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DOI

[10.1016/0550-3213\(87\)90640-7](https://doi.org/10.1016/0550-3213(87)90640-7)

Publication date

1987

Published in

Nuclear Physics B

[Link to publication](#)

Citation for published version (APA):

Schoutens, K. (1987). Structure of d=2 conformal supergravity and covariant actions for strings. *Nuclear Physics B*, 292, 150-180. [https://doi.org/10.1016/0550-3213\(87\)90640-7](https://doi.org/10.1016/0550-3213(87)90640-7)

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STRUCTURE OF $d = 2$ CONFORMAL SUPERGRAVITY AND COVARIANT ACTIONS FOR STRINGS

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Received 17 November 1986

We present a systematic way to construct theories for $d = 2$ conformal supergravity with $N = 0, 1, 2$ or 4 as gauge theories corresponding to infinite dimensional algebras. Coupling these theories to scalar matter multiplets leads to σ -models in a background of conformal supergravity which provide covariant actions for various string models. The construction of $N = 4$ models with quaternionic geometry requires an extension of the $N = 4$ supergravity multiplet containing an additional $\text{sp}(1)$ vector multiplet. We demonstrate the role played by this vector multiplet in a construction of the $N = 4$ quaternionic σ -model with target space $\text{HP}(n - 1)$.

1. Introduction

A convenient starting point for the construction of a (super)string model is a covariant action in two dimensions which describes the dynamics of the string coordinate fields $X^\mu(\sigma, \tau)$ [1, 2, 3]. Such a string action is invariant under general coordinate transformations, local scale transformations and (for the supersymmetric models) local Q - and S -supersymmetry and can therefore be viewed as describing a σ -model in a background of $d = 2$ conformal (super)gravity. The bosonic string, heterotic string and superstring models are described by actions of this type with $N = 0$, $N = \frac{1}{2}$ and $N = 1$ supersymmetry, respectively, but also models with extended supersymmetry (i.e. $N > 1$) exist. In several recent publications supergravity-string actions have been considered with $N = 1, 2$ and 4 [3–10]. It has been argued that for classical models which admit a string interpretation the maximal number of supersymmetries that can be realized is $N = 4$ [11]. This number coincides with the maximal number of supersymmetries of a *globally* supersymmetric $d = 2$ non-linear σ -model [12]. For models with *local* supersymmetry this bound does not apply and it has been shown recently that it is possible to realize $N = 8$ or even $N = 16$ local supersymmetries on a $d = 2$ conformally invariant σ -model [13].

At the classical level the $N = 2, 4$ models can be interpreted as describing strings with “colour” symmetries $U(1)$ and $SU(2)$ respectively. At the quantum level their

interpretation is more problematic since, according to our present understanding, the higher- N strings are plagued by quantum anomalies in all space-time dimensions $d \geq 4$. In fact, the critical dimensions for the $N = 2$ and $N = 4$ string models have been found to be 2 and -2 resp. [14–16, 4].

In this paper we investigate the structure of covariant string actions with $N = 0, 1, 2$ or 4 supersymmetry. Their construction requires a proper understanding of the structure of conformal gravity and supergravity in two dimensions. In dimensions $d > 2$ these theories are well-understood [17–20]. For $d > 2$ conformal gravity can be constructed as the gauge theory corresponding to the conformal algebra of d -dimensional flat space-time, which is $so(d, 2)$, restricted by a set of curvature constraints. The theory describes $\frac{1}{2}(d+1)(d-2)$ degrees of freedom, which is just the number of components needed for the description of a massive spin-2 field in d dimensions [21]. In contrast with the situation for $d > 2$, in $d = 2$ the algebra generated by all regular conformal transformations is an *infinite* dimensional algebra which contains $so(2, 2)$ as a finite dimensional subalgebra. In this paper we will present a formulation of $d = 2$ conformal gravity as the gauge theory corresponding to this algebra^{*}. It turns out that this gauge theory admits an invariant set of conventional curvature constraints which are such that the unconstrained fields represent no off-shell degrees of freedom, i.e. are pure gauge. This is consistent with the fact that in $d = 2$ a massive spin-2 field has no propagating modes. Our scheme for describing conformal supergravity in $d = 2$ admits superextensions to theories with 1, 2 or 4 supersymmetries. We will present a construction for the maximally extended case, i.e. $N = 4$ conformal supergravity, which we base on an infinite dimensional extension of the superalgebra $ssu(1, 1|2) \oplus ssu(1, 1|2)$.

The fact that spinning string models can be constructed by considering gauge theories corresponding to $d = 2$ superconformal algebras was first recognized in [3]. In the literature several such constructions for $N = 1, 2$ models have been presented which are all based on finite dimensional superalgebras containing the bosonic subalgebra $so(2, 2)$: $Osp(N|2) \oplus Osp(N|2)$, $N = 1, 2$, for non-chiral $N = 1, 2$ models [3, 5–8, 10] and corresponding truncations for chiral $(2, 0)$ and $(2, 1)$ models [6, 9]. For the gauge theories based on these finite dimensional algebras there is no set of conventional curvature constraints which reduce the theory to $0 + \underline{0}$ off-shell degrees of freedom. Even after imposing a maximal set of conventional curvature constraints one is left with so-called “non-solvable field components” [6, 7]. There have been different proposals in the literature for how to deal with these components in order to obtain a consistent description with $0 + \underline{0}$ degrees of freedom and a closed gauge algebra on all fields [3, 6–8]. In the formulation of $d = 2$ conformal supergravity based on an infinite dimensional gauge algebra this problem is easily solved: the field components that were previously unsolvable are simply pure gauge and they represent no degrees of freedom.

^{*}This approach has been considered independently by E. Bergshoeff and A. Van Proeyen (private communication). It was discussed in [22].

Covariant actions for spinning strings with $N = 1, 2$ or 4 supersymmetry can be obtained by coupling the $N = 1, 2, 4$ supergravity theories to scalar multiplets which describe the string coordinate fields and their superpartners. The target space of the $N = 1$ spinning strings can be any riemannian manifold [23] but strings with extended supersymmetry can only be defined on a restricted class of manifolds. It was noted in [4] that in the models with $N = 4$ supersymmetry, which describe strings with internal $\text{su}(2)$ symmetry, the target space can be either a hyper-Kähler or a quaternionic manifold. In this paper we argue that $\text{su}(2)$ strings on quaternionic manifolds can be conveniently described in the context of an extension of the $N = 4$ supergravity multiplet containing an additional $8 + 8$ component $\text{sp}(1)$ vector multiplet. The coupling of the extended supergravity multiplet to a flat hyper-Kähler σ -model results in a locally supersymmetric quaternionic σ -model with target space $\text{HP}(n - 1)$. The generalization of this construction to arbitrary hyper-Kähler σ -models leads to a more general class of locally $N = 4$ supersymmetric quaternionic σ -models.

This paper is organized as follows: In sect. 2 we present the formalism for $d = 2$ conformal gravity based on an infinite dimensional gauge algebra and explain its relation to a formulation based on $\text{so}(2, 2)$. In sect. 3 superextensions are discussed. We give a detailed description of the structure of the $N = 4$ theory and we compare our results with formulations which are based on finite dimensional gauge algebras. In sect. 4 we introduce the $\text{sp}(1)$ vector multiplet and discuss its coupling to $N = 4$ conformal supergravity. These results are used in sect. 5 where we discuss $N = 4$ scalar matter multiplets and present an explicit construction of the $\text{HP}(n - 1)$ σ -model. We close with some comments on our results in sect. 6.

2. $d = 2$ conformal gravity as the gauge theory of an infinite dimensional algebra

Conformal transformations in a general d -dimensional space-time are defined as those general coordinate transformations (g.c.t.) that leave the metric $g_{\mu\nu}(x)$ invariant up to a local scale factor

$$\delta g_{\mu\nu}(x) = D_\mu \xi_\nu(x) + D_\nu \xi_\mu(x) = \Lambda(x) g_{\mu\nu}(x). \quad (2.1)$$

The parameters $\xi_\mu(x)$ are called conformal Killing vectors. From (2.1) we derive the conformal Killing equation

$$D_\mu \xi_\nu + D_\nu \xi_\mu = \frac{2}{d} g_{\mu\nu} D_\rho \xi^\rho. \quad (2.2)$$

In general, the number of independent conformal Killing vectors depends on the geometry of space-time. The maximal number (for dimension d) is for instance realized in flat d -dimensional Minkowski space-time where (2.2) reduces to

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d} \delta_{\mu\nu} \partial_\rho \xi^\rho. \quad (2.3)$$

For $d > 2$ the general solution to this equation is given by

$$\xi_\mu(x) = \Lambda_\mu + \Lambda_{\mu\nu}x_\nu + \Lambda x_\mu + \Lambda_\nu^K(2x_\mu x_\nu - \delta_{\mu\nu}x^2). \quad (2.4)$$

It contains $\frac{1}{2}(d+1)(d+2)$ parameters Λ_μ , $\Lambda_{\mu\nu}$, Λ and Λ_μ^K which correspond to translations P , Lorentz transformations M , dilatations D and conformal boosts K respectively. However, in $d=2$ the solution (2.4) is not the most general one. In order to solve (2.3) in $d=2$ it is convenient to introduce the light-cone components ξ^+ and ξ^-

$$\begin{aligned} \xi^+ &= \xi_1 + i\xi_2, & \partial_+ &= \frac{1}{2}(\partial_1 - i\partial_2), \\ \xi^- &= \xi_1 - i\xi_2, & \partial_- &= \frac{1}{2}(\partial_1 + i\partial_2). \end{aligned} \quad (2.5)$$

Eq. (2.3) reduces to

$$\partial_+\xi^- = 0, \quad \partial_-\xi^+ = 0. \quad (2.6)$$

The general solution for conformal Killing vectors that are regular at the origin is

$$\xi^+ = \sum_{n \geq -1} \xi_{(n)}^+ x_+^{n+1}, \quad \xi^- = \sum_{n \geq -1} \xi_{(n)}^- x_-^{n+1}. \quad (2.7)$$

These regular conformal transformations are generated by $L_n^\pm$, $n \geq 1$, where

$$L_n^+ : \delta x_+ = x_+^{n+1}, \quad L_n^- : \delta x_- = x_-^{n+1}. \quad (2.8)$$

We will call the group of all regular conformal transformations G ; H is the stability subgroup of the point $x=0$, which is generated by the generators $L_n^\pm$, $n \geq 0$.

We can realize conformal transformations on fields $\Phi(x)$ by specifying a representation of the subgroup H on the fields $\Phi(0)$ at the origin. Since G/H is isomorphic to $d=2$ Minkowski space we can use the theory of induced representations to extend such a representation of H to a representation of the full group G on the fields $\Phi(x)$ ([24], see also [3]). Denoting the H -generators on $\Phi(0)$ by $H_n^\pm$ and the corresponding parameters by $\zeta_n^\pm$

$$[H_n^\pm, H_m^\pm] = (n-m)H_{n+m}^\pm, \quad (2.9)$$

$$\delta_H \Phi(0) = \sum_{n \geq 0} \zeta_n^\pm H_n^\pm \Phi(0), \quad (2.10)$$

we find the following action of a conformal transformation with parameters $\xi^\pm$ on the field $\Phi(x)$

$$\delta_G \Phi(x) = \sum_{n \geq 0} \zeta_n^\pm H_n^\pm \Phi(x) - \xi^\pm \partial_\pm \Phi(x), \quad (2.11)$$

where $\xi^\pm$ and $\zeta_n^\pm$ are related by

$$\zeta_n^\pm = \frac{1}{(n+1)!} \partial_\pm^{n+1} \xi^\pm. \quad (2.12)$$

This formula establishes a connection between base space transformations with parameters $\xi^\pm$ and H transformations with parameters $\zeta_n^\pm$ which act in the internal space spanned by the fields $\Phi(0)$. This internal space is an extension of the “tangent space” occurring in a formulation of Poincaré supergravity based on the coset structure G'/H' , where G' is the Poincaré group and H' the Lorentz group.

Our aim is here to construct a theory which has *local* conformal invariance. Thus, instead of the global conformal transformations (2.11) we now consider g.c.t. in the base space and *local* H transformations in the internal space. A first step in the realization of these symmetries is to construct the gauge theory corresponding to the algebra H (below we will find that this gauge algebra is actually too small for our purposes). We introduce a gauge field ω_μ with values in the Lie algebra of H

$$\omega_\mu = \sum_{n \geq 0} \omega_{\mu(n)}^+ H_n^+ + \sum_{n \geq 0} \omega_{\mu(n)}^- H_n^-. \quad (2.13)$$

The components $\omega_{\mu(n)}^\pm$ transform as follows

$$\delta_H \omega_{\mu(n)}^\pm = \partial_\mu \zeta_n^\pm - \sum_{p+q=n} (p-q) \omega_{\mu(q)}^\pm \zeta_p^\pm. \quad (2.14)$$

Since we consider the local theory as an extension of a theory with global conformal invariance we will insist on the property that the local theory has a rigid limit which is invariant under the global transformations (2.11). In this limit the metric $g_{\mu\nu}(x)$ reduces to $\delta_{\mu\nu}$ which is indeed invariant under (2.11) if we put

$$\delta_H(\zeta_n^\pm) g_{\mu\nu} = -(\zeta_0^+ + \zeta_0^-) g_{\mu\nu}. \quad (2.15)$$

However, the rigid limit configuration

$$\omega_{\mu(n)}^\pm = 0, \quad n \geq 0 \quad (2.16)$$

is *not* invariant under conformal transformations with parameters (2.12).

Clearly, the element missing in our set-up so far is a connection between the base space and the internal H space. Such a connection is provided by a *vielbein* field e_μ^m which is related to the metric $g_{\mu\nu}$ by

$$g_{\mu\nu} = e_\mu^m e_\nu^n \delta_{mn}. \quad (2.17)$$

Assuming that the vielbein has an inverse $e^{\mu m}$ we derive the following properties for

the lightcone components $e_\mu^\pm$ and $e^{\mu\pm}$

$$\begin{aligned} e_\mu^+ e^{\mu+} = e_\mu^- e^{\mu-} = 0, \quad e_\mu^+ e^{\mu-} = e_\mu^- e^{\mu+} = 2, \\ \varepsilon^{\mu\nu} e_\nu^\pm = \pm i e^{\mu\pm}. \end{aligned} \quad (2.18)$$

The presence of the vielbein field allows us to establish the correct rigid limit for our local theory. Indeed, the configuration

$$e_\mu^m = \delta_\mu^m, \quad \omega_{\mu(n)}^\pm = 0 \quad (2.19)$$

is invariant under conformal transformations with parameters (2.12) provided we put

$$\delta_H e_\mu^\pm = -e_\mu^\pm \zeta_0^\pm \quad (2.20)$$

and we extend (2.14) to

$$\delta \omega_{\mu(n)}^\pm = \delta_H \omega_{\mu(n)}^\pm (2.14) - (n+2) e_\mu^\pm \zeta_{n+1}^\pm. \quad (2.21)$$

We make the following observation: the transformation (2.20) and the extra terms in (2.21) follow from including extra generators $H_{-1}^\pm$ associated with the gauge fields $e_\mu^\pm$ in the gauge algebra, where

$$[H_{-1}^\pm, H_n^\pm] = -(n+1) H_{n-1}^\pm. \quad (2.22)$$

From (2.9) and (2.22) we see that the algebra spanned by $H_n^\pm$, $n \geq -1$, is isomorphic to the Lie algebra of the conformal group G generated by $L_n^\pm$, $n \geq -1$. This means that the correct local theory is really the gauge theory corresponding to the *full* conformal group G .

In this gauge theory the curvature corresponding to the generator $H_n^\pm$ is given by

$$R_{\mu\nu(n)}^\pm = 2 \partial_{[\mu} \omega_{\nu]}^\pm - 2 \sum_{p+q=n} (p-q) \omega_{\mu(q)}^\pm \omega_{\nu(p)}^\pm, \quad (2.23)$$

where n, p, q now take values in $\{-1, 0, 1, \dots\}$ and we identify

$$e_\mu^\pm = \omega_{\mu(-1)}^\pm. \quad (2.24)$$

For the extra generators $H_{-1}^\pm$ we define

$$P_1 = H_{-1}^+ + H_{-1}^-, \quad P_2 = i(H_{-1}^+ - H_{-1}^-). \quad (2.25)$$

These P transformations have been introduced as a technical device and there is no

reason for keeping them as local symmetries in our theory. We therefore consider the following identity which relates P transformations with parameter Λ^m to g.c.t. and local H transformations

$$\delta_P(\Lambda^m)\omega_{\nu(n)}^\pm = \delta_{gct}(\xi^\mu = \Lambda^m e_m^\mu)\omega_{\nu(n)}^\pm - \left[\delta_H(\zeta_n^\pm = \xi^\mu \omega_{\mu(n)}^\pm)\omega_\nu^\pm \right]_{(n)} - \xi^\mu R_{\mu\nu(n)}^\pm. \quad (2.26)$$

Hence, up to curvature terms P transformations are just linear combinations of g.c.t. and local H transformations. This leads us to impose the following constraints on the gauge field ω_μ

$$R_{\mu\nu(n)}^\pm = 0, \quad n \geq -1. \quad (2.27)$$

These constraints are all of conventional type as they can be used to express some of the gauge field components algebraically in terms of the others. In order to see this we note first that

$$\varepsilon^{\mu\nu} R_{\mu\nu(n)}^\pm = \pm 4i(n+2)\omega_{\mu(n+1)}^\pm e^{\mu\pm} + \text{more}. \quad (2.28)$$

Assuming that the vielbein is non-singular we can decompose

$$\omega_{\mu(n)}^\pm = a_n^\pm e_\mu^\pm + b_n^\pm e_\mu^\mp, \quad n \geq 0. \quad (2.29)$$

We then have

$$\varepsilon^{\mu\nu} R_{\mu\nu(n)}^\pm = \pm 8i(n+2)b_{n+1}^\pm + \text{more}, \quad (2.30)$$

so that the constraints (2.27) allow us to express the components $b_n^\pm$, $n \geq 0$, in terms of other gauge field components.

As the constraints (2.30) are gauge invariant (curvatures transform into curvatures), the transformation rules (2.14) are still valid for the dependent gauge fields. A similar conclusion will hold for the supersymmetric extensions to be discussed in sect. 3. In this respect the $d=2$ theories based on infinite dimensional algebras are more transparent than analogous theories in higher dimensions where the constraints are *not* gauge invariant since they do not include all curvatures.

Let us now count the off-shell degrees of freedom in the theory. We have

$$\delta_G \omega_{\mu(n)}^\pm = -(n+2)e_\mu^\pm \zeta_{n+1}^\pm + \text{more}, \quad (2.31)$$

so that

$$\delta_G a_n^\pm = -(n+2)\zeta_{n+1}^\pm + \text{more}. \quad (2.32)$$

This implies that all components $a_n^\pm$, $n \geq 0$, are pure gauge and represent no degrees

of freedom. The components $b_n^\pm$, $n \geq 0$, are dependent fields and do not represent degrees of freedom either. Only the vielbein e_μ^m , which has four components, remains as an independent field. It is easily verified that the vielbein is pure gauge w.r.t. the remaining gauge parameters $\zeta_0^\pm$ and $\xi^\pm$. We conclude that in this formulation $d=2$ conformal gravity has no off-shell degrees of freedom.

To appreciate the above results we will now compare our description of $d=2$ conformal gravity with a formulation based on the finite dimensional algebra $\mathfrak{so}(2,2)$, which is the subalgebra of G spanned by $L_n^\pm$, $n = -1, 0, 1$. The conventional $\mathfrak{so}(2,2)$ generators P (translations), D (dilatations), M (local Lorentz transformations) and K (conformal boosts) are given by

$$\begin{aligned} P_1 &= L_{-1}^+ + L_{-1}^-, & P_2 &= iL_{-1}^+ - iL_{-1}^-, \\ D &= -L_0^+ - L_0^-, & M &= -L_0^+ + L_0^-, \\ K_1 &= L_1^+ + L_1^-, & K_2 &= -iL_1^+ + iL_1^-, \end{aligned} \quad (2.33)$$

and satisfy the following commutation relations

$$\begin{aligned} [P_m, K_n] &= 2\delta_{mn}D - 2i\epsilon_{mn}M, \\ [P_m, M] &= -i\epsilon_{mn}P_n, & [P_m, D] &= P_m, \\ [K_m, M] &= -i\epsilon_{mn}K_n, & [K_m, D] &= -K_m. \end{aligned} \quad (2.34)$$

This algebra is similar to the conformal algebra in dimensions $d > 2$. The corresponding gauge theory, with gauge fields e_μ^m , b_μ , ω_μ and f_μ^m , has the following maximal set of conventional constraints

$$R_{\mu\nu}^m(P) = 0, \quad R_{\mu\nu}(D) = 0, \quad R_{\mu\nu}(M) = 0. \quad (2.35)$$

From these equations the gauge field components ω_μ , f_μ^μ and $f_{[\mu\nu]}$ can be solved in terms of the other gauge fields. Just as before the constraints allow us to discard the P transformations.

Counting the off-shell degrees of freedom for this theory we find that the fields e_μ^m , b_μ and $f_{((\mu\nu))}$ (the symmetric traceless part of f_μ^m) have 8 independent components, whereas only 6 local gauge parameters ξ_μ , Λ_D , Λ_M and Λ_m^K are present. If we use these to make a gauge choice for e_μ^m and b_μ we are left with 2 independent components $f_{((\mu\nu))}$ of the conformal vielbein. Due to the presence of these non-solvable components the gauge theory of $\mathfrak{so}(2,2)$ at this stage does *not* realize conformal gravity as a theory with no off-shell degrees of freedom.

There have been some proposals in the literature on how to deal with the non-solvable components of f_μ^m but we regard none of these as very satisfactory. In

[3] the constraint $f_{((\mu\nu))} = 0$ is proposed, whereas in [6] one introduces a set of compensating gauge transformations in order to decouple the non-solvable components in analogous models with $N = 1, 2$ supersymmetry. In [8] the constraint $R_{\mu\nu}^m(K) = 0$ is proposed in order to force the theory to 0 off-shell degrees of freedom. However, in the theory based on $so(2, 2)$ this constraint is *not* of conventional type and is therefore hard to interpret.

3. Extension to conformal supergravity

In this section we extend the construction presented in sect. 2 to theories for $d = 2$ conformal supergravity. These theories are based on infinite dimensional superalgebras containing the bosonic conformal algebra G as a subalgebra.

In order to find appropriate superextensions of the bosonic algebra G it is sufficient to examine superextensions of the Virasoro algebra V . Extra generators X_m in a superextension of V can be characterized by their conformal weight J which determines their commutation relation with the Virasoro generators L_n according to

$$[L_n, X_m] = ((J - 1)n - m) X_{n+m}. \quad (3.1)$$

We will restrict ourselves to superextensions of V which are generated by the L_n ($n \in \mathbb{Z}$), $J = \frac{3}{2}$ supercharges G_r^a ($r \in \mathbb{Z} + \frac{1}{2}$) and $J = 1$ internal generators T_m^i ($m \in \mathbb{Z}$). In addition we require the usual property that the anticommutator of two supercharges $Q \sim G_{-1/2}$ contains a term proportional to $P \sim L_{-1}$. Algebras of this type can be classified by using Kac's results on the classification of finite dimensional simple superalgebras [25, 26]. One finds [11, 27] that only three such algebras exist which have $N = 1, 2$ and 4 supersymmetry. They all contain finite dimensional sub-superalgebras which are $Osp(1|2)$, $Osp(2|2)$ and $ssu(1, 1|2)$ respectively.

If one relaxes some of the above restrictions other candidate gauge algebras can be found [28, 27]. For example, for every N an extension of G can be found which has $Osp(N|2) \oplus Osp(N|2)$ as a finite dimensional subalgebra and which has generators with $J = 2, \frac{3}{2}, \dots, 2 - \frac{1}{2}N$. It is presently not known whether any of these algebras admits a gauge theory which can be consistently interpreted as a non-standard generalization of $d = 2$ conformal supergravity.

The construction of conformal gravity based on the bosonic algebra G can be extended to the $N = 1, 2, 4$ superextensions of G . These algebras all admit consistent gauge theories with $0 + 0$ degrees of freedom which can be interpreted as theories for conformal supergravity with $N = 1, 2$ and 4. In this section we present a construction for the maximally extended theory (i.e. $N = 4$; the $N = 1, 2$ cases can be treated similarly). The $N = 4$ superconformal algebra is based on the $su(2)$ extended super-Virasoro algebra. This algebra is built from Virasoro generators L_n , $n \in \mathbb{Z}$, supercharges G_r^a and $\tilde{G}_{ra}$, $r \in \mathbb{Z} + \frac{1}{2}$, $a = 1, 2$, and internal generators T_n^i , $n \in \mathbb{Z}$, $i = 1, 2, 3$. The supercharges G_r^a and $\tilde{G}_{ra}$ form complex doublets under the

TABLE 1
su(2)-extended super-Virasoro algebra

$[L_n, L_m] = (n-m)L_{n+m}$	
$[L_n, G_m^a] = (\frac{1}{2}n-m)G_{n+m}^a$	$[L_n, \tilde{G}_{ma}] = (\frac{1}{2}n-m)\tilde{G}_{n+m,a}$
$\{G_n^a, G_m^b\} = \{\tilde{G}_{na}, \tilde{G}_{mb}\} = 0$	
$\{G_n^a, \tilde{G}_{mb}\} = -\delta_b^a L_{n+m} - i(m-n)(\sigma^i)_b^a T_{n+m}^i$	
$[T_n^i, G_m^a] = \frac{1}{2}i(\sigma^i)_b^a G_{n+m}^b$	
$[T_n^i, \tilde{G}_{ma}] = -\frac{1}{2}i(\sigma^i)_a^b \tilde{G}_{n+m,b}$	
$[T_n^i, T_m^j] = \epsilon^{ijk} T_{n+m}^k$	

su(2) subalgebra generated by T_0^i , $i = 1, 2, 3$. The structure of the su(2)-extended super-Virasoro algebra is listed in table 1. The generators

$$L_n, \quad \frac{1}{2}(G_r^a + \tilde{G}_{ra}), \quad \frac{1}{2}i(G_r^a - \tilde{G}_{ra}), \quad T_m^i,$$

$$n = -1, 0, 1, \quad r = -\frac{1}{2}, \frac{1}{2}, \quad m = 0$$

span a finite dimensional sub-superalgebra which can be shown to be isomorphic to $\text{ssu}(1, 1|2)$ by using an explicit matrix representation.

The gauge algebra for $N = 4$ conformal supergravity is the direct sum of two light-cone parts, both of which are isomorphic to a subalgebra of the su(2)-extended super-Virasoro algebra. It is generated by

$$L_n^\pm, \quad G_r^{\pm a}, \quad \tilde{G}_{ra}^\pm, \quad T_m^{\pm i},$$

$$n = -1, 0, 1, 2, \dots, \quad r = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots, \quad m = 0, 1, \dots$$

The construction of the corresponding gauge theory is a straightforward generalization of the bosonic case presented in sect. 2. We introduce gauge fields and parameters by

$$\omega_\mu^\pm = \sum_{n \geq -1} \omega_{\mu(n)}^\pm L_n^\pm + \sum_{r \geq -\frac{1}{2}} [\bar{\psi}_{\mu a(r)}^\pm G_r^{\pm a} - \text{h.c.}] + \sum_{m \geq 0} A_{\mu(m)}^{\pm i} T_m^{\pm i}, \quad (3.2)$$

$$\xi^\pm = \sum_{n \geq -1} \xi_n^\pm L_n^\pm + \sum_{r \geq -\frac{1}{2}} [\bar{\epsilon}_{ra}^\pm G_r^{\pm a} - \text{h.c.}] + \sum_{m \geq 0} \Lambda_m^{\pm i} T_m^{\pm i}, \quad (3.3)$$

where the hermitian conjugate (h.c.) $\bar{G}_{ra}^\pm$ is defined as $\bar{G}_{ra}^\pm = \pm i \tilde{G}_{ra}^\pm$. The transformation rules of the gauge fields follow straightforwardly from the algebra listed in table 1. As in sect. 2 we restrict the gauge theory by requiring that all

curvatures vanish, i.e.

$$R_{\mu\nu(n)}^{\pm} = 0, \quad R_{\mu\nu(r)}^{\pm a} = 0, \quad R_{\mu\nu(m)}^{\pm i} = 0. \quad (3.4)$$

These gauge invariant constraints are of conventional type and we can thus use them to express some of the gauge field components in terms of the others.

In order to examine the field content of the theory it is convenient to decompose the gauge fields as follows

$$\begin{aligned} \omega_{\mu(n)}^{\pm} &= a_n^{\pm} e_{\mu}^{\pm} + b_n^{\pm} e_{\mu}^{\mp}, & n \geq 0, \\ \psi_{\mu(r)}^{\pm a} &= p_r^{\pm} e_{\mu}^{\pm} + q_r^{\pm a} e_{\mu}^{\mp}, & r \geq \frac{1}{2}, \\ A_{\mu(m)}^{\pm i} &= x_m^{\pm i} e_{\mu}^{\pm} + y_m^{\pm i} e_{\mu}^{\mp}, & m \geq 1. \end{aligned} \quad (3.5)$$

From relations analogous to (2.30) and (2.32) we conclude that the components $b_n^{\pm}$, $q_r^{\pm a}$ and $y_m^{\pm i}$ can all be expressed in terms of the other gauge fields, whereas the components $a_n^{\pm}$, $p_r^{\pm a}$ and $x_m^{\pm i}$ are pure gauge. Partial gauge fixing leads to an intermediate formulation of the theory based on $4 + \underline{16} + 12$ gauge field components. One then easily shows that these $16 + \underline{16}$ gauge field components are exactly balanced by the residual local symmetries. This shows that $N=4$ conformal supergravity has no off-shell degrees of freedom.

For applications it is convenient to write the theory in a different form by combining the generators of the left and right light-cone sectors into the usual superconformal generators. The generators of supersymmetry (Q), conformal supersymmetry (S), internal phase transformations (B) and internal chiral transformations (A) are defined as follows

$$\begin{aligned} Q^a &= \begin{pmatrix} G_{-1/2}^{+a} \\ G_{-1/2}^{-a} \end{pmatrix}, & \bar{Q}_a &= (\bar{G}_{-1/2\ a}^{+}, \bar{G}_{-1/2\ a}^{-}), \\ S^a &= \begin{pmatrix} G_{1/2}^{+a} \\ -G_{1/2}^{-a} \end{pmatrix}, & \bar{S}_a &= (-\bar{G}_{1/2\ a}^{+}, \bar{G}_{1/2\ a}^{-}), \\ A^i &= T_0^{+i} - T_0^{-i}, & B^i &= T_0^{+i} + T_0^{-i}, \end{aligned} \quad (3.6)$$

whereas P_m , D , M and K_m are defined as in (2.33). We extend these definitions to the generators which are not in the subalgebra $\text{ssu}(1,1|2) \oplus \text{ssu}(1,1|2)$

$$\begin{aligned} D' &= -L_2^{+} - L_2^{-}, & M' &= -L_2^{+} + L_2^{-}, \\ Q'^a &= \begin{pmatrix} G_{3/2}^{+a} \\ G_{3/2}^{-a} \end{pmatrix}, & \bar{Q}'_a &= (\bar{G}_{3/2\ a}^{+}, \bar{G}_{3/2\ a}^{-}), \\ X^i &= T_1^{+i} - T_1^{-i}, & Y^i &= T_1^{+i} + T_1^{-i}, \end{aligned} \quad (3.7)$$

etc. We combine the spinors Q^a , S^a , Q'^a , ... and their complex conjugates $\bar{Q}_a$, $\bar{S}_a$,

TABLE 2
Commutators for the $N = 4$ superconformal algebra

$$\begin{aligned}
[P_m, K_n] &= 2\delta_{mn}D - 2i\epsilon_{mn}M \\
[P_m, M] &= -i\epsilon_{mn}P_n, \quad [P_m, D] = P_m \\
[K_m, M] &= -i\epsilon_{mn}K_n, \quad [K_m, D] = -K_m \\
[K_m, Q^{Aa}] &= -i\gamma_m S^{Aa}, \quad [P_m, S^{Aa}] = -i\gamma_m Q^{Aa} \\
[Q^{Aa}, D] &= \frac{1}{2}Q^{Aa}, \quad [S^{Aa}, D] = -\frac{1}{2}S^{Aa} \\
[Q^{Aa}, M] &= -\frac{1}{2}\gamma_3 Q^{Aa}, \quad [S^{Aa}, M] = -\frac{1}{2}\gamma_3 S^{Aa} \\
\{Q^{Aa}, Q^{Bb}\} &= -\frac{1}{2}\epsilon^{AB}\epsilon^{ab}\gamma^m C_+^{-1}P_m \\
\{S^{Aa}, S^{Bb}\} &= \frac{1}{2}\epsilon^{AB}\epsilon^{ab}\gamma^m C_+^{-1}K_m \\
\{Q^{Aa}, S^{Bb}\} &= \epsilon^{AB}\left[\frac{1}{2}i\epsilon^{ab}(D - \gamma_3 M)C_+^{-1} + \frac{1}{2}(\sigma_i)^b{}_c \epsilon^{ca}(B^i - \gamma_3 A^i)C_+^{-1}\right] \\
[Q^{Aa}, A^i] &= \frac{1}{2}i(\sigma^i)^a{}_b \gamma_3 Q^{Ab} \\
[Q^{Aa}, B^i] &= -\frac{1}{2}i(\sigma^i)^a{}_b \gamma_3 Q^{Ab} \\
[S^{Aa}, A^i] &= \frac{1}{2}i(\sigma^i)^a{}_b \gamma_3 S^{Ab} \\
[S^{Aa}, B^i] &= -\frac{1}{2}i(\sigma^i)^a{}_b \gamma_3 S^{Ab} \\
[A^i, A^j] &= \epsilon^{ijk}B^k, \quad [A^i, B^j] = \epsilon^{ijk}A^k \\
[B^i, B^j] &= \epsilon^{ijk}B^k \\
[P_1, X^i] &= -A^i, \quad [P_2, X^i] = -iB^i \\
[P_1, Y^i] &= -B^i, \quad [P_2, Y^i] = -iA^i \\
[P_m, Q'^{Aa}] &= -2i(-1)^m \gamma_m S^{Aa} \\
[P_m, D'] &= 3(-1)^m K_m, \quad [P_m, M'] = 3i(-1)^m \epsilon_{mn}K_n \\
[Q^{Aa}, X^i] &= -\frac{1}{2}i(\sigma^i)^a{}_b \gamma_2 S^{Ab} \\
[Q^{Aa}, Y^i] &= -\frac{1}{2}(\sigma^i)^a{}_b \gamma_1 S^{Ab} \\
\{Q^{Aa}, Q'^{Bb}\} &= -\epsilon^{AB}\epsilon^{ab}(\gamma_1 K_1 - \gamma_2 K_2)C_+^{-1} \\
&\quad - \epsilon^{AB}(\sigma_i)^a{}_c \epsilon^{bc}(\gamma_2 X^i - i\gamma_1 Y^i)C_+^{-1}
\end{aligned}$$

We list all the non-zero commutators of the subalgebra $\text{ssu}(1,1|2) \oplus \text{ssu}(1,1|2)$ and some of the commutation relations involving extra generators.

$\bar{Q}'_a, \dots$ into doubly symplectic Majorana spinors $Q^{Aa}, S^{Aa}, Q'^{Aa}, \dots$ (see appendix). The (anti)commutation relations for the finite dimensional part of the algebra and some of the extra generators are listed in table 2. In table 3 the structure of the full $N = 4$ gauge algebra is schematically pictured.

A general gauge parameter is now written as

$$\begin{aligned}
&\Lambda^m P_m + \Lambda_M M + \Lambda_D D + \Lambda_K^m K_m + \Lambda_{M'} M' + \Lambda_{D'} D' + \dots \\
&+ \bar{\epsilon}_{Aa} Q^{Aa} + \bar{\eta}_{Aa} S^{Aa} + \bar{\epsilon}'_{Aa} Q'^{Aa} + \dots \\
&+ \Lambda_A^i A^i + \Lambda_B^i B^i + \Lambda_X^i X^i + \Lambda_Y^i Y^i + \dots
\end{aligned} \tag{3.8}$$

TABLE 3
Structure of the $N = 4$ superconformal algebra

Weyl-weight	3	—	—	—
	2	M', D'	Q'^{Aa}	—
	1	K_m	S^{Aa}	X^i, Y^i
	0	M, D	Q^{Aa}	A^i, B^i
	-1	P_m		

The generators in the frame span the subalgebra $\text{ssu}(1,1|2) \oplus \text{ssu}(1,1|2)$.

and a general gauge field as

$$\begin{aligned}
 & e_\mu^m P_m + \omega_\mu M + b_\mu D + f_\mu^m K_m + \omega'_\mu M' + b'_\mu D' + \dots \\
 & + \bar{\psi}_{\mu Aa} Q^{Aa} + \bar{\varphi}_{\mu Aa} S^{Aa} + \bar{\psi}'_{\mu Aa} Q'^{Aa} + \dots \\
 & + A_\mu^i A^i + B_\mu^i B^i + X_\mu^i X^i + Y_\mu^i Y^i + \dots
 \end{aligned} \quad (3.9)$$

The transformation rules under Q - and S -supersymmetry and the bosonic symmetries D , M , K , A^i and B^i for the most important fields of the $N = 4$ conformal supergravity model read

$$\begin{aligned}
 \delta e_\mu^m &= -\frac{1}{2} \bar{\varepsilon}_{Aa} \gamma^m \psi_\mu^{Aa} + \lambda_D e_\mu^m + i \varepsilon^{mn} \Lambda_M e_{\mu n}, \\
 \delta \psi_\mu^{Aa} &= \partial_\mu \varepsilon^{Aa} - \frac{1}{2} b_\mu \varepsilon^{Aa} - \frac{1}{2} \omega_\mu \gamma_3 \varepsilon^{Aa} - \frac{1}{2} i (\sigma^i)^a_b \varepsilon^{Ab} B_\mu^i \\
 &\quad - \frac{1}{2} i (\sigma^i)^a_b \gamma_3 \varepsilon^{Ab} A_\mu^i - i \gamma_\mu \eta^{Aa} + \frac{1}{2} \Lambda_D \psi_\mu^{Aa} + \frac{1}{2} \Lambda_M \gamma_3 \psi_\mu^{Aa} \\
 &\quad + \frac{1}{2} i \alpha^i (\sigma^i)^a_b \gamma_3 \psi_\mu^{Ab} + \frac{1}{2} i \beta^i (\sigma^i)^a_b \psi_\mu^{Ab}, \\
 \delta b_\mu &= \frac{1}{2} i (\bar{\varepsilon}_{Aa} \varphi_\mu^{Aa}) + \frac{1}{2} i (\bar{\eta}_{Aa} \psi_\mu^{Aa}) + \partial_\mu \Lambda_D + 2 e_\mu^m \Lambda_{Km}, \\
 \delta \omega_\mu &= -\frac{1}{2} i (\bar{\varepsilon}_{Aa} \gamma_3 \varphi_\mu^{Aa}) + \frac{1}{2} i (\bar{\eta}_{Aa} \gamma_3 \psi_\mu^{Aa}) + \partial_\mu \Lambda_M - 2 i \varepsilon_{mn} e_\mu^m \Lambda_K^n, \\
 \delta A_\mu^i &= -\frac{1}{2} (\bar{\varepsilon}_{Aa} \gamma_3 (\sigma^i)^a_b \varphi_\mu^{Ab}) - \frac{1}{2} (\bar{\eta}_{Aa} \gamma_3 (\sigma^i)^a_b \psi_\mu^{Ab}) \\
 &\quad + \partial_\mu \alpha^i + \varepsilon^{ijk} (B_\mu^j \alpha^k + A_\mu^j \beta^k), \\
 \delta B_\mu^i &= \frac{1}{2} (\bar{\varepsilon}_{Aa} (\sigma^i)^a_b \varphi_\mu^{Ab}) - \frac{1}{2} (\bar{\eta}_{Aa} (\sigma^i)^a_b \psi_\mu^{Ab}) \\
 &\quad + \partial_\mu \beta^i + \varepsilon^{ijk} (A_\mu^j \alpha^k + B_\mu^j \beta^k).
 \end{aligned} \quad (3.10)$$

The curvature constraints $R_{\mu\nu}^m(P) = 0$ and $R_{\mu\nu}^{Aa}(Q) = 0$ can be used to derive the following expressions

$$\omega^m = -\frac{1}{2}i\varepsilon^{\mu\nu}(\bar{\psi}_{\nu Aa}\gamma^m\psi_{\mu}^{Aa}) + 2i\varepsilon^{\mu\nu}(\partial_{\mu}e_{\nu}^m - b_{\mu}e_{\nu}^m), \quad (3.11)$$

$$\gamma^{\mu}\varphi_{\mu}^{Aa} = \gamma_3\varepsilon^{\mu\nu}D_{[\mu}\psi_{\nu]}^{Aa}. \quad (3.12)$$

The expressions that follow from the other constraints will not be used in the following. For the applications to be discussed in the sects. 4 and 5 we prefer *not* to specify any gauge condition for the generators $K, D', M', Q', X, Y, \dots$ of the $N = 4$ superconformal algebra. We refrain from imposing gauge conditions because the derivations in the sects. 4 and 5 will be based on the structure of the commutator $[\delta_Q, \delta_Q]$, which is most transparent in a formulation without gauge fixing.

In the literature constructions of $d = 2$ conformal supergravity have been discussed which, in contrast to the above construction, are all based on *finite* dimensional superconformal algebras [3,5–10]. We will here comment on the relation between the two approaches.

In the gauge theory corresponding to the finite dimensional superalgebra $\text{ssu}(1,1|2) \oplus \text{ssu}(1,1|2)$ the following *maximal* set of conventional constraints can be imposed

$$R_{\mu\nu}^m(P) = 0, \quad R_{\mu\nu}(D) = 0, \quad R_{\mu\nu}(M) = 0, \quad R_{\mu\nu}^{Aa}(Q) = 0. \quad (3.13)$$

These constraints leave the following components of f_{μ}^m and φ_{μ}^{Aa} unspecified

$$f_{((\mu\nu))} = f_{\mu\nu} - f_{[\mu\nu]} - \frac{1}{2}g_{\mu\nu}f_{\rho}^{\rho}, \quad (3.14)$$

$$\varphi_{(\mu)}^{Aa} = \varphi_{\mu}^{Aa} - \frac{1}{2}\gamma_{\mu}\gamma^{\rho}\varphi_{\rho}^{Aa}. \quad (3.15)$$

The same phenomenon occurs in the $N = 1, 2$ models where it has been treated differently by various authors [3,6–8]. In the above formulation of the theory this problem is solved by the presence of extra generators in the gauge algebra. Under D', M' and Q' , we have

$$\delta_{D', M'}f_{\mu}^m = 3(-1)^me_{\mu}^m\Lambda_{D'} + 3i(-1)^n\varepsilon_{nm}e_{\mu}^n\Lambda_{M'}, \quad (3.16)$$

$$\delta_{Q'}\varphi_{\mu}^{Aa} = -2i\sum_m(-1)^me_{\mu}^m\gamma_m\varepsilon'^{Aa}. \quad (3.17)$$

These transformations precisely affect the non-solvable components $f_{((\mu\nu))}$ and $\varphi_{(\mu)}^{Aa}$ and it is easily checked that they are in fact such that these components are pure gauge. They can be adjusted to any value by applying D', M' and Q' gauge transformations.

In refs. [29, 30, 3, 4, 6] the $N = 2$ and $N = 4$ models were presented in a formalism with only *one* gauge field $\hat{B}_\mu$ which acts as a gauge connection for *two* different local symmetries A and B . This phenomenon was first noted in [30]. In [3] it was pointed out that the single gauge field $\hat{B}_\mu$ can be viewed as a linear combination of two distinct gauge fields corresponding to the symmetries A and B . From our construction one can easily understand why a formulation with only one gauge field yields consistent results: the linear combinations

$$\hat{A}_\mu^i = B_\mu^i - i\epsilon_\mu{}^\nu A_\nu^i, \quad \bar{B}_\mu^i = B_\mu^i + i\epsilon_\mu{}^\nu A_\nu^i \quad (3.18)$$

transform as follows under the local symmetries X and Y

$$\begin{aligned} \delta_{X,Y} \hat{A}_\mu^i &= -2e_\mu^1 \Lambda_Y^i - 2ie_\mu^2 \Lambda_X^i, \\ \delta_{X,Y} \hat{B}_\mu^i &= 0. \end{aligned} \quad (3.19)$$

Clearly, the field $\hat{A}_\mu^i$ is pure gauge w.r.t. X and Y and we can employ these symmetries to obtain a formulation where only the gauge field $\hat{B}_\mu^i$ is present. The local symmetries A and B transform the field $\hat{B}_\mu^i$ as follows

$$\delta_{A,B} \hat{B}_\mu^i = \partial_\mu \beta^i + \epsilon_\mu{}^\nu \partial_\nu \alpha^i + \epsilon^{ijk} (\hat{B}_\mu^j \beta^k + i\epsilon_\mu{}^\nu \hat{B}_\nu^j \alpha^k). \quad (3.20)$$

We see that indeed the field $\hat{B}_\mu^i$ can be interpreted as a gauge connection for both symmetries A and B simultaneously.

The results for the $N = 4$ model reported in [4] are reproduced if we impose the following gauge conditions for the conformal boosts K and the extra generators D' , M' , Q' , X^i and Y^i

$$b_\mu = 0, \quad f_{((\mu\nu))} = 0, \quad \varphi_{(\mu)}^{Aa} = 0, \quad \hat{A}_\mu^i = 0. \quad (3.21)$$

Combining this with (3.12) we find the following expression for the conformal gravitino φ_μ

$$\varphi_\mu^{Aa} = \frac{1}{2} i \gamma_\mu \gamma^{\rho\nu} D_{[\rho} \psi_{\nu]}^{Aa}. \quad (3.22)$$

The field $\hat{B}_\mu^i$ transforms as follows under Q - and S -supersymmetry

$$\delta_{Q,S} \hat{B}_\mu^i = i\bar{\epsilon} \sigma^i \gamma^\nu D_{[\mu} \psi_{\nu]} - \frac{1}{2} \bar{\eta} \sigma^i \gamma^\nu \gamma_\mu \psi_\nu. \quad (3.23)$$

The results (3.10) and (3.23) agree with those reported in [4].

In order to maintain the gauge conditions (3.21) we need to include compensating transformations in the supersymmetry transformations:

$$\delta_Q \rightarrow \delta_Q + \delta_K + \delta_{D'} + \delta_{M'} + \delta_{Q'} + \delta_X + \delta_Y + \dots \quad (3.24)$$

These compensating gauge transformations do not affect the δ_Q transformation properties of the fields e_μ^m , ψ_μ^{Aa} and $\hat{B}_\mu^i$ since these are all inert under the extra transformations (cf. (3.19)). However, they do affect the commutator algebra obeyed by these transformations. We have worked out the modified algebra structure for a formulation of the $N=2$ theory with only the gauge fields e_μ^m , ψ_μ and $\hat{B}_\mu$ present ($\hat{B}_\mu$ is a gauge field for phase and chiral $U(1)$ transformations). Such a formulation can be obtained from the gauge theory of the infinite dimensional $N=2$ superconformal algebra by imposing the gauge conditions

$$b_\mu = 0, \quad f_{((\mu\nu))} = 0, \quad \varphi_{(\mu)} = 0, \quad \hat{A}_\mu = \hat{B}_\mu. \quad (3.25)$$

The results we find agree with the outcome of an analysis reported in ref. [6].

4. The $\text{sp}(1)$ vector multiplet in the $N=4$ model

The spinning string action, which was first published in [2], describes the coupling of the string coordinate fields $X^\mu(\sigma, \tau)$ $\mu=1, 2, \dots, d$ and their superpartners $\lambda^\mu(\sigma, \tau)$ to the gauge fields of $d=2$, $N=1$ conformal supergravity. If the target space described by the X^μ is not flat this action describes a non-linear σ -model with local $N=1$ conformal supersymmetry. For $N=1$ the curved target space can be any riemannian manifold [23], but the presence of extended supersymmetry ($N=2, 4$) restricts the geometry of the manifold. We will here consider $N=4$ models, which describe spinning strings with internal $\text{su}(2)$ symmetry. It has been shown recently [4] that for these models the target space can be either a hyper-Kähler or a quaternionic manifold. While the $N=4$ hyper-Kähler coupling is a rather straightforward generalization of the $N=1$ result, the extension to quaternionic manifolds is more complicated and requires the inclusion of several non-linear modifications in the action and the transformation rules.

Both hyper-Kähler and quaternionic manifolds have three independent almost complex structures. On a hyper-Kähler manifold these structures are integrable and each of them is covariantly constant w.r.t. the Christoffel connection $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$. This property makes it possible to define a $N=4$ *globally* supersymmetric σ -model on any hyper-Kähler manifold [12]. On a general quaternionic manifold the almost complex structures are not covariantly constant (cf. [31]). Under parallel transport along a closed curve they are not separately invariant but rotate into each other. This is due to the fact that the holonomy group of a general quaternionic manifold contains a $\text{sp}(1)$ factor which acts non-trivially on the almost complex structures.

The geometrical properties of quaternionic manifolds imply that the supersymmetry parameter ε of a $N=4$ (locally) supersymmetric σ -model defined on such a manifold transforms under $\text{sp}(1)$ transformations. This means that the fully covariantized derivatives and transformation rules for the supergravity fields will

contain a non-trivial $\text{sp}(1)$ connection $\omega_I^A{}_B$ which is expressed in the scalar coordinate fields of the quaternionic manifold.

In the gauge theory for $N = 4$ conformal supergravity (cf. sect. 3) it is not possible to define $\text{sp}(1)$ covariantizations since the Weyl multiplet does not include a gauge field for this symmetry. For the description of the quaternionic couplings it is therefore natural to consider first an *extension* of the $N = 4$ supergravity multiplet which includes a $\text{sp}(1)$ vector field V_μ^I . Apart from the fields of $N = 4$ conformal supergravity this multiplet contains the fields of a $N = 4$ $\text{sp}(1)$ vector multiplet, which turns out to have $8 + 8$ components.

The linearized supersymmetry transformations of the $\text{sp}(1)$ vector multiplet can be found by applying dimensional reduction to the fields of the vector multiplet of $N = 2$ conformal supergravity in four dimensions [32]. In a notation with real bosonic fields and symplectic Majorana spinor fields, this multiplet, which has $8 + 8$ components, contains a vector W_μ , scalars X_1 and X_2 , a scalar $\text{su}(2)$ -triplet Y_I and a symplectic Majorana spinor Ω_A . The linearized δ_Q transformations read

$$\begin{aligned}\delta_Q W_\mu &= \bar{\epsilon}_A \Gamma_\mu \Omega^A, \\ \delta_Q X_1 &= i \bar{\epsilon}_A \Omega^A, \quad \delta_Q X_2 = \bar{\epsilon}_A \Gamma_5 \Omega^A, \\ \delta_Q Y_I &= -i \bar{\epsilon}_A (\sigma_I)^A{}_B \Gamma^\mu \partial_\mu \Omega^B, \\ \delta_Q \Omega^A &= -\frac{1}{4} \left[i Y_I (\sigma_I)^A{}_B \epsilon^B - i \Gamma^\mu \partial_\mu X_1 \epsilon^A + \Gamma_5 \Gamma^\mu \partial_\mu X_2 \epsilon^A - \Sigma^{\mu\nu} F_{\mu\nu} \epsilon^A \right].\end{aligned}\quad (4.1)$$

We now perform the dimensional reduction $(x_1, x_2, x_3, x_4) \rightarrow (x_1, x_4)$, which in ref. [4] was done for the fields of a $N = 2$ non-linear σ -model, for the $N = 2$ vector multiplet. We use the explicit representation of the $d = 4$ Dirac matrices given in the appendix. The spinor fields Ω_A reduce to doubly symplectic Majorana spinors Ω_{Aa} in $d = 2$. For the bosonic fields we define

$$E_1 = W_2, \quad E_2 = W_3, \quad E_3 = X_2, \quad P = X_1, \quad D = -i(\partial_1 W_4 - \partial_4 W_1) \quad (4.2)$$

and we introduce a $d = 2$ vector field V_μ^I ($\mu, \nu, \dots = 1, 2$) such that

$$\epsilon^{\mu\nu} \partial_\mu V_\nu^I = i Y_I. \quad (4.3)$$

TABLE 4
Properties of the fields of the $N = 4$ $\text{sp}(1)$ vector multiplet $\mathcal{V}$

Field	Statistics	Weyl	# compts.	Properties
D	boson	2	1	real
P	boson	1	1	real
E_i	boson	1	3	$\text{su}(2)$ triplet
V_μ^I	boson	0	3	$\text{sp}(1)$ gauge field
Ω^{Aa}	fermion	$\frac{3}{2}$	8	$\bar{\Omega}_{Aa} = \epsilon_{AB} \epsilon_{ab} \Omega^{Bb} C_+$

Some properties of these fields are given in table 4. Their linearized δ_Q transformations read

$$\begin{aligned}
 \delta_Q D &= \bar{\epsilon} \gamma_3 \gamma^\mu \partial_\mu \Omega, \\
 \delta_Q P &= i \bar{\epsilon} \Omega, \quad \delta_Q E_i = \bar{\epsilon} \sigma_i \gamma_3 \Omega, \\
 \delta_Q V_\mu^I &= i \bar{\epsilon} \sigma^I \gamma_\mu \gamma_3 \Omega, \\
 \delta_Q \Omega &= -\frac{1}{4} \left[\epsilon^{\mu\nu} \partial_\mu V_\nu^I \sigma_I \epsilon - i \gamma^\mu \partial_\mu P \epsilon + \sigma_i \gamma_3 \gamma^\mu \partial_\mu E_i \epsilon - D \gamma_3 \epsilon \right]. \quad (4.4)
 \end{aligned}$$

Notice that Ω_{Aa} and E_i transform non-trivially under an internal $\text{su}(2)$ algebra which is a remnant of the four-dimensional Lorentz algebra. Following ref. [4] we identify this $\text{su}(2)$ with the $\text{su}(2)$ subalgebra of the $N = 4$ superconformal algebra which is gauged by B_μ^i .

We proceed by coupling the vector multiplet, which we denote by $\mathcal{V}$, to the fields of $N = 4$ conformal supergravity. The action of the generators S , D , M , A and B on the fields of $\mathcal{V}$ is derived by imposing the commutation relations of the $N = 4$ superconformal algebra. We find

$$\begin{aligned}
 \delta D &= -2i\bar{\eta} \gamma_3 \Omega - 2\Lambda_D D, \\
 \delta P &= -\Lambda_D P - \alpha^i E_i, \quad \delta E_i = -\Lambda_D E_i + \alpha_i P + \epsilon_{ijk} \beta^k E^j, \\
 \delta V_\mu^I &= 0, \\
 \delta \Omega &= -\frac{1}{2} \eta P - \frac{1}{2} i \gamma_3 \sigma^i \eta E_i - \frac{3}{2} \Lambda_D \Omega + \frac{1}{2} \Lambda_M \gamma_3 \Omega + \frac{1}{2} i (\alpha_i \gamma_3 + \beta_i) \sigma^i \Omega. \quad (4.5)
 \end{aligned}$$

$\mathcal{V}$ is inert under K , Q' , D' , M' and all higher generators of the $N = 4$ superconformal algebra.

Evaluating the $[\delta_Q^1, \delta_Q^2]$ commutator on the vector field V_μ^I we find the following extra contribution

$$[\delta_Q^1, \delta_Q^2] V_\mu^I = \dots + \delta_{\text{sp}(1)} \left(\Lambda^J = -\xi^\mu V_\mu^J - \xi^J P + \xi^{Ji} E_i \right) \quad (4.6)$$

(for the definition of the composite ξ -parameters, see A.11). The fields Ω_{Aa} and $\psi_{\mu Aa}$ transform as doublets under the $\text{sp}(1)$ algebra

$$\delta_{\text{sp}(1)}\Omega = -\frac{1}{2}ig\Lambda^I\sigma_I\Omega, \quad \delta_{\text{sp}(1)}\psi_\mu = -\frac{1}{2}ig\Lambda^I\sigma_I\psi_\mu \quad (4.7)$$

(g is an arbitrary coupling constant). This implies that, apart from the usual supergravity covariantizations in the transformation rules of $\mathcal{V}$, we need to include $\text{sp}(1)$ covariantizations in the transformation rules of both the fields of $\mathcal{V}$ and the supergravity fields in order to realize the $\delta_{\text{sp}(1)}$ term in (4.6) uniformly on all fields. They lead to extra terms in the $[\delta_Q^1, \delta_Q^2]$ commutator which are of first order in the coupling constant g . The final result is the following set of fully covariantized supersymmetry transformations

$$\begin{aligned} \delta_Q D &= \bar{\epsilon}\gamma_3\gamma^\mu D_\mu\Omega, \\ \delta_Q P &= i\bar{\epsilon}\Omega, \quad \delta_Q E_i = \bar{\epsilon}\sigma_i\gamma_3\Omega, \\ \delta_Q V_\mu^I &= i\bar{\epsilon}\sigma^I\gamma_\mu\gamma_3\Omega + \frac{1}{2}(\bar{\epsilon}\sigma^I\gamma_3\psi_\mu)P - \frac{1}{2}i(\bar{\epsilon}\sigma^I\sigma^i\psi_\mu)E_i, \\ \delta_Q \Omega &= -\frac{1}{4}[\epsilon^{\mu\nu}D_\mu V_\nu^I\sigma^I - i\gamma^\mu D_\mu P - \frac{1}{2}g\gamma_3 P^2 \\ &\quad + \sigma^i\gamma_3\gamma^\mu D_\mu E_i - \frac{1}{2}g\gamma_3 E^2 - \gamma_3 D]\epsilon, \\ \delta_Q e_\mu^m &= -\frac{1}{2}\bar{\epsilon}\gamma^m\psi_\mu, \\ \delta_Q \psi_\mu &= D_\mu\epsilon + \frac{1}{2}ig\gamma_\mu\gamma_3\epsilon P + \frac{1}{2}g\gamma_\mu\sigma^j\epsilon E_j, \\ \delta_Q B_\mu^i &= \frac{1}{2}(\bar{\epsilon}\sigma^i\varphi_\mu) - \frac{1}{2}ig(\bar{\epsilon}\sigma^i\gamma_3\gamma_\mu\Omega) \\ &\quad - \frac{1}{4}g(\bar{\psi}_\mu\gamma_3\sigma^i\epsilon)P + \frac{1}{4}ig(\bar{\psi}_\mu\sigma^j\sigma^i\epsilon)E_j, \\ \delta_Q A_\mu^i &= -\frac{1}{2}(\bar{\epsilon}\gamma_3\sigma^i\varphi_\mu) + \frac{1}{2}ig(\bar{\epsilon}\sigma^i\gamma_\mu\Omega) \\ &\quad - \frac{1}{4}g(\bar{\psi}_\mu\sigma^i\epsilon)P + \frac{1}{4}ig(\bar{\psi}_\mu\sigma^j\sigma^i\gamma_3\epsilon)E_j, \\ \delta_Q b_\mu &= \frac{1}{2}i(\bar{\epsilon}\varphi_\mu) + \frac{1}{4}ig(\bar{\psi}_\mu\gamma_3\epsilon)P + \frac{1}{4}g(\bar{\psi}_\mu\sigma^i\epsilon)E_i, \\ \delta_Q \omega_\mu &= -\frac{1}{2}i(\bar{\epsilon}\gamma_3\varphi_\mu) - \frac{1}{4}ig(\bar{\psi}_\mu\epsilon)P - \frac{1}{4}g(\bar{\psi}_\mu\sigma^i\gamma_3\epsilon)E_i, \end{aligned} \quad (4.8)$$

where we defined the following covariant derivatives

$$\begin{aligned}
D_\mu P &= \partial_\mu P + i\bar{\psi}_\mu \Omega + b_\mu P + E_i A_\mu^i, \\
D_\mu E_i &= \partial_\mu E_i + \bar{\psi}_\mu \sigma_i \gamma_3 \Omega + b_\mu E_i - P A_\mu^i - \varepsilon_{ijk} B_\mu^k E^j, \\
D_\mu V_\nu^I &= \partial_\mu V_\nu^I - i(\bar{\psi}_\mu \sigma^I \gamma_\nu \gamma_3 \Omega) - \frac{1}{2}(\bar{\psi}_\mu \sigma^I \gamma_3 \psi_\nu) P \\
&\quad + \frac{1}{2}i(\bar{\psi}_\mu \sigma^I \sigma^i \psi_\nu) E_i - \frac{1}{2}g\varepsilon_{IJK} V_\mu^J V_\nu^K, \\
D_\mu \Omega &= \left(\partial_\mu - \frac{1}{2}\omega_\mu \gamma_3 + \frac{3}{2}b_\mu - \frac{1}{2}iB_\mu^i \sigma_i - \frac{1}{2}iA_\mu^i \gamma_3 \sigma_i \right) \Omega \\
&\quad + \frac{1}{2}igV_\mu^I \sigma_I \Omega - \delta_Q \Omega(\varepsilon \rightarrow \psi_\mu) + \frac{1}{2}\varphi_\mu P + \frac{1}{2}i\gamma_3 \sigma_i \varphi_\mu E_i, \\
D_\mu \varepsilon &= \left(\partial_\mu - \frac{1}{2}\omega_\mu \gamma_3 - \frac{1}{2}b_\mu - \frac{1}{2}iB_\mu^i \sigma_i - \frac{1}{2}iA_\mu^i \sigma_i \gamma_3 \right) \varepsilon \\
&\quad + \frac{1}{2}igV_\mu^I \sigma_I \varepsilon.
\end{aligned} \tag{4.9}$$

These transformations realize the following $[\delta_Q^1, \delta_Q^2]$ commutator on all fields

$$\begin{aligned}
[\delta_Q^1, \delta_Q^2] &= \delta_{\text{gct}}(\xi^\rho) \\
&\quad + \delta_Q(\varepsilon = -\xi^\rho \psi_\rho) + \delta_S(\eta = -\xi^\rho \varphi_\rho) + \delta_D(\Lambda_D = -\xi^\rho b_\rho) + \dots \\
&\quad + \delta_{\text{sp}(1)}(\Lambda^I = -\xi^\rho V_\rho^I - \xi^I P + \xi^I E_i) \\
&\quad + \delta_M(\Lambda_M = -g\xi P - g\xi^i E_i) \\
&\quad + \delta_S(\eta = \frac{1}{4}g\xi \gamma_3 \Omega + \frac{1}{4}g\xi^I \gamma_3 \sigma_i \sigma_I \Omega \\
&\quad \quad - \frac{1}{4}ig\xi^i \sigma_i \Omega - \frac{1}{4}ig\xi^I \sigma_I \Omega + ig\xi^\rho \gamma_3 \gamma_\rho \Omega).
\end{aligned} \tag{4.10}$$

The remaining commutation relations coincide with those of the $N=4$ superconformal algebra, e.g.

$$\begin{aligned}
[\delta_S, \delta_Q] &= \delta_D(\Lambda_D = -\frac{1}{2}i\bar{\eta}\varepsilon) + \delta_M(\Lambda_M = -\frac{1}{2}i\bar{\eta}\gamma_3 \varepsilon) \\
&\quad + \delta_A(\alpha^i = \frac{1}{2}\bar{\eta}\sigma_i \gamma_3 \varepsilon) + \delta_B(\beta^i = \frac{1}{2}\bar{\eta}\sigma_i \varepsilon).
\end{aligned} \tag{4.11}$$

Under linearized supersymmetry the scalar field D , which has Weyl weight 2, transforms into a total derivative. This suggests that the term eD can be covarian-

tized into an invariant action density. Indeed, the following lagrangian is invariant under all local symmetries of the $N = 4$ superconformal algebra

$$\begin{aligned} \mathcal{L}_D = & eD + e\bar{\psi}_\mu\gamma_3\gamma^\mu\Omega + \frac{1}{4}e\left(\bar{\psi}_\mu\epsilon^{\mu\nu}\psi_\nu\right)P - \frac{1}{2}egP^2 \\ & - \frac{1}{4}ie\left(\bar{\psi}_\mu\gamma_3\sigma_i\epsilon^{\mu\nu}\psi_\nu\right)E_i - \frac{1}{2}egE^2. \end{aligned} \quad (4.12)$$

For the theory of $N = 2$ conformal supergravity an extended supergravity multiplet can be defined which is similar to the $N = 4$ multiplet above. It contains a $4 + 4$ component U(1) vector multiplet (V_μ, Ω, P, E, D) [V_μ is a U(1) vector, P , E and $\bar{D}$ are real scalars and Ω is a Dirac spinor with U(1)-charge $\frac{1}{2}$]. The linearized δ_Q transformations read

$$\begin{aligned} \delta_Q D &= i\bar{\epsilon}\gamma_3\gamma^\mu\partial_\mu\Omega + \text{c.c.}, \\ \delta_Q P &= \bar{\epsilon}\Omega + \text{c.c.}, \quad \delta_Q E = -i\bar{\epsilon}\gamma_3\Omega + \text{c.c.}, \\ \delta_Q V_\mu &= \bar{\epsilon}\gamma_3\gamma_\mu\Omega + \text{c.c.}, \\ \delta_Q \Omega &= \left[\epsilon^{\mu\nu}\partial_\mu V_\nu + \gamma^\mu\partial_\mu P + i\gamma_3\gamma_\mu\partial_\mu E + i\gamma_3 D\right]\epsilon. \end{aligned} \quad (4.13)$$

Just as in the $N = 4$ case these rules can be extended to a fully covariant set of transformation rules.

5. $N = 4$ scalar multiplets and the $\text{HP}(N - 1)$ σ -model

In this section we discuss the coupling of $N = 4$ scalar multiplets to the extended $N = 4$ supergravity multiplet. We will show that this coupling naturally leads to a non-linear σ -model with scalar fields parametrizing a quaternionic projective space $\text{HP}(n - 1)$. The construction is done in two steps. We first couple the extended $8 + 8$ supergravity multiplet to n $N = 4$ scalar multiplets, which represent $4 + 4$ on-shell degrees of freedom each. The resulting model is a locally supersymmetric flat σ -model with local $\text{sp}(1)$ invariance. In the second step we eliminate the fields of the vector multiplet by using their equations of motion. Since some of the fields in the vector multiplet occur as Lagrange multipliers in the action this leads to $1 + 4$ constraints on the matter fields, which together with a $\text{sp}(1)$ gauge choice, reduce the model to $4(n - 1) + 4(n - 1)$ on-shell degrees of freedom. The scalar fields in the resulting model parametrize the space $\text{HP}(n - 1)$.

We introduce the following notation for $N = 4$ scalar and spinor matter fields

$$(\Phi_A^X, \lambda^{Xa}), \quad (5.1)$$

where $A = 1, 2$, $a = 1, 2$ and $X = 1, 2, \dots, r$. In analogy with the doubly symplectic

Majorana conditions for ε^{Aa} , Ω^{Aa} , etc. we require λ^{Xa} to satisfy the following doubly symplectic Majorana condition

$$\bar{\lambda}_{Xa} = (\lambda^\dagger)_{Xa} \gamma_2 = \rho_{XY} \varepsilon_{ab} \lambda^{Yb} C_+ . \quad (5.2)$$

Consistency requires

$$\rho_{XY}(\rho^*)^{YZ} = -\delta_X^Z . \quad (5.3)$$

It can be shown (see appendix B of ref. [33] for a detailed discussion) that by suitably redefining λ the matrix ρ can be brought into the following canonical form

$$(\rho_{XY}) = \begin{pmatrix} & & & 1 \\ & & & \\ -1 & & & \\ & & 1 & \\ & -1 & & \\ & & & \ddots \end{pmatrix}, \quad X = 1, 2, \dots, 2n . \quad (5.4)$$

The linearized δ_Q rules for Φ and λ read

$$\begin{aligned} \delta \Phi_A^X &= -\sqrt{\frac{1}{2}} \bar{\varepsilon}_{Aa} \lambda^{Xa}, \\ \delta \lambda^{Xa} &= \sqrt{\frac{1}{2}} \gamma^\mu \partial_\mu \Phi_A^X \varepsilon^{Aa}. \end{aligned} \quad (5.5)$$

Consistency of these rules with (5.2) requires

$$(\Phi_A^X)^* = \varepsilon^{AB} \Phi_B^Y \rho_{YX}. \quad (5.6)$$

We conclude that, with raising and lowering of A , a and X defined as in the appendix, we have the following reality constraints

$$(\varepsilon^{Aa})^* = \varepsilon_{Aa}, \quad (\Phi_A^X)^* = \Phi_X^A, \quad (\lambda^{Xa})^* = \lambda_{Xa}. \quad (5.7)$$

From the condition (5.6) on Φ_A^X we conclude that the fields Φ describe a vector space of dimension n over the quaternions, with real dimension $4n$, which we denote by $\mathbb{H}^n$.

We will now discuss the coupling of these matter fields to the fields of the extended $N = 4$ conformal supergravity multiplet presented in sect. 4. By imposing the superconformal algebra, including the modified $[\delta_Q^1, \delta_Q^2]$ commutator (4.10), on the fields Φ_A^X we derive the following covariantized transformation rules

$$\begin{aligned} \delta_Q \Phi_A^X &= -\sqrt{\frac{1}{2}} \bar{\varepsilon}_A \lambda^X, \\ \delta_Q \lambda^X &= \sqrt{\frac{1}{2}} \gamma^\mu \partial_\mu \Phi_A^X \varepsilon^A - \frac{1}{2} (\bar{\psi}_\mu^A \lambda^X) \gamma^\mu \varepsilon_A \\ &\quad + \sqrt{\frac{1}{2}} g (\gamma_3 \varepsilon^B \Phi_B^X P - i \sigma_i \varepsilon^B \Phi_B^X E_i) \\ &\quad - \sqrt{\frac{1}{2}} i g \gamma^\mu (\sigma_I)^A{}_B \varepsilon^B V_\mu^I \Phi_A^X. \end{aligned} \quad (5.8)$$

Knowing these fully covariantized transformation rules we can now derive an invariant action for (Φ_A^X, λ^{Xa}) coupled to $N=4$ conformal supergravity and the $\text{sp}(1)$ vector multiplet. In lowest order this action is just the usual string action

$$-\frac{1}{2}e\left(\partial_\mu\Phi_A^X\right)\left(\partial^\mu\Phi_X^A\right)-\frac{1}{2}e\bar{\lambda}^X\gamma^\mu\partial_\mu\lambda_X. \quad (5.9)$$

The local $\text{su}(2)$ invariance for λ^{Xa} and $\text{sp}(1)$ invariance for Φ_A^X require the following covariantizations

$$\begin{aligned} \partial_\mu\Phi_X^A &\rightarrow \partial_\mu\Phi_X^A + \frac{1}{2}igV_\mu^I(\sigma_I)^A{}_B\Phi_X^B, \\ \partial_\mu\lambda^{Xa} &\rightarrow \partial_\mu\lambda^{Xa} - \frac{1}{2}i\left(B_\mu^i - \gamma_3 A_\mu^i\right)(\sigma_i)^a{}_b\lambda^{Xb}. \end{aligned} \quad (5.10)$$

Since we know the fully covariantized transformation rules for all fields it is a straightforward exercise to extend the terms given above to an invariant action. We obtain the following result

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= e\mathcal{L}_0 + e\mathcal{L}_1 + e\mathcal{L}_2, \\ \mathcal{L}_0 &= -\frac{1}{2}\left(\partial^\mu\Phi_A^X\right)\left(\partial_\mu\Phi_X^A\right) - \frac{1}{2}\bar{\lambda}^X\gamma^\mu\partial_\mu\lambda_X \\ &\quad - \frac{1}{2}igV_\mu^I\left(\partial^\mu\Phi_A^X\right)(\sigma_I)^A{}_B\Phi_X^B - \frac{1}{8}g^2V^2\Phi^2 - \frac{1}{4}gD\Phi^2 \\ &\quad - \sqrt{\frac{1}{2}}g\left(\bar{\Omega}^A\gamma_3\lambda^X\right)\Phi_{AX} + \frac{1}{4}i\left(\bar{\lambda}^X\gamma^\mu\sigma_i\lambda_X\right)\hat{B}_\mu^i \\ &\quad + \frac{1}{4}ig\left(\bar{\lambda}^X\gamma_3\lambda_X\right)P - \frac{1}{4}g\left(\bar{\lambda}^X\sigma_i\lambda_X\right)E_i, \\ \mathcal{L}_1 &= -\sqrt{\frac{1}{2}}\left(\bar{\psi}_\mu^A\gamma^\nu\gamma^\mu\lambda_X\right)\partial_\nu\Phi_A^X \\ &\quad - \frac{1}{4}g\left(\bar{\psi}_\mu\gamma_3\gamma^\mu\Omega\right)\Phi^2 + \frac{1}{2}\sqrt{\frac{1}{2}}igV_\mu^I\Phi_A^X\left(\bar{\lambda}_X\gamma^\nu\gamma^\mu(\sigma_I)^A{}_B\psi_\nu^B\right) \\ &\quad + \frac{1}{2}\sqrt{\frac{1}{2}}ig\left(\bar{\psi}_\mu^A\gamma_3\gamma^\mu\lambda^X\right)\Phi_{AX}P + \frac{1}{2}\sqrt{\frac{1}{2}}g\left(\bar{\psi}_\mu^A\sigma_i\gamma^\mu\lambda^X\right)\Phi_{AX}E_i, \\ \mathcal{L}_2 &= -\frac{1}{4}\left(\bar{\psi}_\mu^A\gamma^\nu\gamma^\mu\lambda_X\right)\left(\bar{\psi}_{\nu A}\lambda^X\right) \\ &\quad - \frac{1}{16}g\left(\bar{\psi}_\mu\varepsilon^{\mu\nu}\psi_\nu\right)\Phi^2P + \frac{1}{16}ig\left(\bar{\psi}_\mu\gamma_3\sigma_i\varepsilon^{\mu\nu}\psi_\nu\right)\Phi^2E_i. \end{aligned} \quad (5.11)$$

A useful check on this result is provided by evaluating the $[\delta_Q^1, \delta_Q^2]$ commutator on the spinor fields λ^X . Since we have not included auxiliary fields in our matter multiplet this commutator will pick up terms proportional to the λ^X field equation. We have checked that indeed the commutator (4.10) is realized on λ^X modulo terms that are proportional to the λ^X Dirac equation as it follows from the action (5.11).

We are now ready to identify the correct action for the $\mathbb{H}^n$ σ -model with local $\text{sp}(1)$ invariance. The density (5.11) is of “string-type” but it cannot serve as lagrangian for our model since the field equations for the auxiliary field D lead to the constraint $\Phi^2 = 0$, which is clearly not what we want. However, we have a second action (4.12) at our disposal and it turns out that a linear combination of both actions (4.12) and (5.11) has the right features

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \frac{1}{2} \mathcal{L}_D . \quad (5.12)$$

We proceed by considering the field equations for the fields of the vector multiplet $\mathcal{V}$. The fields D and Ω^{Aa} act as Lagrange multipliers in the action; their field equations amount to the following constraints on Φ and λ

$$\Phi^2 = 2/g, \quad \lambda^{Xa} \Phi_X^A = 0. \quad (5.13)$$

The fields V_μ^I , E_i and P can readily be solved from their field equations. Using (5.13) to simplify the expressions we find the following results

$$\begin{aligned} V_\mu^I &= -i \partial_\mu \Phi^X \sigma_I \Phi_X, \\ E_i &= \frac{1}{2} (\bar{\lambda}^X \sigma_i \lambda_X), \\ P &= -\frac{1}{2} i (\bar{\lambda}^X \gamma_3 \lambda_X). \end{aligned} \quad (5.14)$$

The constraints (5.13) eliminate 1 bosonic degree of freedom and 8 fermionic components which correspond to 4 on-shell fermionic degrees of freedom. The $\text{sp}(1)$ gauge condition will eliminate three more bosonic degrees of freedom, which means that we will be left with $4(n-1) + \underline{4(n-1)}$ on-shell degrees of freedom in the resulting model.

We now wish to parametrize these degrees of freedom by unconstrained fields. We split the range of the $X, Y, \dots$ indices as follows

$$p, q, \dots = 1, 2, \quad X, Y, \dots = 3, 4, \dots, 2n.$$

We fix the local $\text{sp}(1)$ symmetry by imposing the following gauge condition

$$\Phi_A^p = i \delta_A^p (2C)^{-1/2}, \quad (5.15)$$

where

$$C^{-1} = \Phi_A^p \Phi_p^A. \quad (5.16)$$

Next we define the following $\text{sp}(1)$ invariant matter fields

$$\begin{aligned} A_p^X &= 2C \Phi_p^A \Phi_A^X, \\ S^X &= \sqrt{(2C)} (\lambda^X - A_p^X \lambda^p). \end{aligned} \quad (5.17)$$

Using the constraints (5.13) we can invert these relations as follows

$$\begin{aligned}
 \Phi_A^X &= \Phi_A^p A_p^X, & \Phi_A^p &= i\delta_A^p (2C)^{-1/2}, \\
 C &= \frac{1}{2}g \left(1 - \frac{1}{2}A_p^X A_X^p\right), \\
 \lambda^Y &= \left(\delta_X^Y + \frac{g}{2C} A_p^Y A_X^p\right) (2C)^{-1/2} S^X, \\
 \lambda^p &= \frac{g}{2C} (2C)^{-1/2} S^X A_X^p.
 \end{aligned} \tag{5.18}$$

We further define

$$\psi_{\mu p} = -i\sqrt{(2C)} \Phi_p^A \psi_{\mu A}, \quad \varepsilon_p = -i\sqrt{(2C)} \Phi_p^A \varepsilon_A \tag{5.19}$$

such that

$$\bar{\varepsilon}_A \psi_\mu^A = \bar{\varepsilon}_p \psi_\mu^p \quad \text{etc.} \tag{5.20}$$

Let us now consider the action in this second order formulation of the model. We insert the expressions for V_μ^I , E_i and P in (5.11) and, using the definitions given above we express the result in terms of the newly defined unconstrained fields A_p^X and S^X . The resulting action looks as follows

$$\begin{aligned}
 e^{-1}\mathcal{L} &= \frac{1}{2}\Delta_X^Y \partial_\mu A_p^X \partial^\mu A_Y^p - \frac{1}{2}\Delta_X^Y \bar{S}^X \gamma^\mu \partial_\mu S_Y \\
 &\quad - \frac{1}{2}\Delta_X^Y \bar{S}^X \left[\frac{g}{2C} (\gamma^\mu \partial_\mu A_Y^p) A_p^Z \right] S_Z - \frac{1}{2}\Delta_X^Y \bar{S}^X \left[\frac{\gamma^\mu \partial_\mu C}{2C} \right] S_Y \\
 &\quad - \frac{1}{16}g \left[\Delta_X^Y (\bar{S}^X \gamma_3 S_Y) \right]^2 + \frac{1}{16}g \left[\Delta_X^Y (\bar{S}^X \sigma_i S_Y) \right]^2 + \frac{1}{4}i\Delta_X^Y (\bar{S}^X \gamma^\mu \sigma_i S_Y) \hat{B}_\mu^i \\
 &\quad - \sqrt{\frac{1}{2}} i\Delta_X^Y (\bar{\psi}_\mu^p \gamma^\nu \gamma^\mu S^X) \left(\partial_\nu A_p^X + \frac{1}{2}\sqrt{\frac{1}{2}} i\bar{\psi}_{\nu p} S^X \right),
 \end{aligned} \tag{5.21}$$

where we defined

$$\Delta_X^Y = \frac{1}{2C} \left[\delta_X^Y + \frac{g}{2C} A_X^p A_p^Y \right]. \tag{5.22}$$

The fields A^{pX} parametrize the quaternionic projective space $\text{HP}(n-1)$. Its metric Δ_X^Y is the analogue of the Fubini-Study metric on the complex projective space $\text{CP}(n)$.

We proceed by deriving the transformation rules under supersymmetry of the redefined fields. Since the Lagrange multiplier Ω_{Aa} occurs in these rules we need to

know its value in terms of Φ and λ . One might be tempted to derive this expression from the expression (5.14) for P and the δ_Q rule $\delta P \sim \bar{\epsilon}\Omega$. However, this leads to a wrong result since the set of field equations for the fields of the vector multiplet $\mathcal{V}$ is *not* an invariant set under supersymmetry. The correct procedure, which preserves the invariance of the action, is to use the field equations of the constraint variables $C^{Aa} = \Phi_X^A \lambda^{Xa}$. In the action there is a term $\simeq \bar{C}_{Aa} \Omega^{Aa}$ and thus Ω_{Aa} can be evaluated from the $\bar{C}_{Aa}$ field equations. We find

$$\Omega_A = -\frac{1}{g\sqrt{2}} \gamma_3 \gamma^\mu \partial_\mu \Phi_{AX} \lambda^X - \frac{1}{4} i \gamma^\mu \psi_{\mu A} P + \frac{1}{4} \gamma_3 \gamma^\mu \sigma_i \psi_{\mu A} E_i. \quad (5.23)$$

The resulting transformation rules for A_p^X , S^X , ψ_μ^p and $\hat{B}_\mu^i$ are as follows

$$\begin{aligned} \delta_Q A_p^X &= -\sqrt{\frac{1}{2}} i \bar{\epsilon}_p S^X, \\ \delta_Q S^X &= \sqrt{\frac{1}{2}} i \gamma^\mu \partial_\mu A_X^p \epsilon_p - \frac{1}{2} \gamma^\mu \epsilon_p (\bar{\psi}_\mu^p S_X) \\ &\quad - \left[\frac{g}{2C} \delta_Q (A_X^p) A_p^Y \right] S_Y + \left[\frac{\delta_Q C}{2C} \right] S_X, \\ \delta_Q \psi_\mu &= D_\mu \epsilon + \left[\frac{g}{4C} \partial_\mu A_X \sigma_I A^X \right] \sigma_I \epsilon - \left[\frac{g}{4C} \delta_Q (A_X) \sigma_I A^X \right] \sigma_I \psi_\mu \\ &\quad - \frac{1}{4} g \Delta_X^Y (\bar{S}^X \gamma_3 S_Y) \gamma_\mu \gamma_3 \epsilon - \frac{1}{4} g \Delta_X^Y (\bar{S}^X \sigma^I S_Y) \gamma_\mu \sigma_I \epsilon, \\ \delta_Q \hat{B}_\mu^i &= \frac{1}{2} (\bar{\epsilon} \sigma_i \gamma_\mu \gamma^\nu \varphi_\nu) + \sqrt{\frac{1}{2}} \Delta_X^Y (\bar{\epsilon}_p \sigma_i \gamma_\mu \gamma^\nu S^X) \partial_\nu A_Y^p. \end{aligned} \quad (5.24)$$

The results (5.21) and (5.24) are in agreement with those reported in [4] for a general quaternionic manifold (note that the hyper-Kähler curvature S_{XYZW} [4] vanishes for the manifold $HP(n-1)$).

We close this section by a comment on the algebra structure of the transformation rules (5.24). In sect. 4 we derived the form (4.10) for the commutator $[\delta_Q^1, \delta_Q^2]$ in the $N=4$ theory with local $sp(1)$ invariance. For the final result (5.24) this form is further modified due to the redefinitions (5.19) (which induce compensating $sp(1)$ transformations in the $\delta_Q \psi_\mu$ rules) and to the fact that we inserted a set of field equations which is not invariant under supersymmetry.

6. Discussion

The above construction of the $HP(n-1)$ σ -model can be generalized to more general quaternionic σ -models. In such a construction the fields of the $N=4$ extended supergravity multiplet are coupled to a globally supersymmetric hyper-

Kähler σ -model of dimension $4n$. The elimination of the fields of the $\text{sp}(1)$ vector multiplet from such a model will lead to a quaternionic σ -model of dimension $4(n-1)$ in a background of $N=4$ supergravity. By an analysis similar to [34] this construction might yield interesting information about the geometry of hyper-Kähler and quaternionic manifolds.

The construction in sect. 5 has interesting parallels with constructions in $N=2$ and $N=4$ supergravity in *four* dimensions of certain non-linear σ -models coupled to Poincaré supergravity [33,35]. In the $d=4$ constructions an essential step consists in fixing the gauge for the dilatation (D) and conformal supersymmetry (S) invariances in order to set a scale for the non-linear σ -model. In two dimensions, where the scalar fields are Weyl-invariant, the D - and S -invariances are not fixed but remain in the theory as manifest local symmetries. Clearly, two dimensions show a new variation on the four-dimensional theme.

The emergence of the $N=4$ $\text{HP}(n-1)$ σ -model from the flat $\mathbb{H}^n$ model can be pictured as follows

$$\mathbb{H}^n \quad 4n \xrightarrow[D]{} 4n-1 \xrightarrow[\text{sp}(1)]{} 4n-4,$$

$$\text{geometry:} \quad \frac{\text{SP}(n)}{\text{SP}(n-1) \times \text{SP}(1)} \simeq \text{HP}(n-1).$$

It turns out that a similar construction can be done for the $N=2$ conformal supergravity model. The coupling of a flat $\mathbb{C}^n$ σ -model to the extended $4+4$ component supergravity multiplet (cf. sect. 4) leads to a $N=2$ non-linear σ -model with scalar fields parametrizing the complex projective space $\text{CP}(n-1)$. Schematically

$$\mathbb{C}^n \quad 2n \xrightarrow[D]{} 2n-1 \xrightarrow[\text{U}(1)]{} 2n-2,$$

$$\text{geometry:} \quad \frac{\text{SU}(n)}{\text{SU}(n-1) \times \text{U}(1)} \simeq \text{CP}(n-1).$$

For both the $N=2$ and the $N=4$ constructions the radius of the scalar manifold is proportional to $g^{-1/2}$, which is a free parameter in the theory.

There are also parallels with the construction of a $(8,0)$ locally supersymmetric σ -model which was presented very recently in [13]. This construction is based on a $64 + \underline{64}$ component supergravity multiplet which is similar to the $8 + \underline{8}$ and $4 + \underline{4}$ extended supergravity multiplets for $N=4,2$. An important distinction is that the $N=8$ $64 + \underline{64}$ multiplet cannot be separated into a “pure” $(0+0)$ supergravity multiplet and a vector multiplet. Accordingly, there is no free parameter g in the theory. One of the fields of the $N=8$ multiplet, which has a local $\text{so}(8)$ symmetry, is

a scalar field D^{ij} which is analogous to the scalar fields D in the $N = 4, 2$ multiplets and which generates 35 constraints. The construction in [13] can be pictured as

$$\mathbb{R}^{8n} \xrightarrow[D^{ij}]{} 8n \xrightarrow[\text{so}(8)]{} 8n - 35 \xrightarrow{} 8n - 63,$$

$$\text{geometry: } \frac{\text{SO}(n)}{\text{SO}(n-8) \times \text{SO}(8)}, \quad \text{dynamical radius } r.$$

Since there is no free parameter g in the theory the radius r of the scalar manifold is not fixed but appears in the model as a dynamical variable. Due to the presence of this r field and of a compensating field φ the structure of the $N = 8$ model is substantially different from the structure of the $N \leq 4$ models.

Several other extensions can be made. We mention chiral models with supersymmetry of type $(p, 0)$, $p = 1, 2, 4$, which include so-called heterotic fermions [6, 36, 37] and models having a Wess-Zumino term in their action [23, 37, 38]. An interesting problem is to find the critical dimension of these models. Since the manifolds occuring for $N > 1$ are not group manifolds the method that was used in [23] is not applicable to models with extended supersymmetry.

It is a pleasure to thank B. de Wit for stimulating discussions and encouragement. This investigation was performed as part of the research program of the “Stichting voor Fundamenteel Onderzoek der Materie” (F.O.M.).

Appendix

In this appendix we specify our notations and conventions for spinors and internal indices.

The metric is $\delta_{mn} = (+, +)$, $m, n = 1, 2$. We have γ -matrices

$$\gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.1})$$

obeying

$$\gamma^m \gamma^n = \gamma^{mn} - i\epsilon^{mn} \gamma_3, \quad (\text{A.2})$$

where $\epsilon_{12} = \epsilon^{12} = 1$. The Majorana conjugate is defined w.r.t. C_+

$$C_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_+ \gamma_m C_+^{-1} = \gamma_m^T. \quad (\text{A.3})$$

For the dimensional reduction $(x_1, x_2, x_3, x_4) \rightarrow (x_1, x_4)$ we use the following

representation for the $d = 4$ Γ -matrices ([4])

$$\begin{aligned} \Gamma_1 &= \begin{pmatrix} \gamma^1 & 0 \\ 0 & \gamma^1 \end{pmatrix}, & \Gamma_2 &= \begin{pmatrix} 0 & \gamma_3 \\ \gamma_3 & 0 \end{pmatrix}, & \Gamma_3 &= \begin{pmatrix} 0 & -i\gamma_3 \\ i\gamma_3 & 0 \end{pmatrix}, & \Gamma_4 &= \begin{pmatrix} \gamma^2 & 0 \\ 0 & \gamma^2 \end{pmatrix}, \\ \Gamma_5 &= \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 = \begin{pmatrix} \gamma_3 & 0 \\ 0 & -\gamma_3 \end{pmatrix}, & C_+^{(4)} &= \begin{pmatrix} 0 & C_+ \\ -C_+ & 0 \end{pmatrix}. \end{aligned} \quad (\text{A.4})$$

$a, b, \dots$ and $i, j, \dots$ are doublet and triplet indices for the $\text{su}(2)$ subalgebra of the $N = 4$ superconformal algebra, denoting representations of both the $\text{su}(2)$ phase transformations B^i and $\text{su}(2)$ chiral transformations A^i . Raising and lowering: $\chi^a = \epsilon^{ab} \chi_b$, $\chi_a = \chi^b \epsilon_{ba}$. $A, B, \dots$ and $I, J, \dots$ are doublet and triplet indices for the $\text{sp}(1)$ algebra corresponding to V_μ^I , $\chi^A = \epsilon^{AB} \chi_B$, $\chi_A = \chi^B \epsilon_{BA}$. The indices $X, Y, \dots$ on the matter fields Φ_A^X and λ^{Xa} are raised and lowered by the canonical antisymmetric matrix ρ_{XY} : $\lambda^X = \rho^{XY} \lambda_Y$, $\lambda_X = \lambda^Y \rho_{YX}$.

The following conditions on spinor fields carrying internal indices are consistent reality constraints

(i) symplectic Majorana spinor in $d = 4$

$$\bar{\chi}_A = \chi_A^\dagger \Gamma_4 = \epsilon_{AB} \chi^B C_+^{(4)}, \quad (\text{A.5})$$

(ii) doubly symplectic Majorana spinor in $d = 2$

$$\bar{\chi}_{Aa} = \chi_{Aa}^\dagger \gamma_2 = \epsilon_{AB} \epsilon_{ab} \chi^{Bb} C_+, \quad (\text{A.6})$$

$$\bar{\chi}_{Xa} = \chi_{Xa}^\dagger \gamma_2 = \Omega_{XY} \epsilon_{ab} \chi^{Yb} C_+. \quad (\text{A.7})$$

Note that (A.5) and (A.6) are related by dimensional reduction based on the Γ -representation (A.4). A spinor χ^a and its conjugate $\bar{\chi}_a$ can be combined into a doubly symplectic Majorana spinor χ^{Aa} by defining

$$\begin{aligned} \chi^{1a} &= \chi^a, & \chi^{2a} &= \bar{\chi}_b \epsilon^{ba} C_+^{-1}, \\ \bar{\chi}_{1a} &= \bar{\chi}_a, & \bar{\chi}_{2a} &= -\epsilon_{ac} \chi^c C_+. \end{aligned} \quad (\text{A.8})$$

If we suppress indices a, A and X in our notations we will always assume the following conventions for summation

$$\begin{array}{ccc} & a & A \quad X \\ & \swarrow & \swarrow \quad \searrow \\ a & & A \quad X \end{array}$$

e.g.

$$\bar{\epsilon} \sigma_i \psi_\mu = \bar{\epsilon}_{Aa} (\sigma_i)^a_b \psi_\mu^{Ab}, \quad \Phi^2 = \Phi_A^X \Phi_X^A, \quad \text{etc.}$$

For rearranging composite expressions we use Fierz rearrangements and flip-lemmas. For the most complicated case, a contraction of spinors carrying both a and A indices, the *Fierz* rearrangement reads

$$(\bar{\chi}_1 \chi_2) \chi_3 = -\frac{1}{8} \sum_{\mathcal{O}} (\bar{\chi}_1 \mathcal{O} \chi_2) \mathcal{O} \chi_3 \quad (\text{A.9})$$

where $\mathcal{O}$ runs over the set $S = S^+ \cup S^-$,

$$S^+ = \{ \gamma_3, \sigma_i, \sigma_I, \gamma_\mu \sigma_i, \gamma_\mu \sigma_I, \gamma_3 \sigma_i \sigma_I \},$$

$$S^- = \{ 1, \gamma_\mu, \gamma_3 \sigma_i, \gamma_3 \sigma_I, \sigma_i \sigma_I, \gamma_\mu \sigma_i \sigma_I \},$$

and we have for $\mathcal{O}^\pm \in S^\pm$ (*flip-lemma*)

$$(\bar{\chi}_1 \mathcal{O}^\pm \chi_2) = \pm (\bar{\chi}_2 \mathcal{O}^\pm \chi_1). \quad (\text{A.10})$$

For fields carrying fewer indices the fierzing is done w.r.t. a subset of S .

We finally mention a shorthand notation for bilinears in the supersymmetry parameters ε_1 and ε_2 of two successive supersymmetry transformations. We define real parameters

$$\begin{aligned} \xi &= -\frac{1}{2} i (\bar{\varepsilon}_2 \varepsilon_1), & \xi^\rho &= -\frac{1}{2} (\bar{\varepsilon}_2 \gamma^\rho \varepsilon_1), \\ \xi^i &= -\frac{1}{2} (\bar{\varepsilon}_2 \gamma_3 \sigma^i \varepsilon_1), & \xi^I &= -\frac{1}{2} (\bar{\varepsilon}_2 \gamma_3 \sigma^I \varepsilon_1), \\ \xi^{iI} &= -\frac{1}{2} i (\bar{\varepsilon}_2 \sigma^i \sigma^I \varepsilon_1). \end{aligned} \quad (\text{A.11})$$

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