with $z > -b/c$ as $z \rightarrow +\infty$, we see that C^s winds infinitely many times counter clockwise inside Δ . C^u and C^s belong to invariant unstable and stable manifolds and so cannot self intersect. We conclude that C^u and C^s must therefore intersect each other at least twice as each describes a full circuit in Δ . See figure 4b, in which we show the second and third intersections of this sequence.

Arguments similar to those for the case $c < 0$ reveal the structure on planes $\{z = \bar{z} \neq z^*\}$ and, recalling the helical motion of solutions, we likewise obtain the structure in the section $\{y = 0\}$ also shown in figure 4b.

We can summarize these results as

Figure 4. The global structure of $W^s(\Gamma)$ and $W^u(\Gamma)$ shown via intersections with planes $\{z = \text{const.}\}$ and $\{y = 0\}$. (a) $c < 0$, (b) $c > 0$; the numbers 1, 2, 3, ... identify primary and higher order homoclinic orbits to Γ which make 1, 2, 3, ... circuits around $\{x = 1, y = 0\}$.

Proposition 3.3. For each $c \neq 0$, $\delta > 0$ and $\nu > \min\{0, -|\rho|(\frac{3}{2} + \delta)\}$ there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$ (2.14) has a transverse homoclinic orbit $\gamma_{\text{int}}(t) \subset \{W^u(\Gamma) \cap W^s(\Gamma)\}$, the projection of which onto any plane $\{z = \text{const.}\}$ makes one circuit around $\{x = 1, y = 0\}$ as t runs from $-\infty$ to $+\infty$. If $c < 0$ this is the unique homoclinic orbit to Γ , while if $c > 0$ there are additionally countably many other homoclinic orbits $\gamma_{\text{int}}^n(t)$, $n \geq 2$, which make 2, 3, ... circuits around $\{x = 1, y = 0\}$.

Remark 3.4. Kath (1985) has used essentially the same energy arguments as in our Remark above to establish the structures of figure 4 in an oscillator with slowly varying potential energy; see figures 5 and 6 in the paper cited.

We now relax our assumption that $\nu > \min\{0, -|\rho|(\frac{3}{2} + \delta)\}$. If $\nu < 0$ then there are two critical points, $S^+ = (0, 0, \sqrt{-\nu})$ and $S^- = (0, 0, -\sqrt{-\nu})$, on Γ . Both are saddle points of (2.1) in the limit $\varepsilon \rightarrow 0$. The linear analysis of §3a reveals that the stable and unstable manifolds of S^+ and S^- are contained in the stable and unstable manifolds of Γ , however, one has to be careful. S^+ and S^- each have a one-dimensional strong stable and a one-dimensional strong unstable manifold which we denote $W^{ss}(S^+)$, $W^{uu}(S^+)$, $W^{ss}(S^-)$ and $W^{uu}(S^-)$. These manifolds merge, in the limit $\varepsilon \rightarrow 0$, with the stable and unstable manifolds of the saddles $(0, 0, \sqrt{-\nu})$ and $(0, 0, -\sqrt{-\nu})$ of the unperturbed system (2.1). As above, we focus on the ' $x > 0$ '-parts of these manifolds which correspond to solutions approaching $S^\pm$, in the limits $t \rightarrow \pm\infty$, with positive x -coordinates. The manifolds $W^{ss}(S^+)$, etc., divide $W^s(\Gamma)$ and

Figure 5. The local stable and unstable manifolds of Γ and of the saddle points S^+ and S^- which exist for $\nu < 0$.

$W^u(\Gamma)$ into 3 open parts: $W_+^{u,s}(\Gamma)$, $W_m^{u,s}(\Gamma)$ and $W_-^{u,s}(\Gamma)$, the boundaries between these open region are given by $W^{ss}(S^+)$, etc. (see figure 5). Likewise Γ can be divided into Γ_+ (points above S^+), Γ_m (points between S^+ and S^-) and Γ_- (points below S^-). The linear analysis implies that

$$\left. \begin{aligned} W^s(S^+) &= W^{ss}(S^+), \\ W^u(S^+) &= \{W_+^u(\Gamma) \cup W^{uu}(S^+) \cup W_m^u(\Gamma)\} \cup \{\Gamma_+ \cup \Gamma_m\}, \\ W^s(S^-) &= \{W_-^s(\Gamma) \cup W^{ss}(S^-) \cup W_m^s(\Gamma)\} \cup \{\Gamma_- \cup \Gamma_m\}, \\ W^u(S^-) &= W^{uu}(S^-), \end{aligned} \right\} \quad (3.19)$$

again taking only the ' $x > 0$ '-parts. The perturbation calculations at the beginning of this subsection imply that, for $\nu < 0$ and ν outside an $O(\varepsilon)$ neighbourhood of $-(b/c)^2$, we have the following behaviour

$$\left. \begin{aligned} -b/c > \sqrt{-\nu} &\Rightarrow \left\{ \begin{aligned} \lim_{t \rightarrow \infty} \gamma_{\text{int}}(t) &= (0, 0, \infty), \\ \lim_{t \rightarrow -\infty} \gamma_{\text{int}}(t) &= S^+, \end{aligned} \right. \\ -b/c \in (-\sqrt{-\nu}, \sqrt{-\nu}) &\Rightarrow \left\{ \begin{aligned} \lim_{t \rightarrow \infty} \gamma_{\text{int}}(t) &= S^-, \\ \lim_{t \rightarrow -\infty} \gamma_{\text{int}}(t) &= S^+, \end{aligned} \right. \\ -b/c < -\sqrt{-\nu} &\Rightarrow \left\{ \begin{aligned} \lim_{t \rightarrow \infty} \gamma_{\text{int}}(t) &= S^-, \\ \lim_{t \rightarrow -\infty} \gamma_{\text{int}}(t) &= (0, 0, -\infty). \end{aligned} \right. \end{aligned} \right\} \quad (3.20)$$

In the next section we will combine the information on the structure of $W^s(\Gamma)$ and $W^u(\Gamma)$ for $\nu > 0$, obtained earlier, with that on the stable and unstable manifolds of S^+ and S^- summarized above, to study the global bifurcations that occur as ν passes $-(b/c)^2$, but we observe that this already implies that, for fixed $b, c > 0$ and

Figure 6. The structure of the stable and unstable manifolds $W^u(\Gamma)$ and $W^s(\Gamma)$ for $\nu \leq 0$, showing the strong manifolds, $W^{uu,ss}(S^\pm)$, of the saddle points for various $\nu, \rho, b > 0$. (a) $c < 0$; (b) $c > 0$.

ν sufficiently large and negative, there are at least two heteroclinic connections from S^+ to S^- : a slow one in Γ_m and γ_{int} , which consists of both slow and fast parts.

4. Homoclinic and heteroclinic orbits to fixed points

In this section we study the bounded homo- and heteroclinic orbits to fixed points which follow or accompany the creation of those equilibria. We first recall the system under study (2.14):

$$\dot{x} = y, \quad \dot{y} = x - x^2 + \varepsilon y(b + cz), \quad \dot{z} = \varepsilon(\nu + \rho x + z^2).$$

Figure 6. *Cont.*

The major results of this section are theorems 4.8 and 4.9, which characterize the global bifurcations occurring as ν decreases for fixed b , c and $\rho > 0$, $= 0$ and < 0 . The two structures of proposition 3.3 and figure 4, for $c < 0$ and $c > 0$, lead to two basically different scenarios. In the first, *explosions* occur in which countably many homoclinic connections are born from the unique connection $\gamma_{\text{int}}(t) \subset \{W^u(\Gamma) \cap W^s(\Gamma)\}$ which exists for large ν . In the second, *implosions* destroy all but one of the countable set of multi-circuit connections in $W^u(\Gamma) \cap W^s(\Gamma)$. We draw extensively on the analysis of earlier sections, which we begin by summarizing in a series of pictures of the bounded parts of $W^u(\Gamma)$ and $W^s(\Gamma)$ between their zeroth and first

intersections with $\{y = 0\}$. We will draw our fully global deductions from these ‘semi-local’ structures.

(a) *Recapitulation*

Figures 6a, b show the cases $c < 0$ and $c > 0$ respectively: the three columns on each figure refer to $\rho < 0$, $\rho = 0$ and $\rho > 0$, reading from left to right and, reading down, the rows give the behaviour for decreasing values of ν . S^+ and S^- appear in a saddle-node bifurcation at $\nu = 0$ and separate until, at $\nu = -(b/c)^2$ and for $c < 0$ ($\Rightarrow -b/c > 0$) (resp. $c > 0$ ($\Rightarrow -b/c < 0$)) S^+ (resp. S^-) passes through the level $\{z = -b/c\}$ on which, to $O(\varepsilon)$, $P(\Gamma)$ and $P^{-1}(\Gamma)$ intersect. Here the sign of ρ has an influence: for $\rho < 0$ and $x > 0$ the third component of (2.14) shows that $\dot{z} < 0$ on $\{z = \pm\sqrt{-\nu}\}$, so that $W^{uu}(S^\pm)$ go out below the planes $\{z = \pm\sqrt{-\nu}\}$ and $W^{ss}(S^\pm)$ come in from above these planes. For $\rho > 0$ the reverse holds and when $\rho = 0$ the third component decouples and the planes $\{z = \pm\sqrt{-\nu}\}$ are invariant. These observations, along with computations of ΔZ in §§2 and 3, imply that the global bifurcation in which orbits homoclinic to $S^\pm$ are found occur not on $\nu = -(b/c)^2$, but on a curve $\nu = -(b/c)^2 + \varepsilon H(\rho, b, c, \varepsilon)$ where $H > 0$ for $\rho < 0$, $H = 0$ for $\rho = 0$, $H < 0$ for $\rho > 0$, and $H = O(1)$. Note that this does not contradict the first order perturbation analysis, which merely predicts that homoclinic bifurcations occur on $\nu = -(b/c)^2 + O(\varepsilon)$. These observations complete the information we need to draw figure 6 and prove:

Proposition 4.1. *For $b > 0$, $c \neq 0$ the bifurcation set on which (2.14) has a homoclinic orbit to a hyperbolic saddle on Γ is given by*

$$\mathcal{C}_{\text{glob}} = \{\nu = -(b/c)^2 + \varepsilon H(\rho, b, c, \varepsilon)\},$$

where $H(\rho, \cdot) > 0$ for $\rho < 0$, $H(0, \cdot) = 0$, $H(\rho, \cdot) < 0$ for $\rho > 0$ and $H(\rho, \cdot) = O(1)$. If $c < 0$ (resp. $c > 0$) the orbit is homoclinic to $S^+ = (0, 0, +\sqrt{-\nu})$ (resp. $S^- = (0, 0, -\sqrt{-\nu})$). For all $\nu < -(b/c)^2 + \varepsilon H$ there are at least two heteroclinic orbits from S^+ to S^- , including the trivial one contained in Γ .

Remark 4.2. In the above discussion we implicitly assumed $b > 0$. For $b = 0$ an exact solution for ‘fast’ heteroclinic and homoclinic solutions along the strong manifolds $W^{uu}(S^\pm)$, $W^{ss}(S^\pm)$ can be found by seeking a solution of the form

$$x = A \operatorname{sech}^2(rt), \quad y = \dot{x}, \quad z = B \tanh(rt). \quad (4.1)$$

Note that these solutions share the full symmetry $(x, y, z, t) \rightarrow (x, -y, -z, -t)$ of the vector field for $b = 0$. Substitution in (2.14) with $b = 0$ yields four equations for A , B , r and ν (we regard ε , ρ and c fixed). We find that $\nu = -B^2$ and solve the remaining equations for A , B and r . For $\varepsilon = 0$ we recover the solution $A = \frac{3}{2}$, $r = \frac{1}{2}$ (and $B = 0$), which describes the homoclinic solution of the unperturbed problem. The solution (4.1) is valid for all ε but it is awkward to give explicit formulae except for ε small:

$$\left. \begin{aligned} A &= \frac{3}{2} - \frac{3}{2}\varepsilon^2 \rho c + O(\varepsilon^4), & B &= 3\varepsilon \rho + O(\varepsilon^3), \\ r &= \frac{1}{2} - \frac{3}{4}\varepsilon^2 \rho c + O(\varepsilon^4), & \nu &= -9\varepsilon^2 \rho^2 + O(\varepsilon^4). \end{aligned} \right\} \quad (4.2)$$

Note that the sign of ρ determines whether the connection runs from S^+ to S^- as t increases ($\rho < 0$) or vice versa (see figure 7). If $\rho = 0$ then $B \equiv 0$ and thus $\nu \equiv 0$, this is exactly at the saddle-node bifurcation (see (3.1)), therefore, (4.1)

Figure 7. The strong heteroclinic connections for $b = 0$: (a) $c < 0$, $\rho < 0$; (b) $c < 0$, $\rho > 0$; (c) $c > 0$, $\rho < 0$; (d) $c > 0$, $\rho > 0$.

describes a homoclinic solution to the degenerate saddle $S^+ = S^-$. For $\rho > 0$, the fast connection from S^- to S^+ , along with the slow connection Γ_m from S^+ to S^- , forms a heteroclinic cycle. These exact solutions are the analogues for our model problem of the solutions of Bekki & Nozaki (1985) for the Ginzburg–Landau travelling wave problem (see Doelman 1993).

(b) *Multiple heteroclinic solutions to fixed points for $\nu < 0$ and $|\nu|$ sufficiently large*

In §3c we established the global structure of $W^u(\Gamma)$ and $W^s(\Gamma)$ for $\nu > 0$ sufficiently large that solutions move monotonically upward in z . For $\varepsilon > 0$, small, we found a unique orbit $\gamma_{\text{int}}(t) \subset W^u(\Gamma) \cap W^s(\Gamma)$ homoclinic to Γ for $c < 0$ while for $c > 0$ there is a countable infinity of such orbits (proposition 3.3). We now turn to the case that $\nu < 0$ is sufficiently large in magnitude that there exists at least one non-trivial heteroclinic orbit $\gamma_{\text{int}}(t) \subset W^u(S^+) \cap W^s(S^-)$ connecting the two saddle points in Γ , as in proposition 4.1. We find that for $c < 0$ there are countably many further heteroclinic orbits to $S^\pm$ while for $c > 0$ no other such orbits exist. In §4c we will show that for $c < 0$ (resp. $c > 0$) infinitely many of these heteroclinic (homoclinic) orbits appear (disappear) simultaneously at a certain bifurcation value of ν . This observation prompted our title: in case $c < 0$ one observes an *explosion* of heteroclinic orbits as ν decreases, while for $c > 0$ an *implosion* occurs in which all but one of the homoclinic orbits to Γ vanish. In subsequent sections we will give more detailed descriptions of the bifurcations in which this occurs, but it seems simplest first to consider the ‘final product’ for $\nu < 0$. We have:

Theorem 4.3. Let $H = H(\rho, b, c, \varepsilon)$ be the function defined in proposition 4.1 and suppose that $\nu < -(b/c)^2 + \varepsilon H$. For $c < 0$ and $\varepsilon > 0$ sufficiently small, equation (2.14) has, in addition to the primary heteroclinic orbit $\gamma_{\text{int}}(t) \subset W^u(S^+) \cap W^s(S^-)$, a countable set of further heteroclinic orbits connecting S^+ to S^- , projections of which onto the planes $\{z = \text{const.}\}$ make 2, 3, ... circuits around $\{x = 1, y = 0\}$ respectively. For $c > 0$ the primary heteroclinic orbit $\gamma_{\text{int}}(t)$ is the only non-trivial such orbit, a sufficient condition for uniqueness is $\nu < \min\{-(b/c)^2 + \delta, -((b/c)^2 + \delta) - \rho(\frac{3}{2} + \delta)\}$, where $\delta(\varepsilon) \rightarrow 0^+$ as $\varepsilon \rightarrow 0^+$.

Proof. We argue geometrically from the structure of $P(\Gamma)$ and $P^{-1}(\Gamma)$ illustrated in figures 6a, b. First suppose $c < 0$ and $\nu < -(b/c)^2 + \varepsilon H(\rho, b, c, \varepsilon)$ and consider the image $\mathcal{P}(\mathcal{L}_1)$ of an open segment $\mathcal{L}_1 \subset P(\Gamma)$, bounded below by the first intersection point $\gamma_{\text{int}}(0) = (x_\varepsilon^u = x_\varepsilon^s, 0, z^*)$, where x_ε^u , x_ε^s and $z^* = -b/c + O(\varepsilon)$ have been introduced in §3c, under the first return map $\mathcal{P}$ on the cross section $\{y = 0, x > 1\}$ defined by the flow. For points inside $W^s(\Gamma)$, i.e. left of $P^{-1}(\Gamma)$, this map is well-defined. Since $\gamma_{\text{int}}(t) \rightarrow S^-$ as $t \rightarrow +\infty$, orbits of points on $P(\Gamma)$ arbitrarily close to $\gamma_{\text{int}}(0)$ emerge from a neighbourhood of Γ close to $W^{\text{uu}}(S^-)$ and hence next pierce $\{y = 0, x > 1\}$ outside $W^s(\Gamma)$, i.e. to the right of $P^{-1}(\Gamma)$. In contrast, orbits of points on $P(\Gamma)$ sufficiently far above $z = -b/c$ and $z = +\sqrt{-\nu}$ move monotonically upwards inside $W^s(\Gamma)$. It follows that $\mathcal{P}(\mathcal{L}_1)$ must intersect $P^{-1}(\Gamma)$ in at least one, secondary, heteroclinic point. See figure 8a. The same argument may now be applied to a segment $\mathcal{L}_2$ of $\mathcal{P}(\mathcal{L}_1)$ yielding a third heteroclinic point and, inductively, to segments $\mathcal{L}_{n+1}$ of $\mathcal{P}(\mathcal{L}_n)$. By our construction each such heteroclinic point belongs to a distinct orbit which leaves S^+ , makes $n+1$ circuits around $\{x = 1, y = 0\}$ and limits on S^- as $t \rightarrow +\infty$. Similar arguments involving preimages of segments of $P^{-1}(\Gamma)$ yield the schematic picture of intersections of $W^u(\Gamma)$, $W^s(\Gamma)$ with $\{y = 0, x > 1\}$ also given in figure 8a.

Now consider the case $c > 0$, for which we take an open segment $\tilde{\mathcal{L}}_1 \subset P(\Gamma)$ bounded *above* by $\gamma_{\text{int}}(0)$, and thus left of $P^{-1}(\Gamma)$, i.e. inside $W^s(\Gamma)$, so that the return map on $\{y = 0, x > 1\}$ is defined. As for $c < 0$, orbits starting at points of $\tilde{\mathcal{L}}_1$ arbitrarily close to $\gamma_{\text{int}}(0)$ emerge from a neighbourhood of Γ close to $W^{\text{uu}}(S^-)$, but we now claim that all points of $\mathcal{P}(\tilde{\mathcal{L}}_1)$ lie below the level set $\{z = z^* = -b/c + O(\varepsilon)\}$. For this it suffices to show that relevant orbits inside the level set (of the unperturbed system) $k = \frac{3}{4}\delta > 0$ (cf. (3.17): $x < \frac{3}{2} + \delta$) pass downward through $\{z = z^*\}$ with non-zero speed. From the third component of (2.14), and recalling the estimates at the beginning of §3c, this is clearly true if $\nu < -((b/c)^2 + \delta)$ if $\rho \leq 0$, where $\delta = \delta(\varepsilon)$ is chosen such that $(z^*)^2 < (b/c)^2 + \delta$; if $\rho > 0$ then all forward images of $\tilde{\mathcal{L}}_1$ lie below $\{z = z^*\}$ if the stronger condition $\nu < -((b/c)^2 + \delta) - \rho(\frac{3}{2} + \delta)$ holds. Similarly, if the same conditions on ν are satisfied, all backward images of $P^{-1}(\Gamma)$ bounded below by $\gamma_{\text{int}}(0)$, and so inside $P(\Gamma)$, necessarily lie *above* $\{z = z^*\}$, precluding the possibility of further intersections of $W^u(\Gamma)$ and $W^s(\Gamma)$, as claimed. See figure 8b. ■

Remark 4.4. As noted above, if $\rho = 0$ the third component of system (2.14) decouples and the planes $\{z = \pm\sqrt{-\nu}\}$ are invariant. The perturbation disappears in the plane $\{z = -b/c\}$ at the global bifurcation, $\nu = -(b/c)^2$, see proposition 4.1. Therefore, $z^* \equiv -b/c$ at the global bifurcation and, since the flow induced by (2.14) cannot pierce the plane $\{z = z^*\}$, the condition on ν in theorem 4.3, in the case $c > 0$, can be improved to: $\nu < -(b/c)^2$.

Figure 8. Further heteroclinic orbits for $c < 0$ (a), and their absence for $c > 0$ (b).

We now consider orbits heteroclinic to the fixed ‘centre’ points

$$C^{\pm} = (1, 0, \pm\sqrt{-\nu - \rho}).$$

For $\nu < -(b/c)^2 - \rho$ and $c < 0$ the linear analysis of §3a reveals that C^+ is a saddle with a two-dimensional stable manifold and C^- is a saddle with a one-dimensional stable manifold, while for $c > 0$ C^+ is a source and C^- a sink.

The global analysis is especially simple in the case $\rho = 0$ (see the above Remark). All fixed points, $C^{\pm}$ and $S^{\pm}$, lie on the invariant planes $\{z = \pm\sqrt{-\nu}\}$. In these planes the flow is governed by the evolution of k (2.6):

$$\dot{k} = \varepsilon(b \pm c\sqrt{-\nu})y^2.$$

The behaviour of $\dot{k}$ is determined by $\text{sgn}(b \pm c\sqrt{-\nu})$, which, in case $\rho = 0$, is also the sign of the real part of the complex conjugate pair of eigenvalues of $C^{\pm}$ (cf. (3.3)). There are no periodic orbits in the invariant planes and, by simple phase plane methods, we find that, for $c < 0$ $W^u(S^+)$ contains an orbit connecting S^+ to C^+

and $W^s(S^-)$ contains an orbit connecting C^- to S^- (we assume that $\nu < -(b/c)^2$). For $c > 0$ the situation is reversed: $W^s(S^+)$ contains an orbit connecting C^+ to S^+ and $W^u(S^-)$ an orbit connecting S^- to C^- . Moreover, in the full three-dimensional phase space these heteroclinic orbits are all transverse. For $c < 0$ $W^s(C^+)$ lies in $\{z = +\sqrt{-\nu}\}$ and transversely intersects the two-dimensional manifold $W^u(S^+)$, which is tangent to $\Gamma = \{x = y = 0\}$ and a similar situation occurs for C^- and S^- , with stable and unstable reversed. For $c > 0$ the intersections are transverse by definition: $\dim(W^u(C^+)) = 3$, $\dim(W^s(S^+)) = 1$ and $\dim(W^u(S^-)) = 1$, $\dim(W^s(C^-)) = 3$. For small $|\rho|$ this structurally stable situation persists and we can immediately conclude:

Lemma 4.5. *For all $\varepsilon > 0$, $\nu < -(b/c)^2 - \rho$ and ρ sufficiently small there are unique transverse heteroclinic orbits as follows*

$$\begin{aligned} c < 0 : & \quad S^+ \rightarrow C^+ \quad \text{and} \quad C^- \rightarrow S^-, \\ c > 0 : & \quad C^+ \rightarrow S^+ \quad \text{and} \quad S^- \rightarrow C^-. \end{aligned}$$

However, using the computations of ΔK and ΔZ of §§2 and 3 we can generalize this result to all ρ (of $O(1)$), and ε small. First suppose $c > 0$ (this is the simplest case). Along solutions we have

$$\dot{k} = \varepsilon(b + cz)y^2 \quad (4.3)$$

and so, if $z < -b/c$, $\dot{k}$ is strictly negative, provided $y \neq 0$. Thus, $k(t)$ decreases on solutions which remain below $\{z = -b/c\}$ and also satisfy $y(t) \not\equiv 0$. Similarly, $k(t)$ increases on solutions remaining above $\{z = -b/c\}$. As in the proof of theorem 4.3, with z^* replaced by $-b/c$, we find that solutions cannot pass downward through $\{z = -b/c\}$ if $\nu < -(b/c)^2 - \rho(\frac{3}{2} + \delta)$, implying that the regions below and above $\{z = -b/c\}$ are respectively positively and negatively invariant. We conclude that the relevant branch of the one-dimensional $W^u(S^-) = W^{uu}(S^-)$ (cf. (3.2)) is trapped below $\{z = -b/c\}$ and hence that points in its ω -limit set satisfy $k = -\frac{1}{6}$, and so lie on $\{x = 1, y = 0\}$ (note that $k \geq -\frac{1}{6}$ for relevant solutions in $\{x \geq 0\}$). The only limit set on this line below $\{z = -b/c\}$ is the sink C^- . Furthermore, $W^{uu}(S^-)$ must remain bounded since $\dot{z} > 0$ if $|z|$ becomes large (§3c), thus the one-dimensional manifold $W^u(S^-)$ consists of one heteroclinic orbit connecting S^- to C^- . An analogous argument, running time backwards, establishes that C^+ is the α -limit set of $W^s(S^+) = W^{ss}(S^+)$.

For $c < 0$ we find from (4.3) that k increases on solutions remaining below $\{z = -b/c\}$ and decreases on solutions remaining above this plane. The positive invariance of $\{z < -b/c\}$ for $\nu < -(b/c)^2 - \rho(\frac{3}{2} + \delta)$ implies that $W^u(C^-)$ lies entirely below $\{z = -b/c\}$ and so, provided $y(t) \not\equiv 0$, $k(t)$ increases on solutions contained in $W^u(C^-)$. The only limit sets for which $y(t) \equiv 0$ are S^- and C^- itself and so, for every solution except C^- in $W^u(C^-)$ either $k(t)$ increases without bound or $k(t) \rightarrow 0^-$ and the solution is a heteroclinic orbit from C^- to S^- . Suppose that no such heteroclinic orbit exists and consider a solution in $W^u(C^-)$ which, for some $t > 0$ satisfies $k(t) = 1$. The level set $k = 1$ necessarily contains a point on $\{y = 0, x > 1\}$ to the right of $P^{-1}(\Gamma)$ and hence $W^u(C^-) \cap \{y = 0, x > 1\}$ necessarily contains a point right of $P^{-1}(\Gamma)$. But, $W^u(C^-) \cap \{y = 0, x > 1\}$ also contains points arbitrarily close to C^- and hence left of $P^{-1}(\Gamma)$; so, by continuity, and since $W^u(C^-)$ is a two-dimensional manifold consisting of orbits which are on a $O(1)$ time scale $O(\varepsilon)$ close

Figure 9. Heteroclinic connections for $\nu < 0$ sufficiently small (a) $c < 0$; (b) $c > 0$.

to orbits of the unperturbed system, just like $W^s(\Gamma)$, $W^u(C^-) \cap \{y = 0, x > 1\}$ must intersect $P^{-1}(\Gamma)$ in at least one point. This intersection corresponds to a heteroclinic orbit connecting C^- with Γ and thus with S^- . This contradicts the assumption and thus we conclude that at least one heteroclinic orbit from C^- to S^- exists. As in the case $c < 0$, similar arguments running time backwards show that at least one connection from S^+ to C^+ exists. Along with lemma 4.5, these arguments establish:

Proposition 4.6. Suppose $\nu < \min\{0, -(b/c)^2 - \rho(\frac{3}{2} + \delta), -(b/c)^2 - \rho\}$ for some $\delta > 0$ and $\varepsilon > 0$ is sufficiently small. Then for all $b \geq 0$, $c \neq 0$ and ρ the following heteroclinic orbits exist:

$$\begin{aligned} c < 0: \quad & S^+ \rightarrow C^+ \quad \text{and} \quad C^- \rightarrow S^-, \\ c > 0: \quad & C^+ \rightarrow S^+ \quad \text{and} \quad S^- \rightarrow C^-. \end{aligned}$$

Those for $c > 0$ are transverse and unique, and if $|\rho|$ is sufficiently small those for $c < 0$ are also transverse and unique.

The global structures are summarized in figures 9a, b.

Remark 4.7. The proofs of propositions 4.5 and 4.6 in the case $c > 0$ actually imply that C^- is the ω -limit set of any orbit which enters $\{z < -b/c\}$ in the relevant region $\{k < 0, x > 0\}$. In particular, orbits passing through points of $P(\Gamma)$ below $\gamma_{\text{int}}(0)$ approach C^- as $t \rightarrow +\infty$. Thus, there is a one parameter family of (transverse) connections in $W^u(S^+) \cap W^s(C^-)$ if ν satisfies the condition given in proposition 4.6. Similarly, and under the same condition, there is a one parameter family of connections in $W^u(C^+) \cap W^s(S^-)$.

(c) Homoclinic bifurcations as ν decreases

Having established the global structure of solutions in the case ν sufficiently large and positive in §3c, and of bounded solutions in the case ν sufficiently large and negative in §4b, we turn to global bifurcations occurring as ν decreases for fixed $b \geq 0$, $c \neq 0$ and ρ . It is in these bifurcations that the homoclinic explosions and implosions occur. Observe that, for instance, in the explosion case, orbits homoclinic to Γ are replaced by heteroclinic orbits from S^+ to S^- , but since S^+ and S^- belong to Γ we persist in calling these events 'homoclinic'. It is simplest first to consider the

case $\rho = 0$, for which the bifurcation occurs at $\nu = -(b/c)^2$ on the invariant plane $\{z = -b/c\}$. The next theorem, with its rather detailed proof, may help prepare for the more general result for $\rho \neq 0$, which will follow.

Theorem 4.8. *For $b \geq 0$, $\rho = 0$ and $\varepsilon > 0$ sufficiently small, as ν decreases (2.14) undergoes a global homoclinic bifurcation at $\nu = -(b/c)^2$ in which*

(a) *for $c < 0$ a single homoclinic orbit to infinity contained in $W^u(\Gamma) \cap W^s(\Gamma)$ disappears and a countable set of heteroclinic orbits in $W^u(S^+) \cap W^s(S^-)$ appears (a homoclinic explosion);*

(b) *for $c > 0$ a countable set of homoclinic orbits to infinity contained in $W^u(\Gamma) \cap W^s(\Gamma)$ disappears and a unique, non-trivial heteroclinic orbit in $W^u(S^+) \cap W^s(S^-)$ appears (a homoclinic implosion).*

If $b > 0$ and $c < 0$ (resp. $c > 0$), the homoclinic orbit(s) to infinity which disappear lie in $W^u(S^+) \cap W^s(\Gamma)$ (resp. $W^u(\Gamma) \cap W^s(S^-)$).

At $\nu = -(b/c)^2$ the invariant plane $\{z = -b/c\}$ contains a one parameter family of periodic orbits bounded by the homoclinic orbit $W^u(S^\pm) \cap W^s(S^\pm)$ for $c < 0$ and $c > 0$ respectively.

Proof. As in the proof of theorem 4.3, we transport pieces of the local stable and unstable manifolds $W^u(\Gamma)$ and $W^s(\Gamma)$ near the homoclinic orbit $\gamma_{\text{int}}(t) \subset \{W^u(\Gamma) \cap W^s(\Gamma)\}$ by the flow. The resulting structures are indicated in figures 10a, b. First suppose $c < 0$ and $b > 0$. At $\nu = 0$ the plane $\{z = 0\}$ is invariant and the flow on it is given by

$$\dot{x} = y, \quad \dot{y} = x - x^2 + \varepsilon by,$$

and phase plane methods easily yield the structure of the flow on this plane. Note that the intersection of $P(\Gamma)$ and $P^{-1}(\Gamma)$ occurs at $\{z = z^*\}$ which is $O(\varepsilon)$ near the plane $\{z = -b/c\}$ and thus above $\{z = 0\}$ in this case. Above $\{z = 0\}$, $\dot{z} = \varepsilon z^2 > 0$ and so orbits move down monotonically as t decreases. Hence the preimages of $P^{-1}(\Gamma)$ between $z = z^*$ and $z = 0$ accumulate on $\{z = 0\}$, thus $\lim_{t \rightarrow -\infty} \gamma_{\text{int}}(t) = S^\pm = (0, 0, 0)$ (see figure 6a). The tongues of $W^u(\Gamma)$ above $\{z = 0\}$ for $\nu > 0$ (cf. the cross section on $\{y = 0, x > 1\}$ of figure 4a) are unaffected but we claim that these of $W^s(\Gamma)$ below $\{z = 0\}$ rise as $\nu \rightarrow 0^+$ and are sutured into the structure indicated in the top panel of figure 10a at $\nu = 0$. To see this, note that, for $\nu = 0$, all preimages of $P^{-1}(\Gamma)$ must intersect the invariant plane $\{z = 0\}$ since the part of $P^{-1}(\Gamma)$ above this plane can never be mapped, by $\mathcal{P}^{-n}$, downward below the invariant plane. Furthermore, since $\dot{z} = \varepsilon z^2 > 0$ for $z < 0$, the preimages of any segment of $P^{-1}(\Gamma)$ bounded above by $\{z = 0\}$ and below by $\{z = \bar{z} < 0\}$ extends down from $\{z = 0\}$ to a z -level below $\bar{z}$. Repeated application of the inverse return map $\mathcal{P}^{-1}$ on $\{y = 0, x > 1\}$ then yields the structure shown. Note that the intersections $\mathcal{P}^{-n}(\Gamma) \cap \{z = 0\}$ accumulate on the perturbed centre $C^\pm = (1, 0, 0)$ as $n \rightarrow \infty$. Similar arguments give the structure for $\nu \in (-(b/c)^2, 0)$ depicted in the central part of figure 10a.

At $\nu = -(b/c)^2$ the flow on the invariant plane $\{z = -b/c\}$ is that of the unperturbed vector field

$$\dot{x} = y, \quad \dot{y} = x - x^2,$$

and one completes the bottom panel of figure 10a by considering images of segments of $P(\Gamma)$ above $\{z = -b/c\}$ and so inside $W^s(\Gamma)$ and preimages of $P^{-1}(\Gamma)$ below $\{z = -b/c\}$ and so inside $W^u(\Gamma)$. In both these cases the lower (resp. upper) boundary given by $z = z^* \equiv -b/c$ can be regarded as a fixed point of $\mathcal{P}$ (resp. $\mathcal{P}^{-1}$) since $\gamma_{\text{int}}(t) = W^{\text{uu}}(S^+) \cap W^{\text{ss}}(S^+)$ is the homoclinic orbit in the plane $\{z = -b/c\}$.

Therefore, the lower boundaries of all images $P(\Gamma)$ are attached to $\gamma_{\text{int}}(0)$, just as are the upper boundaries of the preimages of $P^{-1}(\Gamma)$. So we obtain the two fans of segments of $W^u(S^+) \subset W^u(\Gamma)$ and $W^s(S^-) \subset W^s(\Gamma)$ as shown. As ν approaches $-(b/c)^2$ from above, the infinite upper boundary $W^{\text{ss}}(S^+)$ of $W^s(S^-)$ collapses onto the homoclinic loop $W^{\text{uu}}(S^+) \cap W^{\text{ss}}(S^+)$, pinching the leaves together to form the lower fan, and the tongues of $W^u(\Gamma)$ descend from above $\{z = -b/c\}$ to form the upper fan as the homoclinic orbit to infinity, $\gamma_{\text{int}}(t)$, collapses onto the homoclinic loop. Note that $\dot{z}$ decreases in the neighbourhood of $\{z = -b/c\}$ as ν approaches $-(b/c)^2$, this mechanism causes the tongues of $W^u(\Gamma)$ to come down (cf. §3c). As ν falls below $-(b/c)^2$, the upper boundary of the $W^s(S^-)$ -fan, $W^{\text{ss}}(S^+) \in \{z = +\sqrt{-\nu}\}$, rises above the lower boundary of the $W^u(S^+)$ -fan, $W^{\text{uu}}(S^-) \in \{z = -\sqrt{-\nu}\}$, yielding the countable set of heteroclinic orbits in $W^u(S^+) \cap W^s(S^-)$ (cf. figure 8a).

For $b > 0$, $c > 0$ the three parts of figure 10b are obtained in a similar manner. As $\nu \rightarrow 0^+$, $\gamma_{\text{int}}(t)$ changes from a homoclinic solution to Γ into an orbit connecting $(0, 0, -\infty)$ to $S^\pm$, see figure 6b. The invariant plane at $\nu = 0$, $\{z = 0\}$ is now above the intersection of $P(\Gamma)$ and $P^{-1}(\Gamma)$. Taking images of segments of $P(\Gamma)$ bounded from above by z^* , we see that $W^u(\Gamma)$ is a fan pinched at $W^{\text{uu}}(S^\pm)$, while the countable set of leaves of $W^s(\Gamma) \cap \{y = 0, x > 1\}$ for $\nu > 0$ of figure 4b is unaffected. Thus a countable set of homoclinic orbits to Γ originating at $(0, 0, -\infty)$ but now terminating at $S^\pm$ survives. For $\nu \in (-(b/c)^2, 0)$ the role of the saddle-node $S^\pm$ is taken by $S^- = (0, 0, -\sqrt{-\nu})$ and the fan is now pinched at $W^{\text{uu}}(S^-)$. As ν approaches $-(b/c)^2$ from above, so $W^{\text{uu}}(S^-)$ and $W^{\text{ss}}(S^-)$ approach and merge in the homoclinic loop and the leaves of $W^s(\Gamma) \cap \{y = 0, x > 1\}$, which pass through distinct points of $W^{\text{ss}}(S^-) \cap \{y = 0, x > 1\}$ for $\nu > -(b/c)^2$, are pinched along the loop $W^{\text{uu}}(S^-) \cap W^{\text{ss}}(S^-)$. Now, as ν drops below $-(b/c)^2$, the lower boundary $W^{\text{uu}}(S^-)$ of the $W^s(\Gamma)$ -fan rises above the upper boundary $W^{\text{uu}}(S^-)$ of the $W^u(\Gamma)$ -fan, furthermore, the pinching at the orbits $W^{\text{uu}}(S^-)$ (resp. $W^{\text{ss}}(S^-)$) disappears, see figure 8b. Thus, there is only one unique non-trivial orbit $\gamma_{\text{int}}(t) \subset \{W^u(S^+) \cap W^s(S^-)\}$.

For $b = \rho = 0$ the range $(-(b/c)^2, 0)$ collapses and one has only the single value $\nu = 0$ at which the saddle node and the homoclinic bifurcations coincide. Pictures analogous to the lower panels of figures 10a, b can easily be deduced. ■

Before stating the result which extends theorem 4.8 to the case $\rho \neq 0$, we recall the function $H = H(\rho, b, c, \varepsilon)$ defined in proposition 4.1.

Theorem 4.9. *For $b > 0$, $\rho \neq 0$ and $\varepsilon > 0$ sufficiently small, the assertions of theorem 4.8 must be modified as follows. As ν decreases:*

(a) *for $c < 0$, $\rho > 0$ a single heteroclinic orbit to infinity contained in $W^u(S^+) \cap W^s(\Gamma)$ disappears at $\nu = -(b/c)^2 + \varepsilon H$ and a countable set of heteroclinic orbits in $W^u(S^+) \cap W^s(S^-)$ appears in a homoclinic explosion.*

(b) *for $c < 0$, $\rho < 0$ at $\nu = -(b/c)^2 + \varepsilon H$ a single heteroclinic orbit to infinity in $W^u(S^+) \cap W^s(\Gamma)$ disappears and countable set of heteroclinic orbits in $W^u(S^+) \cap W^s(S^-)$ exists; these orbits persist for all $\nu < -(b/c)^2 + \varepsilon H$ and for $\nu > -(b/c)^2 + \varepsilon H$ with $|\nu + (b/c)^2 - \varepsilon H|$ sufficiently small. The homoclinic explosion occurs above $-(b/c)^2 + \varepsilon H$ while the unique heteroclinic orbit to infinity contained in $W^u(S^+) \cap W^s(\Gamma)$ still exists.*

(c) *for $c > 0$, $\rho > 0$ a countable set of heteroclinic orbits to infinity contained in $W^u(\Gamma) \cap W^s(S^-)$ persists for $\nu \geq -(b/c)^2 + \varepsilon H$ and disappears for $\nu < -(b/c)^2 + \varepsilon H$. For $\nu < -(b/c)^2 + \varepsilon H$ and $|\nu + (b/c)^2 - \varepsilon H|$ sufficiently small, countably many*

Figure 10. Global bifurcations of $W^u(\Gamma)$ and $W^s(\Gamma)$ for $\rho = 0$ as ν decreases (a) $c < 0$; (b) $c > 0$.

heteroclinic orbits in $W^u(\Gamma) \cap W^s(S^-)$ persist. They vanish in a homoclinic implosion which leaves the unique non-trivial heteroclinic orbit $\gamma_{\text{int}}(t) \subset W^u(S^+) \cap W^s(S^-)$.

(d) for $c > 0$, $\rho < 0$ a countable set of homoclinic orbits to infinity contained in $W^u(\Gamma) \cap W^s(\Gamma)$ disappears at $\nu = -(b/c)^2 + \varepsilon H$ and a unique, non-trivial heteroclinic orbit $\gamma_{\text{int}}(t) \subset W^u(S^+) \cap W^s(S^-)$ appears in a homoclinic implosion.

Remark 4.10. The above theorem does not mention a relatively simple bifurcation which occurs in the implosion cases ((c), (d)) at $\nu = 0$, the value of ν at which the saddle points S^+ and S^- are created. As a consequence of this local bifurcation we observe (see figure 6b):

$$\lim_{t \rightarrow +\infty} \gamma_{\text{int}}(t) = S^- \quad \text{for} \quad \nu \leq 0,$$

instead of being unbounded in z as for $\nu > 0$. This change in character of $\gamma_{\text{int}}(t)$ also changes the character of the countable set of intersections of $W^u(\Gamma)$ and $W^s(\Gamma)$: like

Figure 11. Global structure at $\nu = -(b/c)^2 + \varepsilon H$: (a) $c < 0$, $\rho > 0$; (b) $c < 0$, $\rho < 0$; (c) $c > 0$, $\rho > 0$; (d) $c > 0$, $\rho < 0$.

$\gamma_{\text{int}}(t)$ these intersections are connected to S^- (for $t \rightarrow +\infty$) instead of becoming unbounded in the $+z$ -direction, as is the case for $\nu > 0$ (proposition 3.3).

In the proof that follows the reader may find it helpful to refer to figures 11a–d, the four cases pictured being those of the theorem.

Proof. In the following we will frequently refer to points such as $W^{ss}(S^+) \cap P^{-1}(\Gamma)$; we therefore define:

$$P(S^\pm) = (k_u^\pm, z_u^\pm) = W^{uu}(S^\pm) \cap P(\Gamma),$$

$$P^{-1}(S^\pm) = (k_s^\pm, z_s^\pm) = W^{ss}(S^\pm) \cap P^{-1}(\Gamma).$$

It will be necessary to examine the maps $\mathcal{P}$ and $\mathcal{P}^{-1}$ more closely than before. Specifically, recall that Sternberg's theorem (1958; cf. Guckenheimer & Holmes 1983) asserts a C^r -diffeomorphic equivalence between solutions of (2.14) in fixed neighbourhoods $B(\delta)$ of S^+ , S^- and those of the vector field linearized at the corresponding points, provided finitely many (low-order) non-resonance conditions hold among the eigenvalues ($r = 2$ will suffice for our purposes). Equation (3.2) reveals that such

conditions do hold for all $\varepsilon, b, c \neq 0$. Since the linearized flow possesses an invariant plane spanned by the strong eigenvectors belonging to $\lambda_+^{ss}, \lambda_+^{uu}$ (respectively $\lambda_-^{ss}, \lambda_-^{uu}$), it follows that a two-dimensional invariant surface $W_{loc}^{ss-uu}(S^\pm)$, $O(\varepsilon)$ close to horizontal, exists at each point. Gronwall estimates further imply that images and preimages of $W_{loc}^{ss-uu}(S^\pm)$ under $\mathcal{P}$ and $\mathcal{P}^{-1}$ intersect the section $\{y = 0, x > 1\}$ in four ‘horizontal’ curves of length $O(\delta)$ containing the points $(k_s^+, z_s^+), (k_u^+, z_u^+)$ (resp. $(k_s^-, z_s^-), (k_u^-, z_u^-)$). We denote these curves by $\mathcal{M}_s^+, \mathcal{M}_u^+$ (resp. $\mathcal{M}_s^-, \mathcal{M}_u^-$); they will serve to separate images and preimages of components of $P(\Gamma)$ and $P^{-1}(\Gamma)$ as required, playing a more subtle role in the proof than the invariant planes $\{z = \pm\sqrt{-\nu}\}$ in the case $\rho = 0$.

Case (a): $c < 0, \rho > 0$. In this case the conclusion is the same as in theorem 4.8. Furthermore, the mechanism which causes the explosion is in essence the same as in the $\rho = 0$ case; however, here one has to be more careful in showing that the images of $P(\Gamma)$ (the ‘tongues’) cannot intersect the preimages of $P^{-1}(\Gamma)$ (the ‘sutured tongues’ (for $\nu < 0$)) before ν has reached the bifurcation value $\nu = -(b/c)^2 + \varepsilon H$.

Noting that

$$\dot{z}|_{z=\sqrt{-\nu}} = \varepsilon \rho x > 0 \quad \text{for} \quad x > 0, \quad (4.4)$$

we see that the subspaces $\{x > 0, z > \sqrt{-\nu}\}$ and $\{x > 0, z < \sqrt{-\nu}\}$ are positively and negatively invariant, respectively. Thus $z_u^+ > z_s^+$; in fact, taking the limit $k \rightarrow 0^-$ in (3.5), using (B1), and recalling that ΔZ estimates the change in $z(t)$ incurred in one full circuit around $\{x = 1, y = 0\}$, while $P(\Gamma)$ and $P^{-1}(\Gamma)$ represent half circuits, we find

$$z_u^+ = +\sqrt{-\nu} + 3\varepsilon\rho + O(\varepsilon^2), \quad z_s^+ = +\sqrt{-\nu} - 3\varepsilon\rho + O(\varepsilon^2),$$

so that $z_u^+ - z_s^+ = 6\varepsilon\rho + O(\varepsilon^2) > 0$ and hence $\mathcal{M}_u^+$ lies above $\mathcal{M}_s^+$ (cf. figure 6a, right-hand column and figure 11a).

As in the proof of theorem 4.8 (figure 10a), for $\nu < 0$ $W^s(S^-) \cap \{y = 0, x > 1\}$ forms a sutured structure which becomes a fan at $\gamma_{\text{int}}(0) = (k_s^+, z_s^+)$, for $\nu = -(b/c)^2 + \varepsilon H$. Estimates on the linearized flow at S^+ reveal that the fan is tangent to $P^{-1}(\Gamma)$ at $\gamma_{\text{int}}(0)$ and lies below this point in a vertical wedge of angle $O(\varepsilon)$, and thus all components of $W^s(S^-)$ lie below $\mathcal{M}_s^+$. Similarly, the facts that $\lim_{t \rightarrow \infty} \gamma_{\text{int}}(t) = (0, 0, +\infty)$ for all $\nu > -(b/c)^2 + \varepsilon H$ and $\lim_{t \rightarrow \infty} \gamma_{\text{int}}(t) = S^+$ at $\nu = -(b/c)^2 + \varepsilon H$ imply that the tongues of $W^u(S^+)$ lie above $\mathcal{M}_u^+$ for all $\nu \geq -(b/c)^2 + \varepsilon H$. Thus we conclude that no intersections of $W^u(S^+)$ and $W^s(S^-)$ occur in this range. For all $\nu < -(b/c)^2 + \varepsilon H$, $\lim_{t \rightarrow \infty} \gamma_{\text{int}}(t) = S^-$ and hence the $W^u(S^+)$ - and $W^s(S^-)$ -fans intersect as established in the proof of theorem 4.3 (figure 8a).

Case (b): $c < 0, \rho < 0$. Now, since

$$\dot{z}|_{z=\sqrt{-\nu}} = \varepsilon \rho x < 0 \quad \text{for} \quad x > 0,$$

we find that $\mathcal{M}_u^+$ lies below $\mathcal{M}_s^+$ (cf. figure 6a, left-hand column, and figure 11b). At $\nu = -(b/c)^2 + \varepsilon H$, $\lim_{t \rightarrow \pm\infty} \gamma_{\text{int}}(t) = S^+$ and as above in the proof of theorem 4.3 we consider preimages and images of $P^{-1}(\Gamma) \subset W^s(S^-)$ and $P(\Gamma) \subset W^u(S^+)$ near $\gamma_{\text{int}}(0)$; $\gamma_{\text{int}}(0)$ corresponds to (k_s^+, z_s^+) in (k, z) -space at the bifurcation value. Since the segment of $P^{-1}(\Gamma)$ bounded from above by $\gamma_{\text{int}}(0)$ lies inside $P(\Gamma)$ and contains points both above and below $\mathcal{M}_u^+$, its preimage under the flow passes near S^+ and next intersects $\{y = 0, x > 1\}$ in a curve also containing points above and below $\mathcal{M}_s^+$. In fact a study of the mapping linearized at S^+ reveals that $\mathcal{P}^{-1}(P^{-1}(\Gamma))$ approaches $\gamma_{\text{int}}(0)$ from above tangent to $P^{-1}(\Gamma)$ and that it turns back to pass

through $\mathcal{M}_s^+$. Successive preimages yields the fan pinched at $\gamma_{\text{int}}(0)$ ‘from above’, as shown in figure 11b.

The segment of $P(\Gamma)$ inside $P^{-1}(\Gamma)$, bounded below by $\gamma_{\text{int}}(0)$, however, lies above $\mathcal{M}_s^+$. Consequently, its image $\mathcal{P}(P(\Gamma))$ lies above $\mathcal{M}_u^+$, being a curve inside and tangent to $P(\Gamma)$ at $\gamma_{\text{int}}(0)$ extending up to $+\infty$ (in the z -direction). It therefore intersects $P^{-1}(\Gamma)$ at a point p below $\gamma_{\text{int}}(0)$, and the orbit through p necessarily approaches S^- as $t \rightarrow +\infty$. Successive iterations of $\mathcal{P}$ therefore yield a fan pinched at (k_s^-, z_s^-) and extending (in z) to $+\infty$. The intersections of the fans yield the heteroclinic orbits of the first statement of case (b) of the theorem and those of the second statement follow from the ‘simpler’ arguments of theorem 4.3.

For the third statement, note that for $\nu < 0$ but sufficiently far above $-(b/c)^2 + \varepsilon H$, depending on ρ , $W^s(S^-) \cap \{y = 0, x > 1\}$ is sutured at the preimages $\mathcal{P}^{-n}(k_s^+, z_s^+)$, $n = 0, 1, 2, \dots$ into a structure like that of figure 10a. In this range the disconnected tongues $\mathcal{P}^m(P(\Gamma))$ of $W^u(S^+) \cap \{y = 0, x > 1\}$ still do not intersect this structure. But as we have seen (figure 11b), at $\nu = -(b/c)^2 + \varepsilon H$ there are already countably many intersections between $W^u(S^+)$ and $W^s(S^-)$, and hence between $\mathcal{P}^{-n}(P^{-1}(\Gamma))$ and $\mathcal{P}^m(P(\Gamma))$. Thus, heteroclinic intersections must occur *before* ν reaches $-(b/c)^2 + \varepsilon H$. To complete the argument, we claim that there is no ‘first’ intersection with decreasing ν ; rather, as soon as one orbit exists in $W^u(S^+) \cap W^s(S^-)$ there are infinitely many: a homoclinic explosion still occurs. To see this, suppose that some tongue $\mathcal{P}^n(P(\Gamma))$ of $W^u(S^+)$ intersects $P^{-1}(\Gamma)$. Then, by applying $\mathcal{P}^{-1}$ n times, $P(\Gamma) \cap \mathcal{P}^{-n}(P^{-1}(\Gamma)) \neq \emptyset$ and, since the tongues $\mathcal{P}^n(P(\Gamma))$ of $W^u(S^+)$ and $\mathcal{P}^{-n}(P^{-1}(\Gamma))$ of $W^s(S^-)$ are continuous and unbounded in the $+z$ and $-z$ directions respectively, $\mathcal{P}^n(P(\Gamma)) \cap \mathcal{P}^{-n}(P^{-1}(\Gamma)) \neq \emptyset$. One now repeats the argument starting with $\mathcal{P}^{2n}(P(\Gamma)) \cap P^{-1}(\Gamma)$, $\mathcal{P}^{4n}(P(\Gamma)) \cap P^{-1}(\Gamma)$, $\dots$ obtaining a countable set of heteroclinic orbits, as claimed. In fact, since the tongues $\mathcal{P}^{-k}(P^{-1}(\Gamma))$ and $\mathcal{P}^k(P(\Gamma))$ are ordered from right to left with increasing k , our assumption implies that $\mathcal{P}^n(P(\Gamma))$ intersects $\mathcal{P}^{-k}(P^{-1}(\Gamma))$ for all $0 \leq k \leq n$ and in this way one sees that heteroclinic orbits of every circuit number larger than $n + 1$ exist.

Case (c): $c > 0$, $\rho > 0$. For $c > 0$ we are concerned with S^- and the points (k_u^-, z_u^-) and (k_s^-, z_s^-) . Computations using (3.5) as above reveal that

$$z_u^- = -\sqrt{-\nu} + 3\varepsilon\rho + O(\varepsilon^2), \quad z_s^- = -\sqrt{-\nu} - 3\varepsilon\rho + O(\varepsilon^2),$$

hence, $\mathcal{M}_u^-$ lies above $\mathcal{M}_s^-$ for $\rho > 0$. An argument analogous to that of case (b) shows that, at $\nu = -(b/c)^2 + \varepsilon H$, the components $\mathcal{P}^n(P(\Gamma))$ of $W^u(\Gamma)$ form a fan pinched ‘from above’ at $\gamma_{\text{int}}(t) = (k_u^-, z_u^-)$, which turns back through $\mathcal{M}_u^-$ and extends to $-\infty$ in z . The preimage $\mathcal{P}^{-1}(P^{-1}(\Gamma))$ is pinched at (k_s^-, z_s^-) from above $\mathcal{M}_s^-$ and extends to $+\infty$, but since it intersects $P(\Gamma)$ *below* $\gamma_{\text{int}}(0)$ at a point p and the z -coordinate of the orbit through p tends to $-\infty$ as $t \rightarrow -\infty$, successive preimages $\mathcal{P}^{-n}(P^{-1}(\Gamma))$, $n \geq 2$, are disjoint curves ‘parallel’ to $P^{-1}(\Gamma)$ and extending from $+\infty$ to $-\infty$, as in the $c > 0$ structure for $\nu > 0$ (figure 4b).

For $\nu > -(b/c)^2 + \varepsilon H$, $\lim_{t \rightarrow -\infty} \gamma_{\text{int}}(t)|_z = -\infty$ and so the component $\mathcal{P}^{-1}(P^{-1}(\Gamma))$ is also disjoint from $P^{-1}(\Gamma)$ and extends to $-\infty$ (in z); otherwise the structure is as shown in figure 11c. Thus for $\nu \geq -(b/c)^2 + \varepsilon H$ one has a countable set of heteroclinic orbits which approach S^- for $t \rightarrow +\infty$ and which extend for $t \rightarrow -\infty$ to $-\infty$ (in z), as claimed.

For $\nu < -(b/c)^2 + \varepsilon H$, as for $\nu > -(b/c)^2 + \varepsilon H$ in case (b), one expects (some) of the countable set of intersections $\mathcal{P}^{-n}(P^{-1}(\Gamma)) \cap \mathcal{P}^m(P(\Gamma))$ to persist. (Recall that our uniqueness result of theorem 4.3 relies on a bound $\nu < -((b/c)^2 + \delta) - \rho(\frac{3}{2} + \delta)$.)

We now show that, if one such intersection exists for some $n, m > 0$, then infinitely many do. As above, consider images and preimages of segments of $P(\Gamma) \subset W^u(S^+)$ and $P^{-1}(\Gamma) \subset W^s(S^-)$ bounded below and above respectively by $\gamma_{\text{int}}(0)$ and suppose that $\mathcal{P}^n(P(\Gamma)) \cap P^{-1}(\Gamma) \neq \emptyset$. Then $P(\Gamma) \cap \mathcal{P}^{-n}(P^{-1}(\Gamma)) \neq \emptyset$ and $\mathcal{P}^n(P(\Gamma))$ extends from $-\infty$ (in z) to a point above $\gamma_{\text{int}}(0)$ while $\mathcal{P}^{-n}(P^{-1}(\Gamma))$ extends from $+\infty$ to a point below $\gamma_{\text{int}}(0)$. Thus $\mathcal{P}^n(P(\Gamma)) \cap \mathcal{P}^{-n}(P^{-1}(\Gamma)) \neq \emptyset$ and one can proceed inductively as in the proof of part (b) to conclude the existence of a countable set of heteroclinic orbits. Finally, since components of $\mathcal{P}^{-n}(P^{-1}(\Gamma))$ extend below $\gamma_{\text{int}}(0)$ to (k_s^-, z_s^-) and further to $-\infty$, these orbits must occur for $\nu < -(b/c)^2 + \varepsilon H$ and $|\nu + (b/c)^2 - \varepsilon H|$ sufficiently small.

Case (d): $c > 0, \rho < 0$. This case is similar to case (a): the mechanism which causes the implosion is in essence the same as in the $\rho = 0$ case. It differs essentially from (c) since now $\mathcal{M}_u^-$ lies below $\mathcal{M}_s^-$ and hence, at $\nu = -(b/c)^2 + \varepsilon H$ the $W^u(\Gamma)$ - and $W^s(S^-)$ -fans, bounded above and below by $\mathcal{M}_u^-$ and $\mathcal{M}_s^-$ respectively, are disjoint and $P(\Gamma) \cap P^{-1}(\Gamma) \equiv (k_u^-, z_u^-)$ is a unique point. However, for any $\nu > -(b/c)^2 + \varepsilon H$, $\lim_{t \rightarrow -\infty} \gamma_{\text{int}}(t)|_z = -\infty$ and hence, the preimages $\mathcal{P}^{-n}(P^{-1}(\Gamma))$ continue to form a countable set of ‘parallel’ lines as for $\nu > 0$ (figures 4b, 10b). Only at $\nu = -(b/c)^2 + \varepsilon H$ are these lines pinched at (k_s^-, z_s^-) and its preimages to form the $W^s(S^-)$ -fan as is shown in figure 11d. ■

Remark 4.11. Cases (b) and (c) of this theorem exhibit significantly greater complexity than (a) and (d). In those latter cases the planes $\{z = \pm\sqrt{-\nu}\}$, containing the saddles $S^\pm$, effectively separate the relevant components of $W^u(\Gamma)$ and $W^s(\Gamma)$ so that no multi-circuit homoclinic orbits can appear (or disappear) before (or after) the primary bifurcation value $\nu = -(b/c)^2 + \varepsilon H$. Here we make some observations on case (b), $c < 0, \rho < 0$; analogous remarks would apply to case (c).

Let $G(\rho, b, c, \varepsilon)$ describe the secondary bifurcation value at which multi-circuit homoclinic orbits to Γ first appear, i.e.

$$\nu = -(b/c)^2 + G(\rho, b, c, \varepsilon) = \inf_{\nu} \{ \gamma_{\text{int}}(t) = W^u(\Gamma) \cap W^s(\Gamma) \text{ is unique} \}.$$

Unlike the bifurcation function $\varepsilon H(\rho, b, c, \varepsilon)$ of proposition 4.1 G is not prefaced by an ε . In fact, we claim that

$$G(\rho, b, c, \varepsilon) = O(\rho |\log \varepsilon|^{-1}).$$

The reason for this is as follows. In studying ‘premature’ multi-circuit homoclinic orbits we are concerned with preimages $\mathcal{P}^{-n}(p)$ of points $p \in P^{-1}(\Gamma)$ and images $\mathcal{P}^m(p)$ of points $p \in P(\Gamma)$ with p $O(\varepsilon)$ close to $\gamma_{\text{int}}(0)$. Note that by taking p in a $O(\varepsilon)$ neighbourhood of $\gamma_{\text{int}}(0)$ one can consider the Poincaré map $\mathcal{P}$ – as defined in §2 – to be independent of p , at leading order, as long as we choose p such that orbits through it will not get exponentially close (in ε) to Γ , since in that case the slow flow along Γ would have a leading order effect on map $\mathcal{P}$. For convenience we choose p such that the distance to $\gamma_{\text{int}}(0)$ is exactly $O_s(\varepsilon)$, i.e. *not smaller than* $O(\varepsilon)$. We then observe that the orbits through p will pass Γ at an $O_s(\varepsilon^2)$ distance (this is due to the fact that $p \in W^u(\Gamma) \cup W^s(\Gamma)$ and that $W^u(\Gamma)$ and $W^s(\Gamma)$ intersect with an $O(\varepsilon)$ angle). Now assume $\nu = -(b/c)^2 + S|\log \varepsilon|^{-1}$ for some $S > 0$. We can apply (3.5) to compute $\mathcal{P}(p)$. Note that $z = -b/c + O(\varepsilon)$, so by (B 15) and (B 16) we find that the vertical displacement of $\mathcal{P}$ is $\Delta Z = \varepsilon(2S + 6\rho) + \text{h.o.t.}$ (the 2 appears since $k = O_s(\varepsilon^2)$), while the horizontal displacement is $\Delta K = O(\varepsilon^2 |\log \varepsilon|)$. Thus, points p on $P^{-1}(\Gamma)$ are moved up by $\mathcal{P}^{-1}$ and points of $P(\Gamma)$ are moved down by $\mathcal{P}$ over a

distance $\varepsilon(2S + 6\rho)$, so that, provided $2S + 6\rho > 0$, multi-circuit homoclinic orbits must exist. This establishes the claim.

These homoclinic orbits to Γ clearly all belong to $W^u(S^+)$, but their ω -limit points depend on their positions relative to $W^{ss}(S^+)$. Let $q \in P^{-1}(\Gamma)$ be the last point at which such an orbit intersects $\{y = 0, x > 1\}$. If q lies above (k_s^+, z_s^+) the orbit approaches $+\infty$ (in z) as $t \rightarrow \infty$; if q lies below (k_s^+, z_s^+) it approaches S^- as $t \rightarrow \infty$ and if $q = (k_s^+, z_s^+)$ the orbit is homoclinic to S^+ . For ν below but close to $-(b/c)^2 + G$, (k_s^+, z_s^+) lies $O(|\log \varepsilon|^{-1})$ below $\gamma_{\text{int}}(0)$: hence, since every circuit corresponds to an $O_s(\varepsilon)$ ‘step’ in z , at least $O((\varepsilon|\log \varepsilon|)^{-1})$ of these multi-circuit orbits approach $z = +\infty$ rather than S^- (others may appear as heteroclinic orbits connecting S^+ to S^- directly). However, as ν decreases to the primary bifurcation value $\nu = -(b/c)^2 + \varepsilon H$, (k_s^+, z_s^+) rises to coalesce with $\gamma_{\text{int}}(t)$. We conclude that all these orbits, initially in $W^u(S^+) \cap (W^s(\Gamma) \setminus W^s(S^-))$ must become homoclinic to S^+ and subsequently heteroclinic to S^- before ν reaches $-(b/c)^2 + \varepsilon H$. Thus there is a set of multi-circuit homoclinic bifurcations to S^+ preceding the primary homoclinic bifurcation. By the usual unfolding of homoclinic bifurcations (cf. Guckenheimer & Holmes 1983) it follows that multi-circuit periodic solutions will be created in these events.

Remark 4.12. The $b = 0$ (symmetric) case is somewhat different, but can be analysed by the same type of methods; we leave it to the (obsessive) reader to study the details.

5. Heteroclinic orbits to the periodic orbit

We now turn our attention to the family of periodic orbits found in §§2 and 3*b*. We perform a local analysis to extend the simple Hopf bifurcation results, followed by a global study of orbits near the homoclinic bifurcation of proposition 4.1, so that we can provide a partial characterization of the heteroclinic ‘webs’ connecting the equilibria $S^\pm$ and $C^\pm$ to the periodic orbit. The main results of this section are propositions 5.7 and 5.8, which, as in the previous section, describe how the two cases $c < 0$ and $c > 0$ differ.

We begin by observing that the perturbation calculations of §3*b*, in which we establish the stability types of the periodic orbit in terms of the parameters ρ , b and c , also imply that, provided that $\rho, b, c \neq 0$, this is the only (non-trivial) single-circuit periodic orbit which can be continued from the Hopf bifurcation at C^+ or C^- to the homoclinic bifurcation. Recall that the leading term of ΔK vanishes only on $\{z = -b/c\}$ and that of ΔZ vanishes only on $\{z^2 = -\nu - \rho\mathcal{T}(k)\}$. The fact that these two equations have a unique non-degenerate solution for each pair (ρ, ν) in the sectors bounded by $\mathcal{C}_{\text{Hopf}} = \{\nu = -(b/c)^2 - \rho\}$ and $\mathcal{C}_{\text{glob}}$, with (b, c) fixed, as established by the calculations in the Appendices, then implies uniqueness. The stability results of §3*b* show that, as ν varies, no additional orbits appear in local bifurcations from the periodic orbit, provided it remains bounded away from the slow manifold Γ . As Remark 4.11 indicates, additional multi-circuit periodic orbits can appear near the homoclinic bifurcation in certain cases, but we shall not address them in this section.

To understand how the periodic orbit and its stable/unstable manifolds interact with those of the fixed points $S^\pm$ and $C^\pm$ we have two quasi-local tools at our disposal: an analysis of the codimension two Hopf/saddle node bifurcation in which

$C^\pm$ appear on the line $\mathcal{C}_{\text{Hopf}} = \mathcal{C}_{\text{CN}} = \{\nu + \rho = 0\}$ when $b = 0$, and analysis of the homoclinic bifurcation that occurs on $\mathcal{C}_{\text{glob}} = \{\nu = -(b/c)^2 + \varepsilon H\}$. In the next section we obtain a partial characterization of the connections among stable and unstable manifolds of $S^\pm$, $C^\pm$ and the periodic orbit, hereafter denoted by Π , for parameter values near $\mathcal{C}_{\text{Hopf}}$ and $\mathcal{C}_{\text{glob}}$.

(a) *The periodic orbit near $\mathcal{C}_{\text{Hopf}}$ and $\mathcal{C}_{\text{glob}}$*

When $\nu = -\rho$ and $b = 0$ (2.14) has a degenerate fixed point at $(1, 0, 0)$ with eigenvalues $\{\pm i; 0\}$ and associated eigenvectors $\{(1, \pm i, \mp i\varepsilon\rho); (0, 0, 1)\}$. Thus, via the translation $\bar{x} = x - 1$ and the transformation matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \frac{1}{2}\varepsilon b & 0 \\ -\varepsilon\rho & 0 & 1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} -\frac{1}{2}\varepsilon b & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2}\varepsilon^2 b\rho & \varepsilon\rho & 1 \end{pmatrix}$$

we can put the system into the form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \varepsilon(\rho + \nu) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\varepsilon b & -1 & 0 \\ 1 & \frac{1}{2}\varepsilon b & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} -v^2 + \varepsilon cuw \\ 0 \\ \varepsilon w^2 - \varepsilon\rho v^2 \end{pmatrix} + O(\varepsilon^2),$$

where $(\bar{x}, y, z)^T = Q(u, v, w)^T$. When $\nu + \rho = b = 0$ the linear part becomes

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and near-identity normal form transformations can be made which ‘rectify’ the vector field near the degenerate equilibrium at any (finite) order in u, v, w (cf. Guckenheimer & Holmes 1983, § 3.3). Here it suffices to include only quadratic terms. Specifically, letting

$$u = \bar{u} + \frac{2}{3}\bar{u}\bar{v}, \quad v = \bar{v} - \frac{2}{3}\bar{u}^2 - \frac{1}{3}\bar{v}^2, \quad w = \bar{w} + \frac{1}{2}\varepsilon\rho\bar{u}\bar{v}$$

and dropping the bars, we obtain

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \varepsilon(\rho + \nu) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\varepsilon b & -1 & 0 \\ 1 & \frac{1}{2}\varepsilon b & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \varepsilon \begin{pmatrix} \frac{1}{2}cuw \\ \frac{1}{2}cvw \\ w^2 - \frac{1}{2}\rho(u^2 + v^2) \end{pmatrix} + \text{h.o.t.},$$

where the higher order terms are negligible provided we pick u, v, w, b and $\nu + \rho$ of the same order of magnitude and subsequently choose ε sufficiently small. In polar coordinates, $u = r \cos \theta$, $v = r \sin \theta$, we have

$$\left. \begin{aligned} \dot{r} &= \varepsilon(\tfrac{1}{2}br + \tfrac{1}{2}crw) \\ \dot{\theta} &= 1 \\ \dot{w} &= \varepsilon((\nu + \rho) - \tfrac{1}{2}\rho r^2 + w^2) \end{aligned} \right\} + \text{h.o.t.} \quad (5.1)$$

The azimuthal term decouples at this order and, up to reversal and rescaling of time, $t \rightarrow -\varepsilon t$, this system is identical to the unfolding of the codimension two Hopf/saddle node bifurcation in Guckenheimer & Holmes (1983), § 7.4:

$$\dot{r} = \mu_1 r + \bar{a}rw, \quad \dot{w} = \mu_2 + \bar{b}r^2 - w^2, \quad (5.2)$$

with the following correspondence between the present parameters (b, c, ν, ρ) and those of the ‘standard’ unfolding (5.2):

$$\mu_1 = -\frac{1}{2}b, \quad \mu_2 = -(\nu + \rho), \quad \bar{a} = -\frac{1}{2}c, \quad \bar{b} = \frac{1}{2}\rho.$$

The cases I, II, III, IV of the unfolding given in figs 7.4.3–6 of Guckenheimer & Holmes (1983) therefore correspond to $\{c < 0, \rho > 0\}$, $\{c > 0, \rho > 0\}$, $\{c < 0, \rho < 0\}$, and $\{c > 0, \rho < 0\}$ respectively and the relevant phase portraits are drawn, with the appropriate time reversal, in figure 12. These establish the relations between the periodic orbit Π , its stable and unstable manifolds ($W^s(\Pi)$ and $W^u(\Pi)$), and the fixed points $C^\pm$, near the Hopf bifurcation curve $\mathcal{C}_{\text{Hopf}}$. Note that, when $b = 0$, the circular symmetry imposed by the truncated normal form (5.1) leads to continuous families of invariant tori in the cases $\{c < 0, \rho < 0\}$ and $\{c > 0, \rho > 0\}$. In the former case the tori limit on an invariant sphere $S = W^u(C^-) \cap W^u(C^+)$. As described in Guckenheimer & Holmes (1983) (fig. 7.4.11) we expect the latter to actually yield transverse heteroclinic orbits from C^- to C^+ in the full system. However, some of the invariant tori of the truncated system in both cases $\{c < 0, \rho < 0\}$ and $\{c > 0, \rho > 0\}$ can be expected to persist for the full system for $b = 0$, for in that case we have the reversibility symmetry about the x -axis noted at the beginning of §3, so that for such symmetric orbits the flow preserves volume on average.

We now derive from these local unfoldings some global information on the heteroclinic orbits involving $C^\pm$, $S^\pm \subset \Gamma$ and the periodic orbit, Π , itself, for parameter values near $\mathcal{C}_{\text{Hopf}}$. We first consider the case $\rho < 0$.

Lemma 5.1. *Suppose $b \geq 0, \rho < 0$ and $\nu < -\rho - (b/c)^2$ with $|\nu + \rho + (b/c)^2|$ sufficiently small. Then*

(a) *For $c < 0$ $W^s(C^+)$, $W^u(C^-)$ and $W^u(\Pi)$ (for $b > 0$) do not intersect $W^u(\Gamma)$ or $W^s(\Gamma)$. However, we have heteroclinic connections from Π to C^+ if $b > 0$ and from C^- to C^+ if $b = 0$.*

(b) *For $c > 0$, $W^s(C^-)$, $W^s(\Pi)$, $W^u(S^+)$ and $W^u(\Pi)$ do not intersect $W^u(S^+)$ or $W^s(S^-)$. However, we have heteroclinic connections from C^+ to Π and from Π to C^- .*

Proof. For $c < 0, \rho < 0$ this result follows directly from the codimension two unfoldings since in that case ($\bar{a} > 0, \bar{b} < 0$ in (5.2)) the manifolds $W^u(C^-)$, $W^s(C^+)$ are contained in an invariant cylinder surrounding $\{x = 1, y = 0\}$. See the upper two phase portraits in the left-hand column of figure 12. For $c > 0, \rho < 0$ ($\bar{a} < 0, \bar{b} < 0$) the outer branch of the unstable manifold $W^u(\Pi)$ contains only orbits which remain above $\{z = -b/c\}$ (in which Π lies for the truncated system (5.2)) and moreover climb above the level of $C^+ = (0, 0, +\sqrt{-(\nu + \rho)})$ before they leave a neighbourhood of $\{x = 1, y = 0\}$. Since $S^+ = (0, 0, +\sqrt{-\nu})$ lies below C^+ and $W^{ss}(S^+)$ forms the upper boundary of $W^s(S^-)$, it follows that $W^u(\Pi)$ and $W^s(S^-)$ cannot intersect. The connections $W^u(C^+) \cap W^s(\Pi)$ are immediate from the unfolding and the other statements of (b) follow from similar arguments with time reversed. ■

Remark 5.2. When $\rho < 0, c > 0$, it is possible that $W^u(\Pi)$ intersects $W^s(\Gamma)$ and $W^u(\Gamma)$ intersects $W^s(\Pi)$; however, these intersections must occur for ‘large’ $|z|$ values and the resulting heteroclinic orbits are unbounded, cf. the bottom left-hand panels of figure 12.

We now move to the case $\rho > 0$.

Figure 12. Sections of the S^1 -symmetric phase portraits for the truncated normal form (5.1). (a) $c < 0$, $b > 0$; (b) $c < 0$, $b = 0$; (c) $c > 0$, $b > 0$; (d) $c > 0$, $b = 0$. Dots on the central bifurcation sets indicate typical parameter values to which phase portraits correspond.

Lemma 5.3. Suppose $b > 0$, $\rho > 0 > \nu$ and $\nu < -\rho - (b/c)^2$ with $|\nu + \rho + (b/c)^2|$ sufficiently small. Then

- (a) For $c < 0$ $W^u(C^+)$ intersects $W^s(\Pi)$
- (b) For $c > 0$ $W^u(C^+)$, $W^u(C^-)$ and $W^u(S^-)$ all intersect $W^s(\Pi)$ and $W^u(C^+)$ intersects $W^s(S^-)$ and $W^s(S^+)$.

Proof. Part (a) follows from the local unfolding, as do the statements about $W^u(C^\pm)$ and $W^s(\Pi)$ of (b): see the right-hand column of figure 12. For the rest, note that Π is an attracting orbit and there exist orbits in the three-dimensional manifold $W^s(\Pi)$ below and near $\{z = -b/c\}$. Since $dk/dt = \varepsilon(b + cz)y^2 < 0$ and $\dot{z} = \varepsilon(\nu + \rho x + z^2) > 0$ (for $x \geq 0$) on such orbits, $z(t)$ decreases and $k(t)$ increases as $t \rightarrow -\infty$. Hence we can choose orbits in $W^s(\Pi)$ which expand laterally as $t \rightarrow -\infty$ and meet $P(\Gamma)$ within $O(\varepsilon)$ of $\gamma_{\text{int}}(0)$ and at every point below that. In particular

an orbit meeting $P(\Gamma)$ at $(k_u^-, z_u^-) \in W^u(S^-)$ will exist, as claimed. Similar arguments applied to orbits in the three-dimensional manifold $W^u(C^+)$ establish the connections to $W^s(S^\pm)$. ■

Remark 5.4. In the cases covered by this lemma, the fixed points $C^\pm$ do not coexist with the periodic orbit Π unless $b \neq 0$.

We have been unable to determine the global structure of $W^u(C^-)$, $W^u(\Pi)$ and $W^s(\Pi)$ near $\mathcal{C}_{\text{Hopf}}$ when $c < 0$, $\rho > 0$ and so turn to a consideration of the homoclinic bifurcation at $\mathcal{C}_{\text{glob}}$ to illuminate that case.

Lemma 5.5. Suppose $c < 0$, $b > 0$, $\rho > 0 > \nu$ and $\nu < -(b/c)^2 + \varepsilon H$ with $|\nu + (b/c)^2 - \varepsilon H|$ sufficiently small, where $H = H(\rho, b, c, \varepsilon)$ is the function introduced in proposition 4.1. Then $W^u(\Pi)$ intersects $W^s(S^-)$ and $W^s(\Pi)$ intersects $W^u(S^+)$.

Proof. As ν approaches $\mathcal{C}_{\text{glob}}$ from below, a point on the (saddle type) periodic orbit Π approaches the heteroclinic point $\gamma_{\text{int}}(0) \in P(\Gamma) \cap P^{-1}(\Gamma)$. Thus an open arc contained in Π , carrying with it pieces of its stable and unstable manifolds $W^s(\Pi)$, $W^u(\Pi)$ uniformly approaches a compact segment of $\gamma_{\text{int}}(t)$ (say for $-\infty < -\tau \leq t \leq \tau < \infty$). The structure of $W^s(\Pi)$ and $W^u(\Pi)$ near $\mathcal{C}_{\text{glob}}$ may be obtained by analysis of the flow in the neighbourhood of the heteroclinic orbit $\gamma_{\text{int}}(t)$ following the method of Silnikov (cf. Guckenheimer & Holmes 1983, §6.3). On $\mathcal{C}_{\text{glob}}$ for $c < 0$ $\gamma_{\text{int}}(t)$ becomes a homoclinic orbit to S^+ (cf. figure 6a), the eigenvalues of which are, in general

$$\lambda_+^{\text{slow}} = 2\varepsilon\sqrt{-\nu}, \quad \lambda_+^{\text{uu,ss}} = \pm 1 + \frac{1}{2}\varepsilon(b + c\sqrt{-\nu}) \pm \frac{1}{8}\varepsilon^2(b + c\sqrt{-\nu})^2 + O(\varepsilon^4).$$

Note that, for $\nu < -(b/c)^2 + \varepsilon H$ the inequality $-\lambda_+^{\text{ss}} > \lambda_+^{\text{uu}} > 0$ holds, since $H < 0$ for $\rho > 0$. The vector field near S^+ may be written in a local coordinate system as

$$\left. \begin{aligned} \dot{u} &= \lambda_+^{\text{ss}} u \\ \dot{v} &= \lambda_+^{\text{uu}} v \\ \dot{w} &= \lambda_+^{\text{slow}} w \end{aligned} \right\} + O(|u, v, w|^2), \quad (5.3)$$

and we will approximate the Poincaré map as a composition of a local component $Q_0 : \Sigma_0 \rightarrow \Sigma_1$, where $\Sigma_0 = \{u = \delta\}$, $\Sigma_1 = \{v = \delta\}$ are local cross section near S^+ , and a 'global' part $Q_1 : \Sigma_1 \rightarrow \Sigma_0$ (cf. figure 13). As usual, Gronwall's lemma implies that Q_1 is a bounded perturbation of the identity, since orbits take a finite time to travel from Σ_1 to Σ_0 . It is convenient to consider the inverse map $Q_0^{-1} : \Sigma_1 \rightarrow \Sigma_0$ which takes a point $(u_1, \delta, w_1) \in \Sigma_1$ to $(\delta, v_0, w_0) \in \Sigma_0$. We first estimate the local time of flight by (backward) integration of the linearized equation for u :

$$u_1 = u(t_1) = \delta e^{\lambda_+^{\text{ss}} t_1} \quad \Rightarrow \quad t_1 = \frac{1}{\lambda_+^{\text{ss}}} \log \left| \frac{u_1}{\delta} \right|. \quad (5.4)$$

From (5.4) and integration of the other components of (5.3) we obtain the estimate

$$(v_0, w_0) = Q_0^{-1}(u_1, w_1) \approx (\delta^{1-\alpha} |u_1|^\alpha, \delta^{-\beta} |u_1|^\beta w_1),$$

where

$$\alpha = -\lambda_+^{\text{uu}} / \lambda_+^{\text{ss}} \in (0, 1) \quad \text{and} \quad \beta = -\lambda_+^{\text{slow}} / \lambda_+^{\text{ss}} > 0.$$

Since $\alpha < 1$ we can select a rectangle $D \subset \Sigma_1$ bounded by $u_1 = 0$, $u_1 = \bar{u} > 0$, $w_1 = 0$ and $w_1 = -\bar{w} < 0$ whose preimage $Q_0^{-1}(D) \subset \Sigma_0$ is contracted in the w direction and

Figure 13. The (inverse) return map near C_{glob} in the case $c < 0 < \rho$.

expanded in the v direction. In fact, the boundary of $Q_0^{-1}(D)$ on $v = 0$ is pinched to the point $(\delta, 0, 0)$ and points on the opposite boundary satisfy $v_0 = \delta^{1-\alpha}|\bar{u}|^\alpha \gg \bar{u}$ as $\bar{u} \rightarrow 0$. This, together with the uniformly bounded behaviour of Q_1 and the topological structure of the intersection of $P^{-1}(\Gamma)$ and $P(\Gamma)$ on $\{y = 0, x > 1\}$, permits us to conclude that the inverse return map has the structure indicated in figure 13. In particular, since $W^s(S^+)$ lies outside $W^u(S^+)$, above $\gamma_{\text{int}}(t)$, and below $W^{uu}(S^+)$ for $x > 0$ (cf. figure 6a), the preimage $Q_1^{-1}(Q_0^{-1}(D))$ stretches across and overlaps D 'horizontally'.

Similarly the forward map Q_1 carries D to Σ_0 and the local map expands $Q_1(D)$ vertically and contracts it horizontally to produce an image $Q_0(Q_1(D))$ that extends from above $w = 0$ to a point on $W^u(S^-)$ to overlap D 'vertically' (see figure 13). It follows that the return map has a saddle in $Q_1^{-1}(Q_0^{-1}(D)) \cap Q_0(Q_1(D))$ whose stable and unstable manifolds intersect $W^u(S^+)$ and $W^s(S^-)$ respectively. By uniqueness, this saddle corresponds to the periodic orbit Π and our claim is proved. ■

(b) Heteroclinic connections and bifurcations of the periodic orbit

We now collect and extend the results obtained in §5a and in doing so describe some aspects of the heteroclinic orbits to the periodic orbit Π and the bifurcations in which they are created and destroyed.

Lemma 5.6. *For $c < 0$, $\rho < 0$, $b \leq 0$ and $-(b/c)^2 + \varepsilon H < \nu < 0$, heteroclinic orbits from C^- to S^- exist.*

Proof. First observe that when $\rho < 0$

$$\dot{z} = \varepsilon(\nu + \rho x + z^2) < 0$$

for $-\sqrt{-\nu} < z < \sqrt{-\nu}$ and $x \geq 0$. Thus, the two-dimensional unstable manifold $W^u(C^-)$ is trapped below $\{z = +\sqrt{-\nu}\}$. Moreover, when $c < 0$ we have

$$\frac{dk}{dt} = \varepsilon(b + cz)y^2 > 0$$

for $z < +\sqrt{-\nu}$, so that k increases on orbits in $W^u(C^-)$, implying that $W^u(C^-)$ must intersect $P^{-1}(\Gamma)$ below $\{z = +\sqrt{-\nu}\}$. Earlier estimates of the manifolds $W^s(S^+)$, $W^s(S^-)$ (cf. figure 6a), show that such intersection points lie in $W^s(S^-)$, proving our claim. ■

Proposition 5.7. *As ν decreases for fixed $c < 0$, $b > 0$ (small enough) and $\rho < 0$, heteroclinic bifurcations occur between $\mathcal{C}_{\text{Hopf}}$ and $\mathcal{C}_{\text{glob}}$ in which orbits connecting C^- to S^- and Π to S^+ and S^- are created.*

Proof. The result follows from lemmata 5.1a and 5.6. $W^u(C^-)$ forms the two-dimensional boundary of $W^u(\Pi)$ and so lemma 5.6 implies that, as $W^u(C^-)$ first intersects $W^s(S^-)$ ‘before’ $\mathcal{C}_{\text{glob}}$, points in $W^u(S^+) \cap W^s(S^-)$ are created. But the normal form analysis in the proof of lemma 5.1a shows that, near $\mathcal{C}_{\text{Hopf}}$, $W^u(C^-)$ passes up above $\{z = -b/c\}$, without intersecting $W^s(\Gamma)$ at all (cf. figure 12a); moreover, since $dk/dt > 0$ for solutions below $\{z = -b/c\}$, $W^u(C^-)$ cannot ‘double back’ towards $\{x = 1, y = 0\}$ below this level. In the parameter range of interest, $W^u(S^+) \cap P^{-1}(\Gamma)$ lies below $\{z = -b/c\}$. Hence, by continuity, as a component of $W^u(C^-)$ moves down to intersect points on $W^s(S^-) \subset W^s(\Gamma)$, it must encounter $W^{ss}(S^+)$, the upper boundary of $W^s(S^-)$, creating points in $W^u(\Pi) \cap W^s(S^+)$ also. ■

Proposition 5.8. *As ν decreases for fixed $c > 0$, $b > 0$ (small enough) and $\rho < 0$, a heteroclinic bifurcation occurs between $\mathcal{C}_{\text{Hopf}}$ and $\mathcal{C}_{\text{glob}}$ in which orbits connecting Π to S^- are created. No orbits connecting S^+ to Π exist near $\mathcal{C}_{\text{glob}}$.*

Proof. Near $\mathcal{C}_{\text{Hopf}}$ no connections exist by lemma 5.1b. As we approach $\mathcal{C}_{\text{glob}}$ from above, the saddle-type periodic orbit Π approaches $\gamma_{\text{int}}(0) = P^{-1}(\Gamma) \cap P(\Gamma)$, carrying its stable and unstable manifolds with it. Since distinct (un)stable manifolds cannot intersect, as in the proof of lemma 5.5 we conclude that, sufficiently near $\mathcal{C}_{\text{glob}}$, $W^u(\Pi)$ intersects $W^s(\Gamma)$ and $W^s(\Pi)$ intersects $W^u(\Gamma)$ arbitrarily near $\gamma_{\text{int}}(t)$. But $\lim_{t \rightarrow +\infty} \gamma_{\text{int}}(t) = S^-$ in this case (figure 6b, left panel), and so $W^u(\Pi)$ intersects $W^s(S^-)$, as claimed. The fact that the orbit $W^u(\Gamma) \cap W^s(\Pi)$ lies near $\gamma_{\text{int}}(t)$ and that $\lim_{t \rightarrow -\infty} \gamma_{\text{int}}(t) = (0, 0, -\infty)$ rules out connections from S^+ to Π . ■

These two propositions show that heteroclinic bifurcations involving the saddles $S^\pm$, the perturbed centre points $C^\pm$ and the single-circuit periodic orbit Π occur between $\mathcal{C}_{\text{Hopf}}$ and $\mathcal{C}_{\text{glob}}$ in the case $\rho < 0$ for both $c < 0$ and $c > 0$. For $\rho > 0$ one might expect an analogous situation as ν increases from $\mathcal{C}_{\text{Hopf}}$ to $\mathcal{C}_{\text{glob}}$, but in fact it does not occur. As proposition 4.6 (figure 9a) shows, for $c < 0$ and ν sufficiently far below $\mathcal{C}_{\text{Hopf}}$, one has connections from S^+ to C^+ and from C^- to S^- . As ν increases above $\mathcal{C}_{\text{Hopf}}$ the role of the saddle point C^+ and its stable manifold is simply replaced by the saddle type periodic orbit Π and $W^s(\Pi)$, so that one has connections from S^+ to Π which persist until Π vanishes at $\mathcal{C}_{\text{glob}}$. For $c > 0$ (figure 9b) and ν sufficiently far below $\mathcal{C}_{\text{Hopf}}$ one has connections from C^+ to S^+ and from S^- to C^- . In that case above $\mathcal{C}_{\text{Hopf}}$ the sink C^- is replaced by the stable periodic orbit Π and its three-dimensional stable manifold, so that one has connections from S^+ and S^- to Π which persist until Π vanishes on $\mathcal{C}_{\text{glob}}$.

Figure 14. Schematic summary of connecting orbits for equation (2.14), $b > 0$; (a) $c < 0$; (b) $c > 0$.

6. Conclusions and discussion

This is a long paper and we appreciate that the reader may find it difficult to see the wood for the trees. In an attempt to provide a global summary of the various connecting orbits found in §§ 3, 4 and 5, we offer the schematic pictures of figure 14a, for $c < 0$ and 14b, for $c > 0$. In both cases we assume $b > 0$. In these diagrams we incorporate almost all the global bifurcation results of §§ 3, 4 and 5, and we also illustrate some implications not explicitly derived earlier. However, we do not regard these diagrams as definitive.

The bifurcation diagram is summarized in the centre of each figure and around the periphery we indicate the network of connections among the fixed points $S^\pm$, $C^\pm$, the single circuit periodic orbit Π and the points at $z = \pm\infty$, for typical (ρ, ν) parameter values within each open set between the primary bifurcation curves C_{SN} , C_{CN} , C_{Hopf}

Figure 14. *Cont.*

and $\mathcal{C}_{glob}$. The notation should be almost self-explanatory, but we note that, in the event that a connection may or may not exist depending on fine placement within a region of the (ρ, ν) -plane, we show it as a dashed arrow. When countably many (transverse) connections exist, as in region 5 of figure 14a, we indicate this by the symbol ϕ . The regions between $\mathcal{C}_{Hopf}$ and $\mathcal{C}_{glob}$, in which the periodic orbit Π exists, are indicated by faint shading, and the presence of heteroclinic bifurcations involving $C^\pm$, Π and $S^\pm$, as described in propositions 5.5 and 5.6 in the cases $c < 0$, $\rho < 0$ and $c > 0$, $\rho < 0$, is indicated by hatching. Finally the fact that the homoclinic explosion (resp. implosion) is 'premature' in case $c < 0$, $\rho < 0$ (resp. $c > 0$, $\rho > 0$), as described in theorem 4.9 and Remark 4.11, is indicated by a sketch of the curve $\nu = -(b/c)^2 + G(\rho, b, c, \varepsilon)$, denoted $\mathcal{C}_{inf}$.

An immediately striking finding of this paper is described in theorems 4.8 and 4.9: that homoclinic explosions and implosions occur for $c < 0$ and $c > 0$ respectively on (or near) the homoclinic bifurcation curve $\mathcal{C}_{glob}$. In the explosion case in particular,

we stress that the countable set of ‘fast’ heteroclinic orbits from S^+ to S^- which exist for $\nu < -(b/c)^2 + \varepsilon H$ is structurally stable: its members are all transverse and will therefore persist under small perturbations to the form of equation (2.14). This has interesting implications for the original motivation of our work: the study of travelling wave solutions of certain PDEs. Such multi-circuit heteroclinic orbits correspond to finite, but arbitrarily long sequences of ‘domain walls’ separating two distinct periodic states which exist in the limits $\xi \rightarrow -\infty$ and $\xi \rightarrow +\infty$.

The results of §5 are among the first in which connections between fixed points and periodic orbits are proven to exist in a three-dimensional flow. In the PDE context they correspond to domain walls separating periodic and quasi-periodic states.

For the particular case of the (cubic) complex Ginzburg–Landau equation itself, our results are especially tantalizing, since they indicate that it corresponds to a special degenerate case (the reversible ‘ $b = 0$ ’ case) in which the bifurcation curves $\mathcal{C}_{\text{CN}}$ and $\mathcal{C}_{\text{Hopf}}$ coalesce and $\mathcal{C}_{\text{SN}}$ and $\mathcal{C}_{\text{glob}}$ share the same $O(\varepsilon^2)$ neighbourhood (see Remark 4.2 and below). While they imply that the GL case is, in this sense, structurally unstable, many of our results require $b \neq 0$ and so do not apply directly to the GL system. However, the complex, cubic, GL equation,

$$u_t = u + (1 + iA)u_{xx} \pm (1 + iB)|u|^2u,$$

where A and B are constants determined from the underlying nonlinear stability problem (and $A = \varepsilon a$, $B = \varepsilon b$ in (1.1)), is known to be only the leading order term of an ‘extended’ PDE which involves nonlinear terms of higher order (such as $|u|^4u$ and $|u|^2u_x$; cf. Doelman & Eckhaus 1990; Eckhaus & Iooss 1989; van Saarloos & Hohenberg 1992). When searching for travelling waves, these extended equations yield ODEs similar to (1.1) and thus to our model problem (cf. Jones *et al.* 1990; Doelman & Eckhaus 1990; van Saarloos & Hohenberg 1992; Duan & Holmes 1994) but containing additional terms. The structurally unstable coalescence of the bifurcation curves in (1.1), which as mentioned above corresponds to the cubic GL case, will be unfolded by these additional terms, just as in the model problem studied in this paper.

Furthermore, we believe that after one has obtained a geometrical understanding of the flow governed by the model problem one can develop a similar understanding of (1.1) and analogous ‘extended’ systems in a straightforward fashion. However, the computational effort should not be underestimated. To appreciate this the reader should compare the proof of uniqueness of the single-circuit periodic orbit which continues from $\mathcal{C}_{\text{Hopf}}$ to $\mathcal{C}_{\text{glob}}$ as ν varies – which is more or less the starting point of the present paper – with the proof of the similar result for the GL problem in Doelman (1993): there it is one of the main results and involves much more non-trivial ‘calculus’, due to more complicated dependence of the associated Poincaré map on the integrals of the unperturbed system, combined with the fact that the fundamental periods T_0 and T_1 are less easy to control in that case.

Nevertheless, one would at least like to know whether the GL problem corresponds to a model problem in which there is an explosion of heteroclinic solutions, the outside ($c < 0$) case, or to the inside, implosion, case ($c > 0$). One must perform the Melnikov-type calculations of §3 to determine the relative positions of $P(\Gamma)$ and $P^{-1}(\Gamma)$ in GL system (1.1). Note that in the limit $\varepsilon \rightarrow 0$ we have in (1.1) $\Gamma = \{\Omega^2 = \rho^4 - \rho^6, \rho^2 > \frac{2}{3}, V = 0\}$ instead of $\Gamma = \{x = y = 0\}$. Since the calculations are quite straightforward we give no details. We find that the geometrical structure of $P(\Gamma)$ and $P^{-1}(\Gamma)$, in combination with the relative positions of $W^{\text{uu,ss}}(S^\pm)$ (for

the part of the (a, b) -plane which is near the saddle-node bifurcation at which S^+ and S^- merge), corresponds to the implosion case $c > 0$, $\rho > 0$ in the model problem. This is case (c) of theorem 4.9: the implosion occurs after the primary bifurcation and is significantly more complex than the implosions analysed in theorem 4.8 or theorem 4.9d. We refer to Remark 4.11 for a short discussion. Without going into any detail – which is only possible after applying the full geometrical analysis of this paper to (1.1) – we can make two remarks about this situation.

The first concerns the countably infinite set of multi-circuit homoclinic solutions to Γ which exist before S^+ and S^- are created by the saddle-node bifurcation. In the model problem these solutions are unbounded in the z -direction and are exponentially close to Γ ‘before’ and ‘after’ the finite number of excursions governed by the fast flow have been made. However, in (1.1), Γ has a much more complicated structure. In the limit $\varepsilon = 0$ Γ consists of three pieces: a hyperbolic part (corresponding to the saddle points: $\rho^2 > \frac{2}{3}$); a part with $\rho^2 < \frac{2}{3}$ which consists of centres (like the other slow manifold $\{x = 1, y = 0\}$ in (2.14)), and two degenerate points on Γ ($\rho^2 = \frac{2}{3}$) at which the saddles and the centres of the unperturbed system merge. Since the multi-circuit homoclinic orbits to Γ exist near the hyperbolic part of Γ and are exponentially close to this part of Γ in the part of phase-space which is not too close to $\{V = 0\} \cap \{P(\Gamma) \cap P^{-1}(\Gamma)\}$, one first has to study the *global* character of Γ in (1.1) before any statements can be made about the global character of the homoclinic solutions to Γ . This is a highly non-trivial affair, for there are no general results about the existence of slow manifolds for systems in which the unperturbed flow is non-hyperbolic. The exact solution found by Hocking & Stewartson (1972) is an example which indicates the importance of this aspect of (1.1) for the ‘fate’ of the multi-circuit homoclinic orbits to Γ : for any (a, b) and $\varepsilon \neq 0$ one can find a certain pair (v_c, w_c) such that there exists a homoclinic solution to the (degenerate) origin which is $O(\varepsilon^2)$ close to the curve of saddles and centres of the unperturbed system. Thus, this curve is the slow manifold of the model problem, but it is not unbounded. Since the multi-circuit homoclinic solutions to Γ are exponentially close to Γ one expects that these solutions will also come exponentially close to the origin. In this case it may be possible to create multi-circuit homoclinic loops to the origin for suitable choices of v and w . (Note however that one first has to prove that the multi-circuit homoclinic solutions exist near $v = v_c$, $w = w_c$.) See Doelman (1989) for more details on the Hocking and Stewartson solution interpreted as a solution of (1.1).

The second remark concerns the multi-circuit heteroclinic solutions found by Kopell & Howard (1981) and Doelman (1989) for systems like (1.1). These solutions are connections of ‘strong’ unstable with ‘strong’ stable manifolds; they do not follow the slow flow along Γ . Thus, one expects that these solutions play a role in the symmetric GL case (recall that GL corresponds to $b = 0$), similar to the role of the multi-circuit homoclinic orbits in (2.14) predicted by Remark 4.11 for $b \neq 0$. This correspondence is exactly like the relation between the ‘degenerate’ heteroclinic orbits of Remark 4.2 for $b = 0$ and the (one-circuit) homoclinic solutions which exist at the homoclinic bifurcation for $b \neq 0$ (also see below for a more detailed discussion of these solutions).

These two remarks once more indicate the ‘extra’ complexity of the GL system compared to the model problem (2.14). Observations such as these prompted our derivation and study of the latter.

The model problem also sheds light on the ‘hidden symmetry’ in the GL equation

(see van Saarloos & Hohenberg 1992). In deriving (2.14) we start with a system which has perturbation terms of a very general structure (2.2)–(2.4). After interpreting the conditions imposed on this system in terms of the parameters in the functions f and g we arrive at

$$\dot{x} = y, \quad \dot{y} = x - x^2 + \varepsilon y(b + c\mu + cz), \quad \dot{z} = \varepsilon(\lambda + \rho x + 2\mu z + z^2), \quad (6.1)$$

where λ and μ play roles similar to that of v and w in (1.1). At the end of §2 we then immediately noted that this system is invariant under the shift $\bar{z} = \mu + z$ and derived (2.14). Note that the bar of $\bar{z}$ was dropped and we introduced the new parameter $\nu = \lambda - \mu^2$ which replaces both λ and μ . However, (6.1), *and not* (2.14), is the system which should be compared to (1.1). Thus we observe that in our model of the GL system (1.1) the ‘hidden symmetry’ has been made explicit by the shift $\bar{z} = \mu + z$. For instance, the explicit ‘fast’ heteroclinic solution in (2.14) with $b = 0$ which exists only for a uniquely determined value of ν (cf. Remark 4.2) turns into a one-parameter family of heteroclinic orbits in (6.1) by choosing

$$\lambda - \mu^2 = -9\varepsilon^2 \rho^2 + O(\varepsilon^4) \quad (6.2)$$

(see (4.2)). As in the Bekki and Nozaki family of solutions to the GL equation (see Bekki & Nozaki 1985; van Saarloos & Hohenberg 1992) this family contradicts simple dimension counting arguments (van Saarloos & Hohenberg 1992; Doelman 1993) which suggest that one should find at most a finite number of heteroclinic connections between two one-dimensional manifolds in a three-dimensional system with only two parameters (v and w in (1.1), λ and μ in (6.1)). Thus, the family of solutions of the type (4.1) determined by the relation (6.2) is the analogue in our model problem of the Bekki and Nozaki family in (1.1). In the GL case, the existence of this family can only be explained by conjecturing the existence of a hidden symmetry, while in the model problem there is a natural explanation for the existence of these orbits, as follows. Solutions of type (4.1) exist exactly in the co-dimension one situation $b = 0$ (in 2.27). If b changes sign the homoclinic bifurcation in which the (single-circuit) periodic solution is annihilated changes from taking place at a homoclinic orbit to S^- to a homoclinic orbit at S^+ (for $c > 0$, for $c < 0$ the roles of S^+ and S^- are interchanged, cf. §3); if $b = 0$ the global bifurcation appears at the ‘intermediate’ heteroclinic orbit connecting S^+ to S^- . This exceptional orbit is the exactly the one described by (4.1) and (4.2).

Finally we comment on the number of independent parameters in the GL system (1.1). It is easy to see that one should introduce the parameters $A = (w - a)/(b - a)$ and $B = v/(b - a)$ in (1.1). Introducing $\hat{\varepsilon} = (b - a)\varepsilon/\Delta$ one observes that A and B are the sole parameters appearing in (1.1) in the highest order terms (see Doelman (1993) for more details). They are the analogues of λ and μ in (6.1). Thus, by the above discussion, there remains essentially only one independent parameter, analogous to ν in (2.14). (In Doelman (1993) we find that $\{\nu = 0\} = \mathcal{C}_{\text{SN}} = \mathcal{C}_{\text{glob}} + O(\varepsilon^2)$ – recall that $b = 0$ – corresponds to $\{B^2 - 4A + 4 = 0\}$ in (1.1)). Therefore, the GL system (1.1) should be compared to (2.14) with $b = 0$, $c > 0$ and $\rho > 0$. However, if one wishes to impose further properties of the GL system as extra conditions on the model problem (for instance that GL preserves volume for $B = 0$) one might need to add extra terms to (2.14).

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Appendix A. The derivation of the model problem

In this appendix we present computational details of the derivation of model problem (2.14) of §2. Subsidiary computational details involving elliptic integrals are given in Appendix B. We wish to impose conditions on the general system (2.2) so that its flow behaves like that of the GL system (1.1). Since $\Delta K(k, z) = \Delta Z(k, z) = 0$ determines the location of periodic solutions to (2.2) we can reformulate the first condition (given in §2) as:

For any fixed pair (k, z) , $k \in (k_c, k_s)$, there exists a unique pair $(\lambda(k, z), \mu(k, z))$ such that $\Delta K(k, z) = \Delta Z(k, z) = 0$.

Equating (2.10) and (2.9) to 0 yields a linear (2×2) -system in the unknowns (λ, μ) . The determinant $\mathcal{D}(k, z)$ of the associated matrix $\mathcal{M}$ is given by

$$\frac{\mathcal{D}}{T_0^2} = \frac{S_0(k)}{T_0(k)} [(\beta_0 \gamma_1 - \beta_1 \gamma_0) \mathcal{T}(k) + (2z\beta_0 - \gamma_0)], \quad (\text{A } 1)$$

where $\mathcal{T}(k)$ is defined in (3.6), see also Appendix B (since $T_0 \rightarrow \infty$ as $k \uparrow k_s$ we prefer to divide $\mathcal{D}$ by T_0^2). Note that $\mathcal{T} \rightarrow 1$ for $k \downarrow k_c$ and $\mathcal{T} \rightarrow 0$ for $k \uparrow k_s$ (Appendix B). The (2×2) -system has a unique solution if $\mathcal{D}/T_0^2 \neq 0$ for all $z, k \in (k_c, k_s)$. For given k one can solve $\mathcal{D} = 0$ in terms of z , if $\beta_0 \neq 0$ (recall that $S_0/T_0 > 0$ for $k > k_c$). Hence, in order to satisfy the above condition we impose $\beta_0 = 0$. Since $\mathcal{T}(k) \in [0, 1]$ (see Appendix B) we then find by requiring that $\mathcal{D}/T_0^2 \neq 0$ our first condition on the unfolding parameters of (2.3)–(2.4):

$$\beta_0 = 0 \quad \text{and} \quad \gamma_0 \neq 0, \quad \beta_1 > -1. \quad (\text{A } 2)$$

Note that the problem degenerates if $\beta_0 = \gamma_0 = 0$ since then $\mathcal{D} \equiv 0$.

The next step is determining the curve $\mathcal{C}_{\text{glob}}$ in the (λ, μ) -plane, defined by (2.12). This can be done by solving the linear system for $k = -\delta$, $0 < \delta \ll 1$, and taking the limit $\delta \downarrow 0$. In Appendix B we show that S_0 and S_1 can be expressed in terms of T_0 and T_1 :

$$S_0 = \frac{1}{5}(6kT_0 + T_1), \quad S_1 = \frac{6}{35}(kT_0 + (1 + 5k)T_1). \quad (\text{A } 3)$$

Furthermore, we determine the following approximations ((B 14), (B 15))

$$\mathcal{T}(-\delta) = -\frac{6}{\log \delta} + \text{h.o.t.}, \quad \frac{1}{T_0(-\delta)} = -\frac{1}{\log \delta} + \text{h.o.t.} \quad (\text{A } 4)$$

Thus the matrix $\mathcal{M}$ can be inverted explicitly (at leading order). Using (A 3), (A 1) and (A 4) we find

$$\frac{\mathcal{D}}{T_0^2}(-\delta, z) = -\frac{6}{5 \log \delta} (2\beta_0 z - \gamma_0) + \text{h.o.t.} \quad (\text{A } 5)$$

We can now compute $\mathcal{M}^{-1}$ and take the limit $\delta \rightarrow 0$:

$$\lambda_*(z) = \left(2\frac{b_3}{\gamma_0} - 1\right) z^2 + 2\frac{B}{\gamma_0} z, \quad \mu_*(z) = -\frac{b_3}{\gamma_0} z - \frac{B}{\gamma_0} \quad \text{with} \quad B = b_0 + \frac{6}{7}b_1. \quad (\text{A } 6)$$

As above we assume that $\beta_0 = 0$ in computing (A 6). This is necessary: if $\beta_0 \neq 0$ then both $\lambda_*(z)$ and $\mu_*(z)$ diverge at $z = \gamma_0/2\beta_0$. This agrees with the previous analysis: $\mathcal{D} = 0$ ‘at’ k_c for this value of z (A 5). Note that $z^2 + 2\mu_*(z)z + \lambda_*(z) \equiv 0$, i.e. (2.2) has a (perturbed) saddle point in the ‘correct’ z -plane at the global bifurcation – within $O(\varepsilon)$ accuracy.

A very special feature of the GL system is that the above determined curve $\mathcal{C}_{\text{glob}}$ coincides (up to $O(\varepsilon)$) with the curve $\mathcal{C}_{\text{SN}}$ on which the (perturbed) saddles merge. From (2.5) we find for (2.2):

$$\mathcal{C}_{\text{SN}} = \{\mu^2 - \lambda = 0\}. \quad (\text{A } 7)$$

We can determine the intersections of $\mathcal{C}_{\text{glob}}$ and $\mathcal{C}_{\text{SN}}$ by combining (A 6) and (A 7). We find that $\mathcal{C}_{\text{glob}}$ and $\mathcal{C}_{\text{SN}}$ intersect for z given by

$$((b_3 - \gamma_0)z + B)^2 = 0. \quad (\text{A } 8)$$

Therefore, $\mathcal{C}_{\text{glob}} = \mathcal{C}_{\text{SN}}$ at leading order if

$$b_3 - \gamma_0 = 0 \quad \text{and} \quad B = b_0 + \frac{6}{7}b_1 = 0. \quad (\text{A } 9)$$

Thus we found that, in order to behave as the GL system, the parameters must satisfy the conditions (A 9). In our model problem we will unfold this situation by admitting $B \neq 0$, for which it suffices that $b_1 = 0$, $b_0 \neq 0$. Note that $\mathcal{C}_{\text{glob}}$ and $\mathcal{C}_{\text{SN}}$ do not intersect transversely: such an intersection would imply a region in the (λ, μ) -plane in which there are homoclinic bifurcations *in the absence of (perturbed) saddles in (2.2)*; this is not possible.

Next, we define $\Delta K_*(z) = \Delta K(k_s, z)$ and $\Delta Z_*(z) = \Delta Z(k_s, z)$ for $\lambda = \lambda_*(z)$ and $\mu = \mu_*(z)$: ΔK_* and ΔZ_* are the classical Melnikov functions which measure the splitting distance between the fast (1-D) stable and unstable manifolds of the (perturbed) saddle at $(0, 0, z)$ (z plays the role of a parameter now). Using (A 3) and (A 4) we find $S_0(k_c) = \frac{6}{5}$ and $S_1(k_c) = \frac{36}{35}$, thus by (2.10) and (A 6) we have

$$\Delta K_*(z) = O(\varepsilon^2) \quad \text{for all} \quad z. \quad (\text{A } 10)$$

This implies that the stable manifold of $(0, 0, z)$ returns for all $z = O(\varepsilon^2)$ close to the slow manifold $\{x = y = 0\}$. The GL system has a similar property (although there the formulation is more complicated, see Claim 4.5 in Doelman (1993)). Note that (A 10) fails if $\beta_0 \neq 0$. Analogously we find by (2.9) in general

$$\frac{\Delta Z_*(z)}{6\varepsilon} = \left[a_1 - \frac{B\gamma_1}{\gamma_0} \right] + z \left[\frac{B}{\gamma_0} - \frac{b_3\gamma_1}{\gamma_0} \right] + z^2\beta_1 \left(\frac{2b_3}{\gamma_0} - 1 \right) + O(\varepsilon), \quad (\text{A } 11)$$

using that $z^2 + 2\mu_*(z)z + \lambda_*(z) = 0$. Thus imposing condition (A 9), we have

$$\frac{\Delta Z_*(z)}{6\varepsilon} = a_1 - \gamma_1 z + \beta_1 z^2 + O(\varepsilon). \quad (\text{A } 12)$$

In the GL system we find in the corresponding situation that the quantity similar to ΔZ_* cannot be 0. Therefore, as third, and last, condition we impose $\Delta Z_*(z) \neq 0$ for all z and guarantee this by setting

$$\gamma_1 = 0, \quad \beta_1 = 0 \quad \text{and} \quad a_1 \neq 0. \quad (\text{A } 13)$$

We have now acquired enough information on the behaviour of the flow induced by (2.2) to be able to derive the model problem. First we note that the parts of f and g which are odd with respect to y have not appeared in this analysis. Therefore, we set $a_2 = \beta_2 = \gamma_2 = b_2 = 0$. Recalling that we may take $b_1 = 0$, we apply conditions (A 2), (A 9) and (A 13) to conclude that we may reduce (2.3)–(2.4) to (2.13).

Appendix B. The T_0 and T_1 -calculus

In this appendix we derive some important properties of the complete elliptic integrals $T_0(k)$ and $T_1(k)$ of (2.8). First we compute, by a straightforward perturbation analysis and direct integration of the limiting expressions

$$\lim_{k \downarrow -1/6} T_0(k) = \lim_{k \downarrow -1/6} T_1(k) = 2\pi \quad \text{and} \quad \lim_{k \uparrow 0} T_0(k) = \infty, \quad \lim_{k \uparrow 0} T_1(k) = 6, \quad (\text{B } 1)$$

with $k_c = -\frac{1}{6}$ and $k_s = 0$. In order to express S_0 and S_1 (2.11) as functions of T_0 and T_1 we write

$$S_i = \oint \frac{x^i G_k}{\sqrt{G_k}} dx = 2kT_i + T_{i+2} - \frac{2}{3}T_{i+3}, \quad (\text{B } 2)$$

with G_k as in (2.7). Since the integral around a closed curve of any exact differential vanishes, we have

$$0 = \oint \frac{d}{dx} (x^{j-1} \sqrt{G_k}) dx = [(j-1)S_{j-2}] + [T_j - T_{j+1}] \quad (j \geq 1). \quad (\text{B } 3)$$

In particular, for $j = 1$ we have $T_1 = T_2$. From (B 2) and (B 3) we find

$$T_{j+1} = \frac{3j}{2j+1}T_j + \frac{6k(j-1)}{2j+1}T_{j-2}.$$

Manipulating these expressions yields (A 3).

Next, we need to derive $(d/dk)T_i$. For this we follow the methods presented in Armbruster *et al.* (1988) and Doelman (1993). We define

$$J_i(k) = \oint \frac{x^i}{G_k \sqrt{G_k}} dx \quad (i \geq 0), \quad (\text{B } 4)$$

and thus

$$\frac{d}{dk}T_i = -J_i. \quad (\text{B } 5)$$

As in the above analysis we derive

$$T_0 = 2kJ_0 + J_2 - \frac{2}{3}J_3 \quad \text{and} \quad T_1 = 2kJ_1 + J_3 - \frac{2}{3}J_4. \quad (\text{B } 6)$$

Furthermore

$$0 = \oint \frac{d}{dx} \left(\frac{x^{j-1}}{\sqrt{G_k}} \right) dx = [(j-1)T_{j-2}] - [J_j - J_{j+1}] \quad (j \geq 1), \quad (\text{B } 7)$$

hence, by taking $j = 1, 2, 3$ we find 3 additional equations. So, we end up with a linear (5×5) -system in the ‘unknowns’ $J_0, \dots, J_4$:

$$\begin{pmatrix} 2k & 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 2k & 0 & 1 & -\frac{2}{3} \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} J_0 \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{pmatrix} = \begin{pmatrix} T_0 \\ T_1 \\ 0 \\ T_0 \\ 2T_1 \end{pmatrix}. \quad (\text{B } 8)$$

By solving this system we can express J_0 and J_1 as functions of T_0 and T_1 :

$$J_0 = \frac{6kT_0 + T_1}{6k(6k+1)} \quad \text{and} \quad J_1 = \frac{T_0 - T_1}{6k+1}, \quad (\text{B } 9)$$

from which (B 6) yields (3.11). Using (B 5) we compute $(d\mathcal{T}(k)/dk)$ (see (3.6)):

$$\frac{d\mathcal{T}}{dk} = \frac{(dT_1/dk)T_0 - T_1(dT_0/dk)}{T_0^2} = \frac{\mathcal{T}^2 + 12k\mathcal{T} - 6k}{6k(1 + 6k)}. \quad (\text{B } 10)$$

The quadratic polynomial $P_k(x) = x^2 + 12kx - 6k$ has $24k(6k + 1)$ as discriminant, and hence has no real roots for $k \in (-\frac{1}{6}, 0)$; therefore we deduce

$$\frac{d\mathcal{T}}{dk} \leq 0 \quad \text{for} \quad k \in (-\frac{1}{6}, 0). \quad (\text{B } 11)$$

Thus since

$$\lim_{k \downarrow -\frac{1}{6}} \mathcal{T}(k) = 1 \quad \text{and} \quad \lim_{k \uparrow 0} \mathcal{T}(k) = 0$$

we have $\mathcal{T}(k) \in [0, 1]$. Using the implicit expression for $d\mathcal{T}/dk$, we can determine an explicit formula for $\mathcal{T}(k)$ for k near k_s . We set $k = -\delta$, $0 < \delta \ll 1$ and $\mathcal{T}(k) = \tau(\delta)$ and find

$$\frac{d\tau}{d\delta} = \frac{\tau^2 - 12\delta\tau + 6\delta}{6\delta(1 - 6\delta)}. \quad (\text{B } 12)$$

Now, it is essential to observe that $\delta \ll \tau$ (although $\tau(\delta) \rightarrow 0$ if $\delta \rightarrow 0$), i.e. $\mathcal{T}$ does not depend smoothly on k in the limit $k \rightarrow k_s$ (see a classical text like Lawden (1989), or Doelman (1993)). Therefore,

$$\frac{d\tau}{d\delta} = \frac{\tau^2}{6\delta} + \text{h.o.t.}, \quad (\text{B } 13)$$

which yields

$$\tau(\delta) = \mathcal{T}(-\delta) = -\frac{6}{\log \delta} + \text{h.o.t.} \quad (\text{B } 14)$$

By (B 1) we conclude that

$$T_0(-\delta) = -\log \delta + \text{h.o.t.}, \quad (\text{B } 15)$$

furthermore, by substituting (B 14) into (B 5) and (B 9) we find $dT_1/d\delta = \log \delta + \text{h.o.t.}$, which yields

$$T_1(-\delta) = 6 + O(\delta \log \delta). \quad (\text{B } 16)$$

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