Price discovery with fallible choice

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Appendix C

$\mathcal{P}$ dynamics

In price adjustment process $\mathcal{P}$, as proposed in chapter 2, the auctioneer estimates demand and supply schedules based on a Cobb-Douglas approximation of individual preferences. This appendix details the dynamics of $\mathcal{P}$, proves global convergence for a set of CES economies and applies the price adjustment process to the Scarf economies.

C.1 Preliminaries

Consider an exchange economy, ξ, consisting of n agents and m commodities. Each agent i has preferences that can be represented by a continuous, quasi concave, monotone utility function, $u_i : \mathbb{R}^m_+ \to \mathbb{R}$. Traders also have non-negative endowments, $\mathbf{w}_i \in \mathbb{R}^m_+$. By assumption prices, $\mathbf{p}$, are non-negative and add up to 1, $\mathbf{p} \in S^{m-1} = \left\{\mathbf{p} \in \mathbb{R}^m_+ | \sum_j p_j = 1\right\}$.

After he has called prices $\mathbf{p}^k > 0$ and after each trader has responded with demand $\mathbf{x}_i\left(\mathbf{p}^k\right)$, the auctioneer *estimates* demand schedules by *assuming* that each trader has Cobb-Douglas preferences. By calculating expenditure per commodity as a percentage of each agent's budget, the auctioneer estimates individual demand schedules as:

$$x_{ji}(\mathbf{p}|\mathbf{p}^k) = \frac{p_j^k x_{ji}(\mathbf{p}^k)}{\mathbf{p}^k \cdot \mathbf{w}_i} \frac{\mathbf{p} \cdot \mathbf{w}_i}{p_j}.$$

Individual demand at $\mathbf{p}^k$ suffices to identify the hypothetical preferences (assuming that the auctioneer knows the distribution of endowments). This allows the auctioneer to calculate prices that equilibrate the associated Cobb-Douglas economy, and which feed into the next iteration.[1] Call this price process $\mathcal{P}$.

Let $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) = \sum_i \mathbf{x}_i\left(\mathbf{p}|\mathbf{p}^k\right) - \sum_i \mathbf{w}_i$ be the aggregate excess function of the associated Cobb-Douglas economy. Define $\mathbf{p}^{k+1}$ as the unique vector of equilibrium prices in the associated Cobb-Douglas economy. Hence, for $\mathbf{p}^{k+1}$ we have $\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) = \sum_i \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) - \sum_i \mathbf{w}_i = \mathbf{0}$.

[1] By assumption, trivial cases are excluded; e.g. an economy in which each trader exclusively prefers the commodity of which he is already the sole owner.

Lemma C.1. *If* $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$ *and if* $\mathbf{p}^{k+1} \neq \mathbf{p}^k$, *then* $\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) > 0$.

Proof. The Cobb-Douglas economy has an aggregate excess demand function that is characterized by gross substitution (GS); this implies WARP at the aggregate level, provided that prices are compared with the equilibrium price, i.e. with $\mathbf{p}^{k+1}$, (c.f. Mas-Colell et al. (1995, 17.F.3)):

$$\begin{aligned} \left(\mathbf{p}^{k+1} - \mathbf{p}\right) \cdot \left(\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) - \mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)\right) &> 0 \Leftrightarrow \\ \left(\mathbf{p}^{k+1} - \mathbf{p}\right) \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) &> 0 \Leftrightarrow \\ \mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) &> 0. \end{aligned}$$

Here, we used $\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) = \mathbf{0}$ and Walras' Law. □

Together with $\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) = \mathbf{0}$ lemma C.1 implies that the hyperplane $\mathbf{p}^{k+1} \cdot \mathbf{a} = 0$ is tangential to $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ in $\mathbf{p} = \mathbf{p}^{k+1}$. The aggregate excess demand function $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ does not need to be convex, but for all mixtures $\lambda\mathbf{p} + (1-\lambda)\,\mathbf{p}^{k+1}$ with $0 < \lambda < 1$ we have

Corollary C.2. *If* $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$, $\mathbf{p} \neq \mathbf{p}^{k+1}$ *and* $0 < \lambda < 1$, *then*

$$\lambda\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) + (1-\lambda)\,\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) > \mathbf{z}\left(\lambda\mathbf{p} + (1-\lambda)\,\mathbf{p}^{k+1}|\mathbf{p}^k\right).$$

Proof. Suppose the contrary, then there exists a λ^* such that $0 < \lambda^* < 1$ and

$$\begin{aligned} \lambda^*\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) + (1-\lambda^*)\,\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) &\leq \mathbf{z}\left(\lambda^*\mathbf{p} + (1-\lambda^*)\,\mathbf{p}^{k+1}|\mathbf{p}^k\right) \Rightarrow \\ \lambda^*(1-\lambda^*)\,\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) &\leq 0 \end{aligned}$$

This contradicts lemma C.1. The implication is due to $\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) = 0$ (by definition), multiplying both sides with $\lambda^*\mathbf{p} + (1-\lambda^*)\,\mathbf{p}^{k+1}$ and applying Walras' Law. □

Next, we show that the intersection between the $\mathbf{q} \cdot \mathbf{a} = 0$ hyperplane and $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ is unique.

Lemma C.3. *If* $\mathbf{q} > \mathbf{0}$, $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) \neq \mathbf{0}$ *and* $\mathbf{q} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) = 0$ *then* $\mathbf{q} = \mathbf{p}$.

Proof. Let $\mathbf{p} > \mathbf{0}$, $\mathbf{q} > \mathbf{0}$; furthermore suppose that $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) \neq \mathbf{0}$, $\mathbf{q} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) = 0$ and also that $\mathbf{q} \neq \mathbf{p}$. From the latter and from the fact that $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ satisfies GS we have $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) \neq \mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right)$. Multiplying both sides by $\mathbf{q}$ would then lead to $0 \neq 0$; hence $\mathbf{q} = \mathbf{p}$. □

The following lemma states that if $\mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right)$ lies below the $\mathbf{p} \cdot \mathbf{a} = 0$ hyperplane for some $\mathbf{p}$, then the intersection of $\mathbf{p} \cdot \mathbf{a} = 0$ and $\mathbf{z}\left(\cdot|\mathbf{p}^k\right)$ lies above the $\mathbf{q} \cdot \mathbf{a} = 0$ hyperplane.

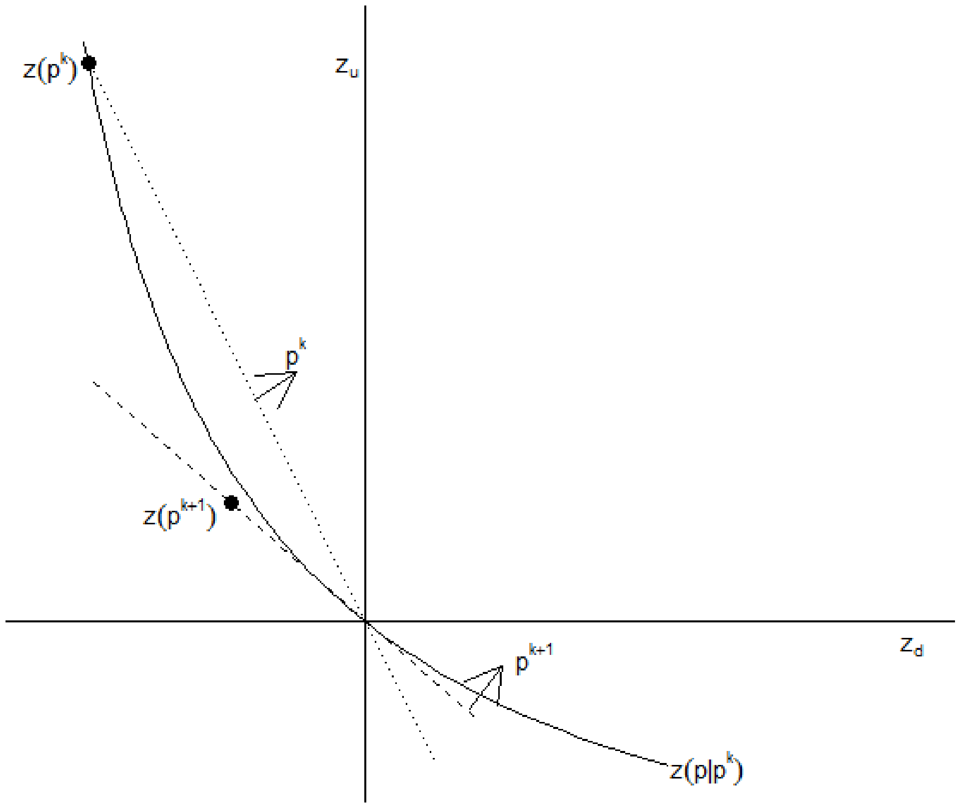

Figure C.1 – *The function* $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ *passes through the origin* $\mathbf{z}=\mathbf{0}$ *if prices are equal to* $\mathbf{p}^{k+1}$*;* $\mathbf{p}^{k+1}\cdot\mathbf{q}=0$ *is a hyperplane that is tangential to* $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$*. The* $\mathbf{p}^k\cdot\mathbf{q}=0$ *hyperplane intersects* $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ *at* $\mathbf{z}\left(\mathbf{p}^k\right)$*, which is a graphical expression of Walras' Law. This is also why* $\mathbf{z}\left(\mathbf{p}^{k+1}\right)$ *lies somewhere on the* $\mathbf{p}^{k+1}\cdot\mathbf{q}=0$ *hyperplane. There is no reason why* $\mathbf{z}\left(\mathbf{p}^{k+1}\right)$ *would be closer to the origin than* $\mathbf{z}\left(\mathbf{p}^k\right)$*, because* $\mathbf{z}(\mathbf{p})$ *does not need to be convex.*

Lemma C.4. *If* $\mathbf{p}\cdot\mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right)<0$ *then* $\mathbf{q}\cdot\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)>0$.

Proof. Let $\mathbf{p}$ be a price such that $\mathbf{p}\cdot\mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right)<0$. From lemma C.1 it follows that $\mathbf{p}^{k+1}\cdot\mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right)>0$; hence there exists a λ^* such that $0<\lambda^*<1$ and $\left(\lambda^*\mathbf{p}+(1-\lambda^*)\,\mathbf{p}^{k+1}\right)\cdot\mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right)=0$. From lemma C.3 we find that $\lambda^*\mathbf{p}+(1-\lambda^*)\,\mathbf{p}^{k+1}=\mathbf{q}$. Combining this with corollary C.2 yields

$$\lambda^*\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)+(1-\lambda^*)\,\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)>\mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right).$$

Multiplying both sides with $\mathbf{q}$, applying Walras' Law and $\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)=\mathbf{0}$ imply the result. □

Figure C.1 depicts a slice of $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ in $\mathbb{R}^m$. Note that while $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ appears as a convex function in figure C.1 we do not assume it to be convex.

Most of the analysis below refers to agents having Constant Elasticity of Substitution (CES) utility functions, ranging from Leontief to Cobb-Douglas utility functions. In that case, for each agent i we have

$$u_i(\mathbf{x}) = \left(\sum_j \alpha_{ji} x_{ji}^{\rho_i} \right)^{1/\rho_i}$$

with $\rho_i < 1$ and $\rho \neq 0$. If $\rho_i \to 0$ then the CES utility function converges to the Cobb-Douglas utility function. It will be convenient to define $\sigma_i = \frac{1}{1-\rho_i}$. If $\sigma_i \to 0$, the CES preferences approximate Leontief preferences. If $\sigma_i = 1$ (i.e. if $\rho_i \to 0$) then CES preferences coincide with Cobb-Douglas preferences.[2] Below, we'll consider CES preferences with $\sigma_i \leq 1$.[3] Given prices $\mathbf{p}$, trader i's optimal demand for commodity j can be written as:

$$x_{ji}(\mathbf{p}) = \left(\frac{\alpha_{ji}}{p_j} \right)^{\sigma_i} \frac{\mathbf{p} \cdot \mathbf{w}_i}{\sum_r \alpha_{ri}^{\sigma_i} p_r^{1-\sigma_i}}.$$

C.2 Price dynamics

Lemma C.5 proves the one-to-one correspondence between equilibria of an exchange economy and its associated Cobb-Douglas economy.

Lemma C.5. *Let $\xi = \{(u_i, \mathbf{w}_i)_{i=1}^n\}$ be an exchange economy with one or more Walrasian equilibria; furthermore, let $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$. If $\mathbf{p}^{k+1} = \mathbf{p}^k$ then $\mathbf{p}^k$ is an equilibrium price vector of ξ. Conversely, each equilibrium price vector $\mathbf{p}^*$ of ξ is also the equilibrium of the associated Cobb-Douglas economy.*

Proof. If $\mathbf{p}^{k+1} = \mathbf{p}^k$ then $\sum_i \mathbf{w}_i = \sum_i \mathbf{x}_i(\mathbf{p}^{k+1}|\mathbf{p}^k) = \sum_i \mathbf{x}_i(\mathbf{p}^k|\mathbf{p}^k) = \sum_i \mathbf{x}_i(\mathbf{p}^k)$. Furthermore, $\sum_i x_i(\mathbf{p}^*|\mathbf{p}^*) = \sum_i x_i(\mathbf{p}^*) = \sum_i \mathbf{w}_i$. □

After starting with strictly positive prices, the price adjustment process keeps generating strictly positive prices.

Lemma C.6. *Let $\mathbf{p}^0 > 0$; if $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$ then $\mathbf{p}^{k+1} > 0$.*

Proof. By definition, prices $\mathbf{p}^{k+1}$ clear an associated Cobb-Douglas economy. Suppose $p_j^{k+1} = 0$, then demand for commodity j in the associated Cobb-Douglas economy would be infinite, which is inconsistent with an equilibrium; hence $\forall j : p_j^{k+1} > 0$. □

With strictly positive prices, $\mathcal{P}$ cannot result in a boundary solution. In the simplex, the process can only move in a particular direction for a limited number of steps, unless $\mathcal{P}$ is converging along a straight line.

As a trivial corollary of lemma C.1 we have the sense in which the law of demand and supply holds:

[2] It is clear that if a trader has a Cobb-Douglas utility function then the estimated demand schedule of the auctioneer will be correct; if all traders have a Cobb-Douglas utility function then price adjustment process $\mathcal{P}$ will find the (unique) Walrasian equilibrium in one step.

[3] Due to GS, tâtonnement can be expected to do well for values $\sigma_i > 1$, but our argument requires $\sigma_i \leq 1$.

Corollary C.7. $\left(\mathbf{p}^{k+1}-\mathbf{p}^{k}\right)\cdot\mathbf{z}\left(\mathbf{p}^{k}\right)>0$.

While tâtonnement theory applies this law to each individual market, here we have an aggregate condition instead that allows some prices to decrease (increase) in the face of excess demand (supply) as long as aggregate demand at the previous prices is no longer affordable at the current prices.

Lemma C.8. *If all agents have CES preferences with $0\le\sigma_i\le1$, $\mathbf{p}^{k+1}=\mathcal{P}\left(\mathbf{p}^{k}\right)$ and if $\mathbf{p}^{k+1}\ne\mathbf{p}^{k}$, then $\mathbf{p}^{k}\cdot\mathbf{z}\left(\mathbf{p}^{k+1}\right)\le0$, with equality applying if and only if all agents have Cobb-Douglas preferences.*

Proof. Define $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$ as:

$$\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)=\frac{\mathbf{p}^{k+1}\cdot\mathbf{w}_i}{\mathbf{p}^{k+1}\cdot\mathbf{x}_i\left(\mathbf{p}^{k}\right)}\mathbf{x}_i\left(\mathbf{p}^{k}\right).$$

This point, $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$, is the demand at $\mathbf{p}^{k+1}$ if preferences are Leontief, i.e. if commodities are treated as pure complements, c.f. figure C.2. In this case, $\mathbf{x}_i\left(\mathbf{p}^{k}\right)$ is the Leontief demand at $\mathbf{p}^{k}$ and $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$ is scaled to lie on the $\mathbf{p}^{k+1}$ budget constraint. We want to show that for each trader i we have

$$\mathbf{p}^{k}\cdot\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)>\mathbf{p}^{k}\cdot\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right).\tag{C.2.1}$$

The inequality can be obtained if there exists a hyperplane $\mathbf{p}^k\cdot\mathbf{q}=\theta$ that separates $\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)$ and $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$ in accordance with the inequality. We can determine this hyperplane by looking for hypothetical endowments $\tilde{\mathbf{w}}_i$ that (i) lie on the $\mathbf{p}^{k+1}$ budget constraint and (ii) that yield $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)=\mathbf{x}_i\left(\mathbf{p}^{k}|\mathbf{p}^{k},\tilde{\mathbf{w}}\right)$ and $\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)=\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^{k},\tilde{\mathbf{w}}\right)$. For a graphical version of the argument below, see figure C.2.

Dropping index i, we can rewrite $x_{ji}^L\left(\mathbf{p}^{k+1}\right)$ as

$$x_j^L\left(\mathbf{p}^{k+1}\right)=\frac{p_j^k x_j\left(\mathbf{p}^{k}\right)}{\mathbf{p}^{k}\cdot\mathbf{w}}\frac{\mathbf{p}^{k}\cdot\frac{\mathbf{p}^{k+1}\mathbf{w}}{\mathbf{p}^{k+1}\mathbf{x}\left(\mathbf{p}^{k}\right)}\mathbf{x}\left(\mathbf{p}^{k}\right)}{p_j^k}.$$

Furthermore, we also have

$$\begin{aligned}x_j\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)&=\frac{p_j^k x_j\left(\mathbf{p}^{k}\right)}{\mathbf{p}^{k}\cdot\mathbf{w}}\frac{\mathbf{p}^{k+1}\mathbf{w}}{p_j^{k+1}}\\&=\frac{p_j^k x_j\left(\mathbf{p}^{k}\right)}{\mathbf{p}^{k}\cdot\mathbf{w}}\frac{\mathbf{p}^{k+1}\cdot\frac{\mathbf{p}^{k+1}\mathbf{w}}{\mathbf{p}^{k+1}\mathbf{x}\left(\mathbf{p}^{k}\right)}\mathbf{x}\left(\mathbf{p}^{k}\right)}{p_j^{k+1}}.\end{aligned}$$

Therefore we can argue that (i) an agent who is endowed with $\tilde{\mathbf{w}}=\mathbf{x}^L\left(\mathbf{p}^{k+1}\right)$ and has a demand function $\mathbf{x}\left(\mathbf{p}|\mathbf{p}^{k},\tilde{\mathbf{w}}\right)$ will demand $\mathbf{x}^L\left(\mathbf{p}^{k+1}\right)$ and $\mathbf{x}\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)$ if prices are equal to $\mathbf{p}^{k}$ and $\mathbf{p}^{k+1}$ respectively; (ii) if prices are equal to $\mathbf{p}^{k+1}$ then both $\mathbf{x}^L\left(\mathbf{p}^{k+1}\right)$ and $\mathbf{x}\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)$ are affordable while $\mathbf{x}\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)$ is preferred; (iii) if prices are equal to $\mathbf{p}^{k}$ then $\mathbf{x}^L\left(\mathbf{p}^{k+1}\right)$ is preferred to other affordable options; (iv) the individual demand function $\mathbf{x}_i\left(\mathbf{p}|\mathbf{p}^{k},\tilde{\mathbf{w}}\right)$ satisfies

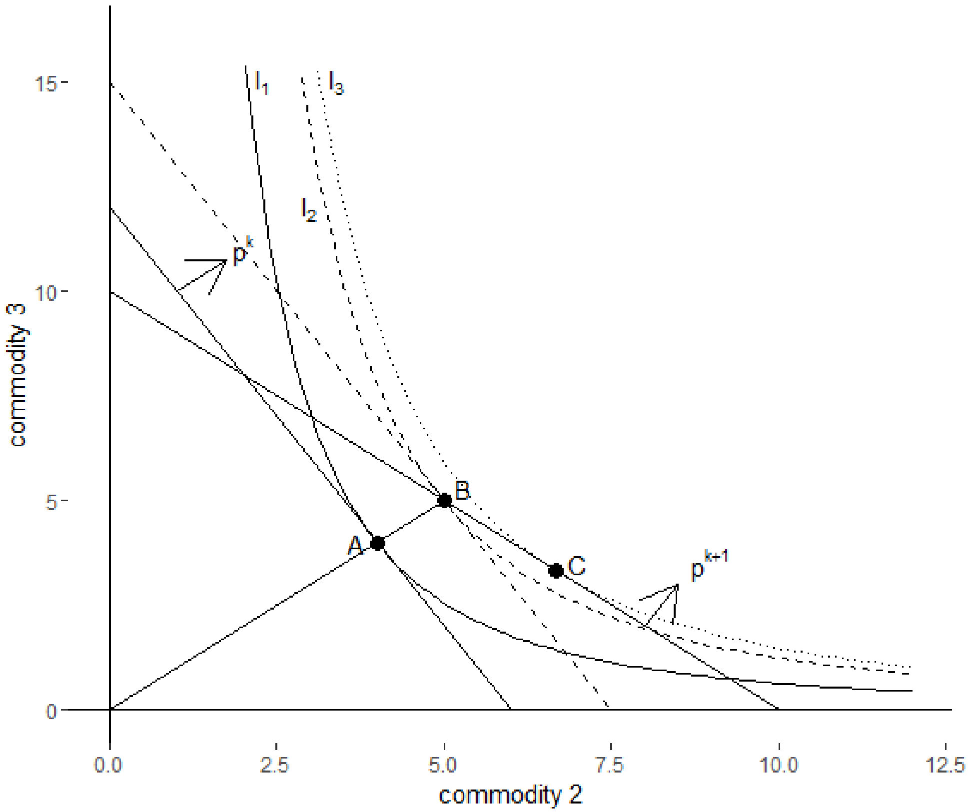

Figure C.2 – *Graphical explanation of the proof of inequality C.2.1. In response to prices* $\mathbf{p}^k$ *a trader demands A. The auctioneer constructs hypothetical Cobb-Douglas preferences that rationalize this choice, i.e. the solid indifference curve* I_1*. Given these preferences, the auctioneer expects that the trader will demand C at the new prices* $\mathbf{p}^{k+1}$*: the dotted indifference curve* I_3 *is tangential to the new budget constraint. Point B is the demand at prices* $\mathbf{p}^{k+1}$ *if the trader has Leontief preferences (instead of the unknown CES or the hypothetical Cobb-Douglas preferences). Inequality C.2.1 expresses the fact that C is not affordable at prices* $\mathbf{p}^k$ *for another trader, with endowments B and the hypothetical Cobb-Douglas preferences. The point is proved by observing that this other trader prefers B at prices* $\mathbf{p}^k$ *and that he prefers C if prices are equal to* $\mathbf{p}^{k+1}$ *(that is, while B is also affordable at* $\mathbf{p}^{k+1}$*). The implication is that C is not affordable at* $\mathbf{p}^k$ *(otherwise this other trader would have chosen C instead of B). Hence, the dashed budget constraint through B is part of a hyperplane that separates B and C.*

WARP; (v) hence $\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$ is not affordable at prices $\mathbf{p}^k$. We have demonstrated the existence of a suitable separating hyperplane and hence for all traders we have $\mathbf{p}^k \cdot \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) > \mathbf{p}^k \cdot \mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$.

If preferences are CES with $0 \leq \sigma_i \leq 1$ then $\mathbf{x}_i\left(\mathbf{p}^{k+1}\right)$ lies between $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$ and $\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$ on the $\mathbf{p}^{k+1}$ budget constraint, i.e.

$$\mathbf{x}_i\left(\mathbf{p}^{k+1}\right) = (1-\theta_i)\,\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right) + \theta_i \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$$

with $0 \leq \theta_i \leq 1$. If $0 \leq \theta_i < 1$ then by virtue of inequality C.2.1 we now have

$$\begin{aligned}\mathbf{p}^k \cdot \mathbf{x}_i\left(\mathbf{p}^{k+1}\right) &= (1-\theta_i)\,\mathbf{p}^k \cdot \mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right) + \theta_i \mathbf{p}^k \cdot \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) \\ &< \mathbf{p}^k \cdot \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right).\end{aligned}$$

Summing over i yields:

$$\begin{aligned}\mathbf{p}^k \cdot \sum_i \mathbf{x}_i\left(\mathbf{p}^{k+1}\right) &< \mathbf{p}^k \cdot \sum_i \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) \Leftrightarrow \\ \mathbf{p}^k \cdot \sum_i \mathbf{x}_i\left(\mathbf{p}^{k+1}\right) &< \mathbf{p}^k \cdot \sum_i \mathbf{w}_i \Leftrightarrow \\ \mathbf{p}^k \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) &< 0.\end{aligned}$$

The second inequality is due to the fact that $\mathbf{p}^{k+1}$ is an equilibrium price vector of the associated Cobb-Douglas economy. If $\forall i : \theta_i = 1$ then the inequality becomes an equality. □

The following lemma shows that constraints on $\mathbf{z}\left(\mathbf{p}^{k+1}\right)$ accumulate, which allows us to prove global convergence in proposition C.10.

Lemma C.9. *If* $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$ *and* $\mathbf{p}^{k+1} \neq \mathbf{p}^k$ *then* $\forall r \leq k :\ \mathbf{p}^r \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$.

Proof. For $r = k$ the result follows directly from lemma C.8. Lemma C.4 combined with $\mathbf{p}^{k-1} \cdot \mathbf{z}\left(\mathbf{p}^k\right) < 0$ (from lemma C.8 after relabeling k) yields $\mathbf{p}^k \cdot \mathbf{z}\left(\mathbf{p}^{k-1}|\mathbf{p}^k\right) > 0$. This says that, if $z_d\left(\mathbf{p}^k\right) < 0$ and $z_u\left(\mathbf{p}^k\right) > 0$ then the $\mathbf{p}^{k-1} \cdot \mathbf{q} = 0$ hyperplane intersects $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ in the (z_d, z_u)-plane above $\left(z_d\left(\mathbf{p}^k\right), z_u\left(\mathbf{p}^k\right)\right)$. Thus we have that (i) $\mathbf{z}\left(\mathbf{p}^k\right)$, lying on the $\mathbf{p}^k \cdot \mathbf{q} = 0$ hyperplane, is strictly below the $\mathbf{p}^{k-1} \cdot \mathbf{q} = 0$ hyperplane; and also that (ii) $\mathbf{z}\left(\mathbf{p}^{k+1}\right)$, lying on the $\mathbf{p}^{k+1} \cdot \mathbf{q} = 0$ hyperplane, is strictly below the $\mathbf{p}^k \cdot \mathbf{q} = 0$ hyperplane. Hence, $\mathbf{z}\left(\mathbf{p}^{k+1}\right)$ must be strictly below the $\mathbf{p}^{k-1} \cdot \mathbf{q} = 0$ hyperplane as well: $\mathbf{p}^{k-1} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$.

This argument can be repeated to obtain $\mathbf{p}^r \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$ for other $r < k-1$ as follows: after relabeling k we know from the previous step that $\mathbf{p}^{k-2} \cdot \mathbf{z}\left(\mathbf{p}^k\right) < 0$, which allows us to apply lemma C.4 to obtain $\mathbf{p}^k \cdot \mathbf{z}\left(\mathbf{p}^{k-2}|\mathbf{p}^k\right) > 0$ and hence also $\mathbf{p}^{k-2} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$. Et cetera. □

Proposition C.10. *Let* $\xi = \{(u_i, \mathbf{w}_i)_{i=1}^n\}$ *be a CES exchange economy with* $\forall i : 0 \leq \sigma_i \leq 1$. *Price adjustment process* $\mathcal{P}$ *almost always converges globally to an equilibrium price vector,* $\mathbf{p}^*$*, of* ξ *(provided it exists).*

Proof. We have $\forall r \leq k :\ \mathbf{p}^r \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$ from lemma C.9. In this case, $\mathbf{p}^{k+1} \neq \sum_{r=0}^k \theta^r \mathbf{p}^r$ for any $\{\theta^r\}_r$ with $0 < \theta^r < 1$ and $\sum_r \theta^r = 1$. To see this, suppose $\mathbf{p}^{k+1} = \sum_r \theta^r \mathbf{p}^r$ for appropriate $\{\theta^r\}_r$; then $\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) = \sum_r \theta^r \mathbf{p}^r \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$, because by assumption each term $\mathbf{p}^r \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$; however, this violates Walras' Law. Let

$$F(k) = \left\{\mathbf{p}|\mathbf{p} = \sum_{r=0}^k \theta^r \mathbf{p}^r,\ 0 < \theta^r < 1,\ \sum_{r=0}^k \theta^r = 1\right\} \subset S^{m-1}$$

be the convex hull of previous prices; this is the set of "forbidden" subsequent prices. We have $F(k) \subseteq F(k+1)$, with equality applying only if $\mathbf{p}^{k+1} = \mathbf{p}^k = \mathbf{p}^*$.

For as long as $\mathcal{P}$ has not yet converged, a new $\mathbf{p}^{k+1} \in S^{m-1} - F(k)$ has to be selected. Due to lemma C.6, the price adjustment process $\mathcal{P}$ cannot move in any particular direction for too long, unless it is converging along a straight line. Therefore $\mathcal{P}$ either moves in a straight line to $\mathbf{p}^*$ or else it keeps removing subsets of non-zero measure from the set of feasible prices.

Suppose that successive increments in the size of $F(k)$ approach zero, then $\mathbf{p}^{k+1}$ converges to either (i) an element in $F(k)$ if $\mathbf{p}^{k+1}$ and $\mathbf{p}^{k+2}$ are becoming arbitrarily close, or else (ii), if $\mathbf{p}^{k+1}$ and $\mathbf{p}^{k+2}$ do not become arbitrarily close then prices converge to a top-cycle, without being able to enter it. Observe that $\forall r \leq k: \mathbf{p}^r \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$ rules out a top-cycle: on the one hand we have $\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}^k\right) > 0$ per lemma C.1 but since $\mathbf{p}^{k+1}$ in a top-cycle is both a successor *and* a predecessor of $\mathbf{p}^k$ then we would also have $\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}^k\right) < 0$ via lemma C.9 and we would have a contradiction. Therefore we have that either (i) prices $\mathbf{p}^{k+1}$ and $\mathbf{p}^{k+2}$ become arbitrarily close and $\mathcal{P}$ is converging to an equilibrium that is covered by $F(k+1)$ or (ii) $\mathcal{P}$ will exhaust the set of feasible prices or else (iii) the very special case that prices orbit around the convex hull of previous prices, expanding its size by ever smaller amounts, without entering into a top-cycle. If such a path around the convex hull of past prices exists then $\mathcal{P}$ can enter it at any point that belongs to this path; since it consists of countable infinitely many points the set where we do not have convergence has measure zero.

If for some k we would have that all equilibria have been covered by $F(k)$ then the process cannot stop even though eventually it has no more prices to choose from. Contradiction; hence, it must be the case that for some k: $\mathbf{p}^k = \mathbf{p}^*$. In light of the exception mentioned above, $\mathcal{P}$ will converge "almost always". □

C.3 Application to the Scarf economies

Figure C.3 shows the convergence of $\mathcal{P}$ in the examples of Scarf, as implemented by Anderson et al. (2004). Here, all three economies have the same, unique, equilibrium with $\mathbf{p}^* = (1, 40, 20)$. Convergence is fast: for instance, starting from $\mathbf{p}^1 = (1, 1, 1)$, it takes 15 iterations to obtain the equilibrium prices in two decimals in the stable case, and 65 iterations in the unstable cases. Observe that the evolution of prices in each of the respective cases is consistent with tâtonnement theory.

C.4 Discussion

Cobb-Douglas approximation provides a new perspective on what information is needed for proving global convergence. Instead of having to know the Jacobian of the aggregate excess demand function, the auctioneer can effectively *estimate* its behavior by *assuming* that demand is generated by Cobb-Douglas preferences. Demand at prices $\mathbf{p}^k$ suffices to identify the hypothetical utility functions.

The convergence in the unstable Scarf economies of price adjustment process $\mathcal{P}$ is probably due to it correctly anticipating income effects, which is a good characteristic to have.

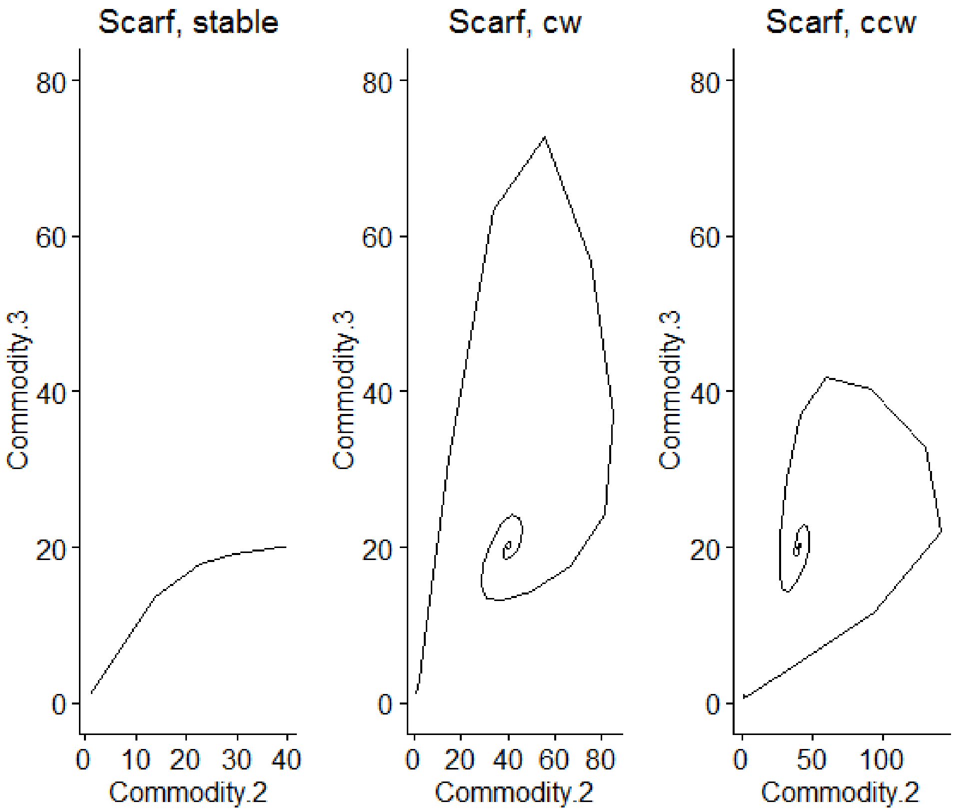

Figure C.3 – *Convergence of price adjustment process $\mathcal{P}$ in the Scarf economies to the equilibrium prices* $\left(p_2^*, p_3^*\right) = (40, 20)$. *Convergence depends on the initial allocation.*

Instead of modeling the law of demand and supply as a differential or as a difference equation that applies to individual markets, this law is better expressed as an aggregate inequality across markets, $\left(\mathbf{p}^{k+1} - \mathbf{p}^k\right) \mathbf{z}\left(\mathbf{p}^k\right) > 0$ if $\mathbf{p}^{k+1} \neq \mathbf{p}^k$, because this formulation allows income effects to dominate price effects.

C.5 Conclusions

In this appendix we have described a price adjustment process in which the auctioneer estimates demand and supply schedules by assuming that traders have Cobb-Douglas preferences. We have proved global convergence in a large set of economies in which agents have CES utility functions, ranging from Leontief to Cobb-Douglas utility functions.