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A NON-LINEAR REPRESENTATION
OF THE $d=2\ so(4)$-EXTENDED SUPERCONFORMAL ALGEBRA

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We present a non-linear representation of the $so(4)$-extended $d=2$ superconformal algebra in terms of one boson and four Majorana fermions. The matter fields and the currents can be grouped into a single $N=4$ superfield. Breaking the supersymmetry to $N=3$ or $N=2$ leads to new representations of the $N=3,2$ superconformal algebras.

1. Introduction. The study of two-dimensional models possessing conformal or superconformal symmetries [1–3] is relevant both for statistical mechanics and high-energy physics. Exponents describing the critical behaviour of certain $d=2$ statistical systems can be related to weights of unitary representations of (super)conformal algebras [4,5]. In string theory (super)conformal algebras are present as gauge algebras in the two-dimensional formulation of (super)string models [6].

In addition to the $d=2$ conformal algebra (Virasoro algebra) and the $N=1$ superconformal algebra (Neveu–Schwarz–Ramond algebra [7,8]) superconformal algebras (SCA) with extended supersymmetry exist. An $N$-extended superconformal algebra consists of Virasoro generators $L_n$, supercharges $G^i_m$, $m=1,2,...,N$, and additional generators that are needed to close the algebra. The most important examples are

(i) The regular series of $N$-extended ($N\geqslant 2$) SCA given by Ademollo et al. [9] which have the $so(N)$ Kac–Moody algebra (KMA) as a subalgebra. For $N\leqslant 4$ these algebras admit a central extension.

(ii) An exceptional $N=4$ SCA with the $su(2)$ KMA as a subalgebra [9,10].

The SCA's exist in different varieties which are in general not equivalent (Neveu–Schwarz, Ramond, twisted, non-twisted, see e.g. ref. [11]). A classification of all SCA’s with generators of conformal spin $2\geqslant J\geqslant \frac{1}{2}$ has been given in refs. [12,13].

It is the purpose of this note to show that the $so(N)$-extended SCA's with $N=2,3$ or 4 can all be realized non-linearly on the Hilbert space spanned by a single real boson $z$ and four Majorana fermions $\psi^i$, $i=1,...,4$, or, alternatively, on the Hilbert space spanned by six Majorana fermions $\chi^i$, $i=1,...,4$, and $\psi^a$, $a=1,2$.

2. $N$-extended superconformal theory. In ref. [9] the $so(N)$-extended SCA is defined as the algebra of superconformal transformations of the coordinates $Z=(z,\theta^i)$, $i=1,...,N$, which describe one light-cone sector of a superspace extension of $d=2$ spacetime. The variation of $z,\theta^i$ under the action of a generator $G^i_{n...iR}$ with parameter $\alpha_n^{i_1...i_R}$ reads

$$\delta z = i^{R(R-1)/2} (2-R) \alpha_n^{i_1...i_R} \theta_i^{i_1...i_R} \psi^{i+1} z^{n+1-R/2},$$

$$\delta \theta^i = -i^{R(R-1)/2} \left( \sum_{i=1}^R (-1)^{R-i} \delta^{i_1...i_R} \alpha_n^{i_1...i_R} \theta_i^{i_1...i_R} \psi^i \psi^{i+1} z^{n+1-R/2} - (n+1-\frac{1}{2}R) \alpha_n^{i_1...i_R} \theta_i^{i_1...i_R} \psi^{i+1} z^{n+1-R/2} \right). \quad (1)$$

Both $G^i_{n...iR}$ and $\alpha_n^{i_1...i_R}$ are completely antisymmetric in $i_1,...,i_R$. The index $n$ is integer (half-integer) if $R$ is even (odd). The algebra of the transformations (1) is
A general superconformal transformation is parametrized by the displacement superfield
\[ E(z, \theta) = 2 \sum_{n} \sum_{(i)} i^{R(R-1)/2} \alpha^{(i)} \theta^{(i)} z^{n+1 - R/2}, \]
where \( (i) \) denotes a general multi-index \( i_1, ..., i_R, 0 \leq R \leq N \). The transformation of a primary superfield \( \Phi_\sigma \) of dimension \( \sigma \) reads (cf. refs. [1, 14])
\[ \delta \Phi_\sigma = \left( \frac{D'}{D} \right) (D^* \Phi_\sigma) + \left( \frac{D}{D'} \right)^* \Phi_\sigma = E \Phi_\sigma + \frac{1}{2} (D'E)(D^* \Phi_\sigma) + \Lambda(\partial \Phi_\sigma). \]
where we used the covariant derivative \( D' = \partial \theta + \theta \partial z \).

Using (3) and (5) we can write the variation of a general superfield under the superconformal transformation parametrized by \( E(z, \theta') \) very concisely as
\[ \delta \Phi(Z_2) = \frac{1}{2} \oint_{C_\infty} \frac{dZ_1}{2\pi i} E(Z_1) J^{(N)}(Z_1) \Phi(Z_2), \]
where \( Z = (z, \theta') \), \( \oint_{C_\infty} \frac{dZ}{2\pi i} = \oint dz/2\pi i \oint d\theta \) and \( C_\infty \) is a curve in the complex plane enclosing the point \( z \). All information about the transformation properties of \( \Phi(z) \) is encoded in the super operator product expansion (SOPE) of \( J^{(N)}(Z_1) \) and \( \Phi(Z_2) \).

3. A representation of the so(4)-extended SCA. Having demonstrated some general features of theories with \( N \)-extended superconformal invariance we now turn to concrete realizations of these theories in terms of bosonic and fermionic quantum fields. If these fields are described by a free action then the central term in the Virasoro subalgebra
\[ [L_n, L_m] = (n - m)L_{n + m} + \frac{1}{2} cm(m^2 - 1) \delta_{m+n} \]
is given by
\[ c = \#(\text{real bosons}) + \frac{1}{2} \#(\text{Majorana fermions}). \]

A number of representations of SCAs's in terms of free bosonic and fermionic fields are known. The \( N=0,1,2 \) and the exceptional \( N=4 \) SCA's admit linear representations with \( c = 1, 3/2, 3 \) and 6 respectively which form the basis of the \( N=0,1,2 \) and 4 (spinning) string models [6]. In ref. [15] a purely bosonic non-linear representation of the \( N=2 \) SCA is given in terms of vertex operators built from a single bosonic field taking values on a circle (see also ref. [16]). Also in ref. [11], where a \( c=3/2 \) representation of the \( N=3 \) SCA is presented, vertex operators are used to construct some of the superconformal generators. Another possibility, which has first been noted in refs. [14,17], is to realize superconformal symmetries non-linearly among free fermions only. For \( N=1 \) the most general construction of this type has been discussed in ref. [18]. In ref. [19] a free fermion construction has been used to construct the discrete series of \( c < 3 \) unitary representations of the \( N=2 \) SCA.
Here we will present a non-linear $c=3$ representation of the so(4)-extended $N=4$ SCA which we obtain by combining the following two $c=3/2$ representations of the $N=1$ SCA.

(i) The linear $N=1$ representation defined in terms of a real boson $\varphi(z)$ and a Majorana fermion $\psi(z)$. Using the elementary contractions

$$\langle \varphi(z)\varphi(w) \rangle = -\log(z-w), \quad \langle \psi(z)\psi(w) \rangle = \frac{1}{z-w}$$

and the Wick theorem it is easily shown that the currents

$$L(z) = -\frac{i}{2} :\partial_z \varphi^2 :, \quad G(z) = -i\psi \partial_z \varphi$$

satisfy the following short distance operator product expansions (OPE) with $c=3/2$:

$$L(z)L(w) = \frac{\frac{1}{2}c}{(z-w)^2} + \frac{2L(w)}{z-w} + \ldots,$$

$$L(z)G(w) = \frac{\frac{1}{2}c}{(z-w)^2} + \frac{\partial_z L(w)}{z-w} + \ldots,$$

$$L(z)\psi(w) = \frac{\frac{1}{2}c}{(z-w)^2} + \frac{\partial_z \psi(w)}{z-w} + \ldots$$

These OPE's are equivalent to the component algebra (2), which for $N=1$ is just the familiar NSR algebra.

(ii) The purely fermionic $N=1$ representation defined in terms of three Majorana fermions $\chi^{i}(z)$, $i=1,2,3$. The currents

$$\chi^{i}(z) = \frac{1}{2}\epsilon^{ijk} \chi^{j}(z) \chi^{k}(z)$$

satisfy the same OPE's (11) with $c=3/2$. Supersymmetry is realized non-linearly on the fields $\chi^{i}(z)$

$$\delta_{Q} \chi^{i}(z) = \frac{1}{2}i\epsilon^{ijk} \epsilon(z) \chi^{j}(z) \chi^{k}(z) .$$

The important observation is now that the combination $(\varphi, \psi, \chi^{i})$ of both $c=3/2$ systems, which trivially realizes the $N=1$ SCA with $c=3$, has a much richer symmetry structure which turns out to be as large as the so(4)-extended SCA! In order to see this we write $(\chi^{i}, \psi)$ as $\chi^{i}$, $i=1, \ldots, 4$, and we define

$$L(z) = -\frac{1}{2} :\chi_{i}^2 :, \quad \Gamma^{i}(z) = -\frac{1}{2}i\epsilon^{ijk} \chi^{j}(z) \chi^{k}(z) - i\chi^{i} \partial_z \varphi ,$$

$$T^{ii}(z) = i\chi^{i} \partial_z \chi^{i} , \quad \Gamma^{i}(z) = \chi^{i} , \quad A(z) = \varphi .$$

These operators satisfy the following OPE's with $c=3$:

$$L(z)L(w) = \frac{\frac{1}{2}c}{(z-w)^2} + \frac{2L(w)}{z-w} + \frac{\partial_w L(w)}{z-w} + \ldots,$$

$$L(z)\psi(w) = \frac{\frac{1}{2}c}{(z-w)^2} + \frac{\partial_w \psi(w)}{z-w} + \ldots,$$

$$G^{i}(z)G^{j}(w) = \frac{\frac{1}{2}c\delta^{ij}}{(z-w)^2} + \frac{2i\epsilon^{ijk} \chi^{k}(z) \chi^{k}(w) - i\epsilon^{ijkl} \Gamma^{l}(z) \Gamma^{l}(w)}{z-w} + \ldots,$$

$$G^{i}(z)T^{jk}(w) = -\epsilon^{ijk}\left( \frac{\Gamma^{l}(w) + \partial_w \Gamma^{l}(w)}{(z-w)^2} \right) - \frac{i}{2}\delta^{ik} G^{j}(z) - \frac{i}{2}\delta^{l} G^{k}(w) + \ldots .$$
The so(4)-extended superconformal algebra. A c=3 realization of this algebra is provided by the Laurent coefficients $L_n = \frac{1}{2}(dz/2\pi i)L(z)z^{n+1}, G_n = \frac{1}{2}(dz/2\pi i)G'(z)z^{n+1/2}, T_n = \frac{1}{2}(dz/2\pi i)T^n(z)z^{n+1/2}, \Gamma_n = \frac{1}{2}(dz/2\pi i)\Gamma^n(z)z^{n+1}$ of the currents (14). Up to central terms this algebra is identical to the algebra (2) for $N=4$ by the identifications $L_n = \frac{1}{2}G_n, G_n = G_n, T_n = T_n, \Gamma_n = (1/3!)(\Gamma_n)^{(3)} \equiv \frac{1}{2} \epsilon^{ijk}G_n^{(i)k}$.

Table 1

| $[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}c(m^2-1)\delta_{m+n}$ |
| $[L_m, G_n] = \frac{1}{12}(m-n)G_{m+n}$ |
| $[L_m, T_n] = -\frac{1}{2}(m-n)T_{m+n}$ |
| $[L_m, \Gamma_n] = -\frac{1}{2}(m-n)\Gamma_{m+n}$ |
| $[L_m, A_n] = (m-n)A_{m+n}$ |
| $[G_m, G_n] = 2\delta^{ij}G_{m+n} - \frac{1}{2}(m-n)T_{m+n}$ |
| $[G_m, T_n] = i(m+n)\delta^i J_j G^J_{m+n} - \frac{1}{2}(m-n)T_{m+n}$ |
| $[G_m, \Gamma_n] = i(m+n)^2 \delta^i J_j \Gamma^J_{m+n} - \frac{1}{2}(m-n)\Gamma_{m+n}$ |
| $[G_m, A_n] = i\frac{1}{2}(m+n)\delta^i J_j A^J_{m+n} - \frac{1}{2}(m-n)A_{m+n}$ |
| $[T_m, T_n] = -(\frac{1}{2}m-n)T_{m+n}$ |
| $[T_m, \Gamma_n] = \frac{1}{2}(\frac{1}{2}m-n)\Gamma_{m+n}$ |
| $[T_m, A_n] = \frac{1}{2}(\frac{1}{2}m-n)A_{m+n}$ |
| $[\Gamma_m, \Gamma_n] = \frac{1}{2}(\frac{1}{2}m-n)\Gamma_{m+n}$ |
| $[\Gamma_m, A_n] = \frac{1}{2}(\frac{1}{2}m-n)A_{m+n}$ |
| $[\Gamma_m, A_n] = \frac{1}{2}(\frac{1}{2}m-n)A_{m+n}$ |

This establishes our central result: the currents (14) form a closed $N=4$ superconformal algebra. The commutator algebra of the Laurent coefficients $L_m, G_n, T_n, \Gamma_n, A_n$ is listed in table 1.

From the representation (14) we can also obtain a purely fermionic representation of the so(4)-extended SCA. By introducing two fermions $\psi^i(z), \psi^i(z)$ and making the replacements

$L(z) \rightarrow -\frac{1}{2} :\psi^i\partial\psi^i: = -\frac{1}{2} :\chi^i\partial\chi^i: , \quad \partial_d A(z) \rightarrow \psi^i \psi^j , \quad G'(z) \rightarrow -\frac{1}{2} \epsilon^{ijkl} \chi^i \chi^j \chi^k - \frac{1}{2} \chi^i \psi^i \psi^j$,

we obtain a realization of the so(4)-extended SCA in terms of fermionic fields only. The presence of six fermionic fields in this representation suggests that it could be possible to have a so(6) symmetry among these fermions and to define as much as $\frac{1}{3!} = 20$ supercharges trilinear in the fermionic fields. However, upon closer inspection it turns out that in order to have a closed algebra without four-fermion terms (as we have in (15)) it is necessary to break the symmetry to so(4) and to keep only $\frac{1}{4} = 4$ of the supercharges.

In the representation (14) the superconformal transformations are realized non-linearly on the matter fields $\psi^i(z), \chi^i(z)$ and making the replacements

$L(z) \rightarrow -\frac{1}{2} :\psi^i\partial\psi^i: = -\frac{1}{2} :\chi^i\partial\chi^i: , \quad \partial_d A(z) \rightarrow \psi^i \psi^j , \quad G'(z) \rightarrow -\frac{1}{2} \epsilon^{ijkl} \chi^i \chi^j \chi^k - \frac{1}{2} \chi^i \psi^i \psi^j$,

we obtain a realization of the so(4)-extended SCA in terms of fermionic fields only. The presence of six fermionic fields in this representation suggests that it could be possible to have a so(6) symmetry among these fermions and to define as much as $\frac{1}{3!} = 20$ supercharges trilinear in the fermionic fields. However, upon closer inspection it turns out that in order to have a closed algebra without four-fermion terms (as we have in (15)) it is necessary to break the symmetry to so(4) and to keep only $\frac{1}{4} = 4$ of the supercharges.

In the representation (14) the superconformal transformations are realized non-linearly on the matter fields $(\phi, \chi')$ but (if we discard the central terms for a moment) they are realized linearly on the currents $(L, G, T, \Gamma', A)$. The current superfield, which for general $N$ is given in (5) reads as follows for $N=4$:

$$J^{(4)} = -A + i\theta^i \Gamma^i - \frac{1}{2} \epsilon^{ijkl} \theta^i \theta^j T^{kl} - \frac{1}{2} \epsilon^{ijkl} \theta^i \theta^j \theta^k G^l + \epsilon^{ijkl} \epsilon^{ijkl} \theta^i \theta^j \theta^k \theta^l \phi + \epsilon^{ijkl} \epsilon^{ijkl} \theta^i \theta^j \theta^k \theta^l \phi L \phi$$

(16)

Apart from central terms, $J^{(4)}$ transforms as a primary superfield of dimension 0. The OPE's (15) for the currents $(L, G, T, \Gamma', A)$ are equivalent to the following SOPE for the current superfield $J^{(4)}$

$$J^{(4)} (Z_1) J^{(4)} (Z_2) = \left( \sum_{k} \frac{\theta^{12}_{12} \theta^{32}_{12}}{Z_{12}^3} D_{23} + 2 \frac{\theta^{12}_{12} \theta^{23}_{12}}{Z_{12}^2} \partial_{23} \right) J^{(4)} (Z_2) - \log (Z_{12})$$

(17)
where \( Z_{12} = z_1 - z_2 - \Sigma \theta_i \theta_i \) and \( \theta_i \theta_i = \theta_i' \theta_i' \). Note that this single formula summarizes all the commutation relations of the so(4)-extended SCA as listed in table 1. Since \( \Delta(x) = \varphi(x) \) and \( \Gamma'(x) = \chi'(x) \) we have the amusing situation that the \( N=4 \) current superfield is at the same time the \( N=4 \) matter superfield, or, stated differently, that the currents and the matter fields are in the same \((8+8)\) component chiral multiplet.

4. Reduction to \( N=3,2 \). Since the so(\( N \))-extended SCA’s with \( N=3,2 \) are subalgebras of the \( N=4 \) algebra, the fields \((\varphi, \chi', \lambda)\) automatically provide representations of these smaller algebras, which are of some interest in their own right. Upon reducing \( N=4 \rightarrow N=3 \) we only keep the superconformal generators \( L(z), G_i(z), T^i(z) = -\frac{1}{2} \epsilon^{ijk} T^{jk}(z) \) and \( \Gamma^4(z), i,j,k = 1,2,3 \). Their Laurent coefficients \( L_n, G_i^n, T_n^i, \Gamma_4^n \) form the so(3)-extended SCA with central charge \( c=3 \) which is a subalgebra of the \( N=4 \) SCA listed in table 1. The superfield \( J^{(4)} \) splits as \( J^{(4)} = -\Phi^{(3)} - \theta^4 J^{(3)} \), where \( J^{(3)} \) is the \( N=3 \) current superfield (cf. (5)) and \( \Phi^{(3)} \) is an \( N=3 \) matter superfield,

\[
\Phi^{(3)} = \varphi - i\theta^i \chi^i - \frac{1}{2} \epsilon^{ijk} \theta^i \theta^j \chi^k + \frac{1}{8} \epsilon^{ijn} \chi^m \chi^m - \psi \partial \phi . \tag{18}
\]

Interestingly, this construction automatically provides an answer to the question of how to obtain an invariant action density for the components \((\varphi, \chi', \psi)\) of a chiral \( N=3 \) multiplet, i.e. a \( N=3 \) scalar superfield \( \Phi(z) \). It has been noted in ref. [9] that the traditional way to obtain an invariant action from a scalar superfield (by constructing a superspace density of the appropriate dimension from \( \Phi \) and its covariant derivatives \( D \Phi \)) fails for the \( N=4 \) algebras with \( N \geq 3 \). From the analysis above we see that for \( N=3 \) this problem can be solved quantum mechanically by expressing the “auxiliary fields” \( F^i \) and \( \lambda \) in \( \varphi, \chi' \) and an additional field \( \psi \) according to

\[
F^k = \chi^k \psi , \quad \lambda = \frac{1}{8} \epsilon^{ijk} \chi^i \chi^j \psi - \psi \partial \phi . \tag{19}
\]

A free action for the fields \((\varphi, \chi', \psi)\) is then invariant under all superconformal transformations. The price one has to pay is that supersymmetry is realized non-linearly on \((\varphi, \chi', \psi)\),

\[
\delta \varphi = i e \chi' , \quad \delta \chi' = -i e \partial \varphi + i \epsilon^{ijk} \chi^k \psi , \quad \delta \psi = -\frac{1}{2} i e^{ijk} \chi^k \psi . \tag{20}
\]

A similar analysis for the breaking \( N=4 \rightarrow N=2 \) leads to a non-linear \( c=3 \) representation of the \( N=2 \) SCA with one real boson and four Majorana fermions, in contrast with the linear \( c=3 \) representation which has a complex scalar field and one single Dirac fermion.

5. Comments.

(i) Superstring compactification. By applying fermionization to compactified coordinates the degrees of freedom of the \( N=1 \) superstring can be organized as follows [20–22]:

\[
(x^\mu, \psi^\mu) \quad \mu = 1, \ldots, 4 , \quad (\chi^I) \quad I = 1, \ldots, 18 .
\]

Since the number of internal fermions is a multiple of 6 it is possible to define a global \( N=4 \) superconformal symmetry on the internal fermionic degrees of freedom. It would be interesting to see whether this symmetry has any consequences for superstring dynamics.

(ii) Highest weight representations. A representation of the so(4)-extended SCA automatically provides a representation of the so(4) Kac–Moody subalgebra, which has level \( k=c/3 \). In a unitary highest weight representation the level \( k \) is a positive integer and consequently \( c \) is a positive multiple of 3. These values for \( c \) can all be realized by taking tensor products of copies of the basic \( c=3 \) representation (14).

(iii) \( N=3,4 \) string models. In principle the field \( \varphi \) in the multiplet \((\varphi, \chi')\) can be interpreted as a string-coordinate. However, due to the non-linearity of the supersymmetry transformations the formulation of “so(3)-or so(4)-strings” is not a straightforward extension of the results for strings with \( \text{u}(1)- \) or \( \text{su}(2)\)-extended supersymmetry [23,10,24].

(iv) Critical central charge. If a SCA appears as the algebra of constraints of a quantum system then the
theory is consistent only for one special value of the central charge. In ref. [25] it is shown that in the BRST-quantization of a system with constraints corresponding to the \( N=3 \) SCA the ghost contribution to \( Q_{\text{BRST}}^2 \) vanishes and that accordingly the value of the critical central charge for this algebra is 0. It turns out that also the \( N=4 \) so(4)-extended SCA has critical central charge \( c=0 \). This value differs from the critical value of \( c \) for the \( N=4 \) su(2)-extended SCA which is \(-12\).

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