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REPRESENTATION THEORY FOR A CLASS OF $\mathfrak{so}(N)$ -EXTENDED SUPERCONFORMAL OPERATOR ALGEBRAS

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We study the representation theory for a class of $\mathfrak{so}(N)$ -extended superconformal operator algebras, which was originally proposed by Knizhnik. Our main result is a set of conditions for consistency and unitarity, which are formulated in terms of the labels $(\Delta; m_1, m_2, \dots, m_{[N/2]})$ that characterize an irreducible highest weight module. We exploit the correspondence of level-1 $\mathfrak{so}(N)$ -extended superconformal theories, which have $c_N = \frac{1}{2}N + 1$, with certain free-field models, to derive character formulas for all representations that are allowed at level 1.

1. Introduction

Although the recent progress in the understanding of two-dimensional conformal field theory is quite spectacular, a complete classification of all possible models is still far off.

In the range $c < 1$, the restrictions coming from unitarity, and from the requirement of modular invariance of the torus partition function, have turned out to be strong enough to fix all possible models [1]. In order to obtain information for the remaining region, $c \geq 1$, various strategies have been developed. In one approach, one tries to relate different models at a certain value for c by continuous deformations (marginal operators) or discontinuous transformations (“modding out”). This method is exhaustive for conformal theories with $c = 1$ [2] and superconformal theories with $\hat{c} = 1$ [3]. In other approaches, a careful analysis of monodromies and one-loop modular transformations leads to further restrictions [4–7].

Meanwhile, it is very instructive to study the structure that arises if one assumes the presence of extra continuous symmetries in addition to conformal invariance. This is not only for the sake of classification; many applications actually require extra symmetry (an example is the $N = 2$ superconformal invariance needed for space-time supersymmetry) or become much more transparent when formulated in terms of some extended symmetry algebra (e.g. the so-called $\mathbf{Z}_N$ models, which are most naturally described in terms of the $\mathfrak{su}(N)$ Casimir algebra [8]).

Some examples of additional symmetries in conformal field theory have been known and explored for a long time, but a systematic study was begun only recently by Zamolodchikov in ref. [9]. In this paper, restrictions on possible extra symmetries coming from conformal Ward-identities, and from the requirement of crossing

symmetry of the four-point function, are investigated. The most interesting example discussed in ref. [9] is a bosonic extension of the Virasoro algebra by a dimension-3 current $W(z)$. This algebra has been generalized to so-called Casimir algebras associated to the affine extensions of the simply laced classical Lie algebras [8]. Extensions with a single dimension- Δ current, with Δ integer or half-integer, were discussed recently in ref. [10].

In the spirit of Zamolodchikov's work, Knizhnik [11] and later Bershadsky [12] considered the possibility of extending the Virasoro algebra with N dimension- $\frac{3}{2}$ supercurrents in the defining representation of an internal $\mathfrak{so}(N)$ or $\mathfrak{su}(N)$ Kac–Moody algebra. They found consistent operator algebras which close on Virasoro currents $T(z)$, supercurrents $G^i(z)$, Kac–Moody currents $J^a(z)$ and multilinear expressions in these currents. The algebras contain a single free parameter which determines the level of the $\mathfrak{so}(N)$ ($\mathfrak{su}(N)$) Kac–Moody subalgebra and the central charge of the Virasoro subalgebra.

With regard to the $\mathfrak{so}(N)$ -extended superconformal algebras proposed by Knizhnik, it is important to note that the (anti-)commutator algebra of the Fourier modes L_n , G_r^i and J_n^a is different (for $N > 2$) from the algebra of $\mathcal{O}(N)$ -extended superconformal transformations as given by Ademollo et al. [13]. Knizhnik's algebras can best be described as (infinitely generated) Lie superalgebras, with the additional structure that some of their generators can be expressed as normal-ordered products of other generators (leaving the generators L_n , G_r^i and J_n^a as a fundamental set). In earlier work on superconformal algebras, the relevance of algebras of this type was not recognized, which explains why they do not occur in the "classification" of all possible superconformal extensions of the Virasoro algebra given in ref. [14]. We recall that the $\mathfrak{o}(N)$ -extended superconformal algebra given by Ademollo et al. are finitely generated Lie superalgebras. They have a natural geometrical interpretation since they correspond to superconformal transformations in an appropriate N -extended superspace. However, their relevance for conformal quantum field theory is limited to the cases $N \leq 4$, since for $N > 4$ the central extensions that are needed do not exist [15].

By construction, the $\mathfrak{so}(N)$ - and $\mathfrak{u}(N)$ -extended superconformal algebras proposed by Knizhnik admit central extensions for general N , and one can therefore expect that they will actually occur as current algebras for certain models of $d = 2$ conformal quantum field theory. It was noticed in ref. [11] that the $\mathfrak{u}(N)$ algebras for $N > 2$ are of little interest since they do not admit unitary representations. Knizhnik's $\mathfrak{u}(2)$ and $\mathfrak{so}(2)$ algebras reduce to the well-known $\mathfrak{su}(2)$ -extended ($N = 4$) and $\mathfrak{u}(1)$ -extended ($N = 2$) superconformal algebras, which have been studied in great detail and have proved to be relevant for many applications. In this paper we report on a study of the remaining algebras, which are the $\mathfrak{so}(N)$ algebras with $N \geq 3$.

In sect. 2, we recall the structure of the $\mathfrak{so}(N)$ operator algebras. We point out some parallels and differences with the current algebras of the $\mathrm{SO}(N)$ Wess–Zumino–Witten (WZW) models.

In sect. 3, the representation theory of these algebras is investigated. The main issue here is to establish the existence of unitary representations and to determine all possible $\mathfrak{so}(N)$ representations and conformal dimensions that can occur at a given level. This analysis leads to *consistency* and *typicality* conditions which are reminiscent of the representation theory of affine Kac–Moody algebras and finite-dimensional classical Lie superalgebras [16, 17]. The typicality conditions distinguish between massless representations, where supersymmetry is unbroken, and massive representations with broken supersymmetry.

In sect. 4, we show that $\mathfrak{so}(N)$ -extended superconformal symmetry (at level 1) is realized in various models with N free Majorana fermions and a single real boson. This correspondence provides independent evidence for the results derived in sect. 3. Furthermore, it allows us to determine explicitly the characters and the supersymmetry indices of all representations at level one. A rather surprising new feature is the fact that for the massless representations in the Ramond sector, the balance between the number of bosonic and fermionic states is broken at all mass-levels simultaneously. As a consequence, the supersymmetry indices of these representations are in a non-trivial representation of the modular group.

2. Structure of the operator algebra

We recall the structure of the $\mathfrak{so}(N)$ -extended superconformal operator algebra as it was given by Knizhnik [11] and Bershadsky [12]. The algebra is generated by the stress-energy tensor $T(z)$, the dimension- $\frac{3}{2}$ supercurrents $G^i(z)$, $i = 1, 2, \dots, N$ and the dimension-1 currents $J^a(z)$, $a = 1, 2, \dots, \frac{1}{2}N(N-1)$, where the indices i and a denote the vector and the adjoint representations of $\mathfrak{so}(N)$, respectively. The operator product algebra reads as follows

$$\begin{aligned}
 T(z)T(w) &= \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots, \\
 T(z)G^i(w) &= \frac{\frac{3}{2}G^i(w)}{(z-w)^2} + \frac{\partial G^i(w)}{(z-w)} + \dots, \\
 T(z)J^a(w) &= \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)} + \dots, \\
 G^i(z)G^j(w) &= \frac{B\delta^{ij}}{(z-w)^3} + \frac{Kt_{ij}^a J^a(w)}{(z-w)^2} + \frac{\frac{1}{2}Kt_{ij}^a \partial J^a(w)}{(z-w)} + \frac{2\delta^{ij}T(w)}{(z-w)} \\
 &\quad + \frac{\gamma \Pi_{ij}^{ab}(J^a J^b)(w)}{(z-w)} + \dots, \\
 J^a(z)G^i(w) &= \frac{t_{ji}^a G^j(w)}{(z-w)} + \dots, \\
 J^a(z)J^b(w) &= \frac{-S\delta^{ab}}{(z-w)^2} + \frac{f^{abc}J^c(w)}{(z-w)} + \dots,
 \end{aligned} \tag{2.1}$$

where t_{ij}^a and f^{abc} satisfy

$$\begin{aligned} [t^a, t^b] &= f^{abc} t^c, & tr(t^a t^b) &= -2\delta^{ab}, \\ t_{ij}^a t_{kl}^a &= \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}, & f^{abc} f^{abd} &= 2(N-2)\delta^{cd}, \end{aligned} \quad (2.2)$$

and the tensor Π_{ij}^{ab} is given by

$$\Pi_{ij}^{ab} = t_{im}^a t_{mj}^b + t_{im}^b t_{mj}^a + 2\delta^{ab} \delta_{ij}. \quad (2.3)$$

It was shown in refs. [11, 12] that this operator algebra is consistent if the constants c , B , K and γ are chosen as follows

$$\begin{aligned} c &= \frac{1}{2} S \frac{6S + N^2 - 10}{S + N - 3}, & B &= KS, \\ K &= \frac{2S + N - 4}{S + N - 3}, & \gamma &= \frac{1}{2} \frac{1}{S + N - 3}, \end{aligned} \quad (2.4)$$

leaving S as a freely adjustable parameter. (It will turn out that unitary representations are possible only if S is a positive integer.)

From the current algebra (2.1) we can extract the (anti-)commutator algebra of the Fourier-modes $L_n = \oint (dz/2\pi i) T(z) z^{n+1}$, $G_r^i = \oint (dz/2\pi i) G^i(z) z^{r+1/2}$ and $J_n^a = \oint (dz/2\pi i) J^a(z) z^n$, where $r \in \mathbf{Z} + \frac{1}{2}$ (Neveu-Schwarz sector) or $r \in \mathbf{Z}$ (Ramond sector)

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{12}c(m^3 - m)\delta_{m+n}, \\ [L_m, G_r^i] &= (\tfrac{1}{2}m - r)G_{m+r}^i, \\ [L_m, J_n^a] &= -nJ_{m+n}^a, \\ \{G_r^i, G_s^j\} &= \tfrac{1}{2}B(r^2 - \tfrac{1}{4})\delta^{ij}\delta_{r+s} + 2\delta^{ij}L_{r+s} + \tfrac{1}{2}K(r-s)t_{ij}^a J_{r+s}^a + \gamma \Pi_{ij}^{ab}(J^a J^b)_{r+s}, \\ [J_m^a, G_r^i] &= t_{ji}^a G_{m+r}^j, \\ [J_m^a, J_n^b] &= -Sm\delta^{ab}\delta_{m+n} + f^{abc}J_{m+n}^c. \end{aligned} \quad (2.5)$$

The main difference with the conventional $N=1, 2, 4$ superconformal algebras is the presence of a quadratic term in the anticommutator $\{G_r^i, G_s^j\}$.

The algebra (2.5) can be viewed as a supersymmetric extension of the symmetry algebra of the $SO(N)$ WZW model, which is the semidirect sum of the Virasoro algebra and the $so(N)$ Kac-Moody algebra [18, 19]. In the analysis of the WZW

models a crucial role is played by the Sugawara relation, which expresses the stress–energy tensor $T(z)$ in terms of the Kac–Moody currents $J^a(z)$. In the present case there is a weaker relation instead

$$\partial T(z) = \frac{1}{N} (G^i G^i)(z) - \gamma \frac{N-2}{N} \partial (J^a J^a)(z). \quad (2.6)$$

It is derived from the operator algebra (2.1) in a straightforward way. As a simple check on eq. (2.6) one can contract both sides with $T(w)$. Equating the central terms ($\sim (z-w)^{-5}$) on both sides yields a relation among the constants c , B , γ and S which is compatible with eq. (2.3).

The “Sugawara relation” (2.6) is not strong enough to fix the conformal dimension Δ of a primary field $\phi_{\Delta}^{(R)}(z)$ in terms of invariants of the $\mathfrak{so}(N)$ -representation (R) carried by $\phi_{\Delta}^{(R)}(z)$. This can be understood by considering the action of L_0 on a highest weight state $|\phi_{\Delta}^{(R)}\rangle$ (see subsect. 3.1 for a definition)

$$\begin{aligned} L_0 |\phi_{\Delta}^{(R)}\rangle &= \frac{1}{2N} \{Q^{i\dagger}, Q^i\} |\phi_{\Delta}^{(R)}\rangle + \gamma \frac{N-2}{N} c_R |\phi_{\Delta}^{(R)}\rangle, \\ L_0 |\phi_{\Delta}^{(R)}\rangle &= \frac{1}{16} B |\phi_{\Delta}^{(R)}\rangle + \frac{1}{N} Q^i Q^i |\phi_{\Delta}^{(R)}\rangle + \gamma \frac{N-2}{N} c_R |\phi_{\Delta}^{(R)}\rangle, \end{aligned} \quad (2.7)$$

where we defined $Q^i = G_{-1/2}^i$ for the Neveu–Schwarz sector and $Q^i = G_0^i$ for the Ramond sector; c_R is the eigenvalue of the quadratic Casimir operator for the representation R , defined by $(\tau^a \tau^a)_{ij} = -c_R \delta_{ij}$. Since the Q^2 -terms in eq. (2.7) are always positive, we obtain lower bounds for the dimension Δ of a primary field $\phi_{\Delta}^{(R)}$, which are $\gamma(N-2)/N$ for the Neveu–Schwarz sector and $(\frac{1}{16}B + \gamma(N-2)/N)$ for the Ramond sector, respectively. However, since, in general, a highest weight state is not annihilated by all supercharges Q^i simultaneously, the lowest possible values for the dimensions Δ are actually higher than the values given above. Precise information on the values for Δ that are allowed, which can not be extracted directly from eq. (2.7), will be derived in sect. 3.

We conclude this section with a remark on the possibility to set up a BRST procedure in order to realize the symmetry (2.1) in *local* form. In ref. [20] an adaptation of the standard BRST scheme to the case of a Zamolodchikov’s spin-3 algebra, which is another example of an algebra with quadratic terms in the operator product expansions, was presented. We checked that a similar adaptation can be made for the present $\mathfrak{so}(N)$ -extended algebras, provided the level of the matter sector is given by $S^{\text{mat}} = -2(N-3)$. (This can be compared to the critical level for the pure $\mathfrak{so}(N)$ Kac–Moody algebra, which is $S^c = -2(N-2)$.) The central charge for a critical representation, which is related to the level S^{mat} through the relation (2.4), is $c^{\text{mat}} = N^2 - 12N + 26$. It is balanced by the ghost contribu-

tions, which are given by

$$\begin{aligned}
 c^{\text{gh}} &= -2 \sum_{\lambda} (-1)^{2\lambda} N_{\lambda} (6\lambda^2 - 6\lambda + 1) \\
 &= -2 \left(13 - \frac{11}{2}N + \frac{1}{2}(N^2 - N) \right) \\
 &= -N^2 + 12N - 26,
 \end{aligned} \tag{2.8}$$

where N_{λ} is the number of dimension- λ generators in the algebra. For all $N \geq 3$, critical representations have $S \leq 0$, which is in conflict with unitarity. Only for $N = 2$, where the algebra reduces to the conventional $N = 2$ superconformal algebra, with critical values $S^{\text{mat}} = 2$, $c^{\text{mat}} = 4$, can local N -extended symmetry be implemented on unitary representations (see, e.g. ref. [21]). The details of this modified BRST construction will be presented in a separate publication [22].

3. Representation theory

3.1. GENERAL CONSIDERATIONS

In this section we study the representation theory of the algebra (2.5). We will restrict ourselves to representations in which L_0 is bounded from below, which are traditionally called highest weight modules (HWM). We recall that the algebra (2.5) can be viewed as a Lie superalgebra, with the additional structure that some of the generators are expressed as normal-ordered products of others. We are interested in representations (in the sense of Lie superalgebras) which respect this extra structure, i.e. in representations ρ which preserve the form of the quadratic terms in the algebra

$$\begin{aligned}
 \{ \rho(G_r^i), \rho(G_s^j) \} &= \frac{1}{2} B \left(r^2 - \frac{1}{4} \right) \delta^{ij} \delta_{r+s} + 2 \delta^{ij} \rho(L)_{r+s} + \frac{1}{2} K(r-s) t_{ij}^a \rho(J^a)_{r+s} \\
 &\quad + \gamma \Pi_{ij}^{ab} \left(\rho(J^a) \rho(J^b) \right)_{r+s}.
 \end{aligned} \tag{3.1}$$

(In ref. [23] representations of this type are called “normal-ordered vacuum representations”.) In order to be able to define a highest weight state we split the generators of the algebra into “Cartan subalgebra” (CSA) generators X^0 , “positive” generators X^+ and “negative” generators X^- . For the $\mathfrak{so}(N)$ subalgebra we can, of course, use the standard CSA and root-space decomposition, giving the basis $H^i, E^{\pm \alpha}$ where the index i labels the CSA generators ($i = 1, 2, \dots, [\frac{1}{2}N]$) and α are the positive roots. For the supercharges G we make a choice of basis which allows a natural split $G^i, G^{\bar{i}}$ ($i, \bar{i} = 1, 2, \dots, l$) for $\mathfrak{so}(2l)$ and $G^i, G^{\bar{i}}, G^0$ ($i, \bar{i} = 1, 2, \dots, 1$) for $\mathfrak{so}(2l+1)$. Some of the technicalities involved in these definitions are collected in the appendix.

The generators of the algebra are now organized as follows

$$\begin{aligned}
 X^+ : \quad & H_{n>0}^i, E_{n>0}^{\pm\alpha}, E_0^\alpha, L_{n>0}, & \begin{cases} G_{r\geq 1/2}^I, \\ G_{r>0}^I, G_0^i, \end{cases} & \begin{aligned} & \text{(NS),} \\ & \text{(R),} \end{aligned} \\
 X^0 : \quad & H_0^i, L_0, & \begin{cases} -, \\ (-1)^F, \end{cases} & \begin{aligned} & \text{(NS),} \\ & \text{(R),} \end{aligned} \\
 X^- : \quad & H_{n<0}^i, E_{n<0}^{\pm\alpha}, E_0^{-\alpha}, L_{n<0}, & \begin{cases} G_{r\leq -1/2}^I, \\ G_{r<0}^I, G_0^i, G_0^0, \end{cases} & \begin{aligned} & \text{(NS),} \\ & \text{(R),} \end{aligned} \end{aligned} \quad (3.2)$$

where the index I denotes $i, \bar{i}, [0]$.

Note that we have extended the algebra in the Ramond sector by including an additional generator, which is the fermion parity $(-1)^F$ defined by

$$[(-1)^F, L_n] = 0, \quad [(-1)^F, J_n^a] = 0, \quad \{(-1)^F, G_r\} = 0. \quad (3.3)$$

It is needed to keep track of the number of bosonic and fermionic states at each level of the representations in the Ramond sector. Using the above decomposition we can now define a highest weight state $|\phi_\Delta^\lambda\rangle$, which is characterized by a $\mathfrak{so}(N)$ weight-vector λ and a conformal dimension Δ

$$X^+ |\phi_\Delta^\lambda\rangle = 0, \quad H_0^i |\phi_\Delta^\lambda\rangle = \langle \lambda, \alpha^i \rangle |\phi_\Delta^\lambda\rangle, \quad L_0 |\phi_\Delta^\lambda\rangle = \Delta |\phi_\Delta^\lambda\rangle, \quad (3.4)$$

where $\langle, \rangle$ denotes the Cartan–Killing form on the dual CSA $\mathfrak{h}^*$ (see appendix). We are free to specify a bosonic $((-1)^F = 1)$ or fermionic $((-1)^F = -1)$ highest weight state.

In the Ramond sector the conditions $X^+ |\phi_\Delta^\lambda\rangle = 0$ include $G_0^i |\phi_\Delta^\lambda\rangle = 0$ and $E^\alpha |\phi_\Delta^\lambda\rangle = 0$. From these we can derive other conditions, such as

$$\{G_0^i, G_0^j\} |\phi_\Delta^\lambda\rangle = 0, \quad [E_0^\alpha, G_0^i] |\phi_\Delta^\lambda\rangle = 0. \quad (3.5)$$

However, by using the explicit form of the tensors t_{ij}^a and Π_{ij}^{ab} , as described in the appendix, one can show that (anti-)commutators like the above do not generate conditions which are not already contained in eq. (3.4). This shows that the split (3.2) and the ensuing definition of $|\phi_\Delta^\lambda\rangle$ are, in fact, consistent.

Starting from the highest weight state $|\phi_\Delta^\lambda\rangle$ we generate a free HWM (the so-called Verma module $M(\Delta; \lambda)$) by repeatedly acting with the generators X^- . Since we are interested in the irreducible quotient modules $L(\Delta; \lambda)$, we will always assume that, if the Verma module contains null states, all submodules generated by these null states have been divided out. Employing the Dynkin labeling for the

$\mathfrak{so}(N)$ -weight λ , defined by $\lambda = \sum m_i \mu_i$, we can characterize the highest weight state $|\phi_\Delta^\lambda\rangle$, and thereby the irreducible HWM, by the labels

$$(\Delta; m_1, m_2, \dots, m_l), \quad \Delta \in \mathbf{R}, \quad m_i \in \mathbf{Z}_{\geq 0}, \quad (3.6)$$

for $\mathfrak{so}(2l)$, $\mathfrak{so}(2l+1)$. We will now discuss some restrictions on these labels which follow from consistency and unitarity.

We recall that there is a natural way to define a non-degenerate contragredient hermitian form $\mathcal{H}$ on $L(\Delta; \lambda)$

$$\begin{aligned} \mathcal{H}(A|\phi_\Delta^\lambda\rangle, B|\phi_\Delta^\lambda\rangle) &= \langle \phi_\Delta^\lambda | B^\dagger A | \phi_\Delta^\lambda \rangle, \\ \mathcal{H}(|\phi_\Delta^\lambda\rangle, |\phi_\Delta^\lambda\rangle) &= \langle \phi_\Delta^\lambda | \phi_\Delta^\lambda \rangle = 1, \end{aligned} \quad (3.7)$$

where A and B stand for arbitrary products of generators in the set X^- and hermitian conjugation is defined by

$$\begin{aligned} L_n^\dagger &= L_{-n}, & (E_n^\alpha)^\dagger &= -E_{-n}^{-\alpha}, & (H_n^i)^\dagger &= H_{-n}^i, \\ (G_r^i)^\dagger &= G_{-r}^i, & (G_r^0)^\dagger &= G_{-r}^0. \end{aligned} \quad (3.8)$$

We impose the condition that the representation $L(\Delta; \lambda)$ is unitary, i.e. that the form $\mathcal{H}$ is positive definite. The positivity of $\mathcal{H}$, which makes $L(\Delta; \lambda)$ into a Hilbert space, is necessary for applications in the context of conformal field theory; it is directly related to reflection positivity, and thereby to unitarity in quantum field theory [24].

The first consequence of the requirement that $\mathcal{H}$ be positive is an *integrability* condition on the labels (m_i) , which is well known from the representation theory of Kac–Moody algebras. It is formulated in terms of the label m_0 , defined as

$$m_0 = S - \sum_1^{[N/2]} a_i^\vee m_i \quad (3.9)$$

(the $a_i^\vee$ are the dual coxeter labels for the affine extension of $\mathfrak{so}(N)$), which is associated with the isospin in the root space in the direction E_{-1}^θ , where θ is the highest root of $\mathfrak{so}(N)$. The condition reads

$$m_0 \in \mathbf{Z}_{\geq 0}. \quad (3.10)$$

The necessity of this condition for applications in conformal field theory was stressed in ref. [19], where it is shown that conformal fields that belong to a non-integrable representation vanish in theories that are invariant under current algebra.

In non-supersymmetric theories, described by pure Kac–Moody current algebra, the integrability condition (3.10) is sufficient to ensure the positivity of $\mathcal{H}$ (theorem 11.7 in ref. [27]). In the present case there are additional restrictions, which are expressed as conditions on the conformal dimension Δ . They will be derived in explicit form in subsections 3.2 (Neveu–Schwarz sector) and 3.3 (Ramond sector). In the remainder of this subsection we shortly summarize the form of the restrictions and we indicate some features of the allowed representations.

It will turn out that the requirement of unitarity leads to a condition of the form

$$\Delta \geq \Delta(S; m_1, m_2, \dots, m_{[N/2]}), \quad (3.11)$$

where $\Delta(S; m_1, \dots)$ is a function of the level S and the labels m_i . In most cases we are free to choose whether Δ saturates the lower bound $\Delta(S; m_1, \dots)$ or not, but for a given level S there are some cases, corresponding to particular values of the labels m_i , where the dimension Δ is necessarily equal to the value $\Delta(S; m_1, \dots)$ (*consistency* conditions). In cases where we can choose whether Δ saturates its lower bound or not, the condition $\Delta = \Delta(S; m_1, \dots)$ is properly called a *typicality* condition. Similar conditions are well-known in the representation theory of finite-dimensional Lie superalgebras [16,17] where they distinguish between atypical (“unbalanced”) and typical (“balanced”) representations. In the context of physics it is convenient to call representations which saturate the lower bound $\Delta = \Delta(S; m_1, \dots)$ *massless* and the others, with $\Delta > \Delta(S; m_1, \dots)$, *massive*. The use of this terminology was motivated in a similar context in ref. [25] by the observation that, in the limit that Δ tends to $\Delta(S; m_1, \dots)$, a massive representation decomposes into a sum of a finite number of massless representations (a similar property holds in the present case).

In massive representations none of the supercharges in the set X^- annihilates the highest weight state, such that supersymmetry is broken. In the Ramond sector a massive representation has equal numbers of bosonic and fermionic states at each mass level. In massless representations at least one of the supersymmetries is unbroken (that is, at least one of the supercharges in the set X^- annihilates the highest weight state), such that a mismatch between the numbers of bosonic and fermionic states can be expected. In all cases that have been known until now, e.g., the conventional $N = 1, 2, 4$ superconformal algebras, this mismatch only occurs at the lowest mass level. We shall see that in the present case the mismatch is much more drastic: it occurs at all mass levels simultaneously. As a consequence, the supersymmetry index $\text{Tr}_R((-1)^{F_q} q^{L_0 - c/24})$ for massless representations in the Ramond sector is no longer a constant but a non-trivial power series in q .

3.2. NEVEU–SCHWARZ SECTOR

In this section we will first describe in some detail the situation for $N = 3$ and later on discuss the generalization to the other values for N .

For convenience we have listed the $N = 3$ algebra for our choice of basis in explicit form in the appendix. A highest weight state $|\phi_\Delta^{(m)}\rangle$ in the Neveu–Schwarz sector with dimension Δ and isospin m satisfies

$$L_0|\phi_\Delta^{(m)}\rangle = \Delta|\phi_\Delta^{(m)}\rangle, \quad H_0|\phi_\Delta^{(m)}\rangle = \frac{1}{4}m|\phi_\Delta^{(m)}\rangle. \quad (3.12)$$

By applying E_0^- we generate the ground-state representation which has dimension $(m+1)$. Since $m_0 = 2S - m^*$, the integrability condition reads

$$m = 1, 2, \dots, 2S. \quad (3.13)$$

The states of the second mass level ($L_0 = \Delta + \frac{1}{2}$) are obtained by applying $G_{-1/2}^+$, $G_{-1/2}^0$ and $G_{-1/2}^-$ to the lowest level states. This results in $3(m+1)$ states, which can be organized in three $\text{so}(3)$ -multiplets with highest weight states

$$\begin{aligned} |+\rangle &= G_{-1/2}^+|\phi_\Delta^{(m)}\rangle, \\ |0\rangle &= \left(\frac{2i}{m+2} E_0^- G_{-1/2}^+ + G_{-1/2}^0 \right) |\phi_\Delta^{(m)}\rangle, \\ |-\rangle &= \left(\frac{2}{m(m+1)} (E_0^-)^2 G_{-1/2}^+ - \frac{2i}{m} E_0^- G_{-1/2}^0 + G_{-1/2}^- \right) |\phi_\Delta^{(m)}\rangle, \end{aligned} \quad (3.14)$$

with isospins $m+2$, m and $m-2$, respectively (the last one only exists for $m \geq 2$ of course). The inner products of these states with themselves (they are mutually orthogonal) are

$$\begin{aligned} \langle + | + \rangle &= 2\Delta - \frac{1}{2}m, \\ \langle 0 | 0 \rangle &= \frac{m}{m+2} (2\Delta + 1) - \frac{m^2}{4S}, \\ \langle - | - \rangle &= \frac{m-1}{m+2} \left(2\Delta + \frac{1}{2}(m+2) \right) \quad (m \geq 2). \end{aligned} \quad (3.15)$$

We see that a necessary and sufficient condition for positivity of the form $\mathcal{H}$ up to this level is given by

$$\Delta \geq \Delta(S; m) = \frac{1}{4}m. \quad (3.16)$$

Note that the difference of the critical value $\Delta(S; m)$ with the lower bound

* The factor of 2 in front of S is related to the fact that $(\text{so}(3), \text{level } S)$ corresponds to $(\text{su}(2), \text{level } k = 2S)$ if the usual normalizations for the so and su series are adopted.

$\gamma((N-2)/N)c_R$ for Δ , which we mentioned in sect. 2, is

$$\Delta(S; m) - \frac{1}{3}\gamma c_m = \frac{m}{24S}((2S-m) + (4(S-\frac{1}{2}))) \geq 0. \quad (3.17)$$

Equality holds only for $m=0$ or $S=\frac{1}{2}$, $m=1$, i.e. for those cases where all the supercharges Q^i annihilate the highest weight state.

For $m=2S$ or $m=2S-1$ there is a consistency condition $\Delta = \Delta(S; m)$, since in those cases we have $m_0=0$ or 1, such that

$$(E_{-1}^+)^2 |\phi_\Delta^{(m)}\rangle = 0 \rightarrow G_{3/2}^-(E_{-1}^+)^2 |\phi_\Delta^{(m)}\rangle = 0 \rightarrow G_{-1/2}^+ |\phi_\Delta^{(m)}\rangle = 0 \rightarrow \Delta = \frac{1}{4}m.$$

For $m=0, 1, \dots, 2S-2$ the condition $\Delta = \Delta(m)$ is a typicality condition which determines whether supersymmetry is broken or not.

For the algebras based on $\mathfrak{so}(N)$ with $N > 3$ the analysis is similar. Let us treat the generic odd $\mathfrak{so}(2l+1)$ ($l \geq 3$) as an example. A highest weight state is now given by

$$|\phi_\Delta^{(m_1, m_2, \dots, m_l)}\rangle.$$

The integrability condition reads

$$m \in \mathbf{Z}_{\geq 0} \rightarrow S - m_1 - 2m_2 - \dots - 2m_{l-1} - m_l \in \mathbf{Z}_{\geq 0}. \quad (3.18)$$

In order to determine the function $\Delta(S; m_1, \dots, m_l)$ we consider the mass level $\Delta + \frac{1}{2}$. The states $|i\rangle = G_{-1/2}^i |\phi_\Delta^\lambda\rangle$ are the analogues of the state $|+\rangle$ for $N=3$. Only one of these, $|i=1\rangle = G_{-1/2}^1 |\phi_\Delta^\lambda\rangle$, is a $\mathfrak{so}(2l+1)$ highest weight state (it has Dynkin labels $(m_1+1, m_2, \dots, m_l)$). We expect (the motivation for this expectation is discussed below) that the form $\mathcal{H}$ is positive on the entire spectrum, once we make sure that the inner product of the state $|i=1\rangle$ with itself is positive. This inner product can be computed by using the explicit expressions for the matrices $t^A, \tilde{t}^A$ given in the appendix. The result is

$$\langle 1|1\rangle = 2\Delta - 2\gamma c_\lambda - 4\gamma(S-1)(2l-1)\langle \lambda, \mu_1 \rangle + 8\gamma(2l-1)^2(\langle \lambda, \mu_1 \rangle)^2, \quad (3.19)$$

where μ_1 is the first fundamental weight and c_λ is given by

$$c_\lambda = 2(2l-1)\langle \lambda, \lambda + 2\delta_p \rangle, \quad (3.20)$$

with $\delta_p = \frac{1}{2}\sum_{\alpha \in p}\alpha$. This gives the following lower bound for the dimension Δ of a highest weight state $|\phi_\Delta^\lambda\rangle$

$$\Delta(S; \lambda) = \gamma c_\lambda + 2\gamma(S-1)(2l-1)\langle \lambda, \mu_1 \rangle - 4\gamma(2l-1)^2(\langle \lambda, \mu_1 \rangle)^2. \quad (3.21)$$

TABLE 1

Results of the representation theory for the Neveu–Schwarz sector at level S . $\Delta(S; \lambda)$ is the lower bound for the conformal dimension Δ of a primary field with $\mathfrak{so}(N)$ highest weight λ . In the right column conditions on the Dynkin labels, that imply $\Delta = \Delta(S; \lambda)$, are listed.

$\mathfrak{so}(3)$	$\Delta(S; m) = \frac{1}{4}m$	$m = 2S$ or $m = 2S - 1$
$\mathfrak{so}(4)$	$\Delta(S; m_1, m_2)$ $= \frac{1}{4}\gamma(m_1 - m_2)^2 + \frac{1}{4}(m_1 + m_2)$	$m_1 = S$ or $m_2 = S$
$\mathfrak{so}(5)$	$\Delta(S; m_1, m_2)$ $= \frac{1}{4}\gamma(m_2^2 + 2m_2) + \frac{1}{4}(2m_1 + m_2)$	$m_1 + m_2 = S$
$\mathfrak{so}(2l)$	$\Delta(S; \lambda)$	
$l \geq 3$	$= \gamma c_\lambda + 4\gamma(S-1)(l-1)\langle\lambda, \mu_1\rangle$ $- 16\gamma(l-1)^2\langle\lambda, \mu_1\rangle^2$	$m_1 + 2m_2 + \dots$ $+ 2m_{l-2} + m_{l-1} + m_l = S$
$\mathfrak{so}(2l+1)$	$\Delta(S; \lambda)$	
$l \geq 2$	$= \gamma c_\lambda + 2\gamma(S-1)(2l-1)\langle\lambda, \mu_1\rangle$ $- 4\gamma(2l-1)^2\langle\lambda, \mu_1\rangle^2$	$m_1 + 2m_2 + \dots$ $+ 2m_{l-1} + m_l = S$

A consistency condition arises when $m_0 = 0$ since then

$$m_0 = 0 \rightarrow E_{-1}^\theta |\phi_\Delta^\lambda\rangle = 0 \rightarrow G_{1/2}^\bar E E_{-1}^\theta |\phi_\Delta^\lambda\rangle = 0 \rightarrow |i=1\rangle = 0 \rightarrow \Delta = \Delta(S; \lambda),$$

(θ is the highest root of $\mathfrak{so}(2l+1)$). We checked that a similar reasoning based on roots $\alpha \neq \theta$ does not give any conditions beyond those for $m_0 = 0$.

In table 1, the dimension formulas $\Delta(S; \lambda)$ and the consistency conditions for the Neveu–Schwarz sector are listed for all $\mathfrak{so}(N)$ -extended algebras, $N \geq 3$.

We conjecture that the representations obeying the conditions described above and listed in table 1 are consistent and unitary. This conjecture is based on explicit calculations for the lowest lying mass levels, some of which were presented above. Proving that the hermitian form $\mathcal{H}$ is actually positive and that our consistency conditions are exhaustive, will probably require the full machinery of a well-developed structure theory (compare with the proof for Kac–Moody algebra representations in ref [27]), which is far beyond the scope of the present paper. Instead we present some arguments in favor of the conjectured results.

A first argument is the close analogy of the present situation with the well-developed representation theory for affine Kac–Moody algebras [27] and finite-dimensional classical Lie superalgebras [16, 17] and with the representation theory of the $\mathfrak{su}(2)$ -extended $N = 4$ superconformal algebra, which has been developed recently [25, 26]. For example, our expectation, that the lower bound on the allowed dimensions Δ can be found by considering the inner product of a single well-chosen state with itself is completely analogous to the fact that the condition $m_0 \in \mathbf{Z}_{\geq 0}$ for affine Kac–Moody algebra representations can be found by considering the inner product of the state that is generated from the highest weight state by E_{-1}^θ with itself.

A second argument is the fact that, for the special case $S = 1$, the results are confirmed by the existence of a correspondence with free-field models. This will be demonstrated in sect. 4, where we will show that the $\text{so}(N)$ -extended superconformal algebras act in a natural way on the Hilbert space of certain free-field models. This observation will allow us to check our $S = 1$ results (both for the Neveu–Schwarz and the Ramond sector) and to extend them to new results such as explicit expressions for characters and supersymmetry indices.

3.3. RAMOND SECTOR

A highest weight state for the $N = 3$ algebra in the Ramond sector satisfies

$$\begin{aligned} X^+ |\phi_\Delta^{(m)}\rangle &= 0, & L_0 |\phi_\Delta^{(m)}\rangle &= \Delta |\phi_\Delta^{(m)}\rangle, \\ H_0 |\phi_\Delta^{(m)}\rangle &= \tfrac{1}{4}m |\phi_\Delta^{(m)}\rangle, & (-1)^F |\phi_\Delta^{(m)}\rangle &= \pm |\phi_\Delta^{(m)}\rangle, \end{aligned} \quad (3.22)$$

with the set X^+ defined as in eq. (3.2). The integrability condition is the same as in the Neveu–Schwarz sector

$$m = 0, 1, \dots, 2S.$$

The ground-state representation is generated from $|\phi_\Delta^{(m)}\rangle$ by applying E_0^- , G_0^0 and G_0^- . It decomposes into four $\text{so}(3)$ -multiplets with highest weight states

$$\begin{aligned} &|\phi_\Delta^{(m)}\rangle, \\ &|0\rangle = G_0^0 |\phi_\Delta^{(m)}\rangle, \\ &|- \rangle = G_0^- - \frac{2i}{m} E_0^- G_0^0 |\phi_\Delta^{(m)}\rangle, \\ &|0 - \rangle = G_0^- G_0^0 - \frac{2i}{m} \left(-\tfrac{1}{16}B + \Delta - \gamma \left(\tfrac{1}{2}m \right)^2 \right) E_0^- |\phi_\Delta^{(m)}\rangle, \end{aligned} \quad (3.23)$$

having isospin m , m , $m - 2$ and $m - 2$, respectively. The inner products of these highest weight states with themselves (they are all orthogonal) are

$$\begin{aligned} \langle \phi_\Delta^{(m)} | \phi_\Delta^{(m)} \rangle &= 1, \\ \langle 0 | 0 \rangle &= -\tfrac{1}{16}B + \Delta - \gamma \left(\tfrac{1}{2}m \right)^2, \\ \langle - | - \rangle &= \left(\frac{m-1}{m} \right) \left(2\Delta - \tfrac{1}{8}B \right), \\ \langle 0 - | 0 - \rangle &= \left(\frac{m-1}{m} \right) \left(-\tfrac{1}{16}B + \Delta - \gamma \left(\tfrac{1}{2}m \right)^2 \right) \left(2\Delta - \tfrac{1}{8}B \right). \end{aligned} \quad (3.24)$$

The condition on Δ , in order that all these inner products are positive, is clearly

$$\Delta \geq \Delta(S; m) = \frac{1}{16}B + \gamma\left(\frac{1}{2}m\right)^2. \quad (3.25)$$

Consistency conditions arise for $m = 0$ or $2S$

$$m = 0, 2S \rightarrow \Delta = \Delta(S; m).$$

For the other isospins m the condition $\Delta = \Delta(S; m)$ is a typicality condition which determines whether supersymmetry is broken or not. If supersymmetry is broken the groundstate has $2m$ bosonic and $2m$ fermionic states; the numbers of bosonic and fermionic states are balanced at all mass levels. If supersymmetry is unbroken the groundstate has $(m+1)$ bosonic and $(m-1)$ fermionic states. In sect. 4, we will determine the supersymmetry indices for the massless representations in the Ramond sector for $S = 1$ (having $m = 0, 1$ or 2). It will turn out that in those cases the balance between the numbers of bosonic and fermionic states is broken at all mass levels simultaneously.

The Ramond sectors of the other ($N > 3$) $\text{so}(N)$ -extended superconformal algebras are treated similarly; we present the case of generic odd $\text{so}(2l+1)$, ($l \geq 3$) as an example. The integrability condition for the labels (m_i) for a highest weight state $|\phi_{\Delta}^{(m_1, m_2, \dots, m_l)}\rangle$ is the same as in the Neveu–Schwarz sector (see eq. (3.18)). There are now $(l+1)$ supercharges $G_0^i, G_0^{\bar{i}}, \bar{i} = 1, 2, \dots, l$, acting on the highest weight state. The lower bound on Δ can be bound by considering the state $|0\rangle = G_0^0|\phi_{\Delta}^{\lambda}\rangle$. From

$$\begin{aligned} 0 &\leq \langle 0|0\rangle \\ &= -\frac{1}{16}B + \Delta - \gamma c_{\lambda} + 4\gamma(2l-1)\langle \lambda, \mu_l \rangle, \end{aligned} \quad (3.26)$$

TABLE 2

Results of the representation theory for the Ramond sector at level S . $\Delta(S; \lambda)$ is the lower bound for the conformal dimension Δ of a primary field with $\text{so}(N)$ highest weight λ . In the right column conditions on the Dynkin labels, that imply $\Delta = \Delta(S; \lambda)$, are listed.

$\text{so}(3)$	$\Delta(S; m) = \frac{1}{16}B + \frac{1}{4}\gamma m^2$	$m = 0$ or $m = 2S$
$\text{so}(4)$	$\Delta(S; m_1, m_2) = \frac{1}{16}B + \frac{1}{4}\gamma(m_1 + m_2)^2 + \frac{1}{2}\gamma(m_1 + m_2)$	$m_1 = 0$ or $m_2 = 0$
$\text{so}(5)$	$\Delta(S; m_1, m_2) = \frac{1}{16}B + \gamma(m_1^2 + m_1 m_2 + \frac{1}{2}m_2^2 + 2m_1 + m_2)$	$m_2 = 0$
$\text{so}(2l)$	$\Delta(S; \lambda) = \frac{1}{16}B + \gamma c_{\lambda} - 8\gamma(l-1)\langle \lambda, \mu_l - \frac{1}{2}\lambda_l \rangle - 16\gamma(l-1)^2(\langle \lambda, \lambda_l \rangle)^2$	$m_{l-1} = 0$ or $m_l = 0$
$l \geq 3$		
$\text{so}(2l+1)$	$\Delta(S; \lambda) = \frac{1}{16}B + \gamma c_{\lambda} - 4\gamma(2l-1)\langle \lambda, \mu_l \rangle$	$m_l = 0$
$l \geq 2$		

we deduce

$$\Delta \geq \Delta(S; \lambda) = \frac{1}{16}B + \gamma c_\lambda - 4\gamma(2l-1)\langle \lambda, \mu_l \rangle. \quad (3.27)$$

Consistency conditions arise when $m_l = 0$.

For the even $\mathfrak{so}(2l)$ algebras, $l \geq 2$, the charge G_0^0 is absent; the bound is here determined through the state $|\bar{l}\rangle = G_0^j |\phi_\Delta^\lambda\rangle$ by the requirement $0 \leq \langle \bar{l} | \bar{l} \rangle$. Consistency conditions arise when $m_{l-1} = 0$ or $m_l = 0$. In table 2 the results for the Ramond sector for all $N \geq 3$ are listed.

We refer to subsect. 3.2 for a discussion on the status of the results collected in tables 1 and 2.

4. Theories at level 1

4.1. CORRESPONDENCE WITH FREE-FIELD MODELS

In this section we focus on the case $S = 1$. This is the lowest level, corresponding to the smallest possible value of the central charge c of the Virasoro algebra, where we expect unitary representations to exist*. For $S = 1$ the central charge takes the value

$$c_N = \frac{1}{2}N + 1,$$

which suggests an interpretation of $S = 1$ models in terms of N free fermions and a single boson. Indeed, starting from a set of N Majorana fermions $\chi^i(z)$, $i = 1, 2, \dots, N$, and a real boson $\varphi(z)$, satisfying

$$\begin{aligned} \chi^i(z) \chi^j(w) &= \frac{\delta^{ij}}{(z-w)} + \dots, \\ \delta\varphi(z) \partial\varphi(w) &= \frac{-1}{(z-w)^2} + \dots, \end{aligned} \quad (4.1)$$

we can construct the following currents

$$\begin{aligned} T(z) &= -\frac{1}{2}(\chi^i \partial\chi^i) - \frac{1}{2}(\partial\varphi)^2, \\ G^i(z) &= i\chi^i \partial\varphi, \\ J^a(z) &= \frac{1}{2}\lambda_{ij}^a \chi^i \chi^j, \end{aligned} \quad (4.2)$$

* For $N = 3$ there is the possibility to have $S = \frac{1}{2}$, $c = 1$. It can be shown that in this case the supercurrents G^i , $i = 1, 2, 3$, all act trivially. As a consequence, the symmetry algebra reduces to $\mathfrak{su}(2)_{k=1}$ and the stress-energy tensor has the Sugawara form. The unique model realizing this symmetry has two primary fields of dimension $0, \frac{1}{4}$, respectively.

which satisfy the operator product algebra (2.1) with $S = 1$, $\gamma = 1/2(N - 2)$, $K = 1$, $B = 1$ and $c = \frac{1}{2}N + 1$. In establishing this result we used the identity

$$\Pi_{ij}^{ab}(J^a J^b)(z) = (N - 2)(\partial \chi^i \chi^j + \partial \chi^j \chi^i + 2\delta^{ij} \chi^c \partial \chi^c)(z), \quad (4.3)$$

which can, for example, be checked by using the lemmas for rearranging normal-ordered expressions given in appendix A of the first paper of ref. [8].

The full N -extended superconformal algebra is the product of two copies of the chiral algebra (2.5), which correspond to the coordinates $z, \bar{z}$, respectively. In the Ramond sector of these algebras we included the extra generators $(-1)^F$ and $(-1)^{\bar{F}}$. In the free-field Hilbert space they can be represented by

$$(-1)^F = \mathcal{P}(-1)^{N_B}, \quad (-1)^{\bar{F}} = \bar{\mathcal{P}}(-1)^{N_{\bar{B}}}, \quad (4.4)$$

where $\mathcal{P}$ and $\bar{\mathcal{P}}$ are operators that change the sign of the bosonic momenta $p, \bar{p}$, respectively,

$$\mathcal{P}: p \leftrightarrow -p, \quad \bar{\mathcal{P}}: \bar{p} \leftrightarrow -\bar{p},$$

and N_B and $N_{\bar{B}}$ are the number operators for the bosonic oscillators $\alpha_n, \bar{\alpha}_n$, $n \neq 0$.

We can conclude from the above that the Hilbert space of a free-field theory with one boson and N $\text{so}(N)$ -symmetric fermion decomposes as a sum of representations of the full $\text{so}(N)$ -extended superconformal algebra. In subsect. 9.2 make this correspondence more explicit by rewriting the partition functions of several free-field models at $c = \frac{1}{2}N + 1$ in terms of characters of representations at $S = 1$ of the $\text{so}(N)$ -extended superconformal algebras. It is instructive to work out the relation with free-field models in some detail, since it provides independent evidence for some of the conjectured results of sect. 3; in particular, it shows that, at level $S = 1$, the dimension formulas, the consistency conditions and the claim that the hermitian form $\mathcal{H}$ is positive, are correct. Furthermore, it provides an illustrative example of how the modular group acts on characters and indices of representations of the $\text{so}(N)$ -extended algebras and how massless and massive representations are combined in the Hilbert space of a physical model. These results can be helpful as a guiding example for the treatment of the levels $S > 1$, where the correspondence with free-field models does not exist.

4.2. DETERMINATION OF CHARACTERS AND INDICES

The results we find when we apply the representation theory of sect. 3 to the representations at level $S = 1$, are listed in table 3. We introduce the following notations for the various characters that we will need as building blocks for the partition functions of theories defined on a torus with modular parameter $q = e^{2\pi i \tau}$. Depending on the choice for the boundary conditions along the two independent

TABLE 3

Results for the conformal dimension Δ for the representations occurring at level $S = 1$. Only the singlet in the Neveu–Schwarz sector and the spinor in the Ramond sector can be massive, all the other representations are necessarily massless, i.e. have supersymmetry unbroken.

	NS	R
Singlet	$\Delta \geq 0$	$\Delta = \frac{1}{16}$
spinor	$\Delta = \frac{1}{16}(N + 1)$	$\Delta \geq \frac{1}{16}N$
vector	$\Delta = \frac{1}{2}$	$\Delta = \frac{9}{16}$

cycles around the torus, we have four characters

$$\begin{aligned}
 \chi^{\text{NS}} &= \chi_{(-, -)} = \text{Tr}_{\text{NS}}(\mathcal{T}), \\
 \tilde{\chi}^{\text{NS}} &= \chi_{(+, -)} = \text{Tr}_{\text{NS}}(\mathcal{T}(-1)^F), \\
 \chi^{\text{R}} &= \chi_{(-, +)} = \text{Tr}_{\text{R}}(\mathcal{T}), \\
 \tilde{\chi}^{\text{R}} &= \chi_{(+, +)} = \text{Tr}_{\text{R}}(\mathcal{T}(-1)^F),
 \end{aligned} \tag{4.5}$$

where $\mathcal{T} = q^{(-c/24 + L_0)}$. The $\text{so}(N)$ -representations that are possible at $S = 1$ will be denoted by subscripts (i, s, c, v) for $\text{so}(2l)$ or (i, s, v) for $\text{so}(2l + 1)$. Massive characters have an extra parameter Δ and carry a $\cdot$ to distinguish them from their massless counterparts.

We have not attempted to compute in a systematic way the characters of arbitrary representations. (Note that for the $\text{su}(2)$ -extended $N = 4$ superconformal algebra, which is rather close to our $N = 3$ algebra, all characters and some of their properties have been determined recently [25].) However, for $S = 1$ the correspondence with free-field models comes to our help and we can quite easily determine all characters explicitly.

We start from the observation of subsect. 4.1 that the Hilbert space of a free-field theory with one real boson and N Majorana fermions, where the fermions have boundary conditions that respect the $\text{so}(N)$ symmetry, can be decomposed as a sum of representations of the full N -extended superconformal algebra, which is the product of two copies of the chiral algebra (2.5). As a consequence, the partition function of such a theory is of the form $Z = \sum_{l, \bar{l}} \chi_l \bar{\chi}_{\bar{l}}$, where $l, \bar{l}$ denote the various labels for $S = 1$ representations that were indicated above. As an example, we consider the partition function describing the tensor product of a bosonic field

compactified on a circle of radius R and N $\text{so}(N)$ -symmetric fermions, which is

$$Z = \frac{1}{2} |\eta|^{-2} \sum_{(p, \bar{p}) \in \Gamma_R} q^{p^2/2} \bar{q}^{\bar{p}^2/2} \left(\left| \frac{\vartheta_3}{\eta} \right|^N + \left| \frac{\vartheta_4}{\eta} \right|^N + \left| \frac{\vartheta_2}{\eta} \right|^N \right), \quad (4.6)$$

where

$$\begin{aligned} \eta(q) &= q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \\ \sqrt{\frac{\vartheta_2(q)}{\eta(q)}} &= \sqrt{2} q^{1/24} \prod_{n=1}^{\infty} (1 + q^n), \\ \sqrt{\frac{\vartheta_3(q)}{\eta(q)}} &= q^{-1/48} \prod_{n=1}^{\infty} (1 + q^{n-1/2}), \\ \sqrt{\frac{\vartheta_4(q)}{\eta(q)}} &= q^{-1/48} \prod_{n=1}^{\infty} (1 - q^{n-1/2}), \end{aligned} \quad (4.7)$$

and

$$\Gamma_R = \left\{ (p, \bar{p}) = \left(\frac{n}{R} + \frac{1}{2}mR, \frac{n}{R} - \frac{1}{2}mR \right), n, m \in \mathbf{Z} \right\}. \quad (4.8)$$

It is easily seen that, for generic values of R , the $(p, \bar{p}) = (0, 0)$ part of this sum is itself an expression of the form $\sum_i \chi_i \bar{\chi}_i$. This observation, together with an explicit inspection of the lowest lying mass levels, suggests that the characters are given by the following expressions

$$\begin{aligned} \chi_i^{\text{NS}} + \chi_v^{\text{NS}} &= q^{-(N+2)/48} \frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})^N}{\prod_{n=1}^{\infty} (1 - q^n)}, \\ \tilde{\chi}_i^{\text{NS}} + \tilde{\chi}_v^{\text{NS}} &= q^{-(N+2)/48} \frac{\prod_{n=1}^{\infty} (1 - q^{n-1/2})^N}{\prod_{n=1}^{\infty} (1 - q^n)}, \\ \chi_{s,c}^{\text{R}} &= 2^{[(N-1)/2]} q^{(N-1)/24} \frac{\prod_{n=1}^{\infty} (1 + q^n)^N}{\prod_{n=1}^{\infty} (1 - q^n)}. \end{aligned} \quad (4.9)$$

The $(p, \bar{p}) = (0, 0)$ terms in Z are then written as

$$\frac{1}{2} |\chi_i^{\text{NS}} + \chi_v^{\text{NS}}|^2 + \frac{1}{2} |\tilde{\chi}_i^{\text{NS}} + \tilde{\chi}_v^{\text{NS}}|^2 + |\chi_s^{\text{R}}|^2 + (s \rightarrow c). \quad (4.10)$$

Here, and in the following, it is understood that the $(s \rightarrow c)$ terms are only present for even values of N .

For generic values of R the states in the Hilbert space described by the other terms in eq. (4.6) correspond to massive representations of the N -extended superconformal algebra. The part of Z corresponding to the Neveu–Schwarz sector is

$$\frac{1}{2} \sum_{\Gamma_R} \chi_i^{\text{NS}, \cdot} \left(\frac{1}{2} p^2 \right) \bar{\chi}_i^{\text{NS}, \cdot} \left(\frac{1}{2} \bar{p}^2 \right) + (\chi \rightarrow \tilde{\chi}), \quad (4.11)$$

where

$$\chi_i^{\text{NS}, \cdot}(\Delta) = q^{\Delta - (N+2)/48} \frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})^N}{\prod_{n=1}^{\infty} (1 - q^n)}. \quad (4.12)$$

The contributions to Z of the Ramond sector can be written in the following form

$$\left(\frac{1}{2} \sum_{\Gamma_R^*} \chi_s^{\text{R}, \cdot} \left(\frac{3}{16} + \frac{1}{2} p^2 \right) \bar{\chi}_s^{\text{R}, \cdot} \left(\frac{3}{16} + \frac{1}{2} \bar{p}^2 \right) + (s \rightarrow c) \right) + (\chi \rightarrow \tilde{\chi}), \quad (4.13)$$

where

$$\Gamma_R^* = \left\{ (p, \bar{p}) = \left(\frac{n}{R} + \frac{1}{2} mR, \frac{n}{R} - \frac{1}{2} mR \right), n > 0, \quad m \in \mathbf{Z} \quad \text{or} \quad n = 0, \quad m > 0 \right\},$$

contains only one of each pair $(p, \bar{p}), (-p, -\bar{p})$ (note that states with opposite bosonic moments $p, -p$ occur in the same irreducible representation of the (extended) chiral Ramond algebra). We included the $(\chi \rightarrow \tilde{\chi})$ terms, which are identically zero for massive representations, to indicate the presence of a projection on states with $(-1)^{F+\bar{F}} = 1$ (note that this same projection is there in the Neveu–Schwarz sector). The character formula for a massive spinor representation in the Ramond sector is

$$\chi_{s,c}^{\text{R}, \cdot}(\Delta) = 2^{[(N+1)/2]} q^{\Delta - (N+2)/48} \frac{\prod_{n=1}^{\infty} (1 + q^n)^N}{\prod_{n=1}^{\infty} (1 - q^n)}. \quad (4.14)$$

We note the following “decomposition formulas” for massive characters in the massless limit, where the conformal dimension Δ approaches the value $\Delta(S; \lambda)$

$$\begin{aligned} \chi_i^{\text{NS}, \cdot}(\Delta) &\xrightarrow{\Delta \downarrow 0} \chi_i^{\text{NS}} + \chi_v^{\text{NS}}, \\ \chi_{s,c}^{\text{R}, \cdot}(\Delta) &\xrightarrow{\Delta \downarrow N/16} 2\chi_{s,c}^{\text{R}}. \end{aligned} \quad (4.15)$$

In the above model the projection on $(-1)^{F+\bar{F}} = 1$ is not realized in the massless part of the Ramond sector of the Hilbert space. Furthermore, not all representations that are allowed at level $S = 1$ do actually occur. In this sense the partition function (4.6) should be viewed as a so-called “exceptional” modular invariant of the N -extended superconformal algebra.

It is not hard to find out what the characters are for the representations that do not occur in the partition function (4.6). Let us first consider the $\tilde{\chi}^R$ characters, which are supersymmetry indices for the massless representations in the Ramond sector. It follows from the expression (4.4) for $(-1)^F$ that the character $\tilde{\chi}_{s,c}^R$ can be obtained from $\chi_{s,c}^R$ by flipping the sign of the contributions of the bosonic oscillators in the free-field representation

$$\tilde{\chi}_{s,c} = \pm 2^{[(N-1)/2]} q^{(N-1)/24} \prod_{n=1}^{\infty} (1 + q^n)^{N-1}. \quad (4.16)$$

In conventional superconformal theories the indices $\tilde{\chi}^R$ are constant and thus modular invariant, but the character (4.16) is in a three-dimensional representation of the modular group. The other elements of this representation can be identified as the indices of the other massless representations in the Ramond sector at level 1

$$\begin{aligned} \tilde{\chi}_i^R - \tilde{\chi}_v^R &= \pm q^{-(N-1)/48} \prod_{n=1}^{\infty} (1 - q^{n-1/2})^{N-1}, \\ \tilde{\chi}_i^R + \tilde{\chi}_v^R &= \pm q^{-(N-1)/48} \prod_{n=1}^{\infty} (1 + q^{n-1/2})^{N-1}. \end{aligned} \quad (4.17)$$

These indices can be combined into a modular-invariant combination

$$Z_2 = |\tilde{\chi}_i^R + \tilde{\chi}_i^R|^2 + |\tilde{\chi}_i^R - \tilde{\chi}_v^R|^2 + |\tilde{\chi}_s^R|^2 + (s \rightarrow c). \quad (4.18)$$

The remaining massless characters are in a six-dimensional representation of the modular group. We list their expressions

$$\begin{aligned} \chi_i^R + \chi_v^R &= q^{-(N-1)/48} \frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})^N}{\prod_{n=1}^{\infty} (1 - q^{n-1/2})}, \\ \chi_i^R - \chi_v^R &= q^{-(N-1)/48} \frac{\prod_{n=1}^{\infty} (1 - q^{n-1/2})^N}{\prod_{n=1}^{\infty} (1 + q^{n-1/2})}, \\ \tilde{\chi}_{s,c}^{NS} &= 2^{[(N-1)/2]} q^{(2N+1)/48} \frac{\prod_{n=1}^{\infty} (1 + q^n)^N}{\prod_{n=1}^{\infty} (1 + q^{n-1/2})}, \\ \chi_{s,c}^{NS} &= 2^{[(N-1)/2]} q^{(2N+1)/48} \frac{\prod_{n=1}^{\infty} (1 + q^n)^N}{\prod_{n=1}^{\infty} (1 - q^{n-1/2})}, \\ \chi_i^{NS} - \chi_v^{NS} &= q^{-(N+2)/48} \frac{\prod_{n=1}^{\infty} (1 - q^{n-1/2})^N}{\prod_{n=1}^{\infty} (1 + q^n)}, \\ \tilde{\chi}_i^{NS} - \tilde{\chi}_v^{NS} &= q^{-(N+2)/48} \frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})^N}{\prod_{n=1}^{\infty} (1 + q^n)}. \end{aligned} \quad (4.19)$$

The modular invariant that can be formed from these characters reads

$$\begin{aligned} Z_3 = & \frac{1}{2} |\chi_i^{\text{NS}} - \chi_v^{\text{NS}}|^2 + \frac{1}{2} |\tilde{\chi}_i^{\text{NS}} - \tilde{\chi}_v^{\text{NS}}|^2 + 2 |\chi_s^{\text{NS}}|^2 + 2 |\tilde{\chi}_s^{\text{NS}}|^2 + (s \rightarrow c) \\ & + |\chi_i^{\text{R}} - \chi_v^{\text{R}}|^2 + |\chi_i^{\text{R}} + \chi_v^{\text{R}}|^2. \end{aligned} \quad (4.20)$$

This completes the list of characters of all possible representations at level $S = 1$.

We should emphasize that our derivations of the above results are not rigorous; in principle there is room for extra massive characters on the l.h.s. of some of the character formulas. For example, from our arguments we cannot exclude the possibility that there are extra terms $\chi_i^{\text{NS}, \cdot}(\Delta)$, where Δ is some large positive integer, on the l.h.s. of the first line of eq. (4.9). Although we have no proof for this, we think it highly unlikely that such terms will actually occur; for $N = 3$ we checked that our expressions are correct for all levels up to q^3 . As an aside, we note that the expressions (4.12) and (4.14) for the massive characters at level 1 are similar to the expressions for the level-1 massive characters of the $\text{su}(2)$ -extended $N = 4$ superconformal algebra presented in ref. [25], which is another indication that they are probably correct.

Having found all characters, we can now easily recognize other free-field models with N -extended superconformal symmetry. A model which is very natural from the point of view of the superconformal structure is the model where the projection $(-1)^{F+\bar{F}} = 1$ is realized for all states. Its partition function is given by

$$\begin{aligned} Z = & \frac{1}{2} (Z(R) + Z_2 + Z_3) \\ = & \frac{1}{2} (\chi_i^{\text{NS}} \bar{\chi}_i^{\text{NS}} + \chi_s^{\text{R}} \bar{\chi}_s^{\text{R}} + (s \rightarrow c) + \chi_v^{\text{NS}} \bar{\chi}_v^{\text{NS}}) + (\chi \rightarrow \tilde{\chi}) \\ & + (\chi_i^{\text{R}} \bar{\chi}_i^{\text{R}} + \chi_s^{\text{NS}} \bar{\chi}_s^{\text{NS}} + (s \rightarrow c) + \chi_v^{\text{R}} \bar{\chi}_v^{\text{R}}) + (\chi \rightarrow \tilde{\chi}) \\ & + \left(\frac{1}{2} \sum_{\Gamma_{\text{R}}} \chi_i^{\text{NS}, \cdot} \left(\frac{1}{2} p^2 \right) \bar{\chi}_i^{\text{NS}, \cdot} \left(\frac{1}{2} \bar{p}^2 \right) \right) + (\chi \rightarrow \tilde{\chi}) \\ & + \left(\frac{1}{2} \sum_{\Gamma_{\text{R}}^*} \chi_s^{\text{R}, \cdot} \left(\frac{3}{16} + \frac{1}{2} p^2 \right) \bar{\chi}_s^{\text{R}, \cdot} \left(\frac{3}{16} + \frac{1}{2} \bar{p}^2 \right) + (s \rightarrow c) \right) + (\chi \rightarrow \tilde{\chi}). \end{aligned} \quad (4.21)$$

Note that the representations $(s, c)_{\text{NS}}$, $(i)_{\text{R}}$ and $(v)_{\text{R}}$ occur with multiplicity two in eq. (4.21).

It can be checked that Z decomposes as (we use the notations of ref. [2])

$$\frac{1}{2}(\Gamma(R) + Z_{0,1/2} + Z_{1/2,0} + Z_{1/2,1/2}) \frac{1}{2} \left(\left| \frac{\partial_3}{\eta} \right|^N + \left| \frac{\partial_4}{\eta} \right|^N + \left| \frac{\partial_2}{\eta} \right|^N \right). \quad (4.22)$$

It describes the tensor product of a $c=1$ Z_2 orbifold model of radius R with $\mathfrak{so}(N)$ -symmetric fermions.

By applying continuous deformations ($R \rightarrow R + \delta R$), or modding out discrete symmetries, we can construct many more free-field models with N -extended superconformal symmetry. For example, the partition function

$$Z = \frac{1}{2}(Z(R) - Z_2 + Z_3), \quad (4.23)$$

describes another “spin-model”, having $(-1)^{F+\bar{F}} = -1$ in the Ramond sector, which does not factorize as a simple tensor product (boson) $\otimes$ (fermions). We expect that the moduli space of all N -extended superconformal models at $S=1$, $c = \frac{1}{2}N + 1$ is, qualitatively, very similar to the $\hat{c}=1$ picture for ordinary superconformal theories as described in ref. [3], i.e. that it consists of one-parameter families of theories, interconnected at multicritical points, that occur for special values of the parameter R .

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Appendix

Before we specify our choice of basis for the matrices of the adjoint representation of $\mathfrak{so}(N)$ we recall some of the structure theory. We adopt the conventions of ref. [28], pp. 297–305.

We use a basis $\{e^I\}$ for the vector representation of $\mathfrak{so}(N)$ which is given by

$$e^I = \begin{cases} e^i, e^{\bar{i}}, & \mathfrak{so}(2l), \\ e^0, e^i, e^{\bar{i}}, & \mathfrak{so}(2l+1), \end{cases}$$

where $i, \bar{i} = 1, 2, \dots, l$. The Cartan subalgebra (CSA) $\mathfrak{h}$ is spanned by the matrices

$$E_{ii} - E_{\bar{i}\bar{i}}, \quad i = 1, 2, \dots, l,$$

for $\mathfrak{so}(2l), \mathfrak{so}(2l+1)$. The linear functions λ_r , $r = 1, 2, \dots, l$, defined by

$$\lambda_r(E_{ii} - E_{\bar{i}\bar{i}}) = \delta_{ri},$$

form a basis for the dual CSA $\mathfrak{h}^*$. The root systems are given by simple roots S

$$\mathfrak{so}(2l): \quad \alpha_i = \lambda_i - \lambda_{i+1}, \quad i \neq l,$$

$$\alpha_l = \lambda_l,$$

$$\mathfrak{so}(2l+1): \quad \alpha_i = \lambda_i - \lambda_{i+1}, \quad i \neq l,$$

$$\alpha_l = \lambda_{l-1} + \lambda_l,$$

positive roots P

$$\mathfrak{so}(2l): \quad \lambda_i - \lambda_j, \lambda_p - \lambda_q,$$

$$\mathfrak{so}(2l+1): \quad \lambda_i - \lambda_j, \lambda_p - \lambda_q, \lambda_r,$$

where $1 \leq i < j \leq l$, $1 \leq p < q \leq l$, $1 \leq r \leq l$,

highest root θ

$$\mathfrak{so}(2l), \mathfrak{so}(2l+1): \quad \theta = \lambda_1 + \lambda_2.$$

The Cartan–Killing form $\langle, \rangle$ on the dual CSA $\mathfrak{h}^*$ is given as

$$\mathfrak{so}(2l): \quad \langle \lambda_i, \lambda_j \rangle = \frac{\delta^{ij}}{4(l-1)},$$

$$\mathfrak{so}(2l+1): \quad \langle \lambda_i, \lambda_j \rangle = \frac{\delta^{ij}}{2(2l-1)}.$$

(Note that in these conventions the long roots for $\mathfrak{so}(2l), \mathfrak{so}(2l+1)$ have length squared $1/2(l-1), 1/2l-1$ respectively.) The fundamental weights μ_i are defined through

$$\langle \mu_i, \alpha_j \rangle = \frac{1}{2} \langle \alpha_j, \alpha_j \rangle \delta_{ij}.$$

To the elements $\alpha_i \in \mathfrak{h}^*$, $\mu_i \in \mathfrak{h}^*$, we can associate elements H_{α_i}, H_{μ_i} of the CSA through the identification

$$\alpha_i(H) = \langle H_{\alpha_i}, H \rangle,$$

$$\mu_i(H) = \langle H_{\mu_i}, H \rangle, \quad \forall H \in \mathfrak{h}.$$

We now define a basis $t^A = t^i, t^\alpha, t^{-\alpha}$ and a dual basis $\tilde{t}^A = \tilde{t}^i, t^{-\alpha}, t^\alpha$ for the matrices of the adjoint representation of $\mathfrak{so}(N)$, such that

$$\mathrm{tr}(\tilde{t}^A t^B) = -2\delta^{AB}.$$

$\mathfrak{so}(2l)$:

$$t^i = H_{\alpha_i},$$

$$\tilde{t}^i = -16(l-1)^2 H_{\mu_i},$$

$$\alpha = \lambda_i - \lambda_j: \quad t^\alpha = E_{ij} - E_{\bar{j}\bar{i}},$$

$$t^{-\alpha} = -E_{ji} + E_{\bar{i}\bar{j}},$$

$$\alpha = \lambda_p - \lambda_q: \quad t^\alpha = E_{p\bar{q}} - E_{q\bar{p}},$$

$$t^{-\alpha} = E_{\bar{p}q} - E_{\bar{q}p},$$

$\mathfrak{so}(2l+1)$:

$$t^i = H_{\alpha_i},$$

$$i \neq l: \quad \tilde{t}^i = -4(2l-1)^2 H_{\mu_i},$$

$$\tilde{t}^l = -8(2l-1)^2 H_{\mu_l},$$

$$\alpha = \lambda_i - \lambda_j: \quad t^\alpha = E_{ij} - E_{\bar{j}\bar{i}},$$

$$t^{-\alpha} = -E_{ji} + E_{\bar{i}\bar{j}},$$

$$\alpha = \lambda_p - \lambda_q: \quad t^\alpha = E_{p\bar{q}} - E_{q\bar{p}},$$

$$t^{-\alpha} = E_{\bar{p}q} - E_{\bar{q}p},$$

$$\alpha = \lambda_r: \quad t^\alpha = E_{0\bar{r}} - E_{r0},$$

$$t^{-\alpha} = E_{0r} - E_{\bar{r}0}.$$

With respect to the bases e^I, t_{IJ}^A we define current components $G^i(z), G^{\bar{i}}(z), [G^0(z),]H^i(z), E^{\pm\alpha}(z)$

$$\sum_{i=1}^N G^i(z) e^i = \sum_{i=1}^l G^i(z) e^{\hat{i}} + \sum_{\bar{i}=1}^l G^{\bar{i}}(z) e^{\bar{i}} [+ G^0(z) e^0],$$

$$\sum_a J^a(z) t^a = \sum_i H^i(z) \tilde{t}^i + \sum_{\alpha \in \mathbb{P}} (E^\alpha(z) t^{-\alpha} + E^{-\alpha}(z) t^\alpha).$$

The commutator algebra (2.5) is covariant under this change of bases, provided we write the anticommutator $\{G_r^i, G_s^j\}$ in the covariant form $\{(G_{-r}^i)^\dagger, G_s^j\}$.

As an example, we list the non-trivial (anti-)commutators of the $N = 3$ algebra in explicit form (for this case we slightly adapt the notation: $G^{i=1} \rightarrow G^+$, $G^{i=-1} \rightarrow G^-$, $H^{i=1} \rightarrow H$, $E^{\pm \alpha_1} \rightarrow E^\pm$)

$$\begin{aligned}
 [H_m, H_n] &= \frac{1}{4} S m \delta_{m+n}, \\
 [H_m, E_n^\pm] &= \pm \frac{1}{2} E_{m+n}^\pm, \\
 [E_m^\pm, E_n^\mp] &= -S m \delta_{m+n} \mp 2 H_{m+n}, \\
 [H_m, G_n^\pm] &= \pm \frac{1}{2} G_{m+n}^\pm, \\
 [E_m^\pm, G_n^0] &= \pm i G_{m+n}^\pm, \\
 [E_m^\pm, G_n^\mp] &= \mp i G_{m+n}^0, \\
 \{G_r^+, G_s^-\} &= \frac{1}{2} B \left(r^2 - \frac{1}{4} \right) \delta_{r+s} + 2 L_{r+s} + K(r-s) H_{r+s} \\
 &\quad + \gamma (E^+ E^- + E^- E^+)_{r+s}, \\
 \{G_r^\pm, G_s^\pm\} &= 2 \gamma (E^\pm E^\pm)_{r+s}, \\
 \{G_r^\pm, G_s^0\} &= \mp \frac{1}{2} i K(r-s) E^\pm - 2 i \gamma (E^\pm H + H E^\pm)_{r+s}, \\
 \{G_r^0, G_s^0\} &= \frac{1}{2} B \left(r^2 - \frac{1}{4} \right) \delta_{r+s} + 2 L_{r+s} - 8 \gamma (H H)_{r+s}.
 \end{aligned}$$

For general N the algebra has a similar form; the troublesome “IIJJ” term in the $\{G, G\}$ anticommutator is expressed as a sum of terms of the form $H^i H^j$, $E^\pm \alpha H^i$ and $E^\pm \alpha E^\pm \beta$, which can easily be evaluated in a given highest weight module.

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