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Exclusion Statistics in Conformal Field Theory Spectra

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We propose a new method for investigating the exclusion statistics of quasiparticles in conformal field theory (CFT) spectra. The method leads to one-particle distribution functions, which generalize the Fermi-Dirac distribution. For the simplest SU(n) invariant CFTs we find a generalization of Gentile parafermions, and we obtain new distributions for the simplest Z_k-invariant CFTs. In special examples, our approach reproduces distributions based on “fractional exclusion statistics” in the sense of Haldane. We comment on applications to fractional quantum Hall effect edge theories.

We here propose a new method for studying the exclusion statistics of CFT quasiparticles. At the heart of our method is what can be called a “transfer matrix for truncated chiral spectra.” We shall present a number of examples where CFT spectra are completely encoded in one-particle distribution functions (generalizing the Fermi-Dirac distribution). In special examples, the statistics that we find are of the type proposed by Haldane, while in other cases we find more general results.

One check on the distributions that we propose here is the coefficient γ in the low-temperature specific heat,
which is known to be related to the central charge $c_{\text{CFT}}$ of the CFT according to [7]

$$\frac{C}{L} = \gamma k_B \rho_0 T, \quad \gamma = \frac{\pi}{6} c_{\text{CFT}},$$

(1)

where $\rho_0 = (\hbar v_F)^{-1}$ is the density of states per unit length.

To introduce our new method we focus on the simplest SU(2) invariant CFT, which is the SU(2)$_1$ Wess-Zumino-Witten model. For this CFT, a quasiparticle formulation has been proposed in [8] and worked out in great detail in [9]. The formulation uses operators $\phi^\pm$, which create quasiparticles called spinons. The spinons form a doublet under SU(2) and carry (dimensionless) energy $L_0 = s$. The chiral spectrum of the SU(2)$_1$ CFT may be built in the following manner. One starts by writing the following polarized $N$-spinon states:

$$\phi^{+} - [2N-1/4] - n_0 \cdots - n_1 [1/4] - n_1 [1/4] - n_1 [0],$$

(2)

with $n_N \geq \cdots \geq n_2 \geq n_1 \geq 0$. One then uses the Yangian symmetry algebra to construct multispinon states with mixed $+$ and $-$ indices. The collection of all these states forms a basis of the full chiral spectrum. The SU(2) content of the Yangian multiplet labeled by $n_1, \ldots, n_N$ follows from the generalized commutation relations satisfied by the spinon modes, or, equivalently, from the representation theory of the Yangian [8,9].

Comparing the allowed spinon modes $\phi^\pm$ with free fermion modes $\psi^{-(1/2) \pm l}$, we observe that the fermion mode $l + \frac{1}{2}$ has “split” into an odd mode $s = l + \frac{1}{2}$ and an even mode $s = l + \frac{1}{2}$. To maintain a level spacing of one unit, we view these two spinon modes as forming a single “one-particle level” in the spectrum. What we would like to do is to factorize the full chiral partition sum of the CFT into a product over these one-particle levels, so that the free energy becomes a sum of one-particle contributions.

While we cannot straightforwardly extract the contribution of the $l$th level, we may proceed as follows. We

$$\pi^{(l)}(q, x, z) = x z \partial_x \ln \lambda^{(l)}(q, x, z) = \frac{2}{\sqrt{1 + 4 q^{l-2} x^{-2}}},$$

(7)

for the expected total number of spinons in level $l$, and

$$\overline{Q}^{(l)}(q, z) = \frac{e}{2} z \partial_z \ln \lambda^{(l)}(q, x = 1, z) = \frac{e q^{l-(1/2)}(z - z^{-1})}{\sqrt{q^{l-1}(z + z^{-1})^2 + 4(1 - q^{l-1})}}$$

(8)

for the expected charge at level $l$ (we assume charges $\pm \frac{e}{2}$ for the $\pm$ spinons). The distributions (7) and (8) agree with the distributions obtained from fractional exclusion statistics with

$$G = \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right).$$

Note that (7) implies that for $z = 1$ (zero voltage) the thermodynamics of the spinon system is identical to that of two free fermions and the central charge is $c_{\text{CFT}} = 2 \times \frac{1}{2} = 1$. Note also that, with $z = e^{(l/2) / 2 e V}$, the integrated charge $\frac{1}{L} \sum_{l=1}^{\infty} \overline{Q}^{(l)}(q, z) = \frac{1}{4 \pi} e^2 V \rho_0$, which
is half of the value obtained for two charge \( \pm e \) free fermions.

The correspondence with Haldane statistics is satisfying since the spinons of the associated SU(2) HS spin chain satisfy these same statistics [4]. This confirms the validity of our new approach, which in no way relied on the exact solution of the HS chain.

We now present some special examples.

(1) SU\((n)\) CFTs.—The first generalization of the SU(2) results concerns the SU\((n)\) spinons. The Yangian symmetry of the SU\((n)\) CFT was established in [10] while the spinon formulation was presented in [3]. There are \( n \) fundamental spinon species \( \phi^i \), transforming in the representation \( \mathbf{\Pi} \) of SU\((n)\). Repeating the analysis shown above, finding explicitly the SU\((n)\) analog of the recursion matrices (4) and (6) (see [11]), we find that (i) a single spinon species \( \phi^i \) (in absence of any others) satisfies Haldane statistics with \( g = n/\pi \), (ii) when exciting all \( n \) spinon species symmetrically (choosing all \( x_i \) equal to \( x = e^{\beta \mu} \), the expected total occupation of the \( l \)th level is given by

\[
\bar{\pi}^{(l)}(q, x) = x \partial_x \ln[1 + q^lx + \ldots + (q^lx)^{n-1}]^n. \tag{9}
\]

Comparing these results, one finds that for \( n > 2 \) there is negative mutual exclusion among different spinons.

Interestingly, the statistics going with the distribution (9) were proposed by Gentile as early as 1940 [12]. One finds that a single "Gentile parafermion" contributes the amount \( n^{-1} / n \) to the central charge, so that the full result becomes

\[
c_{\text{CFT}} = n^{-1} / n = n - 1, \tag{10}
\]
as expected.

Our results are consistent with [13], where, by different methods, the link between SU\((n)\) HS spin chains and Gentile parafermions has also been established.

(2) \( Z_N \) parafermions.—Within the context of CFT, the simplest generalization of the Majorana fermion is the so-called \( Z_N \) parafermion. It features in a CFT of central charge \( c_{\text{CFT}} = 2(N-1)/N \) as a primary field of dimension \( h_N = N-1 \) and \( Z_N \) charge 1 [14]. By applying the method outlined above, we obtained a distribution function for the \( Z_N \) parafermion and established that the full CFT spectrum is reproduced by a gas of noninteracting quasiparticles of this type. For the purpose of explaining these results, we focus on the case \( N = 3 \).

It is well known that the chiral spectrum of the \( Z_3 \) parafermion CFT can be interpreted in terms of two parafermions \( \psi^\pm \) of opposite \( Z_3 \) charge. However, by exploiting the generalized commutation relations obeyed by the modes of \( \psi^\pm \) [14], one easily shows that the modes of \( \psi^+ \) alone can generate the full spectrum, and that the \( \psi^- \) quasiparticle can be viewed as a composite of two \( \psi^+ \) quasiparticles. Having understood how the \( \psi^+ \) modes alone build the chiral spectrum (see [11]), one may define truncated partition sums. We found the following recursion matrix between the \( l \)th and the \((l - 1)\)th truncated sums (which each have three components)

\[
\begin{pmatrix}
(1 - y^3) & y^2 & y \\
y(1 - y^3) & 1 & 2y^2 \\
2y^2(1 - y^3) & y(1 + y^3) & 1 + 2y^3
\end{pmatrix}, \tag{10}
\]

with \( y = x q^l \). The \( Z_3 \) parafermion distribution function is expressed in terms of the largest eigenvalue \( \lambda_+ \) of this matrix

\[
\bar{\pi}^{(l)}(q, x) = (y \partial_y \ln \lambda_+) (y = x q^l). \tag{11}
\]

From the asymptotics \( \lambda_+(y) \approx y^3 \), one finds that the maximum occupation per level equals 3. Using the Cardano formula one may write \( \bar{\pi}^{(l)} \) in closed form. We here present a plot (Fig. 1), which displays the function \( \bar{\pi} \) as a function of energy. As a check we (numerically) evaluated the coefficient \( \gamma \) of the specific heat, reproducing the expected value \( \gamma = \pi^4 / 3 \).

For general \( N \), the distribution for \( Z_N \) parafermions allows a maximum of \( 1 \) \( N(N - 1) \) particles per level.

(3) Quantum Hall effect edge theories.—As a further application we briefly discuss edge theories for the fractional quantum Hall effect (FQHE). We shall come back to this topic in a separate publication [15]. For the \( \nu = 1/m \) FQHE (with \( m \) an odd integer), the edge theory is a chiral

![FIG. 1. Distribution functions for \( Z_3 \) parafermions (solid line), for ordinary fermions (dashed line), and for particles satisfying \( g = 1/\pi \) exclusion statistics (dotted line). All distributions are at \( \mu = 0 \); the energy is given in units \( \beta^{-1} \).](image)
\(c_{\text{CFT}} = 1\) CFT at compactification radius \(R^2 = m\). The natural quasiparticles to consider are the edge electron (of charge \(-e\)) and the edge quasihole (of charge \(\frac{e}{m}\)). Writing the spectrum in terms of these quasiparticles, and applying the above procedure, we find that the fundamental quasiparticles are independent, and obey Haldane exclusion statistics with \(g = m\) and \(g = 1/m\), respectively. This is consistent with the result of bosonization applied to \(g\)-ons [16].

The (known) results for the specific heat \((c_{\text{CFT}} = 1)\) and the response to voltage \((\frac{Q}{V} = \frac{1}{m} \frac{1}{2\pi} e^2 V \rho_0)\) are easily reproduced by exploiting the duality between these edges.

These quantum Hall results can be appreciated on the basis of the analogy with CS quantum mechanics with inverse square interactions (see, e.g., [17]).

When applied to composite edges for the FQHE in the Jain series, at filling fraction \(n\), and the response to voltage \((\frac{Q}{V} = \frac{1}{m} \frac{1}{2\pi} e^2 V \rho_0)\) are easily reproduced by exploiting the duality between these edges.

The potential applications of our new approach to CFT spectra are manifold, especially when the extension to boundary CFTs is considered. We mention edge state scattering, state counting, and a variety of finite-\(T\) characteristics (including tunneling characteristics) of these edges.

The method presented here can successfully be applied to many CFTs other than those mentioned here [11]. An interesting example is the \(c_{\text{CFT}} = -2\) CFT for the Yang-Lee edge singularity where the Virasoro generators take on the role of \(g = 2\) quasiparticles, giving the correct effective central charge \(c_{\text{CFT}} = 2\) [18].

To conclude, we should stress that the CFT quasiparticles considered in this Letter should not be confused with the asymptotic states of the “massless S matrix” approach to CFT [1]. The two are very different, and they give rise to different distribution functions in the thermodynamic limit.

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