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## $O(N)$ -EXTENDED SUPERCONFORMAL FIELD THEORY IN SUPERSPACE

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We discuss the superspace formalism for the description of  $O(N)$ -extended superconformal quantum field theory in two dimensions. Such theories exist for  $N \leq 4$ . In  $O(4)$ -extended superconformal transformations two independent anomalous terms can be present. Under the assumption that in physical models a specific combination of these anomalies is absent, we show that the allowed values of the  $N = 4$  Virasoro central charge  $c$  are  $c(n_+, n_-) = 6n_+n_-/(n_+ + n_-)$ .

### 1. Introduction and summary

Already in 1976 Ademollo et al. [1] have indicated that the conformal (Virasoro) and the  $N = 1$  superconformal (Neveu-Schwarz-Ramond) algebras can be generalized to  $N$ -extended superconformal algebras with arbitrary  $N$ . They presented the following examples:

- (i) a regular series of  $N$ -extended superconformal algebras having an  $\mathfrak{o}(N)$  Kac-Moody algebra as a subalgebra;
- (ii) an exceptional  $N = 4$  superconformal algebra having a  $\mathfrak{su}(2)$  Kac-Moody subalgebra.

Systematic searches [2, 3] for other superconformal algebras that are Lie superalgebras with generators of conformal dimension  $\Delta$  in the range  $\frac{1}{2} \leq \Delta \leq 2$  did not lead to interesting candidates apart from those listed above. The  $\mathfrak{o}(N)$ -extended superconformal algebras do not admit central extensions for  $N \geq 5$  [4–7]. Since in any quantum field theory anomalous terms in the transformation laws, corresponding to central terms in the current algebra, occur, this restricts the algebras that are of physical interest to the  $\mathfrak{o}(N)$ -extended algebras with  $N \leq 4$  and the exceptional  $\mathfrak{su}(2)$ -extended  $N = 4$  algebra.

Quantum field theories with  $O(N)$ -extended superconformal invariance are naturally formulated in the context of a  $N$ -extended superspace with coordinates  $(z, \theta^i; \bar{z}, \bar{\theta}^i)$ . In such a formulation superconformal transformations correspond to a special class of (anti)analytic reparametrizations of superspace. In this paper we study the superspace formalism for  $O(N)$ -extended superconformal field theory. We

derive a complete list of all possible anomalous terms in the transformation rule of the current superfield  $J^{(N)}(Z)$ . Such terms exist for  $N \leq 4$ ;  $N = 4$  is of special interest since in that case two independent anomalous terms turn out to be possible. Mathematically, these terms provide generalizations of the schwarzian derivative [8] in  $N$ -extended superspace. We examine the superconformal transformations that are singled out by the requirement that they are anomaly-free, i.e. have vanishing super-schwarzian derivative. Our results are in agreement with results that are known in the literature for specific values of  $N$  (see e.g. [9, 10] for  $N = 1$ , [11–13] for  $N = 2$ , [6] for  $N = 3$ ). We refer to [5, 6, 12, 14] for previous work on the superspace formalism for general  $N$ .

The formulation of physical models for  $N$ -extended superconformal field theory and, eventually, the classification of all such models, require

- (i) the determination of all highest weight (positive energy) unitary irreducible representations of the  $N$ -extended superconformal algebra;
- (ii) the determination of consistent ways to combine irreducible representations into a physical Hilbert space;
- (iii) for applications in string theory, investigation of local superconformal invariance e.g. in a BRST formalism.

For  $N = 0, 1$  most of the program has been successfully completed; it has resulted in many applications in (super)string theory and statistical mechanics. For  $N = 2$  the representation theory is well-understood [11, 15, 16]. String models based on  $N = 2$  local superconformal invariance can consistently be formulated but have no dynamical content [17]. Recently global  $N = 2$  superconformal symmetry has found beautiful applications in the study of the compactification  $10 \rightarrow 4$  of  $N = 1$  superstrings [15, 18, 19]. It is also expected that  $N = 2$  superconformal invariance will show up in certain  $d = 2$  statistical systems at criticality [20].

In view of the failure of the attempts to construct string models based on local  $N > 2$  extended superconformal invariance, the  $N = 3, 4$  algebras have received little attention until quite recently. Recent work has focussed on the representation theory of the  $N = 3$  and the  $N = 4$   $\text{su}(2)$ -extended algebras [21–25]. Since these algebras have non-abelian Kac-Moody subalgebras, the  $c$ -values allowed by unitarity do not fall into discrete series and continuous regions but are multiples of some constant  $c_0$ , which is equal to  $\frac{3}{2}$  for the  $N = 3$  and 6 for the  $\text{su}(2)$ -extended  $N = 4$  algebra. The representation theory of the  $\text{su}(2)$ -extended  $N = 4$  superconformal algebra has interesting applications in the study of  $\sigma$ -models on  $4n$ -dimensional hyper-Kähler manifolds where it provides information about geometric and topological properties of the  $\sigma$ -model target spaces. Attempts to formulate a string theory based on local  $\text{su}(2)$ -extended superconformal symmetry have been unfruitful due to a negative ( $c = -12$ ) critical central charge [26, 27].

In this paper we focus on the  $\mathfrak{o}(4)$ -extended superconformal algebra, which, so far, has received little attention in the literature. Under the assumption that in physical theories the so-called twisted  $N = 4$  anomaly in the current algebra is

absent, we argue that the allowed  $c$ -values for applications of  $O(4)$ -extended superconformal symmetry are

$$c(n_+, n_-) = \frac{6n_+n_-}{n_+ + n_-}, \quad n_+, n_- \in \mathbf{Z}_{>0}. \quad (1.1)$$

A “basic” representation of the  $\mathfrak{o}(4)$ -extended algebra, with  $n_+ = n_- = 1$ ,  $c = 3$ , was presented in [5]. Recently it has been found that this representation is actually realized in a model with defected Ising chains [28].

It is an interesting question whether or not  $O(4)$ -extended superconformal symmetry can be of any help for the construction of string models. It has been pointed out already in [14, 26] that the formulation of  $O(N)$  strings with  $N > 2$  is problematic; this is in agreement with the observation that both the  $\mathfrak{o}(3)$ - and the  $\mathfrak{o}(4)$ -extended algebras have critical central charge 0 [5, 30]. An interesting possibility would be to employ, in the spirit of [15, 18, 19], global  $N = 4$  superconformal invariance for  $10 \rightarrow 4$  superstring compactification. The  $c$ -value appropriate for this, which is  $c = 9$ , occurs in the list (1.1) as  $c(3, 3)$ ,  $c(6, 2)$  and  $c(2, 6)$ . We expect that the Liouville-WZW  $\sigma$ -models discussed in [29], which provide a link between  $O(4)$ -extended superconformal symmetry and geometry, will be relevant for such compactification schemes. Also, in a fermionic formulation, where the internal space is represented by 18 free fermions, the implementation of  $O(4)$ -extended superconformal symmetry is possible [5].

In the above we have stressed that within the class of Lie superalgebras there are no good candidates for the description of  $N$ -extended superconformal symmetry with  $N > 4$ . Still, it may very well be possible to go beyond  $N = 4$  by dropping the condition that the superconformal operator algebra is actually a Lie superalgebra.  $N$ -extended operator algebras which allow central extensions for all  $N$  have been proposed in [4, 31]. So far no representations of these algebras have been found. Recently it has been established [32, 33] that the representation theory of certain non-supersymmetric extensions of the Virasoro algebra that are not Lie algebras, can be worked out explicitly. We therefore expect that the fact that the  $N$ -extended algebras proposed in [4, 31] are not Lie superalgebras is not a serious obstruction for obtaining results on their representation theory and connecting these algebras with physics.

This paper is organized as follows. In sect. 2 we introduce  $N$ -extended superspace and define superconformal transformations. Sect. 3 describes (finite) superconformal transformations on general primary fields  $\Phi_\Delta^\alpha(Z)$  (characterized by their dimension  $\Delta$  and  $O(N)$ -representation  $M^{\alpha\beta}$ ) and on the current superfield  $J^{(N)}(Z)$ . The transformation rule of the latter involves a central term  $S^{(N)}(Z, \tilde{Z})$  which is a generalization of the  $N = 0$  Schwarzian derivative. In sects. 4 and 5 all possible terms  $S^{(N)}(Z, \tilde{Z})$  consistent with the group property of  $N$ -extended superconformal transformations are determined. The case  $N = 4$  is special; we find that  $S^{(4)}(Z, \tilde{Z})$  is a sum of two independent terms corresponding to a *regular* and a *twisted*

extension of the  $o(4)$ -extended superconformal algebra. The  $N < 4$  Schwarzian derivatives are easily obtained by reducing the  $N = 4$  result. In sect. 6 we examine the condition  $S^{(N)}(Z, \bar{Z}) = 0$ , which selects a non-anomalous subgroup of the conformal group of crucial importance in the analysis of superconformal quantum field theory. For the regular extensions for  $0 \leq N \leq 4$  the condition is solved by superprojective transformations corresponding to the “little conformal group”  $OSP(N/2)$ ; the condition corresponding to the twisted  $N = 4$  extension selects another finite dimensional extension of the  $N = 4$  super-Poincaré group. In sect. 7 we further analyse the  $N = 4$  superconformal algebra. We introduce a real parameter  $\alpha$  to specify the embedding of a Virasoro subalgebra in the  $N = 4$  algebra. The  $sl(2)$  subalgebra of this Virasoro algebra extends to a finite dimensional subalgebra of the  $N = 4$  algebra which is isomorphic to  $D(2/1; \alpha - \frac{1}{2})$ . Tuning the parameter  $\alpha$  such that the twisted anomaly is absent leads to the set (1.1) of  $c$ -values for which integrable unitary representations exist.

**2.  $N$ -extended superspace; superconformal transformations**

In sects. 2 and 3 we describe the general formalism for  $N$ -extended superconformal field theory in superspace. Our treatment follows the exposition of the  $N = 1$  formalism as presented in [9, 10].

We write the coordinates of  $N$ -extended superspace as  $(Z, \bar{Z})$ , where  $Z = (z, \theta^i)$ ,  $\bar{Z} = (\bar{z}, \bar{\theta}^i)$ ,  $i = 1, 2, \dots, N$ . The left covariant derivative w.r.t. supertranslations  $D^i = \theta^i \partial_z + \partial_{\theta^i}$  satisfies

$$\{D^i, D^j\} = 2\delta^{ij} \partial_z. \tag{2.1}$$

Analytic functions  $f(Z)$  on this superspace obey relations which are direct generalizations of the Cauchy integral theorem and the Taylor expansion for ordinary complex functions. We introduce the following notation

$$\begin{aligned}
 [i] &= i_1 i_2 \dots i_R, & i_j \neq i_k, & & 0 \leq R \leq N, \\
 \theta^{[i]} &= \theta^{i_1} \theta^{i_2} \dots \theta^{i_R} & \text{etc.}, & & \\
 \theta^{N-[i]} &= \frac{1}{(N-R)!} \epsilon^{j_1 \dots j_{N-R} i_1 \dots i_R} \theta^{j_1} \dots \theta^{j_{N-R}}, \\
 f_R &= \frac{1}{2} R(R-1), \\
 \theta_{12}^i &= \theta_1^i - \theta_2^i, \\
 Z_{12} &= z_1 - z_2 - \theta_1^i \theta_2^i.
 \end{aligned} \tag{2.2}$$

The Cauchy integral theorem reads

$$(-1)^{f_R} \frac{1}{n!} \partial_{z_2}^n D_2^{[i]} f(Z_2) = \oint_{C_2} \frac{dz_1}{2\pi i} \int d^N \theta_1 \frac{\theta_{12}^{N-[i]}}{Z_{12}^{n+1}} f(Z_1) \tag{2.3}$$

(where  $C_2$  is a curve in the complex plane enclosing the point  $z_2$ ) and the Taylor expansion is given by

$$f(Z_1) = \sum_{n \geq 0} \sum_{[i]} (-1)^{f_R} \frac{1}{n!} \frac{1}{R!} Z_{12}^n \theta_{12}^{[i]} D_2^{[i]} \partial_{z_2}^n f(Z_2). \tag{2.4}$$

The expansion (2.4) can be derived from (2.3) by using the identity

$$\sum_{[i]} (-1)^R \frac{1}{R!} \theta_{12}^{[i]} \theta_{32}^{N-[i]} = \theta_{31}^N. \tag{2.5}$$

For  $N = 0$  conformal transformations are just the (anti)analytic transformations  $z \rightarrow \tilde{z}(z)$ ,  $\bar{z} \rightarrow \tilde{\bar{z}}(\bar{z})$ . For  $N > 0$  the superconformal transformations are the (anti)analytic transformations  $Z \rightarrow \tilde{Z}$ ,  $\bar{Z} \rightarrow \tilde{\bar{Z}}$  that transform the covariant derivatives  $D^i$  and  $\bar{D}^i$  homogeneously:

$$D^i = (D^i \tilde{\theta}^j) \tilde{D}^j, \quad \bar{D}^i = (\bar{D}^i \tilde{\theta}^j) \tilde{\bar{D}}^j. \tag{2.6}$$

We focus on the analytic superconformal transformations; the antianalytic ones are treated similarly. From (2.6) we derive

- (i)  $D^i \tilde{z} = \tilde{\theta}^j D^i \tilde{\theta}^j,$
- (ii)  $(D^i \tilde{\theta}^j)(D^k \tilde{\theta}^l) = \delta^{ik} (\partial_z \tilde{z} + \tilde{\theta}^j \partial_z \tilde{\theta}^j),$
- (iii)  $\det \left[ \frac{D \tilde{\theta}}{(\partial_z \tilde{z} + \tilde{\theta}^j \partial_z \tilde{\theta}^j)^{1/2}} \right] = \pm 1. \tag{2.7}$

The property (ii) just states that the matrices  $(D \tilde{\theta})$  are orthogonal up to an overall multiplicative factor. The  $\pm$  sign in (iii) distinguishes between orientation preserving (+) and twisted (-) transformations [12].

For infinitesimal conformal transformations the condition (2.6) can be solved in terms of a single unrestricted superfield  $E(Z)$ :

$$\delta z = E - \frac{1}{2} \theta^i D^i E, \quad \delta \theta^i = \frac{1}{2} D^i E. \tag{2.8}$$

These transformations were first described by Ademollo et al. [1]. The superfield

$E(Z)$  can be expanded in components according to

$$E(z, \theta^i) = 2 \sum_n \sum_{[i]} i^{f_R} \alpha_n^{[i]} \theta^{[i]} z^{n+1-R/2}. \tag{2.9}$$

The  $\alpha_n^{[i]}$  are completely antisymmetric in  $i_1, i_2, \dots, i_R$ ;  $n$  is integer (half-integer) if  $R$  is even (odd) (Neveu-Schwarz sector). The variation of  $(z, \theta^i)$  under the superconformal generator  $J_n^{[i]}$  corresponding to the parameter  $\alpha_n^{[i]}$  reads

$$\begin{aligned} \delta z &= i^{f_R} (2 - R) \alpha_n^{i_1 \dots i_R} \theta^{i_1} \dots \theta^{i_R} z^{n+1-R/2}, \\ \delta \theta^i &= -i^{f_R} \left[ \sum_{l=1}^R (-1)^{R-1} \delta^{i, i_l} \alpha_n^{i_1 \dots i_R} \theta^{i_1} \dots \hat{\theta}^{i_l} \dots \theta^{i_R} z^{n+1-R/2} \right. \\ &\quad \left. - (n+1 - \frac{1}{2}R) \alpha_n^{i_1 \dots i_R} \theta^{i_1} \dots \theta^{i_R} \theta^i z^{n-R/2} \right]. \end{aligned} \tag{2.10}$$

The  $J_n^{[i]}$  generate the classical  $\mathfrak{o}(N)$ -extended superconformal algebra. The (anti)commutation relations are

$$\begin{aligned} [J_m^{i_1 \dots i_R}, J_n^{j_1 \dots j_S}] &= i^{-RS} \left\{ [m(2-S) - n(2-R)] J_{n+m}^{i_1 \dots i_R j_1 \dots j_S} \right. \\ &\quad \left. - i \sum_{l=1}^R \sum_{k=1}^S (-1)^{l+k+S} \delta^{i_l, j_k} J_{n+m}^{i_1 \dots \hat{i}_l \dots i_R j_1 \dots \hat{j}_k \dots j_S} \right\}. \end{aligned} \tag{2.11}$$

The generators  $L_n = \frac{1}{2} J_n$  generate a Virasoro subalgebra

$$[L_m, L_n] = (m - n) L_{m+n}. \tag{2.12}$$

The relation

$$[L_m, J_n^{[i]}] = ((1 - \frac{1}{2}R)m - n) J_{m+n}^{[i]} \tag{2.13}$$

shows that the generator  $J_n^{[i]}$  has conformal dimension  $(2 - \frac{1}{2}R)$ .

### 3. Field theory in $N$ -extended superspace

$N$ -extended superconformal quantum field theory can be formulated in the superspace described above along the same lines as ordinary conformal field theory is described in the complex plane. Let us consider the transformation law of a primary superfield  $\Phi(Z)$  in such a theory (we focus on the  $Z$ -dependence, the  $\bar{Z}$ -dependence is treated similarly). By inspecting the properties (2.6) and (2.7) we

find that we are free to include a non-trivial representation

$$D\tilde{\theta} \rightarrow [\det(D\tilde{\theta})]^{2\Delta/N} M^{\alpha\beta} \left[ \frac{D\tilde{\theta}}{(\partial_z \tilde{z} + \tilde{\theta}^j \partial_z \tilde{\theta}^j)^{1/2}} \right]$$

in the transformation law

$$\Phi_\Delta^\alpha(Z) = [\det(D\tilde{\theta})]^{2\Delta/N} M^{\alpha\beta} \left[ \frac{D\tilde{\theta}}{(\partial_z \tilde{z} + \tilde{\theta}^j \partial_z \tilde{\theta}^j)^{1/2}} \right] \tilde{\Phi}_\Delta^\beta(\tilde{Z}). \tag{3.1}$$

$\Delta$  is the conformal dimension of  $\Phi_\Delta^\alpha$ ; the  $M^{\alpha\beta}$  form a representation of  $O(N)$ .

For the infinitesimal conformal transformation (2.8) we have

$$[\det(D\tilde{\theta})]^{2\Delta/N} \sim 1 + \Delta \partial_z E, \quad M^{\alpha\beta} [ \ ] \sim \delta^{\alpha\beta} + \frac{1}{2} i (D^i D^j E) (T^{ij})^{\alpha\beta}, \tag{3.2}$$

(where the  $T^{ij}$  span a representation of the  $\mathfrak{o}(N)$  Lie algebra) and correspondingly

$$\delta_E \Phi_\Delta^\alpha = E (\partial_z \Phi_\Delta^\alpha) + \frac{1}{2} (D^i E) (D^i \Phi_\Delta^\alpha) + \Delta (\partial_z E) \Phi_\Delta^\alpha + \frac{1}{2} i (D^i D^j E) (T^{ij})^{\alpha\beta} \Phi_\Delta^\beta. \tag{3.3}$$

These transformations are generated by a current superfield  $J^{(N)}(Z)$  through the relation

$$\delta_E \Phi(Z_2) = \frac{1}{2} \oint_{\mathcal{C}_2} \frac{dz_1}{2\pi i} \int d^N \theta_1 E(Z_1) J^{(N)}(Z_1) \Phi(Z_2). \tag{3.4}$$

All information about the transformation properties of  $\Phi(Z)$  is encoded in the singular behaviour of the product  $J^{(N)}(Z_1) \Phi(Z_2)$  for  $Z_1 \rightarrow Z_2$ . For the primary fields  $\Phi_\Delta^\alpha$  the super operator product expansion (SOPE) reads

$$J^{(N)}(Z_1) \Phi_\Delta^\alpha(Z_1) = \left[ 2\Delta \frac{\theta_{12}^N}{Z_{12}^2} + \frac{\theta_{12}^{N-i}}{Z_{12}} D_2^i + 2 \frac{\theta_{12}^N}{Z_{12}} \partial_{z_2} \right] \Phi_\Delta^\alpha(Z_2) - i (T^{ij})^{\alpha\beta} \frac{\theta_{12}^{N-ij}}{Z_{12}} \Phi_\Delta^\beta(Z_2). \tag{3.5}$$

We read off from (3.5) that the conformal dimension of  $J^{(N)}(Z)$  is  $(2 - \frac{1}{2}N)$ .  $J^{(N)}$  is not primary; it is a descendant field in the conformal family of the identity. Its transformation law is given by

$$J^{(N)}(Z) = [\det(D\tilde{\theta})]^{(4-N)/N} \tilde{J}^{(N)}(\tilde{Z}) + S^{(N)}(Z, \tilde{Z}). \tag{3.6}$$

$S^{(N)}(Z, \tilde{Z})$  is a generalization of the schwarzian derivative present in the transfor-



mation law for  $N = 0$  [34]

$$T(z) = \left( \frac{d\tilde{z}}{dz} \right)^2 \tilde{T}(\tilde{z}) + S(z, \tilde{z}), \tag{3.7}$$

$$S(z, \tilde{z}) = \frac{c}{12} \left\{ \frac{\tilde{z}'''}{\tilde{z}'} - \frac{3}{2} \left( \frac{\tilde{z}''}{\tilde{z}'} \right)^2 \right\}. \tag{3.8}$$

Consistency of the transformation rule (3.6) requires that  $S^{(N)}(Z, \tilde{Z})$  satisfies the following group property

$$S^{(N)}(Z, \tilde{\tilde{Z}}) = [\det(D\tilde{\theta})]^{(4-N)/N} S^{(N)}(\tilde{Z}, \tilde{\tilde{Z}}) + S^{(N)}(Z, \tilde{Z}). \tag{3.9}$$

The fact that this equation cannot be satisfied for every value of  $N$  puts restrictions on the  $N$ -values for which physical quantum field theories can be formulated. In the next two sections we will analyze eq. (3.9) and show that it has solutions only for  $N \leq 4$  ( $N = 4$  is of special interest since in that case the equation admits two independent non-trivial solutions). The upper bound  $N = 4$  is a natural one since for  $N > 4$  the current superfield  $J^{(N)}$  has components with negative conformal dimension. It may well be possible to formulate classical superconformal field theories for general  $N$  [29], but in theories of this type there is definitely no quantum physics beyond  $N = 4$ .

#### 4. Central extensions

Since it is rather hard to solve the equation (3.9) directly, we perform the analysis in two steps. We first consider (3.9) for infinitesimal transformations  $Z \rightarrow \tilde{Z}$  of the form (2.8) and determine the infinitesimal expressions corresponding to  $S^{(N)}(Z, \tilde{Z})$ . We then use the invariance of operator product expansions to derive the full expression for finite transformations  $Z \rightarrow \tilde{Z}$ .

Under the infinitesimal superconformal transformation (2.8) with parameter superfield  $E(Z)$   $J^{(N)}$  transforms as

$$\delta_E J^{(N)} = E(\partial_z J^{(N)}) + \frac{1}{2}(D^i E)(D^i J^{(N)}) + (2 - \frac{1}{2}N)(\partial_z E)J^{(N)} + O^{(N)}E, \tag{4.1}$$

where

$$(O^{(N)}E)(Z) \equiv S^{(N)}\left(Z, \left(z + E - \frac{1}{2}\theta^i D^i E, \theta^i + \frac{1}{2}D^i E\right)\right). \tag{4.2}$$

The group property of  $S^{(N)}$  translates into the following condition on the operator  $O^{(N)}$  [6]

$$\begin{aligned} & \{EO^{(N)}\partial_z F + (2 - \frac{1}{2}N)(\partial_z E)O^{(N)}F + \frac{1}{2}(D^i E)(D^i F)\} - \{E \leftrightarrow F\} \\ & = O^{(N)}\{E\partial_z F - F\partial_z E + \frac{1}{2}(D^i E)(D^i F)\}. \end{aligned} \tag{4.3}$$

In order to examine the solutions to eq. (4.3) it is convenient to switch to yet another notation and to rephrase (4.1) as a SOPE

$$J^{(N)}(Z_1)J^{(N)}(Z_2) = \left[ (4-N)\frac{\theta_{12}^N}{Z_{12}^2} + \frac{\theta_{12}^{N-i}}{Z_{12}}D_2^i + 2\frac{\theta_{12}^N}{Z_{12}}\partial_{z_2} \right] J^{(N)}(Z_2) + c^{(N)}(Z_1, Z_2), \tag{4.4}$$

where  $c^{(N)}$  is related to  $O^{(N)}$  through

$$\frac{1}{2}\oint_{C_2} \frac{dz_1}{2\pi i} \int d^N\theta_1 E(Z_1)c^{(N)}(Z_1, Z_2) = (O^{(N)}E)(Z_2). \tag{4.5}$$

By inspecting superconformal Ward-identities for 2-point correlation functions and using dimensional arguments we obtain the following list of candidate solutions for  $c^{(N)}(Z_1, Z_2)$

$$\begin{aligned} \text{regular} \quad & \frac{1}{Z_{12}^{4-N}}, \quad N=1, 2, 3, \quad \log(Z_{12}), \quad N=4; \\ \text{twisted} \quad & \frac{\theta_{12}^N}{Z_{12}^{4-(1/2)N}}, \quad N=2, 4, 6, \quad \theta_{12}^8 \log(Z_{12}), \quad N=8. \end{aligned} \tag{4.6}$$

Checking the condition (4.3) we find that the  $N \leq 4$  regular terms are all solutions, whereas of the twisted terms only the  $N=4$  term is a solution. Table 1 gives the complete list of central terms  $c^{(N)}$  and the corresponding operators  $O^{(N)}$ . Notice that the operator  $O^{(N)}$  for the regular  $N=4$  extension is non-local. This is related to the fact that the conformal dimension of  $J^{(4)}(Z)$  is zero; its superconformal properties are similar to the conformal properties of a dimension zero scalar field

TABLE 1  
Results for the central terms  $c^{(N)}(Z_1, Z_2)$  and the operators  $O^{(N)}$ , which are both related to the anomalous terms in infinitesimal transformations of  $J^{(N)}(Z)$

	$c^{(N)}(Z_1, Z_2)$	$O^{(N)}$	type
$N = 0, 1, 2, 3$	$1/Z_{12}^{4-N}$	$\partial_z^{3-N} D^N$	regular
$N = 4$	$\log(Z_{12})$	$\epsilon^{ijkl} D^i D^j D^k / D^l$	regular
$N = 4$	$\theta_{12}^4 / Z_{12}^2$	$\partial_z$	twisted

$\varphi(z)$  in ordinary conformal field theory. We can avoid non-localities by formulating the theory in terms of  $J^i(Z) \equiv D^i J^{(4)}(Z)$  which transforms as

$$\delta_E J^i = E \partial_z J^i + \frac{1}{2}(D^j E)(D^i J^j) + \frac{1}{2}(D^i D^j E)J^j - \frac{1}{6}cD^{4-i}E + \frac{1}{6}c'D^i \partial_z E, \quad (4.7)$$

where we normalized the central terms as

$$c^{(4)}(Z_1, Z_2) = -\frac{1}{3}c \log(Z_{12}) + \frac{c'}{3} \frac{\theta_{12}^4}{Z_{12}^2}. \quad (4.8)$$

In the transformation rule (4.7) both operators  $O^{(4)}$  act in a local way.

### 5. The schwarzian derivatives $S^{(N)}(Z, \tilde{Z})$ for $N \leq 4$

Having found all possible operators  $O^{(N)}$ , which describe the behaviour of  $S^{(N)}(Z, \tilde{Z})$  for  $Z - \tilde{Z}$  infinitesimal, we proceed by deriving the expressions for  $S^{(N)}(Z, \tilde{Z})$ , which give the anomalous terms in finite superconformal transformations of  $J^{(N)}(Z)$ .

Notice that once we have derived the expression for  $S^{(4)}(Z, \tilde{Z})$  corresponding to the regular  $N = 4$  extension we can easily obtain the expressions for  $S^{(N)}(Z, \tilde{Z})$ ,  $N < 4$ . This is done by simply inserting a special transformation  $Z \rightarrow \tilde{Z}$  which is a superconformal transformation for some  $N < 4$  and extracting from  $S^{(4)}(Z, \tilde{Z})$  the part which corresponds to the lower- $N$  current superfield  $J^{(N)}(Z)$ . We therefore focus on the case  $N = 4$ .

The derived  $N = 4$  current superfield  $J^i(Z)$  transforms as follows under finite superconformal transformations

$$J^i(Z) = (D^i \tilde{\theta}^j) \tilde{J}^j(\tilde{Z}) + D^i S^{(4)}(Z, \tilde{Z}). \quad (5.1)$$

Since  $S^{(4)}(Z, \tilde{Z})$  is a non-local expression we pass to  $S^i \equiv D^i S^{(4)}$ . In order to derive an expression for  $S^i(Z, \tilde{Z})$  we consider the SOPE  $J^i(Z_1)J^i(Z_2)$  for  $Z_1 \rightarrow Z_2$

$$\begin{aligned} J^i(Z_1)J^i(Z_2) &= \frac{1}{Z_{12}^2} \left\{ \delta^{ii} \theta_{12}^{4-j} J^j - \theta_{12}^{4-i} J^i - \theta_{12}^{4-l} J^l + \theta_{12}^4 D_2^l J^i \right\} \\ &+ \frac{1}{Z_{12}} \left\{ \theta_{12}^{4-ii} J^j + \theta_{12}^{4-lj} D_2^l J^i - 2\theta_{12}^{4-l} \partial_{z_2} J^i \right\} \\ &- \frac{c}{3} \left\{ \frac{\delta^{ii}}{Z_{12}} - \frac{\theta_{12}^i}{Z_{12}^2} \right\} + \frac{c'}{3} \left\{ 2 \frac{\delta^{ii} \theta_{12}^4}{Z_{12}^3} + \frac{\theta_{12}^{4-ii}}{Z_{12}^2} \right\} + \text{regular}. \quad (5.2) \end{aligned}$$

Suppose now we have a finite superconformal transformation  $Z(\tilde{Z})$  sending  $Z_1$  to

$\tilde{Z}_1$  and  $Z_2$  to  $\tilde{Z}_2$ . We put  $l = i$  in (5.2) and rewrite the SOPE in terms of  $\tilde{J}^m$  and  $\tilde{Z}_{1,2}$ . Transforming the l.h.s. of (5.2) results in

$$J^i(Z_1)J^i(Z_2) = (D_1^i\tilde{\theta}_1^j)(D_2^i\tilde{\theta}_2^k)\tilde{J}^j(\tilde{Z}_1)\tilde{J}^k(\tilde{Z}_2) + \text{regular}. \tag{5.3}$$

Transforming the r.h.s. gives

$$J^i(Z_1)J^i(Z_2) = \left\{ \frac{2\theta_{12}^{4-j}}{Z_{12}^2} + \frac{\theta_{12}^4}{Z_{12}^2}D_2^j + \frac{\theta_{12}^{4-jl}}{Z_{12}}D_2^l - \frac{2\theta_{12}^{4-j}}{Z_{12}}\partial_{z_2} \right\} \\ \times \left\{ (D^j\tilde{\theta}^m)\tilde{J}^m(\tilde{Z}_2) + S^j(Z_2, \tilde{Z}_2) \right\} - \frac{c}{3}\frac{4}{Z_{12}} + \frac{c'}{3}\frac{8\theta_{12}^4}{Z_{12}^3} + \text{regular}. \tag{5.4}$$

The expression (5.3) can be worked out by using the SOPE (5.2) for the transformed field  $\tilde{J}^i(\tilde{Z})$  (notice that for twisted transformations  $Z \rightarrow \tilde{Z}$  (with the minus sign in the relation (2.7), (iii)) the coefficient of the twisted central term in the SOPE  $\tilde{J}(\tilde{Z}_1)\tilde{J}(\tilde{Z}_2)$  has an extra minus sign w.r.t. (5.2)). Equating the resulting expression to (5.4) and extracting the terms independent of  $\tilde{J}^i(\tilde{Z})$  gives an identity which determines the form of  $S^j(Z, \tilde{Z})$  completely. This identity expresses the fact that the ‘‘anomaly’’ in the transformation properties under  $Z \rightarrow \tilde{Z}$  of the central terms in the SOPE (5.2) is just compensated by terms coming from the anomalous term  $S^j(Z, \tilde{Z})$  in the transformation rule of the current  $J^i(Z)$ . A straightforward calculation leads to the following result

$$S^i(Z, \tilde{Z}) = \frac{2c}{9}e^{ijkl}\frac{(D^jD^k\tilde{\theta}^m)(D^l\tilde{\theta}^m)}{(D^p\tilde{\theta}^q)(D^p\tilde{\theta}^q)} \\ + 4c'\frac{(\partial_z\tilde{\theta}^j)(D^i\tilde{\theta}^j)}{(D^p\tilde{\theta}^q)(D^p\tilde{\theta}^q)} - \frac{2c'}{3}\frac{(D^iD^j\tilde{\theta}^k)(D^j\tilde{\theta}^k)}{(D^p\tilde{\theta}^q)(D^p\tilde{\theta}^q)}. \tag{5.5}$$

It is easily checked that if we substitute an infinitesimal superconformal transformation  $\tilde{Z}(Z)$  of the form (2.8) into the super schwarzian derivative (5.5) we recover the central terms of (4.7)

$$S^i(Z, \tilde{Z}) \xrightarrow[\delta\theta^l = \frac{1}{2}D^lE]{\delta z = E - \frac{1}{2}\theta^l D^l E} - \frac{1}{6}cD^{4-i}E + \frac{1}{6}c'D^i\partial_z E. \tag{5.6}$$

The schwarzian derivatives  $S^{(N)}(Z, \tilde{Z})$  for the  $N < 4$  superconformal transformations can be obtained from (5.5) by applying the reduction procedure described above. The reduction of the twisted term (proportional to  $c'$ ) to lower  $N$  gives a vanishing result; reducing the regular term (proportional to  $c$ ) provides us with a

TABLE 2  
Schwarzian derivatives for the  $O(N)$ -extended superconformal transformations (3.6).

$N = 4$	(regular)	$S^i(Z, \tilde{Z}) = \frac{2c}{9} e^{ijkl} \frac{(D^j D^k \tilde{\theta}^m)(D^l \tilde{\theta}^m)}{(D^p \tilde{\theta}^a)(D^p \tilde{\theta}^a)}$
	(twisted)	$S^i(Z, \tilde{Z}) = 4c' \frac{(\partial_z \tilde{\theta}^j)(D^i \tilde{\theta}^j)}{(D^p \tilde{\theta}^a)(D^p \tilde{\theta}^a)} - \frac{2c'}{3} \frac{(D^i D^j \tilde{\theta}^k)(D^j \tilde{\theta}^k)}{(D^p \tilde{\theta}^a)(D^p \tilde{\theta}^a)}$
$N = 3$		$S^{(3)}(Z, \tilde{Z}) = \frac{c}{6} e^{ijkl} \frac{(D^i D^j \tilde{\theta}^l)(D^k \tilde{\theta}^l)}{(D^p \tilde{\theta}^a)(D^p \tilde{\theta}^a)}$
$N = 2$		$S^{(2)}(Z, \tilde{Z}) = \frac{c}{6} e^{ij} \frac{2(D^i \partial_z \tilde{\theta}^k)(D^j \tilde{\theta}^k) - (D^i D^j \tilde{\theta}^k)(\partial_z \tilde{\theta}^k)}{(D^p \tilde{\theta}^a)(D^p \tilde{\theta}^a)}$
$N = 1$		$S^{(1)}(Z, \tilde{Z}) = \frac{c}{3} \left[ \frac{D^4 \tilde{\theta}}{D \tilde{\theta}} - 2 \frac{(D^3 \tilde{\theta})(D^2 \tilde{\theta})}{(D \tilde{\theta})^2} \right]$
$N = 0$		$S(z, \tilde{z}) = \frac{c}{12} \left[ \frac{\tilde{z}'''}{\tilde{z}'} - \frac{3}{2} \left( \frac{\tilde{z}''}{\tilde{z}'} \right)^2 \right]$

In all cases the normalization has been chosen such that the corresponding central term in the Virasoro subalgebra (cfr. (2.12)) has the usual form  $\frac{1}{12} cm(m^2 - 1) \delta_{m+n}$ .

complete list of the schwarzian derivatives for  $0 \leq N \leq 3$  (see table 2). The  $N = 0, 1, 2$  results agree with the results already known in the literature [9, 12, 13, 34].

### 6. Anomaly-free superconformal transformations

The actual application of the full group of superconformal transformations in quantum field theory is troubled by the occurrence of central terms in general superconformal transformations. It is therefore interesting to consider superconformal transformations  $\tilde{Z}(Z)$  that obey the condition

$$S^{(N)}(Z, \tilde{Z}) = 0. \tag{6.1}$$

For  $N = 0$  the solutions to the condition (6.1) that are connected to the identity are the projective transformations

$$\tilde{z}(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1. \tag{6.2}$$

Symmetry of conformal field theory under these anomaly-free transformations allows one to reduce a general  $n$ -point amplitude to an expression depending on  $n - 3$  variables only.

In this section we consider the analogues of the transformations (6.2) in  $N$ -extended superspace, which provide solutions to the condition (6.1) for the regular schwarzian derivatives  $S^{(N)}$  for  $0 \leq N \leq 4$ . We also indicate a class of  $N = 4$  superconformal transformations with vanishing twisted schwarzian derivative.

A convenient characterization of the  $N = 0$  transformations (6.2) is the following: they are the fractional linear transformations  $z \rightarrow \tilde{z}$  that satisfy

$$\tilde{z}_1 - \tilde{z}_2 = \left( \frac{d\tilde{z}_1}{dz_1} \right)^{1/2} \left( \frac{d\tilde{z}_2}{dz_2} \right)^{1/2} (z_1 - z_2). \tag{6.3}$$

As a natural generalization we define  $N$ -extended superprojective transformations to be the fractional linear superconformal transformations that obey the condition

$$\tilde{Z}_{12} = \left[ \frac{1}{N} (D_1^i \tilde{\theta}_1^i) (D_1^i \tilde{\theta}_1^i) \right]^{1/2} \left[ \frac{1}{N} (D_2^i \tilde{\theta}_2^i) (D_2^i \tilde{\theta}_2^i) \right]^{1/2} Z_{12}. \tag{6.4}$$

From this condition we derive the following transformation property of the covariant differential [12, 14]  $dZ = dz - \theta^i d\theta^i$

$$d\tilde{Z} = \left[ \frac{1}{N} (D^i \tilde{\theta}^j) (D^i \tilde{\theta}^j) \right] dZ \tag{6.5}$$

which is equivalent to the superconformal condition (2.6). This shows that the superprojective transformations as defined above are actually superconformal as they should.

As an aside, notice that under a superprojective transformation  $Z \rightarrow \tilde{Z}$  with real parameters the following form is left invariant

$$d\Omega = \frac{dZ d\bar{Z}}{(\text{Im } Z)^2}, \tag{6.6}$$

where  $\text{Im } Z = \frac{1}{2}(Z - \bar{Z} - \theta^i \bar{\theta}^i)$ . For  $N = 0$   $d\Omega$  is just the measure for the Poincaré upper half-plane. For  $N = 1$  it is possible [12] to define a square root of  $d\Omega$  which defines an invariant measure [35]

$$(d\Omega)^{1/2} = \frac{dz d\bar{z} d\theta d\bar{\theta}}{\text{Im } Z} \quad (N = 1). \tag{6.7}$$

For general  $N$  the superprojective transformations are a convenient starting point for the study of the moduli spaces of  $N$ -extended super Riemann surfaces [12, 35].

In order to determine the superprojective transformations explicitly we introduce homogeneous coordinates  $(x, y, \xi^i)$  such that  $z = xy^{-1}$  and  $\theta^i = \xi^i y^{-1}$ . On  $(x, y, \xi^i)$

a superprojective transformation acts as a linear map with matrix say  $A$ . The condition (6.5) translates into the following condition on  $A$

$${}^tAMA = M, \quad M = \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & -\mathbf{1} \end{bmatrix}, \tag{6.8}$$

which is just the statement that  $A$  is an element of the group  $OSP(N/2)$ . The matrices  $A$  solving eq. (6.7) can conveniently be parametrized in terms of the following set of parameters

$$a, b, c, d, t^{ij}, \epsilon_1^i, \epsilon_2^i$$

where  $\epsilon_1^i, \epsilon_2^i$  are Grassmann,  $ad - bd = 1$ ,  $t^{ij}t^{ik} = \delta^{ik}$ :

$$A = \begin{bmatrix} X & Y \\ Z & O \end{bmatrix},$$

where

$$\begin{aligned} X &= \gamma \begin{bmatrix} a & b \\ c & d \end{bmatrix}, & Y^j &= \frac{1}{\gamma} \begin{bmatrix} b\epsilon_2^i - a\epsilon_1^i \\ d\epsilon_2^i - c\epsilon_1^i \end{bmatrix} (t^{ij} - \epsilon_2^i \epsilon_1^k t^{kj}), \\ Z^i &= [\epsilon_2^i, \epsilon_1^i], & O^{ij} &= t^{ij} - \epsilon_2^i \epsilon_1^k t^{kj} \end{aligned} \tag{6.9}$$

and  $\gamma = \sqrt{1 - \epsilon_1^i \epsilon_2^i}$ . On the original coordinates  $(z, \theta^i)$  the superprojective transformation reads

$$\tilde{z} = \frac{az + b + (1/\gamma)Y^{1j}\theta^j}{cz + d + (1/\gamma)Y^{2j}\theta^j}, \quad \tilde{\theta}^i = \frac{1}{\gamma} \frac{\epsilon_2^i z + \epsilon_1^i + O^{ij}\theta^j}{cz + d + (1/\gamma)Y^{2j}\theta^j}. \tag{6.10}$$

We have

$$\det(D\tilde{\theta}) = \det(t) [\gamma(cz + d) + Y^{2j}\theta^j]^{-N}. \tag{6.11}$$

In infinitesimal form the superprojective transformations reduce to the form (2.8) with non-zero parameters

$$\alpha_{-1}, \alpha_0, \alpha_1; \quad \alpha_{-1/2}^i, \alpha_{1/2}^i; \quad \alpha_0^{ij} \quad i, j = 1, \dots, N. \tag{6.12}$$

These generate a  $3 + 2N + \frac{1}{2}N(N - 1)$  parameter subalgebra of the  $N$ -extended superconformal algebra which is isomorphic to the superalgebra  $osp(N/2)$ .

It is easily verified that the  $N = 4$  regular schwarzian derivative vanishes for a superprojective transformation of the form (6.9) (in fact, the regular term in

$S^i(Z, \tilde{Z})$  vanishes for general fractional linear transformations). By reduction, the same holds for the  $N < 4$  schwarzian derivatives.

From the twisted  $N = 4$  schwarzian derivative as given in (5.5) we can extract a covariant derivative without losing the locality of the expression. This leads to

$$S_{\text{twisted}}^{(4)} = \frac{c'}{12} \log[\pm \det(D\tilde{\theta})] \tag{6.13}$$

(with  $\pm$  as in (2.7)), which is consistent with the group property (3.9). For infinitesimal transformations the vanishing of (6.13) gives the condition  $\partial_z E = 0$  which allows for  $8 + \underline{8}$  non-zero parameters

$$\alpha_{-1}, \alpha_{-1/2}^i, \alpha_0^{ij}, \alpha_{1/2}^{ijk}, \alpha_1^{ijkl}. \tag{6.14}$$

Thus the regular and the twisted  $N = 4$  schwarzian derivatives select different finite dimensional extensions of the  $N = 4$  Poincaré algebra. The finite superconformal transformations  $Z \rightarrow \tilde{Z}$  corresponding to (6.14) are parametrized by  $8 + \underline{8}$  parameters

$$b, \epsilon^i, t^{ij}, \gamma^i, s,$$

where  $\epsilon^i, \gamma^i$  are Grassmann,  $t^{ij}t^{ik} = \delta^{ik}$ :

$$\begin{aligned} \tilde{z} &= z + b - (\epsilon^i t^{ij} \theta^j) (1 + 2(\theta^k \gamma^k)^2) - (\epsilon^i t^{ij} \epsilon^{jklm} \gamma^k) (\theta^l \theta^m) \\ &\quad + (6\epsilon^i t^{ij} s + 2\gamma^j) \theta^{4-j} - 12s\theta^4, \\ \tilde{\theta}^i &= \epsilon^i + t^{ij} \theta^j (1 + 2(\theta^k \gamma^k)^2) + t^{ij} \epsilon^{jklm} (\gamma^k + s\theta^k) \theta^l \theta^m, \\ \det(D\tilde{\theta}) &= \det(t). \end{aligned} \tag{6.15}$$

### 7. Central charges for $N = 4$

We define components  $J_n^{[i]}$  of the current superfield  $J^{(N)}(Z)$  by

$$J_n^{[i]} = \oint_{C_0} \frac{dz}{2\pi i} \int d^N \theta i^j \theta^j \theta^{[i]} z^{n+1-R/2} J^{(N)/2}(Z). \tag{7.1}$$

The SOPE  $J^{(N)}(Z_1)J^{(N)}(Z_2)$  is equivalent to the (anti)commutator algebra of the component fields  $J_n^{[i]}$ . Up to central terms this component algebra coincides with the classical  $\mathfrak{o}(N)$ -extended superconformal algebra (2.11). In this section we focus on the algebra for  $N = 4$  which we denote by SCA(4).



We introduce a real parameter  $\alpha$  and we define the following set of generators

$$\begin{aligned}
 L_n &= \frac{1}{2}J_n + \alpha n(n+1) \frac{1}{4!} \epsilon^{ijkl} J_n^{ijkl} . \\
 G_n^i &= J_n^i + 2i\alpha(n + \frac{1}{2}) \frac{1}{3!} \epsilon^{ijkl} J_n^{jkl} . \\
 T_n^{ij} &= J_n^{ij} , \\
 \Gamma_n^i &= -\frac{1}{3!} \epsilon^{ijkl} J_n^{jkl} , \\
 \Delta_n &= \frac{1}{4!} \epsilon^{ijkl} J_n^{ijkl} .
 \end{aligned}
 \tag{7.2}$$

The (anti)commutator algebra of these generators is listed in table 3. The charges  $c_\alpha, c'_\alpha$  are related to  $c, c'$  by

$$c_\alpha = c(1 + 4\alpha^2) - 4\alpha c' , \quad c'_\alpha = c' - 2\alpha c .
 \tag{7.3}$$

Note that the (anti)commutators depend non-trivially on  $\alpha$ , not only through the value of the central charges but also through some of the structure constants. The generators  $L_{-1,0,1}, G_{-1/2,1/2}^i, T_0^{ij}$  span a finite dimensional subalgebra of SCA(4). These subalgebras are not equivalent for different values of  $\alpha$ ; they are isomorphic to the superalgebras  $D(2/1; \alpha - \frac{1}{2})$  occurring in Kac classification [36, 37] of simple finite dimensional superalgebras.

The structure of the (anti)commutator algebra as listed in table 3 looks complicated but it can quite easily be understood in the following way. The  $\mathfrak{o}(4)$  Kac-Moody subalgebra is the sum of two commuting  $\mathfrak{su}(2)$  Kac-Moody algebras, corresponding to the selfdual and anti-selfdual combinations of the generators  $T_n^{ij}$ , which have level  $\frac{1}{3}(c + c')$  and  $\frac{1}{3}(c - c')$  resp. The central extensions of the  $\mathfrak{su}(2)$  Kac-Moody subalgebras both extend to the full superconformal algebra which explains why SCA(4) admits two independent central extensions. The parameter  $\alpha$  measures the asymmetry between the occurrences of the two  $\mathfrak{su}(2)$  Kac-Moody subalgebras in (anti)commutators involving odd generators. For  $\alpha = \pm \frac{1}{2}$  one of the  $\mathfrak{su}(2)$  Kac-Moody subalgebras together with the generators  $\Gamma_n^i, \Delta_n$  decouple and we are left with a  $\mathfrak{su}(2)$ -extended  $N = 4$  superconformal algebra, which was already described in the original paper by Ademollo et al. [1].

For integrable highest weight representations of SCA(4) both  $\frac{1}{3}(c + c')$  and  $\frac{1}{3}(c - c')$ , which occur as levels of  $\mathfrak{su}(2)$  Kac-Moody subalgebras, are necessarily positive integers. This leads to the following set of allowed values for the central

TABLE 3  
The o(4)-extended superconformal algebra

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$$\begin{aligned}
 [L_m, L_n] &= (m - n)L_{m+n} + \frac{1}{12}c_\alpha m(m^2 - 1)\delta_{m+n} \\
 [L_m, G_n^i] &= (\frac{1}{2}m - n)G_{m+n}^i & [L_m, T_n^{ij}] &= -nT_{m+n}^{ij} \\
 [L_m, \Gamma_n^i] &= (-\frac{1}{2}m - n)\Gamma_{m+n}^i & [L_m, \Delta_n] &= (-m - n)\Delta_{m+n} - \frac{1}{6}c'_\alpha(m + 1)\delta_{m+n} \\
 \{G_m^i, G_n^j\} &= 2\delta^{ij}L_{m+n} - i(m - n)(T_{m+n}^{ij} + 2\alpha\tilde{T}_{m+n}^{ij}) + \frac{1}{3}c_\alpha(m^2 - \frac{1}{4})\delta^{ij}\delta_{m+n} \\
 [G_m^i, T_n^{jk}] &= n\epsilon^{ijkl}\Gamma_{m+n}^l + 2i\delta^{il}G_{m+n}^k - 4n\alpha\delta^{il}\Gamma_{m+n}^k \\
 \{G_m^i, \Gamma_n^j\} &= i(m + n)\delta^{ij}\Delta_{m+n} - \tilde{T}_{m+n}^{ij} + \frac{1}{3}ic'_\alpha(m + \frac{1}{2})\delta^{ij}\delta_{m+n} \\
 [G_m^i, \Delta_n] &= i\Gamma_{m+n}^i \\
 [T_m^{ij}, T_n^{kl}] &= -i(\delta^{ik}T_{m+n}^{jl} - \delta^{il}T_{m+n}^{jk} - \delta^{jk}T_{m+n}^{il} + \delta^{jl}T_{m+n}^{ik}) \\
 &\quad + \frac{1}{3}c(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})m\delta_{m+n} + \frac{1}{3}c'\epsilon^{ijkl}m\delta_{m+n} \\
 [T_m^{ij}, \Gamma_n^k] &= -2i\delta^{ki}\Gamma_{m+n}^j & [T_m^{ij}, \Delta_n] &= 0 \\
 \{\Gamma_m^i, \Gamma_n^j\} &= \frac{1}{3}c\delta^{ij}\delta_{m+n} & [\Gamma_m^i, \Delta_n] &= 0 \\
 [\Delta_m, \Delta_n] &= \frac{1}{3}c\delta_{m+n}/m
 \end{aligned}$$


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The parameter  $\alpha$  labels the choice of the generators  $L_n, G_n^i, T_n^{ij}, \Gamma_n^i$  and  $\Delta_n$  as in (7.2).

charges  $c, c', c_\alpha, c'_\alpha$

$$\begin{aligned}
 c &= \frac{3}{2}(n_+ + n_-), & c_\alpha &= \frac{3}{2}(1 - 2\alpha)^2 n_+ + \frac{3}{2}(1 + 2\alpha)^2 n_-, \\
 c' &= \frac{3}{2}(n_+ - n_-), & c'_\alpha &= \frac{3}{2}(1 - 2\alpha)n_+ - \frac{3}{2}(1 + 2\alpha)n_-, \\
 & & & n_+, n_- \in \mathbb{Z}_{>0}. \quad (7.4)
 \end{aligned}$$

We expect that in physical models with O(4)-extended superconformal symmetry the generators  $\Delta_n$  are actually primary w.r.t. the Virasoro algebra, i.e. that the coefficient  $c'_\alpha$  of the anomaly in  $[L_m, \Delta_n]$  vanishes. This criterion leads to the following quantization of the parameter  $\alpha$

$$\alpha = \frac{1}{2} \frac{n_+ - n_-}{n_+ + n_-}, \quad n_+, n_- \in \mathbb{Z}_{>0} \quad (7.5)$$

and of the Virasoro central charge  $c_\alpha$

$$c_\alpha(n_+, n_-) = \frac{6n_+n_-}{n_+ + n_-}, \quad n_+, n_- \in \mathbf{Z}_{>0}. \quad (7.6)$$

Remarkably, the same quantization condition of the parameter  $\alpha$  comes out of an analysis of a classical Liouville-WZW  $\sigma$ -model based on the algebra SCA(4) [29].

For the symmetric case ( $n_+ = n_-$ ,  $\alpha = 0$ ) the  $c_\alpha$  values (7.6) are the positive multiples of 3. These values can all be realized by taking the tensor product of copies of a basic  $c = 3$  representation, which involves a real scalar field and four Majorana fermions [5]. The su(2)-extended algebra, which is obtained in the limit  $n_\pm \rightarrow \infty$ ,  $\alpha \rightarrow \pm \frac{1}{2}$ , allows multiples of 6 for its central charge. These can all be realized in a linear free field representation.

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