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Schoutens, K.

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A NON-LINEAR REPRESENTATION
OF THE \( d=2 \) so(4)-EXTENDED SUPERCONFORMAL ALGEBRA

K. SCHOUTENS
Institute for Theoretical Physics, Princetonplein 5, PO Box 80.006, 3508 TA Utrecht, The Netherlands

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We present a non-linear representation of the so(4)-extended \( d=2 \) superconformal algebra in terms of one boson and four Majorana fermions. The matter fields and the currents can be grouped into a single \( N=4 \) superfield. Breaking the supersymmetry to \( N=3 \) or \( N=2 \) leads to new representations of the \( N=3,2 \) superconformal algebras.

1. Introduction. The study of two-dimensional models possessing conformal or superconformal symmetries [1–3] is relevant both for statistical mechanics and high-energy physics. Exponents describing the critical behaviour of certain \( d=2 \) statistical systems can be related to weights of unitary representations of (super)conformal algebras [4,5]. In string theory (super)conformal algebras are present as gauge algebras in the two-dimensional formulation of (super)string models [6].

In addition to the \( d=2 \) conformal algebra (Virasoro algebra) and the \( N=1 \) superconformal algebra (Neveu–Schwarz–Ramond algebra [7,8]) superconformal algebras (SCA) with extended supersymmetry exist. An \( N \)-extended superconformal algebra consists of Virasoro generators \( L_n \), supercharges \( G^i_m, m=1,2,\ldots,N \), and additional generators that are needed to close the algebra. The most important examples are

(i) The regular series of \( N \)-extended \( (N\geq2) \) SCA given by Ademollo et al. [9] which have the so(\( N \)) Kac–Moody algebra (KMA) as a subalgebra. For \( N\leq4 \) these algebras admit a central extension.

(ii) An exceptional \( N=4 \) SCA with the su(2) KMA as a subalgebra [9,10].

The SCA's exist in different varieties which are in general not equivalent (Neveu–Schwarz, Ramond, twisted, non-twisted, see e.g. ref. [11]). A classification of all SCA's with generators of conformal spin \( 2\geq J\geq\frac{1}{2} \) has been given in refs. [12,13].

It is the purpose of this note to show that the so(\( N \))-extended SCA's with \( N=2,3 \) or 4 can all be realized non-linearly on the Hilbert space spanned by a single real boson \( \phi \) and four Majorana fermions \( \chi^i, i=1,\ldots,4 \), or, alternatively, on the Hilbert space spanned by six Majorana fermions \( \chi^i, i=1,\ldots,4 \), and \( \psi^a, a=1,2 \).

2. \( N \)-extended superconformal theory. In ref. [9] the so(\( N \))-extended SCA is defined as the algebra of superconformal transformations of the coordinates \( Z=(z,\theta^i), i=1,\ldots,N \), which describe one light-cone sector of a superspace extension of \( d=2 \) spacetime. The variation of \( z,\theta^i \) under the action of a generator \( G^{i_1\ldots i_R}_n \) with parameter \( \alpha^{i_1\ldots i_R}_n \) reads

\[
\delta z = i^{R(R-1)/2} (2-R) \alpha^{i_1\ldots i_R}_n \theta^{i_1} \ldots \theta^{i_R} z^{n+1-\frac{R}{2}},
\]

\[
\delta \theta^i = -i^{R(R-1)/2} \left( \sum_{l=1}^{R} (-1)^{R-l} \delta^i_{i_l} \alpha^{i_1\ldots i_l i_{l+1}\ldots i_R}_n \theta^{i_1} \ldots \theta^{i_l} \theta^{i_{l+1}} \ldots \theta^{i_R} z^{n+1-\frac{R}{2}} - (n+1-\frac{1}{2} R) \alpha^{i_1\ldots i_R}_n \theta^{i_1} \ldots \theta^{i_{R-1}} \theta^{i_R} z^{n+1-\frac{R}{2}} \right).
\] (1)

Both \( G^{i_1\ldots i_R}_n \) and \( \alpha^{i_1\ldots i_R}_n \) are completely antisymmetric in \( i_1\ldots i_R \). The index \( n \) is integer (half-integer) if \( R \) is even (odd). The algebra of the transformations (1) is
A general superconformal transformation is parametrized by the displacement superfield

$$E(z, \theta') = 2 \sum_{n=1}^{R} i^{R(R-1)/2} \alpha_{n}^{(i)} \theta^{(i)} z^{n+1-\frac{R+1}{2}} ,$$

(3)

where \((i)\) denotes a general multi-index \(i_1 \ldots i_R, 0 \leq R \leq N\). The transformation of a primary superfield \(\Phi_\alpha\) of dimension \(d\) reads (cf. refs. [1,14])

$$\delta \Phi_\alpha = \frac{1}{2} \frac{dZ}{2\pi i} E(Z, \theta') \frac{dZ}{2\pi i} \Phi_\alpha ,$$

(6)

where \(E(z, \theta')\) is the super operator product expansion (SOPE) of \(J^{(N)}(Z_1)\) and \(\Phi(Z_2)\).

### 3. A representation of the so(4)-extended SCA.

Having demonstrated some general features of theories with \(N\)-extended superconformal invariance we now turn to concrete realizations of these theories in terms of bosonic and fermionic quantum fields. If these fields are described by a free action then the central term in the Virasoro subalgebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12} c(m^2-1) \delta_{m+n} ,$$

(7)

is given by

$$c = \# \text{(real bosons)} + \frac{1}{2} \# \text{(Majorana fermions)} ,$$

(8)

A number of representations of SCA’s in terms of free bosonic and fermionic fields are known. The \(N=0,1,2\) and the exceptional \(N=4\) SCA’s admit linear representations with \(c=1, 3/2, 3\) and 6 respectively which form the basis of the \(N=0,1,2\) and 4 (spinning) string models [6]. In ref. [15] a purely bosonic non-linear representation of the \(N=2\) SCA is given in terms of vertex operators built from a single bosonic field taking values on a circle (see also ref. [16]). Also in ref. [11], where a \(c=3/2\) representation of the \(N=3\) SCA is presented, vertex operators are used to construct some of the superconformal generators. Another possibility, which has first been noted in refs. [14,17], is to realize superconformal symmetries non-linearly among free fermions only. For \(N=1\) the most general construction of this type has been discussed in ref. [18]. In ref. [19] a free fermion construction has been used to construct the discrete series of \(c<3\) unitary representations of the \(N=2\) SCA.
Here we will present a non-linear $c=3$ representation of the so(4)-extended $N=4$ SCA which we obtain by combining the following two $c=3/2$ representations of the $N=1$ SCA.

(i) The linear $N=1$ representation defined in terms of a real boson $\phi(z)$ and a Majorana fermion $\psi(z)$. Using the elementary contractions
\begin{equation}
\langle \phi(z)\phi(w) \rangle = -\log(z-w), \quad \langle \psi(z)\psi(w) \rangle = \frac{1}{z-w}
\end{equation}
and the Wick theorem it is easily shown that the currents
\begin{equation}
L(z) = -\frac{i}{2}(\partial_z\phi)^2 - \frac{i}{2}\psi\partial_z\psi, \quad G(z) = -i\psi\partial_z\phi
\end{equation}
satisfy the following short distance operator product expansions (OPE) with $c=3/2$:
\begin{align}
L(z) L(w) &= \frac{\frac{3}{2}c}{(z-w)^2} + \frac{2L(w)}{z-w} + \cdots,
L(z) G(w) &= \frac{\frac{3}{2}c}{(z-w)^2} + \frac{2L(w)}{z-w} + \cdots, \quad G(z) G(w) &= \frac{\frac{3}{2}c}{(z-w)^3} + \frac{2L(w)}{z-w} + \cdots.
\end{align}
These OPE's are equivalent to the component algebra (2), which for $N=1$ is just the familiar NSR algebra.

(ii) The purely fermionic $N=1$ representation defined in terms of three Majorana fermions $\chi^i(z)$, $i=1,2,3$. The currents
\begin{equation}
L(z) = -\frac{i}{4}\chi^i\partial_z\chi^i, \quad G^i(z) = i\chi^i\chi^j\chi^k
\end{equation}
satisfy the same OPE's (11) with $c=3/2$. Supersymmetry is realized non-linearly on the fields $\chi^i(z)$
\begin{equation}
\delta_\omega \chi^i(z) = \frac{i}{2}ie^{ijk}\epsilon(z)\chi^j(z)\chi^k(z).
\end{equation}
The important observation is now that the combination $(\phi,\psi,\chi^i)$ of both $c=3/2$ systems, which trivially realizes the $N=1$ SCA with $c=3$, has a much richer symmetry structure which turns out to be as large as the so(4)-extended SCA! In order to see this we write $(\chi^i,\psi)$ as $\chi^i$, $i=1,...,4$, and we define
\begin{equation}
L(z) = -\frac{1}{4}(\partial_z\phi)^2 - \frac{1}{4}\chi^i\partial_z\chi^i, \quad G^i(z) = -\frac{i}{4}ie^{ijk}\chi^j\chi^k - i\chi^i\partial_z\phi,
\end{equation}
\begin{equation}
T^i(z) = i\chi^i, \quad \Gamma^i(z) = \chi^i, \quad A(z) = \phi.
\end{equation}
These operators satisfy the following OPE's with $c=3$:
\begin{align}
L(z)L(w) &= \frac{\frac{3}{2}c}{(z-w)^2} + \frac{2L(w)}{z-w} + \frac{\partial_w L(w)}{z-w} + \cdots, \quad L(z)G^i(w) = \frac{\frac{3}{2}c}{(z-w)^2} + \frac{\partial_w G^i(w)}{z-w} + \cdots, \\
L(z)T^{ij}(w) &= \frac{\Gamma^{ij}(w)}{(z-w)^2} + \frac{\partial_w T^{ij}(w)}{z-w} + \cdots, \\
L(z)\Gamma^i(w) &= \frac{\frac{3}{2}c\delta^{ij}}{(z-w)^3} + \frac{2iT^{ij}(w)}{(z-w)^2} - \frac{i\partial_w T^{ij}(w)}{z-w} + \frac{2\delta^{ij}L(w)}{z-w} + \cdots, \\
G^i(z)G^j(w) &= -\epsilon^{ijk}\left[\frac{\Gamma^i(w)}{(z-w)^2} + \frac{\partial_w \Gamma^i(w)}{z-w}\right] - \frac{i[\delta^{ik}G^j - \delta^{ij}G^k(w)]}{z-w} + \cdots.
\end{align}
Table 1

The so(4)-extended superconformal algebra. A c=3 realization of this algebra is provided by the Laurent coefficients \( L_n = \frac{\partial}{\partial z} \), \( G_n = \frac{\partial}{\partial z} \), \( T_n = \frac{\partial}{\partial z} \), \( F_n = \frac{\partial}{\partial z} \), \( J_n = \frac{\partial}{\partial z} \). Up to central terms this algebra is identical to the algebra (2) for \( N=4 \) by the identifications \( L_n \rightarrow L_n \), \( G_n \rightarrow G_n \), \( T_n \rightarrow T_n \), \( F_n \rightarrow F_n \), \( J_n \rightarrow J_n \).

\[
\begin{align*}
[L_n, L_n] &= (m-n) \delta_{m+n}, \\
[L_n, G_n] &= L_n, \\
[L_n, T_n] &= -T_n, \\
[L_n, F_n] &= -F_n, \\
[L_n, J_n] &= -J_n,
\end{align*}
\]

\[
\begin{align*}
[L_n, G_m] &= 2 \delta^{nm} T_n, \\
[L_n, T_m] &= 0, \\
[L_n, F_m] &= 0, \\
[L_n, J_m] &= 0.
\end{align*}
\]

\[
\begin{align*}
[T_n, T_m] &= -i \delta^{nk} T_{n+k}, \\
[T_n, F_m] &= -i \delta^{nk} F_{n+k}, \\
[T_n, J_m] &= -i \delta^{nk} J_{n+k}.
\end{align*}
\]

\[
\begin{align*}
\{G_n, G_m\} &= 2 \delta^{nm} T_n, \\
\{G_n, T_m\} &= -i \delta^{nk} T_{n+k}, \\
\{G_n, F_m\} &= -i \delta^{nk} F_{n+k}, \\
\{G_n, J_m\} &= -i \delta^{nk} J_{n+k}.
\end{align*}
\]

\[
\begin{align*}
\{T_n, F_m\} &= -i \delta^{nk} T_{n+k}, \\
\{T_n, J_m\} &= -i \delta^{nk} J_{n+k}.
\end{align*}
\]

This establishes our central result: the currents (14) form a closed \( N=4 \) superconformal algebra. The commutator algebra of the Laurent coefficients \( L_n, G_n, T_n, F_n, A_n \) is listed in table 1.

From the representation (14) we can also obtain a purely fermionic representation of the so(4)-extended SCA. By introducing two fermions \( \psi^I(z), \chi^I(z) \) and making the replacements

\[
L(z) \rightarrow -\frac{1}{2} :\chi^I \partial \chi^I : - \frac{1}{2} :\chi^I \partial \chi^I : \partial \psi^I \partial \psi^I : \partial^2 \psi^I \partial \psi^I : \psi^I \chi^I \chi^I : \chi^I \chi^I \psi^I \psi^I : ,
\]

we obtain a realization of the so(4)-extended SCA in terms of fermionic fields only. The presence of six fermionic fields in this representation suggests that it could be possible to have a so(6) symmetry among these fermions and to define as much as \( \frac{3}{2} = 20 \) supercharges trilinear in the fermionic fields. However, upon closer inspection it turns out that in order to have a closed algebra without four-fermion terms (as we have in (15)) it is necessary to break the symmetry to so(4) and to keep only \( \frac{1}{2} = 4 \) of the supercharges.

In the representation (14) the superconformal transformations are realized non-linearly on the matter fields \( (\phi, \chi^I) \) but (if we discard the central terms for a moment) they are realized linearly on the currents \( (L_n, G_n, T_n, F_n, A_n) \). The current superfield, which for general \( N \) is given in (5) reads as follows for \( N=4 \):

\[
J^{(4)} = -\Delta + i \theta^I \Gamma^I - \frac{1}{2} \epsilon^{ijkl} \epsilon^{ijkl} G^I + \gamma^I \epsilon^{ijkl} \theta^I \theta^j \theta^k \theta^l L.
\]

Apart from central terms, \( J^{(4)} \) transforms as a primary superfield of dimension 0. The OPE's (15) for the currents \( (L_n, G_n, T_n, F_n, A_n) \) are equivalent to the following SOPE for the current superfield \( J^{(4)} \):

\[
J^{(4)}(Z_1)J^{(4)}(Z_2) = \left( \sum_{l} \frac{\theta^l_1}{Z} \frac{\theta^l_2}{Z} D_l + 2 \frac{\theta^l_1}{Z} \frac{\theta^l_2}{Z} \partial \right) J^{(4)}(Z_2) - \log(Z_{12}),
\]

This is the superconformal transformation non-linearly on the matter fields \( (\phi, \chi^I) \) but (if we discard the central terms for a moment) they are realized linearly on the currents \( (L_n, G_n, T_n, F_n, A_n) \). The current superfield, which for general \( N \) is given in (5) reads as follows for \( N=4 \):
where $Z_{12} = z_1 - z_2 - \Sigma, \theta_1 \theta_2$ and $\theta_{12} = \theta_1 - \theta_2$. Note that this single formula summarizes all the commutation relations of the so(4)-extended SCA as listed in table 1. Since $A(z) = \phi(z)$ and $I'(z) = \chi'(z)$ we have the amusing situation that the $N=4$ current superfield is at the same time the $N=4$ matter superfield, or, stated differently, that the currents and the matter fields are in the same $(8+8)$ component chiral multiplet.

4. Reduction to $N=3,2$. Since the so($N$)-extended SCA's with $N=3,2$ are subalgebras of the $N=4$ algebra, the fields $(\phi, \chi')$ automatically provide representations of these smaller algebras, which are of some interest in their own right. Upon reducing $N=4\rightarrow N=3$ we only keep the superconformal generators $L(z), G'(z), T'(z) = -\frac{1}{8} \epsilon^{ijk} T^{jk}(z)$ and $I'(z), i,j,k = 1,2,3$. Their Laurent coefficients $L_n, G'_n, T'_n, I'_n$ form the so(3)-extended SCA with central charge $c=3$ which is a subalgebra of the $N=4$ SCA listed in table 1. The superfield $J^{(4)} = -\Phi^{(3)} = \Phi^{(3)}$, where $J^{(3)}$ is the $N=3$ current superfield (cf. (5)) and $\Phi^{(3)}$ is an $N=3$ matter superfield,

$$\Phi^{(3)} = \phi - i\theta^i \chi^i - \frac{1}{2} \epsilon^{ijk} \theta^i \theta^j \chi^k \psi + \frac{1}{2} \epsilon^{ijk} \theta^i \theta^j \theta^k (\frac{1}{6} \epsilon^{mnk} \chi^m \chi^n - \psi \partial \phi).$$

Interestingly, this construction automatically provides an answer to the question of how to obtain an invariant action density for the components $(\phi, x', F_1, \lambda)$ of a chiral $N=3$ multiplet, i.e. a $N=3$ scalar superfield $\phi(z)$. It has been noted in ref. [9] that the traditional way to obtain an invariant action from a scalar superfield (by constructing a superspace density of the appropriate dimension from $\Phi$ and its covariant derivatives $D\Phi$) fails for the $N$-extended algebras with $N>3$. From the analysis above we see that for $N=3$ this problem can be solved quantum mechanically by expressing the "auxiliary fields" $F_1$ and $\lambda$ in $\phi, \chi'$ and an additional field $\psi$ according to

$$F_k = \chi^k \psi, \quad \lambda = \frac{1}{2} \epsilon^{ijk} \chi^j \chi^k \psi - \psi \partial \phi.$$   (19)

A free action for the fields $(\phi, \chi', \psi)$ is then invariant under all superconformal transformations. The price one has to pay is that supersymmetry is realized non-linearly on $(\phi, \chi', \psi)$,

$$\delta \phi = ie^{i\theta} \chi', \quad \delta \chi' = -ie^{i\theta} \chi', \quad \delta \chi = -ie^{i\theta} \psi + ie^{i\theta} \chi^k \psi, \quad \delta \psi = -\frac{1}{2} ie^{i\theta} \chi^k \chi^k.$$  (20)

A similar analysis for the breaking $N=4\rightarrow N=2$ leads to a non-linear $c=3$ representation of the $N=2$ SCA with one real boson and four Majorana fermions, in contrast with the linear $c=3$ representation which has a complex scalar field and one single Dirac fermion.

5. Comments.

(i) Superstring compactification. By applying fermionization to compactified coordinates the degrees of freedom of the $N=1$ superstring can be organized as follows [20–22]:

$$(x^\mu, \psi^\mu) \quad \mu = 1, \ldots, 4, \quad \chi^I = 1, \ldots, 18.$$  

Since the number of internal fermions is a multiple of 6 it is possible to define a global $N=4$ superconformal symmetry on the internal fermionic degrees of freedom. It would be interesting to see whether this symmetry has any consequences for superstring dynamics.

(ii) Highest weight representations. A representation of the so(4)-extended SCA automatically provides a representation of the so(4) Kac–Moody subalgebra, which has level $k = c/3$. In a unitary highest weight representation the level $k$ is a positive integer and consequently $c$ is a positive multiple of 3. These values for $c$ can all be realized by taking tensor products of copies of the basic $c=3$ representation (14).

(iii) $N=3,4$ string models. In principle the field $\phi$ in the multiplet $(\phi, \chi')$ can be interpreted as a string-coordinate. However, due to the non-linearity of the supersymmetry transformations the formulation of "so(3)- or so(4)-strings" is not a straightforward extension of the results for strings with $u(1)$- or su(2)-extended supersymmetry [23,10,24] .

(iv) Critical central charge. If a SCA appears as the algebra of constraints of a quantum system then the
theory is consistent only for one special value of the central charge. In ref. [25] it is shown that in the BRST-quantization of a system with constraints corresponding to the \( N=3 \) SCA the ghost contribution to \( \mathcal{Q}^2 \) vanishes and that accordingly the value of the critical central charge for this algebra is 0. It turns out that also the \( N=4 \) so(4)-extended SCA has critical central charge \( c=0 \). This value differs from the critical value of \( c \) for the \( N=4 \) su(2)-extended SCA which is \(-12\).

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