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Integrable quantum mappings

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A quantum R -matrix structure is presented for a family of exactly integrable multidimensional rational mappings related to lattice versions of the Korteweg-de Vries equation. It is shown that these mappings possess a commuting family of invariants.

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The study of discrete-time integrable systems is in our view of great importance for the understanding of both classical as well as quantum integrable systems. A starting point for this investigation are integrable lattices, which are nonlinear partial *difference* equations, i.e., equations in which both the spatial as well as the time variables are discrete. Such systems were systematically studied in a number of papers [1,2]. The lattice models are fundamental in the sense that their various continuum limits give rise to entire hierarchies of integrable partial differential equations (PDE's) [3]. Furthermore, in contrast to most continuous equations, space and time variables appear on the same footing, exhibiting a covariance between the lattice directions. More specifically, dynamical *mappings* (i.e., finite-degree-of-freedom systems with discrete-time dynamics), have been applied to a variety of physical problems [4], including, e.g., the stability of beam-beam collisions in storage rings, Anderson localization, and incommensurate structures. Being computationally simple they can be accessed by numerical approaches and as such are used for the numerical integration of PDE's and the study of numerically induced chaos [5]. *Exactly integrable* mappings are useful for the study of bifurcations or transition to chaos, using perturbation techniques. Such integrable mappings have been the subject of recent studies [6–11]. On the quantum level, mappings have been studied in connection with the theory of quantum chaos [12,13]. In that context also it would be useful to have at one's disposal exactly integrable quantum mappings in order to study the phenomena of transition to chaos within a perturbation approach in the quantum regime.

Previously we have shown that special reductions of integrable lattice versions of soliton equations to give rise to integrable mappings [10] which carry a classical r -matrix structure [11]. Here we present an integrable quantization scheme for these mappings, namely within a quantum R -matrix formulation. We show that as a consequence of this scheme the mappings possess a com-

plete family of commuting quantum invariants, thus making way for a full diagonalization scheme in order to perform the exact integration of the quantum dynamics.

We are investigating mappings that arise from compatibility equations of the form

$$L'_j(k)M_j(k) = M_{j+1}(k)L_j(k), \quad (1)$$

in which the matrices L_j and M_j are the discrete analogues of Zakharov-Shabat (ZS) type of systems, k being a spectral parameter and the prime denoting a discrete-time shift. Such systems arise from integrable lattices by considering periodic initial value problems on “staircases” on a two-dimensional (space-time) lattice [10]. As a consequence of (1), a complete family of invariants of the mapping can be obtained, namely by introducing the associated *monodromy matrix* $T(k)$, obtained as an ordered product of the elementary translation matrices L_j along the staircase over one period P (imposing periodicity conditions $L_{j+P} = L_j, M_{j+P} = M_j$)

$$T(k) \equiv \prod_{j=1}^P L_j(k). \quad (2)$$

On the classical level, the trace of the monodromy matrix is invariant under the mapping (1) as a consequence of (1) and the periodicity condition, thus leading to P independent invariants by expanding the trace in powers of k . Furthermore, the mappings we consider here possess a canonical structure (symplectic Poisson structure) with respect to which the integrals exhibit the involutivity property, whence the systems are integrable in the sense of Liouville [8]. Clearly, in the discrete-time case, there is not a Hamiltonian structure in the usual sense. However, instead of a Hamiltonian, one can introduce a generating function of the mapping viewed as a canonical transformation [10,11].

In this Brief Report we are interested in a particular example of integrable mappings that exhibit the structure

outlined above, namely mappings of Korteweg–de Vries (KdV) type (i.e., mappings arising from the periodic initial value problem of lattice versions of the KdV equation [10]). These are rational mappings $\mathbb{R}^{2P} \rightarrow \mathbb{R}^{2P}$: $(\{v_j\}) \mapsto (\{v'_j\})$ of the form

$$v'_{2j-1} = v_{2j}, \quad v'_{2j} = v_{2j+1} + \frac{\epsilon\delta}{v_{2j}} - \frac{\epsilon\delta}{v_{2j+2}} \quad (j=1, \dots, P), \quad (3)$$

imposing the periodicity condition $v_{i+2P} = v_i$. The mapping (3) has the Casimir invariants

$$\sum_{j=1}^P v_{2j} = \sum_{j=1}^P v_{2j-1} = c,$$

where c is chosen to be invariant under the mapping, in which case we obtain a $(2P-2)$ -dimensional generalization of the McMillan mapping [6]. Equation (3) arises as the compatibility condition of a ZS system (1) with

$$L_j = V_{2j} V_{2j-1}, \quad M_j = \begin{bmatrix} u_j & 1 \\ \lambda_{2j} & 0 \end{bmatrix}, \quad V_i = \begin{bmatrix} v_i & 1 \\ \lambda_i & 0 \end{bmatrix}, \quad (4)$$

setting $u_j = u_{2j-1} - \epsilon\delta/v_{2j}$, and $\lambda_{2j} = k^2 - q^2$, $\lambda_{2j+1} = k^2 - p^2$, p and q being parameters corresponding to the lattice spacing of the original KdV lattice [1,10], and $\epsilon\delta = p^2 - q^2$. The corresponding classical invariants, obtained by expanding the trace of the monodromy matrix (2) in powers of k^2 , are in involution with respect to the Poisson structure

$$\{v_j, v_{j'}\} = \delta_{j+1, j'} - \delta_{j, j'+1}, \quad (5)$$

which was obtained using a Legendre transformation on an appropriately chosen Lagrangian [10]. Thus, on the classical level we show integrability of the mapping (3) in the sense of Liouville [10,11].

We now describe a novel quantum R -matrix structure that is suitable to demonstrate that the mapping (3) remains integrable on the quantum level. In that case the variables v_j become Hermitian quantum operators obeying Heisenberg type of commutation relations, noting that the natural quantization of the Poisson relations (5) is provided by ($\hbar = i\hbar$)

$$[v_j, v_{j'}] = \hbar(\delta_{j, j'+1} - \delta_{j+1, j'}). \quad (6)$$

It follows immediately that the mapping (3) is a canonical transformation with respect to these commutation relations.

We impose on the quantum L operator the commutation relations

$$\begin{aligned} R_{12}^+ L_{j,1} L_{j,2} &= L_{j,2} L_{j,1} R_{12}^-, \\ L_{j+1,1} S_{12} L_{j,2} &= L_{j,2} L_{j+1,1}, \end{aligned} \quad (7)$$

in which we adopt the standard abbreviations for tensorial products [14]. These relations are the quantum analogues of so-called *nonultralocal* Poisson brackets [15], which have been studied in a number of papers [16,17]. A similar quantization scheme was recently proposed in connection with the quantum Toda theory [18]. We

show that the quantum mappings provide us with a new class of examples of such a nonultralocal quantum R -matrix structure.

The compatibility of the set of Eqs. (7) leads to the following consistency conditions on $R^\pm$ and S :

$$R_{12}^\pm R_{13}^\pm R_{23}^\pm = R_{23}^\pm R_{13}^\pm R_{12}^\pm, \quad R_{23}^\pm S_{12}^\pm S_{13}^\pm = S_{13}^\pm S_{12}^\pm R_{23}^\pm, \quad (8)$$

where $S_{12}^+ = S_{12}$, $S_{12}^- = S_{21}$. Equations (8) are the quantum Yang-Baxter equations for $R^\pm$ coupled with additional equations for $S^\pm$. In order to establish that the structure given by the commutation relations (7) allows for suitable commutation relations for the monodromy matrix, we need to impose in addition to (8)

$$R_{12}^- S_{21} = S_{12} R_{12}^+, \quad R_{12}^+ S_{12} = S_{21} R_{12}^-. \quad (9)$$

Taking into account the periodic boundary conditions, we obtain for the monodromy matrix the commutation relations

$$R_{12}^+ T_1 S_{12} T_2 = T_2 S_{21} T_1 R_{12}^-. \quad (10)$$

A commuting parameter family of operators is constructed in a similar way as in Ref. [19] by taking

$$\tau(k) = \text{tr}[T(k)K(k)], \quad (11)$$

in which we introduce a family of matrices $K(k)$ obeying the relations

$$K_1 S_{21}^{-1} K_2 R_{12}^+ = R_{12}^- K_2 S_{12}^{-1} K_1. \quad (12)$$

Expanding (11) in powers of the spectral parameter, we obtain a complete set of commuting observables of the quantum system in terms of which we can find a common basis of eigenvectors in the associated Hilbert space. A matrix $K(k)$ is commonly introduced in connection with quantum boundary conditions other than periodic ones [19], but here it is essential in the periodic case as well.

The special solution of the quantum relations (8) and (9), which constitutes the R, S -matrix structure for the quantum mapping (3), together with the commutation relations (6) is given by

$$\begin{aligned} R_{12}^+ &= R_{12}^- + \frac{\hbar}{\mu_2} Q_{21} - \frac{\hbar}{\mu_1} Q_{12}, \\ R_{12}^- &= 1 \otimes 1 + \hbar \frac{P_{12}}{\mu_1 - \mu_2}, \\ S_{12} &= 1 \otimes 1 - \frac{\hbar}{\mu_2} Q_{21}, \end{aligned} \quad (13)$$

where $Q_{12} = E \otimes F$, $\mu_\alpha = k_\alpha^2 - q^2$, and where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We mention the useful identity

$$R_{12}^+ = \Lambda_1 \Lambda_2 R_{12}^- \Lambda_1^{-1} \Lambda_2^{-1},$$

where $\Lambda_\alpha = \mu_\alpha F + E$, ($\alpha=1,2$), from which it is evident

that it is not strictly necessary to introduce two different R -matrices $R^\pm$. However, working with a symmetrized version of the basic equations using only $\tilde{R}_{12} \equiv R_{12}^-$ and a symmetric matrix $\tilde{S} \equiv \Lambda_2^{-1} S_{12}^+ \Lambda_2 = 1 \otimes 1 - hF \otimes F$ has the disadvantage that the relations (9) and the definition of the monodromy matrix become less natural.

Using the identities

$$V_{i+1,1} S_{12} V_{i,2} = V_{i,2} V_{i+1,1}, \quad R_{12}^+ V_{i,1} V_{i,2} = V_{i,2} V_{i,1} R_{12}^-$$

[in which $R_{12}^\pm, S_{12}$ are given by (13) with μ replaced by λ_i], which follow from the commutation relations (3), it is straightforward that the quantum L operator (4) constitutes an exact solution of Eqs. (7) using the R, S matrices of (13). We remark that this special solution of the quantum L -matrix relations (7) has the nice feature that no

nontrivial modifications have to be introduced in the transition from the classical theory to the quantum theory, provided that the quantum operators v_j are normally ordered, i.e., in accordance with j increasing from the right to the left.

The solution (13) of the quantum relations given above allows for the construction of various families of commuting operators [e.g., by taking the solution $K \equiv 1$ of (12)]. However, it turns out that a particular family of commuting operators lead to exact *invariants* of the quantum mapping (3). These can be constructed by using (11) together with the following solution of Eq. (12):

$$K(k) = 1 + \frac{h}{\mu} FE$$

($\mu = k^2 - q^2$). The expression for $\tau(k)$ becomes

$$\tau(k) =: \left[\prod_{j=1}^{2P} v_j \right] \left[1 + \sum_{\substack{1 \leq J_1 < \dots < J_N \leq 2P \\ J_{v+1} - J_v \geq 2, J_1 - J_N = 2P \geq 2}} \prod_{v=1}^N \frac{\hat{\lambda}_{J_v}}{v_{J_v+1} v_{J_v}} \right] :, \quad (14)$$

leading one to find a full family of commuting invariants. In (14) $::$ denotes the normal ordering of the operators v_j in accordance with their enumeration, and $\hat{\lambda}_J = \lambda_J$ for $J \neq 2P$, $\hat{\lambda}_{2P} = \lambda_{2P} + h$. The quantum integrability of the mapping (3) follows consequently, and by noting that the mapping (3) is a canonical transformation, i.e., it preserves the quantum commutation relations (6). Because of (1) also the form of the quantum L operator is preserved under the mapping, hence also the quantum structure given by (7).

Let us give as a concrete example the quantum invariant of the original McMillan mapping [6], i.e., (3) for $P=2$, namely,

$$x' = y, \quad y' = -x - \frac{2\epsilon\delta y}{\epsilon^2 - y^2}, \quad (15)$$

for $x = v_1 - \epsilon$, $y = v_2 - \epsilon$ (choosing $c = 2\epsilon$) and where $[y, x] = h$, having the invariant

$$\mathcal{J} = (\epsilon^2 - y^2)(\epsilon^2 - x^2) - (\epsilon\delta + h)(2yx - \epsilon\delta). \quad (16)$$

The invariant $\mathcal{J}$ can be viewed as a Hamiltonian generating a continuous-time interpolating flow by $\dot{x} = (1/h)[\mathcal{J}, x]$, $\dot{y} = (1/h)[\mathcal{J}, y]$, whose solutions can be considered to be parametrized in terms of what we could call a quantum version of the Jacobi elliptic functions. It would be interesting to investigate whether such functions and their multidimensional generalizations correspond to the quasiperiodic meromorphic functions on a "quantum Riemann surface."

The classical limit of the quantum structure (7)–(11) is easily obtained by considering the quasiclassical expansion

$$S_{12}^\pm = 1 \otimes 1 - h s_{12}^\pm + O(h^2),$$

$$R_{12}^\pm = 1 \otimes 1 + h r_{12}^\pm + O(h^2),$$

which in the present case truncates at first order in h . In this limit the quantum commutation relations (7) yield the nonultralocal Poisson bracket structure of Refs. [11,17].

The results presented here can be extended to any member of a lattice Gel'fand-Dikii (GD) hierarchy [20], to which the same quantization scheme given above applies. This is of particular interest in connection with the study of lattice analogues of the W algebras [21] and consequently for the quantization of conformal field theories [22]. One may anticipate that these quantum mappings and lattices play a similar role as the GD hierarchy in the string equations and two-dimensional quantum gravity [23]. Furthermore we expect that discrete-time integrable systems will yield new insights in the quantum group structure associated with nonultralocal models [24]. Finally, the study of perturbations of these integrable mappings could yield some fundamental new insight in questions related to the transition to quantum chaos [12,13].

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