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Abstract

We study the description of the $SU(2)$, level $k = 1$, Wess–Zumino–Witten conformal field theory in terms of the modes of the spin-1/2 affine primary field ϕ^α. These are shown to satisfy generalized 'canonical commutation relations', which we use to construct a basis of Hilbert space in terms of representations of the Yangian $Y(sl_2)$. Using this description, we explicitly derive so-called 'fermionic representations' of the Virasoro characters, which were first conjectured by R. Kedem, T. Klassen, B. McCoy and E. Melzer [Phys. Lett. B 307 (1993) 68, and references therein]. We point out that similar results are expected for a wide class of rational conformal field theories.

1. Introduction

Recently a number of independent studies have pointed at a novel description of the structure of the Hilbert space of certain rational conformal field theories (RCFT), which is totally different from the conventional one in terms of representations of a chiral conformal algebra: roughly speaking, the Hilbert space can be built up from fundamental 'quasiparticles', in a way that is reminiscent of the Fock space construction for free fermions. A rule, generalizing the Pauli exclusion principle, governs the allowed ways the 'quasiparticles' can occupy single momentum states. Clearly, such a novel description of RCFT is expected to lead to a number of new insights, both at the mathematical level and in applications. Early results in this spirit can be found in the work of Faddeev and Takhtajan [1], and of Zamolodchikov and Zamolodchikov [2].

In this Letter, we make a connection between two developments in this area. The first goes back to Haldane et al. [3], who propose a description of the $SU(2)$, level-1 Wess–Zumino–Witten (WZW) conformal field theory in terms of 'spinons' (spin-1/2 doublets) and so-called *Yangian symmetry*. This description, which has its origin in the structure of the Haldane–Shastry spin chains with $1/r^2$ exchange, stresses the fact that the fundamental fields in this theory are 'spinon' fields, which may be viewed as free fields apart from purely statistical (in this case: semionic) interactions that may be taken into account by a rule generalizing the Pauli principle. The second development, initiated by the Stony Brook group [4], is the observation that the fundamental characters of many RCFT's can be written in what has been called a *'fermionic representation'*. So far, the 'fermionic representations' have not been related to an underlying algebraic structure in the CFT.

Here, following [3], we construct a basis of the Hilbert space of the $SU(2)_1$ WZW model in terms of modes of the spin-1/2 affine primary field and relate it to representations of the Yangian $Y(sl_2)$ [1]. In the main part of this Letter, we shall then use this spinon description to directly derive the 'fermionic representations' of the Virasoro characters in the model. We thus identify the common structure underlying both developments mentioned above. We will be able to give a straightforward derivation, which uses little more than the generalized commutation relations of the spinon modes.

In our closing section, commenting on our results, we argue that, if a RCFT can be described by different (bosonic) chiral algebras, one expects a 'fermionic representation' of the characters and a generalized notion of 'Yangian symmetry' for each of those descriptions. We also comment on the $SU(2)$ WZW models with level $k > 1$.

2. Spinon fields: OPE's and generalized commutators

In this section we begin our discussion of the $SU(2)_1$ WZW model by specifying the properties of the modes of the (chiral) spinon field $\phi^\pm(z)$, which is nothing else than the $j = \frac{1}{2}$ primary field of the affine symmetry algebra $A_1^{(1)}$. If one were to base the description of the theory on affine symmetry alone, one would have the vacuum and the lowest spinon state (of conformal dimension $L_0 = \frac{1}{4}$) as the fundamental primary states, and one would use affine currents to generate all other states in the theory. Here we shall focus instead on a description where all states in the spectrum are written as multi-spinon states, i.e., as products of modes of the spin-1/2 affine primary.

In order to clarify conventions, we give a complete set of OPE's. The current algebra is

$$J^a(z)J^b(w) = \frac{d^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{(z-w)} + \dots \quad (1)$$

The adjoint index takes the values $a = +, 3, =$; the metric is $d^{+=} = 1$, $d_{=+} = 1$, $d^{33} = 2$, $d_{33} = \frac{1}{2}$ and the structure constants follow from $f^{+=}{}_3 = 1$ and satisfy

$f_a{}^{bc} f_{dbc} = -4 d_{ad}$. The spinon fields $\phi^\alpha(z)$, $\alpha = \pm$, transform as

$$J^a(z)\phi^\alpha(w) = (t^a)^\alpha{}_\beta \frac{\phi^\beta(w)}{(z-w)} + \dots, \quad (2)$$

with $(t^+)^-{}_+ = (t^=)^+{}_- = 1$, $(t^3)^\pm{}_\pm = \pm 1$. Furthermore, we have the following spinon–spinon OPE's

$$\phi^\alpha(z)\phi^\beta(w) =$$
$$(-1)^q (z-w)^{-\frac{1}{2}} \epsilon^{\alpha\beta} \left(1 + \frac{1}{2}(z-w)^2 T(w) + \dots \right)$$
$$- (-1)^q (z-w)^{\frac{1}{2}} (t_a)^{\alpha\beta}$$
$$\times \left(J^a(w) + \frac{1}{2}(z-w)\partial J^a(w) + \dots \right), \quad (3)$$

where $\epsilon^{+-} = -\epsilon^{-+} = 1$, and $(t_a)^{\alpha\beta} = d_{ab}\epsilon^{\beta\gamma}(t^b)^\alpha{}_\gamma$. In these formulas, q depends on the sector that the OPE's are acting on [2]: $q = 0$ on states that are created by an even number of spinons (i.e., the states in the vacuum module of $A_1^{(1)}$) and $q = 1$ if the number of spinons is odd (the $j = \frac{1}{2}$ module of $A_1^{(1)}$).

The occurrence of a factor $(z-w)^{1/2}$ in the spinon–spinon OPE's clearly shows that the braiding properties of the spinons are those of semions or 'half-fermions'. The mode expansions of the spinons are

$$\phi^\alpha(z)\chi_q(0) = \sum_m z^{m+\frac{q}{2}} \phi^\alpha_{-m-\frac{q}{2}-\frac{1}{4}} \chi_q(0),$$

$$\phi^\alpha_{-m-\frac{q}{2}+\frac{1}{4}} \chi_q(0) = \oint \frac{dz}{2\pi i} z^{-m-\frac{q}{2}} \phi^\alpha(z)\chi_q(0), \quad (4)$$

where $\chi_q(0)$ is an arbitrary state in the sector indicated by the value of q. On states with $q = 0$ we can apply modes $\phi^\pm_{-1/4-n}$ with n integer, and on states with $q = 1$ we can apply $\phi^\pm_{-3/4-n}$.

By following a standard procedure (see, e.g. [5]) we can derive the following generalized commutation relations for the modes of the spinon fields

$$\sum_{l \geq 0} C_l^{(-\frac{1}{2})} \left(\phi^\alpha_{-m-\frac{q+1}{2}-l+\frac{3}{4}} \phi^\beta_{-n-\frac{q}{2}+l+\frac{3}{4}} - \binom{\alpha \leftrightarrow \beta}{m \leftrightarrow n} \right)$$
$$= (-1)^q \epsilon^{\alpha\beta} \delta_{m+n+q-1} \quad (5)$$

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[1] Related results were obtained independently in Ref. [9].

[2] The appearance of explicit factors $(-1)^q$ is due to our convention to write upper indices on all spinon fields. The natural convention would be to write lower indices whenever a spinon field acts on a $q = 1$ state; raising these indices by using $\phi^\alpha = \epsilon^{\alpha\beta}\phi_\beta$ leads to an explicit relative minus sign between the OPE's in the two sectors $q = 0, 1$.

$$\sum_{l \geq 0} C_l^{(-\frac{3}{2})} \left(\phi^\alpha_{-m-\frac{q+1}{2}-l-\frac{1}{4}} \phi^\beta_{-n-\frac{q}{2}+l+\frac{3}{4}} + \begin{pmatrix} \alpha \leftrightarrow \beta \\ m \leftrightarrow n \end{pmatrix} \right)$$

$$= (-1)^q \left(-\epsilon^{\alpha\beta}(m+\frac{q}{2})\delta_{m+n+q} - (t_a)^{\alpha\beta} J^a_{-m-n-q} \right) \tag{6}$$

$$\sum_{l \geq 0} C_l^{(-\frac{5}{2})} \left(\phi^\alpha_{-m-\frac{q+1}{2}-l-\frac{5}{4}} \phi^\beta_{-n-\frac{q}{2}+l+\frac{3}{4}} - \begin{pmatrix} \alpha \leftrightarrow \beta \\ m \leftrightarrow n \end{pmatrix} \right)$$

$$= (-1)^q \Big(\frac{1}{2} \epsilon^{\alpha\beta} (m+\frac{q}{2})(m+1+\frac{q}{2})\delta_{m+n+q+1}$$

$$- \frac{1}{2}(t_a)^{\alpha\beta}(n-m)J^a_{-m-n-q-1} + \frac{1}{2}\epsilon^{\alpha\beta}L_{-m-n-q-1} \Big) \tag{7}$$

In these relations, the coefficients $C_l^{(\alpha)}$ are defined by the expansion

$$(1-x)^\alpha = \sum_{l \geq 0} C_l^{(\alpha)} x^l . \tag{8}$$

The relations (5) can be interpreted as *generalized canonical commutation relations* of the fundamental spinon fields. The other relations can be used to express the current modes J^a_n and the Virasoro generators L_n as bilinears in spinon modes. We would like to stress that, up to the complication of the infinite series in the mode index l, these relations are very reminiscent of the anticommutation relations for free fermions and of the formulas that express affine and Virasoro currents as fermion bilinears.

3. Yangian symmetry

We recall from [3] that the $SU(2)_1$ WZW RCFT provides a realization of the Yangian $Y(sl_2)$. The Yangian generators are

$$Q_0^a = J_0^a , \quad Q_1^a = \frac{1}{2} f^a{}_{bc} \sum_{m>0} J^b_{-m} J^c_m . \tag{9}$$

It is easy to see that the Virasoro generator L_0 commutes with the generators in (9). When acting on integrable highest weight representations of $A_1^{(1)}$ at level-1, these generators satisfy the defining relation of Drinfel'd's Yangian ([6]). The Yangian is endowed with a *comultiplication* Δ, which can be viewed as a generalization of angular momentum coupling. When the action of the Yangian is defined on two spaces, it acts on the tensor product as follows:

$$\Delta(Q_0^a) = 1 \otimes Q_0^a + Q_0^a \otimes 1,$$

$$\Delta(Q_1^a) = 1 \otimes Q_1^a + Q_1^a \otimes 1 + \frac{1}{2} f^a{}_{bc} Q_0^b \otimes Q_0^c . \tag{10}$$

The paper [3] also identified a number of operators (beyond the Virasoro zero-mode L_0) that commute with the Yangian generators. The first non-trivial example is the operator H_2 given by

$$H_2 = \sum_{m>0} d_{ab} \, m \, J^a_{-m} J^b_m . \tag{11}$$

If the group $SU(2)$ is replaced by $SU(N)$, the corresponding expressions for Q_1^a and H_2 have to be modified by additional terms that involve the 3-index d-symbol of $SU(N)$, $N \geq 3$ [7].

4. Constructing the Hilbert space

We now come to the description of the full space of states of the theory in terms of multi-spinon states. There are actually two ways to set this up, and we shall discuss both in this section.

4.1. Basis I

Rephrasing the prescription of [3], we claim that a complete basis for all the (chiral) states in the Hilbert space of the $SU(2)_1$ WZW model is given as follows. We first construct the following *fully polarized N-spinon states*

$$\phi^+_{-\frac{(2N-1)}{4}-n_N} \cdots \phi^+_{-\frac{5}{4}-n_3} \phi^+_{-\frac{3}{4}-n_2} \phi^+_{-\frac{1}{4}-n_1} |0\rangle ,$$
$$\text{with } n_N \geq n_{N-1} \geq \ldots \geq n_2 \geq n_1 \geq 0 . \tag{12}$$

It is easily seen that the eigenvalue of the Virasoro zero mode L_0 on these states is

$$L_0 = \frac{N^2}{4} + \sum_{i=1}^N n_i . \tag{13}$$

In the second step we construct a collection of irreducible *Yangian multiplets*: we first construct a Yangian highest weight state (YHWS) [annihilated by Q_0^+ and Q_1^+] by taking suitable linear combinations of the states in (12), with fixed N (number of 'spinons') and fixed L_0 eigenvalue (see, e.g. (19)). Then we repeatedly apply the generators Q_0^a and Q_1^a. The union

of all these Yangian multiplets precisely forms a basis for the Hilbert space.

The structure of the irreducible Yangian multiplets, described in [3], following [8], reads in our present language as follows: (i) Each Yangian multiplet is characterized by a set of non-decreasing integers $\{n_i\}_{i=1,...,N}$ as in (12). (ii) The eigenvalue of L_0 (commuting with the Yangian) on the states in the Yangian multiplet specified by $\{n_i\}_{i=1,...,N}$ is given by (13). (iii) When acting on a Yangian highest weight state (which will be a linear combination of 'fully polarized' states of the form (12)), the Yangian generators (9) create states of a similar form, which however have some of the + indices replaced by −, and which have different coefficients in the linear combination. If we were to act only with Q_0^a we would find a total of $N + 1$ such states; the maximal possible number when acting with the full Yangian is 2^N. This maximal number is only realized if the mode indices n_i are all different. If some of the n_i's are equal, the corresponding product of doublets is projected on the symmetric combination. For example, a 2-spinon Yangian multiplet will have $3 + 1 = 4$ states if $n_2 \neq n_1$, but only 3 states if $n_2 = n_1$. (For comparison, note that for free spinful fermions, the Pauli principle forces 2 fermions with equal momentum to be in a singlet state, the triplet being dropped. In this sense the Yangian generalizes the Pauli principle.)

In [3], the above description of the space of states of the WZW model was obtained as an extrapolation of results based on the exact solution of so-called Haldane–Shastry spin chains. However, we can easily see that this result is a rather direct consequence of the generalized commutation relations given above, and in particular of the relations (5). Using this relation with $\alpha = +$, $\beta = +$, one can show that each of the ϕ^+ modes can be applied only once, and that mode indices $n_1, n_2, \ldots$, of the state (12) can be chosen in a preferred order. The relation (5) with $\alpha = +$, $\beta = -$ can be used to show the existence of null states which reduce the number of states in the Yangian multiplets. We will further illustrate this in the next section.

4.2. Basis II

A second way to write a multi-spinon basis for the $SU(2)_1$ WZW model is as follows. One considers the states

$$\phi^-_{-\frac{2(N^++N^-)-1}{4}-n^-_{N^-}} \cdots \phi^-_{-\frac{2(N^++1)-1}{4}-n^-_1}$$

$$\times \phi^+_{-\frac{2N^+-1}{4}-n^+_{N^+}} \cdots \phi^+_{-\frac{1}{4}-n^+_1}|0\rangle \,,$$

with $n^+_{N^+} \geq \ldots \geq n^+_2 \geq n^+_1 \geq 0$,

$$n^-_{N^-} \geq \ldots \geq n^-_2 \geq n^-_1 \geq 0 \,. \tag{14}$$

Once again, it can easily be checked that the generalized commutation relations (5) allow one to write every mixed-index multispinon state as a sum of states of the form (14). The eigenvalue of L_0 is now given as

$$L_0 = \frac{(N^+ + N^-)^2}{4} + \sum_{i=1}^{N^+} n^+_i + \sum_{i=1}^{N^-} n^-_i \,. \tag{15}$$

5. The action of Q_1^a and H_2 on N-spinon states

In this section we study the explicit action of the Yangian generators Q_0^a and Q_1^a and of the operator H_2 on multi-spinon states. For simplicity we shall first discuss the action on two-spinon states and later give some more general formulas.

We introduce the following notations for general 2-spinon states (t, s refer to triplet and singlet states, respectively)

$$\Phi^{t,a}_{n_2,n_1} = (t^a)_{\alpha\beta}\, \phi^\alpha_{-3/4-n_2}\phi^\beta_{-1/4-n_1}|0\rangle,$$

$$\Phi^s_{n_2,n_1} = \epsilon_{\alpha\beta}\, \phi^\alpha_{-3/4-n_2}\phi^\beta_{-1/4-n_1}|0\rangle \,. \tag{16}$$

We can now use

$$(J^a\phi^\alpha)(z) = 2(t^a)^\alpha{}_\beta\partial\phi^\beta \,,$$

$$(\partial J^a\phi^\alpha)(z) = \tfrac{4}{3}(t^a)^\alpha{}_\beta\partial^2\phi^\beta \tag{17}$$

to show that

$$Q_1^a\Phi^{t,b}_{n_2,n_1} = -(n_2 + n_1 + \tfrac{1}{2})f^{ab}{}_c\Phi^{t,c}_{n_2,n_1},$$

$$+ (n_2 - n_1 + 1)d^{ab}\Phi^s_{n_2,n_1} + d^{ab}\sum_{l>0}\Phi^s_{n_2+l,n_1-l}$$

$$Q_1^a\Phi^s_{n_2,n_1} = 2(n_2 - n_1)\Phi^{t,a}_{n_2,n_1} - 2\sum_{l>0}\Phi^{t,a}_{n_2+l,n_1-l} \,. \tag{18}$$

Notice that the action of Q_1^a is not diagonal in the indices (n_2, n_1) but rather lower-triangular in the sense

that (n_2, n_1) gets mapped into $(n_2 + l, n_1 - l)$ with $l \geq 0$ and $n_1 - l \geq 0$.

From the action of $Q_1^=$ it is easily seen that the space of all two-spinon states with $n_1 + n_2 = n$ fixed can be decomposed into multiplets of the Yangian, whose highest weight states are of the form

$$\Phi_{n_2, n_1}^{t, +} + \sum_{l > 0} a_{n_2, n_1}^{(l)} \Phi_{n_2 + l, n_1 - l}^{t, +} , \qquad (19)$$

where the $a_{n_2, n_1}^{(l)}$ are real coefficients.

These multiplets contain a triplet and a singlet of $SU(2)$, i.e., a total of four states, except if $n_1 = n_2$, when the relation (5) can be used to show that the singlet is absent.

We remark that Q_1^a acts by comultiplication (10) on the 2-spinon YHWS, given its action on the 1-spinon states (which are YHWS).

The action of H_2 on the 2-spinon states can be evaluated in a similar fashion. The result is

$$H_2 \Phi_{n_2, n_1}^{t, a}$$
$$= 2 \left((n_2 + 1)(n_2 + \tfrac{1}{2}) + (n_1 + \tfrac{1}{2})n_1 \right) \Phi_{n_2, n_1}^{t, a}$$
$$+ \sum_{l > 0} l \, \Phi_{n_2 + l, n_1 - l}^{t, a} ,$$

$$H_2 \Phi_{n_2, n_1}^{s}$$
$$= 2 \left((n_2 + 1)(n_2 + \tfrac{1}{2}) + (n_1 + \tfrac{1}{2})n_1 \right) \Phi_{n_2, n_1}^{s}$$
$$- 3 \sum_{l > 0} l \, \Phi_{n_2 + l, n_1 - l}^{s} . \qquad (20)$$

Since this action is again lower triangular, the eigenvalues of the operator H_2 are immediately seen to equal $2((n_2 + 1)(n_2 + \tfrac{1}{2}) + (n_1 + \tfrac{1}{2})n_1)$. It can be checked that these values are in agreement with the prescription given in [3], and also with the formula that was recently given in [9].

To be completely explicit, we present the example where $n_1 + n_2 = 4$, which are the 2-spinon states with $L_0 = 5$ (from (13)). In the following formula we list the labels (n_2, n_1) of the Yangian representation, the H_2-eigenvalues, and the states

$$(4, 0) \quad H_2 = 45 \quad \Phi_{4,0}^{t, a} , \quad \Phi_{4,0}^{s}$$

$$(3, 1) \quad H_2 = 31 \quad \Phi_{3,1}^{t, a} - \tfrac{1}{14}\Phi_{4,0}^{t, a} , \quad \Phi_{3,1}^{s} + \tfrac{3}{14}\Phi_{4,0}^{s}$$

$$(2, 2) \quad H_2 = 25 \quad \Phi_{2,2}^{t, a} - \tfrac{1}{6}\Phi_{3,1}^{t, a} - \tfrac{11}{120}\Phi_{4,0}^{t, a} . \qquad (21)$$

When acting on the $(2, 2)$ Yangian highest weight state, $Q_1^=$ produces a multiple of the $SU(2)$ descendant plus a state proportional to $\Phi_{2,2}^{s} + \tfrac{1}{2}\Phi_{3,1}^{s} + \tfrac{3}{8}\Phi_{4,0}^{s}$, which vanishes as a consequence of (5).

To conclude this section, we turn to the action of Q_1^a and H_2 on a general N-spinon state. It is easily seen that the action of Q_1^a is again 'lower triangular' with respect to a natural ordering of the multi-indices $(n_1, n_2, \ldots, n_N)$. This implies that in general the space of N-spinon states can be decomposed into irreducible multiplets of the Yangian. Closed form expressions for general Yangian highest weight states can, e.g. be obtained by using the full machinery of the exact solution of the Calogero–Sutherland models of N-body quantum mechanics and of the Haldane–Shastry spin chains, and this leads to expressions involving so-called Jack polynomials [9].

Let us give an explicit expression for the action of H_2 on a fully polarized N-spinon state. Denoting by $|\chi^{(N-1)}\rangle$ a fully polarized $(N - 1)$-spinon state, we have from (11,17)

$$[H_2, \phi^+_{-\frac{2N-1}{4} - n_N}]|\chi^{(N-1)}\rangle$$
$$= 2 (n_N + \tfrac{N-1}{2})(n_N + \tfrac{N}{2})) \, \phi^+_{-\frac{2N-1}{4} - n_N}|\chi^{(N-1)}\rangle$$
$$+ \sum_{l > 0} l \, \phi^+_{-\frac{2N-1}{4} - n_N - l} J_l^3 |\chi^{(N-1)}\rangle . \qquad (22)$$

Since this action is again 'lower triangular', the eigenvalue of H_2 on a general fully polarized Yangian highest weight state with mode-indices $\{n_1, n_2, \ldots, n_N\}$ is

$$H_2 = \sum_{i=1}^{N} 2(n_i + \tfrac{1}{2}(i - 1))(n_i + \tfrac{1}{2}i) . \qquad (23)$$

This value agrees with the results of [3,9], which were derived by different means.

6. Deriving the Virasoro characters

We now turn to a derivation of the Virasoro characters in the theory by using the characterization of the Hilbert space that we called 'Basis I'. The Virasoro primary fields in the theory are characterized by their $SU(2)$ spin j and their conformal dimension $\Delta =$

j^2. The corresponding characters are of course well known

$$\chi_{j^2}^{\text{Vir}}(q) = \frac{q^{j^2}(1 - q^{2j+1})}{\prod_{m=1}^{\infty}(1 - q^m)} \ . \tag{24}$$

Our goal here is to write this character in a way that is natural from the point of view of the spinon picture. We shall illustrate our approach by first computing the vacuum character, $j = 0$, which is the generating function of all $SU(2)$ singlets in the spectrum.

Before proceeding, we introduce a slightly simplified version of the *motif* notation [3] for Yangian multiplets as follows (simplified in the sense that we will not explicitly write the singlet motif '(1)'). Starting with n_1, we write the symbol ( ) for each n_i that is not equal to one of its neighbours. If a string of l consecutive n_i's are all equal we write the symbol $(00\ldots0)$ (i.e., $l - 1$ zeros). The Yangian highest weight states are thus in 1–1 correspondence with *motif sequences* and the $SU(2)$ content of each multiplet is the free tensor product of the motifs in the sequence.

For example, the Yangian multiplet specified by $\{n_1 = n_2 < n_3 < n_4 = n_5 = n_6 < n_7\}$ ($N = 7$ spinons) has motif $(0)( )(00)( )$. Note that the motif notation only indicates which of the n_i's coincide – there is an infinite number of Yangian multiplets which have the same motif, differing by the actual values of the mode indices $n_1 < n_3 < n_4 < n_7$. However, the motif uniquely specifies the $SU(2)$ content: ( ) has spin $\frac{1}{2}$, (0) spin 1, (00) spin $\frac{3}{2}$, etc., and the total $SU(2)$ content is the free tensor product, for the above motif $1 \otimes \frac{1}{2} \otimes \frac{3}{2} \otimes \frac{1}{2}$.

Our strategy for the computation of the vacuum character is as follows. Clearly, this character will only get contributions from sectors with an even number m_1 of spinons. To find the contribution from the m_1-spinon sector, we proceed in two steps. (1) We draw all path (Bratteli) diagrams, which encode possible ways to extract a singlet from the m_1-fold tensor product of the $SU(2)$ doublet. (2) To each diagram we then associate all allowed motif-sequences from which a singlet can be extracted according to the diagram and we sum the corresponding q-series.

Starting with step (2), let us assume that we have a specific Bratteli diagram, which we can also view as a sequence of uparrows (uaw's) and downarrows (daw's). To this we associate a *leading motif sequence*

according to the following rule. First write $(00\ldots0)$ [i.e., $(l-1)$ zeros] for the first sequence of l uaw's. Then follow the diagram down (daw's) and up again (uaw's) until the next top. If this dip (which can be asymmetric) has a total of l arrows, write $(00\ldots0)$ with $l - 1$ zeros. Finally, write $(00\ldots0)$ [i.e., $l - 1$ zeros] for the final l daw's. For example, the diagram with first l uaw's and then l daw's gives $(00\ldots0)(00\ldots0)$. The diagram with l times the pattern uaw, daw, gives $( )(0)(0)\ldots(0)( )$ with $l - 1$ times (0) in the middle, etc.

From the leading motif sequence we construct *fragmented sequences* by making the replacement '$0 \rightarrow)($' in all possible places.

All this is easily explained in an example. Pick the Bratteli diagram [step (2)]

$$\begin{array}{c} \nearrow\searrow \\ \nearrow \qquad \searrow \end{array} \tag{25}$$

describing $m_1 = 4$ spinons. (The diagram starts and ends at $j = 0$, appropriate for the vacuum character.) The leading motif sequence for this diagram is $(0)(0)$ and the possible fragmentations are $( )( )(0)$, $(0)( )( )$ and $( )( )( )( )$.

Recall that the motif sequences only indicate whether some of the n_i in (12) coincide or not. This means that to each motif sequence there corresponds an infinite number of choices of the labels $\{n_i\}$, which contributes a certain q-series to the character.

In the case of the example one obtains, using $L_0 = 4 + \sum_{i=1}^{4} n_i$ (from (13)), the values shown in Table 1. The sum of all four contributions to the character equals $q^6/(q)_4$, where we use the notation (for $a \in \mathbb{Z}_{>0}$)

$$(q)_a = \prod_{n=1}^{a}(1 - q^n) \tag{26}$$

with $(q)_0 = 1$ and $(q)_{-a} = 0$.

It is easy to show that in general all contributions that correspond to any given Bratteli diagram always add up to

$$\frac{q^{L_0(\text{diagram})}}{(q)_{m_1}} \ . \tag{27}$$

In this formula $L_0(\text{diagram})$ is defined to be the lowest L_0 value for a state corresponding to the leading motif

Table 1

motif	mode integers	character
$(0)(0)$	$0 \leq n_1 = n_2 < n_3 = n_4$	$\dfrac{q^6}{(1-q^2)(1-q^4)}$
$()()(0)$	$0 \leq n_1 < n_2 < n_3 = n_4$	$\dfrac{q^9}{(1-q^2)(1-q^3)(1-q^4)}$
$(0)()()$	$0 \leq n_1 = n_2 < n_3 < n_4$	$\dfrac{q^7}{(1-q^1)(1-q^2)(1-q^4)}$
$()()()()$	$0 \leq n_1 < n_2 < n_3 < n_4$	$\dfrac{q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$

sequence of that diagram $[(0)(0)$ for the example
(25), giving $L_0 = 6$].

Our remaining task is to sum $q^{L_0(\text{diagram})}$ over all
possible Bratteli diagrams [step (1)]. Let us start by
evaluating the sum at $q = 1$, i.e., by simply counting
the number of such diagrams. It is well known that
this number can be written in terms of binomial coef-
ficients

$$\#(\text{singlets in } 2^{m_1})$$

$$= \sum_{m_2, m_3, \ldots \in 2\mathbb{Z}} \prod_{a \geq 2} \binom{\frac{1}{2}(m_{a-1} + m_{a+1})}{m_a} . \quad (28)$$

To a given set $\{m_1, m_2, \ldots\}$ of even integers corre-
sponds a set of Bratteli diagrams as follows (the m_a's
must satisfy $\frac{1}{2}(m_{a-1} + m_{a+1}) \geq m_a$ and only a fi-
nite number of them are non-zero; the non-zero ones
are strictly descending). Suppose that m_l is the high-
est m_i not equal to zero. We start by drawing a pat-
tern that has first l uaw's, then l daw's, etc., repeat-
ing this pattern $\frac{1}{2}m_l$ times (m_l is even). Next we in-
sert $(\frac{1}{2}m_{l-1} - m_l)$ times a similar pattern, of lenght
$2(l-1)$ instead of $2l$. In principle, the pattern can be
inserted as a 'top' (first uaw's, then daw's) or as a
'dip' (first daw's, then uaw's). The rule is that if the
arrow on the left of the insertion points up you put a
'dip', else a 'top'. It may be checked that, since all
allowed diagrams are to be counted precisely once,
there are $m_l + 1$ positions where the insertions (which
may be multiple) can be done. This means that we
are separating m_l objects by $(\frac{1}{2}m_{l-1} - m_l)$ separators,
and this can be done in $\binom{\frac{1}{2}m_{l-1}}{m_l}$ ways. Next, we in-
sert $(\frac{1}{2}(m_{l-2} + m_l) - m_{l-1})$ times a top or a dip of
length $2(l-2)$. There are $m_{l-1} + 1$ spots where the
insertions can be done and the corresponding factor is
$\binom{\frac{1}{2}(m_{l-2} + m_l)}{m_{l-1}}$. Continuing, we build up the full prod-

uct in (28).

We should now compute L_0 for each of the diagrams
and sum q^{L_0} over all diagrams associated with a set
$\{m_1, m_2, \ldots\}$. We claim that the result is

$$q^{\frac{1}{2}(m_1^2 + m_2^2 + \ldots - m_1 m_2 - m_2 m_3 - \ldots)}$$

$$\times \prod_{a \geq 2} \begin{bmatrix} \frac{1}{2}(m_{a-1} + m_{a+1}) \\ m_a \end{bmatrix}_q , \quad (29)$$

where
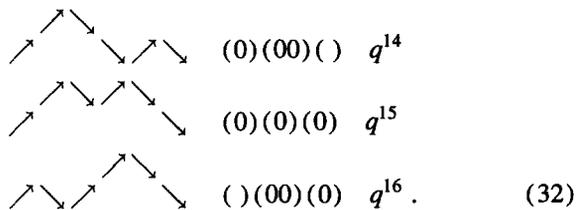
$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(q)_a}{(q)_{a-b}(q)_b}, \quad \text{for } a \geq b , \quad (30)$$

(and zero otherwise). This formula can be understood
as a q-deformation of the combinatorical expression in
(28), and it can be derived by going through the same
steps, this time keeping track of the q dependence of
all factors.

As an example, let us choose $m_1 = 6$, $m_2 = 2$ and
the rest 0. Going through the construction of diagrams
as described above ($l = 2$ here), we first draw the
diagram

$$\downarrow \nearrow \searrow \downarrow$$
$$\nearrow \qquad \searrow \downarrow \qquad\qquad\qquad (31)$$

The vertical arrows indicate the positions where, in the
second step, we can insert the pattern $\searrow \nearrow$ or $\nearrow \searrow$.
We thus find a total of $\binom{3}{2} = 3$ diagrams. With their
corresponding leading motif sequence and q^{L_0} value
they are

$$\nearrow \quad \nearrow \searrow \quad \searrow \nearrow \searrow \qquad (0)(00)() \quad q^{14}$$

$$\nearrow \quad \nearrow \searrow \nearrow \searrow \quad \searrow \qquad (0)(0)(0) \quad q^{15}$$

$$\nearrow \searrow \nearrow \quad \nearrow \searrow \quad \searrow \qquad ()(00)(0) \quad q^{16} . \quad (32)$$

These add up to $q^{14}(1+q+q^2)$ which is indeed equal to $q^{14} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q$.

Putting together all ingredients, we find the following result for the Virasoro vacuum character

$$\chi_0^{\mathrm{Vir}} = \sum_{m_1,m_2,\ldots \in 2\mathbb{Z}} q^{\frac{1}{2}(m_1^2+m_2^2+\ldots -m_1 m_2 - m_2 m_3 - \ldots)}$$
$$\times \frac{1}{(q)_{m_1}} \prod_{a \geq 2} \begin{bmatrix} \frac{1}{2}(m_{a-1}+m_{a+1}) \\ m_a \end{bmatrix}_q . \quad (33)$$

This expression is the $c = 1$ limit of a 'fermionic representation' of the Virasoro characters for unitary minimal models, as first given by Kedem et al. [4] and proven in [10].

Note that the sum over $m_2, m_3, \ldots$, at fixed m_1 gives the contribution of the m_1-spinon states to the character.

The Virasoro characters corresponding to non-zero $SU(2)$ spin j can be obtained in an identical way, this time using Bratteli diagrams that start at $j = 0$ and arrive at j. The character formula obtained reads as follows

$$\chi_{j^2}^{\mathrm{Vir}} = q^{-j/2} \sum_{m_1,m_2,\ldots}^{*} q^{\frac{1}{2}(m_1^2+m_2^2+\ldots -m_1 m_2 - m_2 m_3 - \ldots)}$$
$$\times \frac{1}{(q)_{m_1}} \prod_{a \geq 2} \begin{bmatrix} \frac{1}{2}(m_{a-1}+m_{a+1}+\delta_{a,2j+1}) \\ m_a \end{bmatrix}_q , \quad (34)$$

where the $*$ on the summation symbol indicates that $m_{2j}, m_{2j-2}, \ldots$, are odd and the other m_i are even.

7. $A_1^{(1)}$ characters and generalizations

From the basis of states which we called Basis II, it immediately follows that the (level-1) affine characters that occur in this theory have the following fermionic representation

$$\chi_{j=0}^{A_1^{(1)}} = \sum_{n^+ + n^- \text{even}} \frac{q^{(n^+ + n^-)^2/4}}{(q)_{n^+}(q)_{n^-}},$$

$$\chi_{j=\frac{1}{2}}^{A_1^{(1)}} = \sum_{n^+ + n^- \text{odd}} \frac{q^{(n^+ + n^-)^2/4}}{(q)_{n^+}(q)_{n^-}}. \quad (35)$$

These formulas were first written in [11] and they were related to the spinon picture in [9] (see also [12] for closely related results).

We thus see that for both choices of chiral algebra in this CFT (Virasoro and $A_1^{(1)}$) the characters have a fermionic representation, and that in both cases we can derive this from the generalized commutation relations of the fundamental spinon fields.

Looking at other RCFT's, we make a number of observations. First of all, it seems clear that in general, the possibility to write 'fermionic representations' of characters in a RCFT directly points at a specific chiral algebra for which those characters are appropriate. (Interesting speculations on the connection of this with the possibility to find integrable massive perturbations of the CFT were put forward in Ref. [13].) Secondly, there is a rather clear picture of how such a description changes if we pass from a WZW model to corresponding (diagonal) coset conformal field theories: in the character formulas such as (33) for the nth minimal coset model only a finite number of labels $m_1, m_2, \ldots, m_n$ are allowed, which implies that the coset CFT's can be viewed as *interacting* spinon theories in which some of the multi-spinon states are lost.

In the language of the Bethe Ansatz equations, the labels $\{m_2, m_3, \ldots\}$ in (33) refer to so-called 'ghost' excitations, which do not have a macroscopic range for their momenta. This is in contrast to m_1, which corresponds to a true quasi-particle. We can now describe the role of the Yangian symmetry in the $SU(2)_1$ WZW model as follows: starting from the states generated by the single true quasi-particle (which we take to be ϕ^+), the Yangian adds 'descendant states', which in the Bethe Ansatz language would be associated with the 'ghost' degrees of freedom. In this formulation, the Yangian symmetry is characterized in an operational way, and one can try to find its analogue in other RCFT's.

Applying this approach to the level-2 $SU(2)$ WZW model, we observe the following. The character formula for the $SU(2)$ singlets in the (unprojected) Neveu–Schwarz vacuum sector of the super Virasoro algebra takes a form [4] which is identical to (33), with, however, for a given m_1 an additional factor that has the form

$$\sum_{m_0 \in \mathbb{Z}} q^{\frac{1}{2}m_0^2 - \frac{1}{2}m_0 m_1} \begin{bmatrix} \frac{1}{2}m_1 \\ m_0 \end{bmatrix}_q . \tag{36}$$

(we shifted indices as compared to [4]). This level-2 character clearly suggests an algebraic structure where, in addition to generators that are analogous to the Yangian generators for level-1, there are generators that create 'replicas' of the level-1 Yangian multiplets. For example, if $m_1 = 2$ the additional factor equals $(1 + q^{-\frac{1}{2}})$, and one expects a two-fold degeneracy of all two-spinon states. The details of this algebraic structure, which can be studied for general level $k > 1$, will be published elsewhere [14].

Acknowledgement

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