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FLUCTUATION PROPERTIES AND COLLECTIVE MODES OF QUANTUM PLASMAS IN A MAGNETIC FIELD

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A complete set of equilibrium fluctuation formulas for the charge density, the momentum density and the energy density of a magnetized one-component quantum plasma is presented. The derivation is based on the use of equations of motion for Fourier-transformed imaginary-time Green functions. The resulting formulas depend both on the strength and the orientation of the magnetic field. They are a basic ingredient for the derivation of the collective-mode spectrum in the long-wavelength limit. Projection-operator methods are used to establish explicit expressions for the mode frequencies of a magnetized quantum plasma up to second order in the wave number.

1. INTRODUCTION

In classical theory the influence of a uniform external magnetic field on the properties of a Coulomb system is in principle well understood. Equilibrium properties of classical plasmas are independent of a magnetic field, while in the dynamic behaviour the cyclotron motion of the particles shows up as can be seen for instance in the collective modes. For a quantum plasma the situation is different, since already the static properties are modified in the presence of a magnetic field. It is the purpose of this short paper to present results from which the influence of a magnetic field on both the static and the dynamic properties of a quantum plasma can be read off. A more detailed account will appear elsewhere[1].

The system we shall consider is a homogeneous one-component plasma (OCP) in a uniform magnetic field. The OCP consists of N quantum particles of charge e and mass m in a volume V. The particles move in an inert neutralizing background of charge density $-q_v = -en = -eN/V$. The Hamiltonian reads

$$H = \frac{\hbar^2}{2mV}\sum_{\mathbf{k}}\psi^\dagger(\mathbf{k})\big(\mathbf{k} - \frac{ie}{2\hbar c}\mathbf{B}\wedge\nabla_{\mathbf{k}}\big)^2\psi(\mathbf{k}) + \frac{1}{2V^3}\sum_{\mathbf{k}\mathbf{k}'}\sum_{\mathbf{q}\neq\mathbf{0}}\frac{e^2}{q^2}\psi^\dagger(\mathbf{k}+\mathbf{q})\psi^\dagger(\mathbf{k}'-\mathbf{q})\psi(\mathbf{k}')\psi(\mathbf{k}), \tag{1}$$

where $\mathbf{B}$ denotes the magnetic field and where $\psi^\dagger(\mathbf{k})$ and $\psi(\mathbf{k})$ are the creation and annihilation operators satisfying the standard (anti)commutation relations.

2. FLUCTUATION PROPERTIES

Fluctuation formulas are in general defined as relations for Fourier-transformed imaginary-time-dependent Green functions for small wave numbers. The imaginary-time-dependent Green function for two local operators Ω and Ω' is defined by

$$\frac{1}{V}\langle\Omega_\tau(\mathbf{k})\Omega'(-\mathbf{k})\rangle_T = \frac{1}{V}\int d\mathbf{r}_1 d\mathbf{r}_2 e^{-i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\langle(e^{\tau H}\Omega(\mathbf{r}_1)e^{-\tau H} - \langle\Omega\rangle)(\Omega'(\mathbf{r}_2) - \langle\Omega'\rangle)\rangle, \tag{2}$$

where the average is taken in the canonical ensemble and where the subscript T denotes truncation. The Green function satisfies the KMS condition:

$$\frac{1}{V}\langle\Omega_\tau(\mathbf{k})\Omega'(-\mathbf{k})\rangle_T = \frac{1}{V}\langle\Omega'_{\beta-\tau}(-\mathbf{k})\Omega(\mathbf{k})\rangle_T. \tag{3}$$

The standard equal-time fluctuation formulas follow by putting $\tau = 0$. Alternatively, one may consider so-called Kubo-transformed fluctuation formulas that are obtained by averaging the Green functions over all values of the imaginary time between 0 and $\beta = 1/k_B T$.

In the following we shall discuss the fluctuation formulas for the charge density, the momentum density and the energy density. The charge density and the momentum density in Fourier language are

$$Q(\mathbf{k}) = \frac{e}{V}\sum_{\mathbf{k}'}\psi^\dagger(\mathbf{k}'-\mathbf{k})\psi(\mathbf{k}'), \tag{4}$$

$$\mathbf{G}(\mathbf{k}) = \frac{\hbar}{2V}\sum_{\mathbf{k}'}\psi^\dagger(\mathbf{k}'-\mathbf{k})(2\mathbf{k}'-\mathbf{k} - \frac{ie}{\hbar c}\mathbf{B}\wedge\nabla_{\mathbf{k}'})\psi(\mathbf{k}'). \tag{5}$$

The energy density $E(\mathbf{k})$ is the sum of a kinetic part and a potential part. It reads:

$$E(\mathbf{k}) = \frac{\hbar^2}{8mV}\sum_{\mathbf{k}'}\psi^\dagger(\mathbf{k}'-\mathbf{k})\big(2\mathbf{k}'-\mathbf{k}-\frac{ie}{\hbar c}\mathbf{B}\wedge\nabla_{\mathbf{k}'}\big)^2\psi(\mathbf{k}')$$
$$-\frac{1}{2V^3}\sum_{\mathbf{q}(\neq\mathbf{0},\neq\mathbf{k})}\frac{e^2\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})}{q^2(\mathbf{k}-\mathbf{q})^2}\sum_{\mathbf{k}'\mathbf{k}''}\psi^\dagger(\mathbf{k}'-\mathbf{k}+\mathbf{q})\psi^\dagger(\mathbf{k}''-\mathbf{q})\psi(\mathbf{k}'')\psi(\mathbf{k}'). \tag{6}$$

For the potential part of the energy density we have adopted the expression $\frac{1}{2}E^2$, with $\mathbf{E}$ the electric field, and have omitted self-terms. Of course, one has $H = E(\mathbf{k}=\mathbf{0})$.

To derive the fluctuation formulas one starts from the equations of motion for the charge density and the momentum density. The commutation relations of these densities with the Hamiltonian read

$$[H, Q(\mathbf{k})] = -\frac{e\hbar}{m}\mathbf{k}\cdot\mathbf{G}(\mathbf{k}), \tag{7}$$
$$[H, \mathbf{G}(\mathbf{k})] = -\hbar q_v\frac{\mathbf{k}}{k^2}Q(\mathbf{k}) - i\hbar\omega_c\mathbf{G}(\mathbf{k})\wedge\hat{\mathbf{B}} - \hbar\mathbf{k}\cdot T(\mathbf{k}), \tag{8}$$

where $\omega_c = (e/mc)|\mathbf{B}|$ is the cyclotron frequency and $\hat{\mathbf{B}}$ a unit vector in the direction of the magnetic field. Furthermore, $T(\mathbf{k})$ is the pressure tensor. It is the sum of a kinetic and a potential part for both of which explicit expressions are available[1].

From the commutator relations (7)–(8) one may derive a set of coupled differential equations for the Green functions involving either Q or $\mathbf{G}$ and an arbitrary local operator Ω. The KMS condition yields a boundary condition to these differential equations. Solution leads to the following expressions for the Green functions:

$$\frac{1}{V}\langle Q_\tau(\mathbf{k})\Omega(-\mathbf{k})\rangle_T = k\sum_{\lambda\rho}c_{\lambda\rho}(\tau)e^{\hbar\rho\omega_\lambda\tau}, \tag{9}$$
$$\frac{1}{V}\langle \mathbf{G}_\tau(\mathbf{k})\Omega(-\mathbf{k})\rangle_T = -\sqrt{nm}\sum_{\lambda\rho}\mathbf{v}_{\lambda\rho}c_{\lambda\rho}(\tau)e^{\hbar\rho\omega_\lambda\tau}. \tag{10}$$

Here $\rho\omega_\lambda$, with $\lambda,\rho=\pm1$, are the fundamental frequencies of the so-called 'gyro-plasmon modes' that have been discussed before[2]:

$$\omega_\lambda = \tfrac{1}{2}\sqrt{\omega_p^2+\omega_c^2+2\omega_p\omega_c\cos\theta} + \tfrac{1}{2}\lambda\sqrt{\omega_p^2+\omega_c^2-2\omega_p\omega_c\cos\theta}, \tag{11}$$

with ω_p the plasma frequency and θ the angle between the wave vector and the magnetic field. Furthermore, the vectors $\mathbf{v}_{\lambda\rho}$ are defined as

$$\mathbf{v}_{\lambda\rho}(\hat{\mathbf{k}}) = \frac{\rho\omega_p\omega_\lambda}{\omega_\lambda^2-\omega_c^2}\hat{\mathbf{k}}_\perp + \frac{\rho\omega_p}{\omega_\lambda}\hat{\mathbf{k}}_\parallel - \frac{i\omega_p\omega_c}{\omega_\lambda^2-\omega_c^2}\hat{\mathbf{k}}\wedge\hat{\mathbf{B}}, \tag{12}$$

in which $\hat{\mathbf{k}}_\parallel = \cos\theta\hat{\mathbf{B}}$ and $\hat{\mathbf{k}}_\perp = \hat{\mathbf{k}} - \hat{\mathbf{k}}_\parallel$. The coefficients $c_{\lambda\rho}(\tau)$ are linear combinations of the averaged commutators of Ω with Q, $\mathbf{G}$ and of integrals over the Green function of Ω with the pressure tensor[1].

The expressions (9)-(10) for the Green functions of Ω with Q and $\mathbf{G}$ are still completely general. By specializing to small values of the wave number one derives the fluctuation formulas we are interested in. Choosing for Ω either Q or $\mathbf{G}$ one finds that in leading order of k the Green functions involving the pressure tensor drop out from the coefficients $c_{\lambda\rho}$. Evaluating the remaining commutators one arrives at the following fluctuation formulas:

$$\frac{1}{V}\langle Q_\tau(\mathbf{k})Q(-\mathbf{k})\rangle_T^{(2)} = \tfrac{1}{2}\hbar\sum_{\lambda\rho}\rho\omega_\lambda\frac{\omega_\lambda^2-\omega_c^2}{\omega_\lambda^2-\omega_{-\lambda}^2}\frac{e^{\hbar\rho\omega_\lambda\tau}}{e^{\hbar\rho\omega_\lambda\beta}-1}, \tag{13}$$
$$\frac{1}{V}\langle \mathbf{G}_\tau(\mathbf{k})Q(-\mathbf{k})\rangle_T^{(1)} = -\tfrac{1}{2}\hbar\sqrt{nm}\sum_{\lambda\rho}\rho\omega_\lambda\frac{\omega_\lambda^2-\omega_c^2}{\omega_\lambda^2-\omega_{-\lambda}^2}\mathbf{v}_{\lambda\rho}\frac{e^{\hbar\rho\omega_\lambda\tau}}{e^{\hbar\rho\omega_\lambda\beta}-1}, \tag{14}$$
$$\frac{1}{V}\langle \mathbf{G}_\tau(\mathbf{k})\mathbf{G}(-\mathbf{k})\rangle_T^{(0)} = \tfrac{1}{2}\hbar nm\sum_{\lambda\rho}\rho\omega_\lambda\frac{\omega_\lambda^2-\omega_c^2}{\omega_\lambda^2-\omega_{-\lambda}^2}\mathbf{v}_{\lambda\rho}\mathbf{v}_{\lambda\rho}^*\frac{e^{\hbar\rho\omega_\lambda\tau}}{e^{\hbar\rho\omega_\lambda\beta}-1}, \tag{15}$$

where a superscript (n) denotes the n-th order term in an expansion with respect to the wave number. All these fluctuation formulas depend explicitly on the magnetic field and on the orientation of the wave vector, in contrast to their classical counterparts. The first formula is the quantum version of the Stillinger-Lovett rule for a magnetized OCP; it has already been given in ref. 3. The other two formulas have been derived recently[1]. The fact that the momentum-momentum fluctuation formula depends on $\hat{\mathbf{k}}$ although it is of zeroth order in the wave number points to the fact that the Fourier transform is only conditionally convergent. In position space the Green function is a slowly decaying fuction of the separation between the observation points. For the unmagnetized OCP this fact has been noted before[3,4].

Fluctuation formulas involving the energy density may be obtained as well from (9)–(10). The results in leading order of k are:

$$\frac{1}{V}\langle Q_\tau(\mathbf{k})E(-\mathbf{k})\rangle_T^{(2)} = \frac{\hbar}{2q_v}\sum_{\lambda\rho}\rho\omega_\lambda\frac{\omega_\lambda^2-\omega_c^2}{\omega_\lambda^2-\omega_{-\lambda}^2}(h_v^B-\tfrac{3}{2}\frac{\omega_\lambda^2-\omega_p^2}{\omega_c^2}\delta p^B)\frac{e^{\hbar\rho\omega_\lambda\tau}}{e^{\hbar\rho\omega_\lambda\beta}-1}+\frac{1}{q_v}\frac{\partial p^B}{\partial\beta}, \tag{16}$$

$$\frac{1}{V}\langle \mathbf{G}_\tau(\mathbf{k})E(-\mathbf{k})\rangle_T^{(1)} = -\frac{\hbar}{2\omega_p}\sum_{\lambda\rho}\rho\omega_\lambda\frac{\omega_\lambda^2-\omega_c^2}{\omega_\lambda^2-\omega_{-\lambda}^2}(h_v^B-\tfrac{3}{2}\frac{\omega_\lambda^2-\omega_p^2}{\omega_c^2}\delta p^B)\mathbf{v}_{\lambda\rho}\frac{e^{\hbar\rho\omega_\lambda\tau}}{e^{\hbar\rho\omega_\lambda\beta}-1}-\frac{3i}{2\omega_c}\hat{\mathbf{k}}\wedge\hat{\mathbf{B}}\frac{\partial\delta p^B}{\partial\beta}, \tag{17}$$

where $p^B=\hat{\mathbf{B}}\cdot\langle T\rangle\cdot\hat{\mathbf{B}}$ is the pressure in the direction of the field, $\delta p^B=\hat{\mathbf{B}}\cdot\langle T\rangle\cdot\hat{\mathbf{B}}-\frac{1}{3}\,\mathrm{tr}\langle T\rangle$ the anisotropic part of the pressure tensor and $h_v^B=e_v+p^B$ the enthalpy density. It may be argued that in the fluid phase the equilibrium quantum OCP cannot sustain shear forces, so that δp^B vanishes. However, it is not clear whether this argument still holds in the presence of a magnetic field.

The energy-energy fluctuation formula is the only one that we have not come across as yet. However, since it is independent of the imaginary time in zeroth order of the wavelength, we immediately get:

$$\frac{1}{V}\langle E_\tau(\mathbf{k})E(-\mathbf{k})\rangle_T^{(0)} = \frac{nc_v}{k_B\beta^2}. \tag{18}$$

We have obtained now the complete set of fluctuation formulas for the charge density, the momentum density and the energy density. The parameter τ in these formulas can be chosen arbitrarily. In particular, taking $\tau=0$ one gets the equal-time fluctuation formulas. These still depend on the eigenfrequencies $\rho\omega_\lambda$, and hence on orientation and strength of the magnetic field. Much simpler versions of the fluctuation formulas are found by taking the Kubo transform. Denoting this transform by the symbol $\mathcal{K}$ we find from (13)–(17):

$$\mathcal{K}\frac{1}{V}\langle Q(\mathbf{k})Q(-\mathbf{k})\rangle_T^{(2)} = \frac{1}{\beta}, \tag{19}$$

$$\mathcal{K}\frac{1}{V}\langle \mathbf{G}(\mathbf{k})Q(-\mathbf{k})\rangle_T^{(1)} = 0, \tag{20}$$

$$\mathcal{K}\frac{1}{V}\langle \mathbf{G}(\mathbf{k})\mathbf{G}(-\mathbf{k})\rangle_T^{(0)} = \frac{nm}{\beta}U, \tag{21}$$

$$\mathcal{K}\frac{1}{V}\langle Q(\mathbf{k})E(-\mathbf{k})\rangle_T^{(2)} = \frac{1}{\beta q_v}(h_v^B+\beta\frac{\partial p^B}{\partial\beta}), \tag{22}$$

$$\mathcal{K}\frac{1}{V}\langle \mathbf{G}(\mathbf{k})E(-\mathbf{k})\rangle_T^{(1)} = -\frac{3i}{2\beta\omega_c}\hat{\mathbf{k}}\wedge\hat{\mathbf{B}}\frac{\partial\beta\delta p^B}{\partial\beta}. \tag{23}$$

The right-hand side of the last formula vanishes if the pressure tensor is isotropic.

3. COLLECTIVE MODES

In the long-wavelength limit the collective modes are given by suitable linear combinations $a_i(\mathbf{k})$ of the conserved quantities of the system, which in the present case are $Q(\mathbf{k})/k$, $\mathbf{G}(\mathbf{k})$ and $E(\mathbf{k})$. The charge density has been divided by k to account for the fact that the system does not support large-scale charge fluctuations. By employing projection operator techniques one proves that the modes follow by solving an eigenvalue problem of the form

$$\det[z\mathcal{K}\frac{1}{V}\langle a_i^\dagger(\mathbf{k})a_j(\mathbf{k})\rangle_T-\Omega_{ij}(\mathbf{k},z)]=0, \tag{24}$$

with a frequency matrix that is the sum of a direct and an indirect part given as

$$\Omega_{ij}(\mathbf{k},z)=-\mathcal{K}\frac{1}{V}\langle a_i^\dagger(\mathbf{k})La_j(\mathbf{k})\rangle_T+\mathcal{K}\frac{1}{V}\langle a_i^\dagger(\mathbf{k})LQ\frac{1}{z+QLQ}QLa_j(\mathbf{k})\rangle_T. \tag{25}$$

The Liouville (super)operator L is defined by $L\Omega=\hbar^{-1}[H,\Omega]$, while Q is a projection operator that projects an operator on the complement of the space of conserved operators.

Evaluating the frequency matrix in leading order of the wave number, with the help of the fluctuation formulas given above, one arrives at expressions for the modes and the corresponding frequencies. In zeroth order of k there are five frequencies, namely four gyro-plasmon frequencies $z_{\lambda\rho}^{(0)}=\rho\omega_\lambda$ as defined in (11) and one frequency $z_T^{(0)}=0$, corresponding to a heat mode. In second order of k these frequencies get additional terms that decribe damping and dispersion effects. These second-order terms are found to have the following form:

$$\begin{aligned}z_{\lambda\rho}^{(2)} &= \frac{\omega_\lambda^2-\omega_c^2}{2(\omega_\lambda^2-\omega_{-\lambda}^2)}\{\frac{\rho\omega_\lambda T}{nc_vq_v^2}[\frac{\partial}{\partial T}(p^B-\tfrac{3}{2}\frac{\omega_\lambda^2-\omega_p^2}{\omega_c^2}\delta p^B)]^2\\ &+\frac{1}{\sqrt{nm}k_BT}\mathcal{K}\frac{1}{V}\langle[\frac{Q(\mathbf{k})}{k}+\frac{\mathbf{v}_{\lambda\rho}}{\sqrt{nm}}\cdot\mathbf{G}(\mathbf{k})]^\dagger\hat{\mathbf{k}}\cdot T(\mathbf{k})\cdot\mathbf{v}_{\lambda\rho}\rangle_T^{(1)}\\ &+\frac{1}{nmk_BT}\mathcal{K}\frac{1}{V}\langle[\hat{\mathbf{k}}\cdot T(\mathbf{k})\cdot\mathbf{v}_{\lambda\rho}]^\dagger Q\frac{1}{\rho\omega_\lambda+QLQ}Q\hat{\mathbf{k}}\cdot T(\mathbf{k})\cdot\mathbf{v}_{\lambda\rho}\rangle_T^{(0)}\}, \qquad (26)\\ z_T^{(2)} &= \frac{1}{nc_vk_BT^2}\mathcal{K}\frac{1}{V}\langle[\hat{\mathbf{k}}\cdot\mathbf{J}_E(\mathbf{k})]^\dagger Q\frac{1}{QLQ}Q\hat{\mathbf{k}}\cdot\mathbf{J}_E(\mathbf{k})\rangle_T^{(0)}, \qquad (27)\end{aligned}$$

with $\mathbf{J}_E$ the heat current. In the second-order terms of the gyro-plasmon eigenfrequencies contributions arising from both the direct and the indirect part of the frequency matrix show up. In the former a fluctuation expression involving the pressure tensor is encountered; since it has not been evaluated above we have left it in its present form. In classical theory such a fluctuation expression is proportional to the inverse compressibility. The indirect contributions in (26) and (27) can be analyzed further in terms of heat conductivities and viscosities. Since the system possesses cylinder symmetry the number of these transport coefficients is higher than for an isotropic system (where there are three independent coefficients). The analysis is similar to that for a classical plasma as given in ref. 2.

The modes read up to first order in the wave number:

$$\begin{aligned}a_{\lambda\rho}(\mathbf{k}) &= C_{\lambda\rho}[\frac{Q(\mathbf{k})}{k}+\frac{\mathbf{v}_{\lambda\rho}}{\sqrt{nm}}\cdot\mathbf{G}(\mathbf{k})+\frac{k}{nc_vq_v}\frac{\partial}{\partial T}(p^B-\tfrac{3}{2}\frac{\omega_\lambda^2-\omega_p^2}{\omega_c^2}\delta p^B)E(\mathbf{k})], \qquad (28)\\ a_T(\mathbf{k}) &= C_T[E(\mathbf{k})-\frac{h_v^B}{q_v}Q(\mathbf{k})-\tfrac{3}{2}\frac{i}{nm\omega_c}\delta p^B(\mathbf{k}\wedge\hat{\mathbf{B}})\cdot\mathbf{G}(\mathbf{k})], \qquad (29)\end{aligned}$$

with $C_{\lambda\rho}$ and C_T normalization factors. Both these mode expressions contain explicit contributions depending on the anisotropic part of the pressure tensor, as was the case in the second-order terms of the gyro-plasmon mode frequency. Apart from these terms the expressions for the modes have the same form as those obtained in classical theory[2].

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