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EDGE EFFECTS IN MAGNETOPLASMAS

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ABSTRACT
Edge effects in magnetized charged-particle systems are discussed with the help of a multiple-reflection expansion for the Green function. The profiles of the density and the electric current are determined both for the non-degenerate and the highly degenerate case. The asymptotic form of the profiles near the bulk is found to be exponentially decaying in both cases.

1. Introduction

Edge effects play an important role in systems of charged particles that are embedded in a magnetic field. In fact, the response of such a system to a field is determined by diamagnetic currents that start to circulate near the surface. As is well-known, this diamagnetic response, first described by Landau[1], is a purely quantummechanical effect that is present even if the interactions between the charged particles are neglected. Hence, it can be studied conveniently for a system of particles without mutual interaction.

One of the topics of interest in these systems is the spatial dependence of the diamagnetic currents, and of other physical quantities like the density or the pressure. These spatial profiles have been studied by several authors[2−6], and in particular the relation between the total current and the magnetization has been established. Nevertheless, there are still some open questions, for instance on the precise form of the profiles for degenerate quantum systems. Some authors[2,3] have found algebraically decaying tails in these profiles, whereas others[4] have derived exponentially decreasing bounds.

An interesting approach to the study of edge effects in finite systems is due to Balian and Bloch[7]. They present a so-called multiple-reflection expansion that relates the Green function for a finite system to that for an infinite system. Some time ago their method, which was originally derived for a free-particle system, has been extended to the case of magnetized systems[8]. The results found in this way, in particular for the finite-size corrections in the diamagnetic response (which are closely related to the current profiles), turn out to be different from those derived through other methods[2,6,9].

In the present paper we shall show how the multiple-relection expansion can be used to determine the profiles for a magnetized electron gas without interaction in a finite enclosure. In particular, we will obtain the asymptotic behaviour of the profiles of the density and the current, both for weakly and strongly degenerate systems. It will become clear why the results in ref. 8 are different from those derived by the other authors. A more detailed account of the results will be published elsewhere[10].

2. Multiple-reflection expansion for the Green function

We consider a free electron gas in a uniform magnetic field. The electron gas is confined to a three-dimensional cylinder-shaped domain D with hard walls. The base of the cylinder is arbitrary and the direction of the magnetic field is perpendicular to the base manifold. The Hamiltonian for a single particle in a magnetic field is given by

$$H = -\frac{\hbar^2}{2m}\Delta_\perp + i\hbar\omega_c x\frac{\partial}{\partial y} + \tfrac{1}{2}m\omega_c^2 x^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} \equiv H_\perp + H_\parallel \; . \tag{1}$$

The vector potential for the magnetic field $\mathbf{B} = (0,0,B)$, with associated cylotron frequency $\omega_c = eB/mc$, is chosen as $\mathbf{A} = (0, Bx, 0)$. In the following we will only deal with the transverse part of the motion and write $\mathbf{r}$ to denote (x, y).

The Green function for arbitrary complex z is defined as

$$G_z(\mathbf{r},\mathbf{r}') = \sum_n \psi_n(\mathbf{r})\psi_n^*(\mathbf{r}')\frac{1}{z - E_n} \; , \tag{2}$$

with ψ_n the eigenfunctions and E_n the eigenvalues. The Green function vanishes at the walls; it satisfies the equation

$$(H_\perp - z)G_z(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \; . \tag{3}$$

Following ref. 7 and 8 we split the Green function in a part G_z^0, which is the Green function for the system without confinement, and a correction part G_z^c, so that $G_z = G_z^0 + G_z^c$. The equation for the correction part is

$$(H_\perp - z)G_z^c(\mathbf{r},\mathbf{r}') = 0 \; , \tag{4}$$

for all $\mathbf{r}, \mathbf{r}' \in D/\partial D$, with the boundary condition

$$G_z^c(\mathbf{r},\mathbf{r}') = -G_z^0(\mathbf{r},\mathbf{r}'), \quad \mathbf{r} \in \partial D \quad \text{and/or} \quad \mathbf{r}' \in \partial D \; . \tag{5}$$

The multiple-reflection expansion for the correction part of the Green function in terms of G_z^0 can now be derived as follows. Firstly, from (4), (5) and the Green equality one finds

$$G_z^c(\mathbf{r},\mathbf{r}') = -\frac{\hbar^2}{2m}\int_{\partial D} d\sigma''^{\mathrm{W}}\mathbf{n}'' \cdot \left[\nabla_{\mathbf{r}''}G_z(\mathbf{r},\mathbf{r}'')\right]_{\mathbf{r}''\to\mathbf{r}''^{\mathrm{W}}} \left[G_z^0(\mathbf{r}'',\mathbf{r}')\right]_{\mathbf{r}''\to\mathbf{r}''^{\mathrm{W}}} \; , \tag{6}$$

where $\mathbf{n}''$ is the normal vector, directed outwards, at the point $\mathbf{r}''$ of the boundary. In the limiting process the position vector $\mathbf{r}''$ is taken to the boundary (denoted by the symbol W) from the inside. Subsequently, by differentiating (6) and using the asymptotic formula for the infinite-domain Green function at small $|\mathbf{r}-\mathbf{r}'|$, one may derive

$$\mathbf{n}'\cdot[\nabla_{\mathbf{r}'}G_z(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'\to\mathbf{r}'W}=2\mathbf{n}'\cdot\left[\nabla_{\mathbf{r}'}G_z^0(\mathbf{r},\mathbf{r}')\right]_{\mathbf{r}'\to\mathbf{r}'W}$$
$$-\frac{\hbar^2}{m}\int_{\partial D}d\sigma''^{\mathrm{W}}\mathbf{n}''\cdot[\nabla_{\mathbf{r}''}G_z(\mathbf{r},\mathbf{r}'')]_{\mathbf{r}''\to\mathbf{r}''\mathrm{W}}\,\mathbf{n}'\cdot\left[\left[\nabla_{\mathbf{r}'}G_z^0(\mathbf{r}'',\mathbf{r}')\right]_{\mathbf{r}''\to\mathbf{r}''\mathrm{W}}\right]_{\mathbf{r}'\uparrow\downarrow\mathbf{r}'\mathrm{W}}\,. \quad (7)$$

Here the symbol $\mathbf{r}'\uparrow\downarrow\mathbf{r}'^{\mathrm{W}}$ indicates an average of two limits, with $\mathbf{r}'$ approaching the wall either from the inside or the outside of the system.

One can solve (7) iteratively for $\mathbf{n}'\cdot[\nabla_{\mathbf{r}'}G_z(\mathbf{r},\mathbf{r}')]_{\mathbf{r}'\to\mathbf{r}'\mathrm{W}}$. The solution can be substituted in (6) so as to get the desired expansion for the Green function of the confined system in terms of the infinite-domain Green function. It can be understood in terms of multiple-reflection processes via the walls.

For the case of a free-particle system without a magnetic field it has been shown[7] that the higher-order terms in the multiple-reflection expansion are important only for curved boundaries, whereas the leading term is the only one that survives for nearly planar walls. For a magnetized system it has been assumed[8] that likewise the leading term yields a reasonable approximation for the Green function near a planar wall. However, it is not obvious that these higher-order terms can be thrown away for a magnetized system as well. Indeed, because the particles follow curved trajectories, they can scatter several times off the same boundary.

In the following we will study the edge effects for a system with half-space geometry $x \geq 0$, with a boundary located at $x = 0$. For this geometry we will determine the profiles both for the non-degenerate and the degenerate case.

3. Profiles for a non-degenerate electron gas

If the temperature is sufficiently high, degeneracy effects play no role. In that case all space-dependent properties of the system, like the profiles of the particle density or the electric current density, follow directly from the temperature Green function, which is defined as

$$G_\beta(\mathbf{r},\mathbf{r}')=\frac{i}{2\pi}\int_0^\infty dE\,\mathrm{e}^{-\beta E}[G_{z=E+i0}(\mathbf{r},\mathbf{r}')-G_{z=E-i0}(\mathbf{r},\mathbf{r}')]\,. \quad (8)$$

The expansion (6), with (7), for the correction part of the Green function can be rewritten in terms of temperature Green functions. Subsequently, upon insertion of the infinite-domain temperature Green function[11] we find the correction part of the temperature Green function as a multiple-reflection series

$$G_\beta^{\mathrm{c}}(\mathbf{r},\mathbf{r}')=\sum_{n=1}^{\infty}G_\beta^{(n)}(\mathbf{r},\mathbf{r}')\,. \quad (9)$$

Here $G_\beta^{(n)}(\mathbf{r},\mathbf{r}')$ is a multiple integral over intermediate inverse temperatures, with a rather complicated integrand that depends on $\mathbf{r}$ and $\mathbf{r}'$. The explicit form of the integrand can be found in ref. 10.

For small fields the series (9) can be seen as a series in powers of the cyclotron frequency. In fact, for increasing n the terms $G_\beta^{(n)}(\mathbf{r},\mathbf{r}')$ in the expansion are of growing order in $\omega_{\rm c}$. For $\mathbf{r}=\mathbf{r}'$ one has for example

$$G_\beta^{(n)}(\mathbf{r},\mathbf{r}) = \begin{cases} \mathcal{O}(\omega_{\rm c}^{n-1}) & \text{if } n \text{ odd,} \\ \mathcal{O}(\omega_{\rm c}^{n}) & \text{if } n \text{ even.} \end{cases} \tag{10}$$

Having found a series expansion in $\omega_{\rm c}$ for the temperature Green function we can derive series expansions for the profiles. To evaluate the density profile up to second order in the cylotron frequency one has to evaluate the Green function up to third order in the multiple-reflection expansion. The result is

$$n(x) = n(1-e^{-\xi^2}) + \tfrac{1}{24} n\beta^2\hbar^2\omega_{\rm c}^2\xi^4\left[e^{-\xi^2} - \sqrt{\pi}\,\xi\mathrm{Erfc}(\xi)\right] , \tag{11}$$

with $\xi^2 := 2mx^2/(\beta\hbar^2)$. Likewise, we can calculate the profile of the y-component of the electric current density up to first order in $\omega_{\rm c}$:

$$J^y(x) = \frac{en\hbar\omega_{\rm c}}{4}\left(\frac{2\pi\beta}{m}\right)^{1/2}\xi^2\mathrm{Erfc}(\xi) . \tag{12}$$

As can be seen from (11) and (12) both profiles decay like Gaussians (with algebraic prefactors) for large x. The result (12) has been obtained before[2,3] from linear response theory, while both (11) and (12) have been derived recently with a perturbation method[6]. In the systematic expansion given here higher-order terms in the cyclotron frequency can be included easily, while systems with other geometries can be treated as well. It should be remarked that it was essential to take the higher-order terms in the multiple-reflection expansion into account, even if the boundaries are flat. Hence, the truncation of the series as employed in ref. 8 is not justified for a magnetized system.

4. Profiles for a degenerate electron gas

For low temperatures degeneracy effects will play a role in the profiles near the wall. We shall consider in the following the extremely degenerate case with Fermi-Dirac statistics at $T=0$. To determine the Green function for that case we return to (6) and (7). We use the fact that the Green functions are all translation invariant in the y-direction and introduce the Fourier representation

$$G_z(\mathbf{r},\mathbf{r}') = \int_{-\infty}^{\infty} dk\, \mathrm{e}^{ik(y-y')} G_z(k,x,x') . \tag{13}$$

In this representation the multiple-reflection expansion can be summed analytically, with the result

$$G_z^{\rm c}(k,x,x') = \frac{2\pi\hbar^2}{m}\left[\frac{\partial G_z^0(k,x,x'')}{\partial x''}\right]_{x''\to 0}\left\{1 - \frac{2\pi\hbar^2}{m}\left[\frac{\partial G_z^0(k,0,x'')}{\partial x''}\right]_{x''\uparrow\downarrow 0}\right\}^{-1} G_z^0(k,0,x') . \tag{14}$$

Furthermore, by solving (3) the infinite-domain Green function can be found in terms of parabolic cylinder functions[12]

$$G_z^0(k,x,x') = -\left(\frac{m}{4\pi^3\hbar^3\omega_c}\right)^{1/2}\Gamma(-\bar{z}+\tfrac{1}{2})D_{\bar{z}-1/2}(\sqrt{2}(\bar{x}-\bar{k}))D_{\bar{z}-1/2}(-\sqrt{2}(\bar{x}'-\bar{k}))\ , \tag{15}$$

for $x > x'$, and a similar expression for $x < x'$. Here we introduced the abbreviations $\bar{z} = z/\hbar\omega_c$, $\bar{x} = (m\omega_c/\hbar)^{1/2}x$ and $\bar{k} = (\hbar/m\omega_c)^{1/2}k$. If we substitute (15) into (14) we arrive at an explicit expression for the correction part of the Green function

$$\begin{aligned} G_z^c(k,x,x') &= \left(\frac{m}{4\pi^3\hbar^3\omega_c}\right)^{1/2}\Gamma(-\bar{z}+\tfrac{1}{2})\frac{D_{\bar{z}-1/2}(\sqrt{2}\bar{k})}{D_{\bar{z}-1/2}(-\sqrt{2}\bar{k})}\times \\ &\quad \times D_{\bar{z}-1/2}(\sqrt{2}(\bar{x}-\bar{k}))D_{\bar{z}-1/2}(\sqrt{2}(\bar{x}'-\bar{k}))\ , \end{aligned} \tag{16}$$

valid for all non-negative x and x'. Adding this result to (15) one obtains the complete Green function for the confined system. Incidentally, it may be remarked that once again the complete multiple-reflection expansion was needed to arrive at the correct G_z^c. Truncation of the expansion would have led to an altogether wrong expression.

The profiles of the particle density and the electric current density at temperature $T = 0$ and thermodynamical potential μ can be written as double integrals over the energy Green function $G_E = (i/2\pi)[G_{z=E+i0} - G_{z=E-i0}]$. The singularities of (15) and (16), which determine G_E, originate from the poles of the gamma function and the zeros of the parabolic cylinder function. The contributions of the former cancel exactly. The zeros are found[5,12] to lie on curves $\bar{z}_n(\bar{k})$ that tend to $n+1/2$ for $\bar{k} \to \infty$, and to ∞ for $\bar{k} \to -\infty$, while they pass through $2n+3/2$ for $\bar{k} = 0$. For intermediate values they can be obtained numerically.

Having found the singularities of G_z one gets for the profiles:

$$n_{T=0}(x) = \frac{m^{3/2}\omega_c^{3/2}}{2^{1/2}\pi^3\hbar^{3/2}}\sum_{n=0}^{N}\int_{\bar{k}_n(\bar{\mu})}^{\infty} d\bar{k}(\bar{\mu}-\bar{z}_n)^{1/2}F_n(\bar{k},\bar{x})\ , \tag{17}$$

$$J^y_{T=0}(x) = -\frac{em\omega_c^2}{2^{1/2}\pi^3\hbar}\sum_{n=0}^{N}\int_{\bar{k}_n(\bar{\mu})}^{\infty} d\bar{k}\,(\bar{x}-\bar{k})(\bar{\mu}-\bar{z}_n)^{1/2}F_n(\bar{k},\bar{x})\ . \tag{18}$$

Here we defined $\bar{\mu} = \mu/(\hbar\omega_c)$ and $N = \mathrm{Ent}[\bar{\mu}-1/2]$, while $\bar{k}_n(\bar{\mu})$ is determined from $\bar{z}_n(\bar{k}_n) = \bar{\mu}$. Furthermore, we introduced

$$F_n(\bar{k},\bar{x}) = \left|\frac{d\bar{z}_n}{d\bar{k}}\right|\left[\Gamma(-\bar{z}_n+\tfrac{1}{2})D_{\bar{z}_n-1/2}(\sqrt{2}\bar{k})D_{\bar{z}_n-1/2}(\sqrt{2}(\bar{x}-\bar{k}))\right]^2\ . \tag{19}$$

For general values of x the profiles have to be evaluated numerically. As an example we give in Figure 1 the dimensionless current profiles for a few values of μ, chosen such that only the first term in the sum contributes. (In the figure the integral in (18) is plotted as a function of $\bar{x}$.) The asymptotic form of the profiles for large x

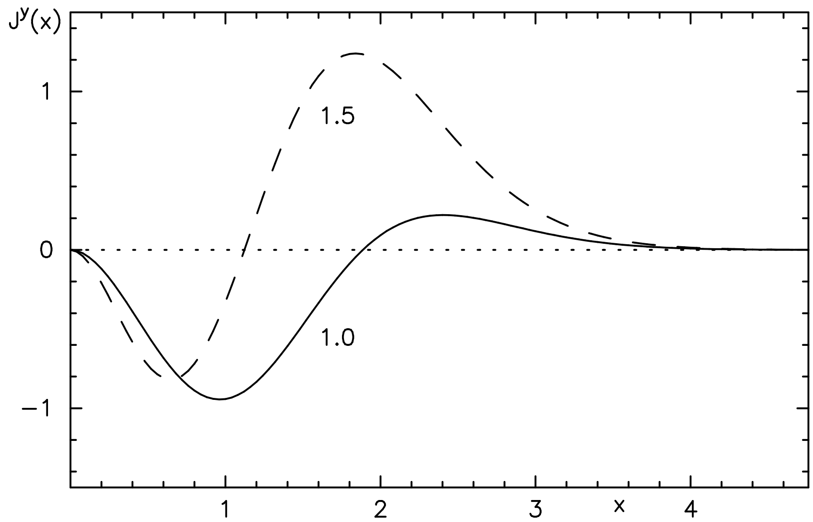

Figure. 1. The dimensionless current profiles at $T = 0$ for (a) $\bar{\mu} = 1.0$, (b) $\bar{\mu} = 1.5$.

is obtained by remarking that asymptotically the integrand is peaked around $\bar{k} = \frac{1}{2}\bar{x}$. One finds:

$$n_{T=0}(x) - n \simeq C\bar{x}^{3/2} e^{-\bar{x}^2/2} , \tag{20}$$

$$J^y_{T=0}(x) \simeq C'\bar{x}^{5/2} e^{-\bar{x}^2/2} , \tag{21}$$

with constants C and C'. Hence, once again a Gaussian decay is found, as in the non-degenerate case. This is consistent with the general result found in ref. 4.

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