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We propose a new method for investigating the exclusion statistics of quasiparticles in conformal field theory (CFT) spectra. The method leads to one-particle distribution functions, which generalize the Fermi-Dirac distribution. For the simplest $SU(n)$ invariant CFTs we find a generalization of Gentile parafermions, and we obtain new distributions for the simplest Z_N-invariant CFTs. In special examples, our approach reproduces distributions based on "fractional exclusion statistics" in the sense of Haldane. We comment on applications to fractional quantum Hall effect edge theories.
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Conformal field theory (CFT) in two dimensions is an invaluable tool in the analysis of (among other things) the low-temperature properties of a variety of condensed matter systems. In the literature on CFT (which is vast), there is a certain dichotomy between, on the one hand, descriptions based on *bosonization* and, on the other, descriptions which give a central role to *quasiparticles*.

In the standard approach to rational CFT, the spectrum is described in terms of representations of a bosonic current algebra called the *chiral algebra*. Examples are affine Kac-Moody (KM) algebras and higher spin extensions (called $\mathcal{W}$ algebras) of the Virasoro algebra. In applications to condensed matter systems, a similar description is often used. Examples are the Luttinger liquids for 1D interacting electrons, which have a U(1) affine KM symmetry and are usually treated in bosonized form. Other examples are the low-temperature theories for the multichannel Kondo effect, which have been analyzed on the basis of their $SU(2)_k$ affine KM symmetry.

When dealing with the CFT for noninteracting electrons, one clearly does not need bosonization, but uses free fermions (satisfying canonical anticommutation relations) instead. While interacting electrons give rise to more general CFTs, it is entirely natural to look for descriptions that mimic the treatment of free electrons. The idea is to identify fundamental excitations (quasiparticles) over the many-body ground state and to study their properties. For integrable models (analyzed using Bethe ansatz and factorizable scattering) such an approach is by now standard.

Until now, a general approach to "CFT quasiparticles" has been lacking. In special cases, progress has been made by viewing specific CFTs as massless limits of integrable particle theories, leading to "massless S matrices for CFT quasiparticles" [1]. Related to this is a new "integrable" approach to CFT; see [2]. Other special examples are CFTs which can be viewed as continuum limits of integrable models of lattice electrons. Examples are the $SU(n)_1$ CFTs, which can be cast in a spinon formulation analogous to that of the $SU(n)$ Haldane-Shastry (HS) spin chains [3].

In this Letter, we propose an approach to CFT quasiparticles which is intrinsic to the CFT, i.e., which does not make reference to associated integrable particle or lattice models. The starting point is the finite size spectrum of a CFT defined on a cylinder. In particular, we focus on the chiral Hilbert spaces, which together build up a CFT partition function. In a quasiparticle formulation, a chiral Hilbert space is viewed as a collection of multiparticle (quasiparticle) states (we shall write "particle" for "quasiparticle" where no confusion can arise). For *fermionic* quasiparticles, the systematics of multiparticle states are simply given by the Pauli principle, resulting in the Fermi-Dirac distribution function. For more general quasiparticles one may try to understand the spectrum in terms of more general distributions that correspond to various forms of *exclusion statistics*.

The notion of exclusion statistics was introduced by Haldane [4], in the context of integrable theories with inverse square interactions. The main idea to study the way one-particle levels in the spectrum are filled to form allowed many-particle states. The simplest scenario [4] is to assume that the act of filling an available one-particle state of type i reduces the dimension of the available Hilbert space for particles of type j by an amount g_{ij}. In the absence of other interactions, the statistics matrix $G = (g_{ij})$ completely determines the thermodynamics (see, e.g., [5,6]). Concrete examples of this type of exclusion statistics are the Calogero-Sutherland (CS) models of quantum mechanics with inverse square two-body interactions (with adjustable g).

We here propose a new method for studying the exclusion statistics of CFT quasiparticles. At the heart of our method is what can be called a "transfer matrix for truncated chiral spectra." We shall present a number of examples where CFT spectra are completely encoded in one-particle distribution functions (generalizing the Fermi-Dirac distribution). In special examples, the statistics that we find are of the type proposed by Haldane, while in other cases we find more general results.

One check on the distributions that we propose here is the coefficient γ in the low-temperature specific heat,

which is known to be related to the central charge c_{CFT} of the CFT according to [7]

$$\frac{C}{L} = \gamma k_B^2 \rho_0 T, \qquad \gamma = \frac{\pi}{6} c_{CFT}, \qquad (1)$$

where $\rho_0 = (\hbar v_F)^{-1}$ is the density of states per unit length.

To introduce our new method we focus on the simplest SU(2) invariant CFT, which is the SU(2)$_1$ Wess-Zumino-Witten model. For this CFT, a quasiparticle formulation has been proposed in [8] and worked out in great detail in [9]. The formulation uses operators $\phi_{-s}^{\pm}$ which create quasiparticles called spinons. The spinons form a doublet under SU(2) and carry (dimensionless) energy $L_0 = s$. The chiral spectrum of the SU(2)$_1$ CFT may be built in the following manner. One starts by writing the following *polarized N-spinon states*:

$$\phi_{-[(2N-1)/4]-n_N}^{+} \cdots \phi_{-(3/4)-n_2}^{+} \phi_{-(1/4)-n_1}^{+} |0\rangle, \qquad (2)$$

with $n_N \geq \cdots \geq n_2 \geq n_1 \geq 0$. One then uses the Yangian symmetry algebra to construct multispinon states with mixed $+$ and $-$ indices. The collection of all these states forms a basis of the full chiral spectrum. The SU(2) content of the Yangian multiplet labeled by $n_1, \ldots, n_N$ follows from the generalized commutation relations satisfied by the spinon modes, or, equivalently, from the representation theory of the Yangian [8,9].

Comparing the allowed spinon modes $\phi_{-s}^{\pm}$ with free fermion modes $\psi_{-(1/2)-l}$, we observe that the fermion mode $l + \frac{1}{2}$ has "split" into an odd mode $s = l + \frac{1}{4}$ and an even mode $s = l + \frac{3}{4}$. To maintain a level spacing of one unit, we view these two spinon modes as forming a single "one-particle level" in the spectrum. What we would like to do is to factorize the full chiral partition sum of the CFT into a product over these one-particle levels, so that the free energy becomes a sum of one-particle contributions.

While we cannot straightforwardly extract the contribution of the lth level, we may proceed as follows. We truncate the chiral spectrum by only allowing the spinon modes of the first l levels. We denote by $P_l(q, x_+, x_-)$, $Q_l(q, x_+, x_-)$ the partition functions of the first l levels, where in P_l the highest occupied mode should be odd while in Q_l it should be even. In formula

$$P_l(q, x_+, x_-) = \text{trace}_{\text{odd}}^{(\leq l)} (q^{L_0} x_+^{N_+} x_-^{N_-}) \qquad (3)$$

and similar for $Q_l(q, x_+, x_-)$, where we introduce chemical potentials $\mu_\pm$, write $x_\pm = e^{\beta \mu_\pm}$, $q = e^{-\beta(2\pi/L)(1/\rho_0)}$, and denote by $N_\pm$ the number of $\pm$ spinons in a state.

Restricting to fully polarized states (putting $x_- = 0$), we obtain the following recursion relation

$$\begin{pmatrix} P_l \\ Q_l \end{pmatrix} = \begin{pmatrix} 1 & q^{l-(3/4)} x_+ \\ q^{l-(1/4)} x_+ & 1 + q^{2l-1} x_+^2 \end{pmatrix} \begin{pmatrix} P_{l-1} \\ Q_{l-1} \end{pmatrix}, \qquad (4)$$

with $P_0 = 0$ and $Q_0 = 1$. We now make the important step of approximating these exact expressions for $P_l(q, x_+)$ and $Q_l(q, x_+)$ by the product from $j = 1$ to l of the largest eigenvalue $\lambda_+^{(j)}(q, x)$ of the jth recursion matrix. Clearly, this brings the partition sums in the desired factorized form and reduces thermodynamic quantities to sums of independent single-level contributions. The average occupation of the lth level is found to be

$$\bar{n}^{(l)}(q, x) = x \partial_x \ln \lambda_+^{(l)}(q, x) = \frac{2}{\sqrt{1 + 4 q^{1-2l} x^{-2}}}$$
$$= \frac{2}{\sqrt{1 + 4 e^{-2\beta(\mu_+ - \epsilon_l)}}}, \qquad (5)$$

with $\epsilon_l = (l + \frac{1}{2}) \frac{2\pi}{L} \frac{1}{\rho_0}$. Note that the maximal occupation of a given level is 2. Interestingly, this distribution function for polarized SU(2) spinons is identical to that of particles with $g = \frac{1}{2}$ (semionic) exclusion statistics [5].

If we now include the multispinon states with mixed indices, we find that the lth recursion matrix becomes

$$\begin{pmatrix} 1 - q^{2l-2} x^2 & q^{l-(3/4)} x(z + \frac{1}{z}) \\ q^{l-(1/4)} x(z + \frac{1}{z})(1 - q^{2l-2} x^2) & 1 + q^{2l-1} x^2 (z^2 + 1 + \frac{1}{z^2}) \end{pmatrix}, \qquad (6)$$

where we put $x_\pm = x z^{\pm 1}$. Following the same logic, we derive the distribution functions $\bar{n}^{(l)}(q, x, z)$. Interesting special cases are

$$\bar{n}^{(l)}(q, x) = x \partial_x \ln \lambda_+^{(l)}(q, x, z = 1) = \frac{2}{1 + q^{(1/2)-l} x^{-1}} \qquad (7)$$

for the expected total number of spinons in level l, and

$$\overline{Q}^{(l)}(q, z) = \frac{e}{2} z \partial_z \ln \lambda_+^{(l)}(q, x = 1, z)$$
$$= \frac{e q^{l-(1/2)} (z - z^{-1})}{\sqrt{q^{2l-1}(z + z^{-1})^2 + 4(1 - q^{2l-1})}} \qquad (8)$$

for the expected charge at level l (we assume charges $\pm\frac{e}{2}$ for the $\pm$ spinons). The distributions (7) and (8) agree with the distributions obtained from fractional exclusion statistics with

$$G = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Note that (7) implies that for $z = 1$ (zero voltage) the thermodynamics of the spinon system is identical to that of two free fermions and the central charge is $c_{CFT} = 2 \times \frac{1}{2} = 1$. Note also that, with $z = e^{(1/2)\beta eV}$, the integrated charge $\frac{1}{L} \sum_{l=1}^{\infty} \overline{Q}^{(l)}(q, z) = \frac{1}{4\pi} e^2 V \rho_o$, which

is half of the value obtained for two charge $\pm e$ free fermions.

The correspondence with Haldane statistics is satisfying since the spinons of the associated SU(2) HS spin chain satisfy these same statistics [4]. This confirms the validity of our new approach, which in no way relied on the exact solution of the HS chain.

We now present some special examples.

(1) The $\mathrm{SU}(n)_1$ *CFTs.*—The first generalization of the $\mathrm{SU}(2)_1$ results concerns the $\mathrm{SU}(n)_1$ spinons. The Yangian symmetry of the $\mathrm{SU}(n)_1$ CFT was established in [10] while the spinon formulation was presented in [3]. There are n fundamental spinon species ϕ^i, transforming in the representation $\overline{\mathbf{n}}$ of $\mathrm{SU}(n)$. Repeating the analysis shown above, finding explicitly the $\mathrm{SU}(n)$ analog of the recursion matrices (4) and (6) (see [11]), we find that (i) a single spinon species ϕ^{i_0} (in absence of any others) satisfies Haldane statistics with $g = \frac{n-1}{n}$, (ii) when exciting all n spinon species symmetrically (choosing all x_i equal to $x = e^{\beta\mu}$), the expected total occupation of the lth level is given by

$$\overline{n}^{(l)}(q, x) = x\partial_x \ln[1 + q^l x + \dots + (q^l x)^{n-1}]^n. \quad (9)$$

Comparing these results, one finds that for $n > 2$ there is negative mutual exclusion among different spinons.

Interestingly, the statistics going with the distribution (9) were proposed by Gentile as early as 1940 [12]. One finds that a single "Gentile parafermion" contributes the amount $\frac{n-1}{n}$ to the central charge, so that the full result becomes $c_{\mathrm{CFT}} = n\frac{n-1}{n} = n - 1$, as expected.

Our results are consistent with [13], where, by different methods, the link between $\mathrm{SU}(n)$ HS spin chains and Gentile parafermions has also been established.

(2) Z_N *parafermions.*—Within the context of CFT, the simplest generalization of the Majorana fermion is the so-called Z_N parafermion. It features in a CFT of central charge $c_N = \frac{2(N-1)}{N+2}$ as a primary field of dimension $h_N = \frac{N-1}{N}$ and Z_N charge 1 [14]. By applying the method outlined above, we obtained a distribution function for the Z_N parafermion and established that the full CFT spectrum is reproduced by a gas of noninteracting quasiparticles of this type. For the purpose of explaining these results, we focus on the case $N = 3$.

It is well known that the chiral spectrum of the Z_3 parafermion CFT can be interpreted in terms of two parafermions $\psi^\pm$ of opposite Z_3 charge. However, by exploiting the generalized commutation relations obeyed by the modes of $\psi^\pm$ [14], one easily shows that the modes of ψ^+ alone can generate the full spectrum, and that the ψ^- quasiparticle can be viewed as a composite of two ψ^+ quasiparticles. Having understood how the ψ^+ modes alone build the chiral spectrum (see [11]), one may define truncated partition sums. We found the following recursion matrix between the lth and the $(l-1)$th truncated sums (which each have three

components)

$$\begin{pmatrix} (1 - y^3) & y^2 & y \\ y(1 - y^3) & 1 & 2y^2 \\ 2y^2(1 - y^3) & y(1 + y^3) & 1 + 2y^3 \end{pmatrix}, \quad (10)$$

with $y = x\,q^l$. The Z_3 parafermion distribution function is expressed in terms of the largest eigenvalue λ_+ of this matrix

$$\overline{n}^{(l)}(q, x) = (y\partial_y \ln \lambda_+)(y = xq^l). \quad (11)$$

From the asymptotics $\lambda_+(y) \propto y^3$, one finds that the maximum occupation per level equals 3. Using the Cardano formula one may write $\overline{n}^{(l)}$ in closed form. We here present a plot (Fig. 1), which displays the function $\overline{n}$ as a function of energy. As a check we (numerically) evaluated the coefficient γ of the specific heat, reproducing the expected value $\gamma = \frac{\pi}{6}\frac{4}{5}$.

For general N, the distribution for Z_N parafermions allows a maximum of $\frac{1}{2}N(N-1)$ particles per level.

(3) Quantum Hall effect edge theories.—As a further application we briefly discuss edge theories for the fractional quantum Hall effect (FQHE). We shall come back to this topic in a separate publication [15]. For the $\nu = \frac{1}{m}$ FQHE (with m an odd integer), the edge theory is a chiral
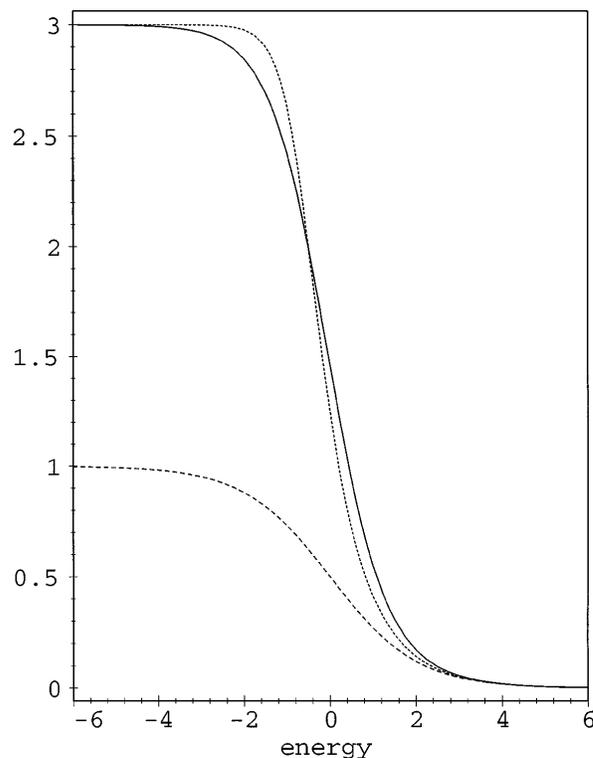

FIG. 1. Distribution functions for Z_3 parafermions (solid line), for ordinary fermions (dashed line), and for particles satisfying $g = \frac{1}{3}$ exclusion statistics (dotted line). All distributions are at $\mu = 0$; the energy is given in units β^{-1}.

$c_{CFT} = 1$ CFT at compactification radius $R^2 = m$. The natural quasiparticles to consider are the edge electron (of charge $-e$) and the edge quasihole (of charge $\frac{e}{m}$). Writing the spectrum in terms of these quasiparticles, and applying the above procedure, we find that the fundamental quasiparticles are independent, and obey Haldane exclusion statistics with $g = m$ and $g = 1/m$, respectively. This is consistent with the result of bosonization applied to g-ons [16].

The (known) results for the specific heat ($c_{CFT} = 1$) and the response to voltage ($\frac{Q}{L} = \frac{1}{m} \frac{1}{2\pi} e^2 V \rho_0$) are easily reproduced by exploiting the duality between $g = m$ and $g = 1/m$ statistics [15]. The central charge arises as a sum $c_m + c_{1/m}$. For $m = 2$ we find $c_2 = \frac{2}{5}$, $c_{1/2} = \frac{3}{5}$, while for $m = 3$, $c_3 = 0.343\ldots$, $c_{1/3} = 0.655\ldots$.

These quantum Hall results can be appreciated on the basis of the analogy with CS quantum mechanics with inverse square interactions (see, e.g., [17]). We stress, however, that our derivation does not rely on this analogy.

When applied to composite edges for the FQHE in the Jain series, at filling fraction $\nu = \frac{n}{2np+1}$, the combined results of this paper lead to a formulation in terms of (i) a single charged mode, satisfying Haldane statistics with $g = \nu$, and (ii) a set of n spinons for $SU(n)_1$, satisfying the generalized Gentile statistics described above. This new quasiparticle formulation forms a suitable starting point for studying finite-T features (including tunneling characteristics) of these edges.

The potential applications of our new approach to CFT spectra are manifold, especially when the extension to boundary CFTs is considered. We mention edge state scattering, state counting, and a variety of finite-T characteristics of (non-Abelian) FQHE edges, and non-Fermi liquid features in quantum impurity problems such as the multi-channel Kondo effect.

The method presented here can successfully be applied to many CFTs other than those mentioned here [11]. An interesting example is the $c_{CFT} = -\frac{22}{5}$ CFT for the Yang-Lee edge singularity where the Virasoro generators take on the role of $g = 2$ quasiparticles, giving the correct effective central charge $\tilde{c}_{CFT} = \frac{2}{5}$ [18].

To conclude, we should stress that the CFT quasiparticles considered in this Letter should not be confused with the asymptotic states of the "massless S matrix" approach to CFT [1]. The two are very different, and they give rise to different distribution functions in the thermodynamic limit.

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