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## Exclusion Statistics for Non-Abelian Quantum Hall States

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We determine the exclusion statistics properties of the fundamental edge quasiparticles over a specific  $\nu = \frac{1}{2}$  non-Abelian quantum Hall state known as the  $q = 2$  Pfaffian state. The fundamental excitations are the edge electrons of charge  $-e$  and the edge quasiholes of charge  $+e/4$ . We explicitly determine thermodynamic distribution functions and establish a duality which generalizes the duality for fractional exclusion statistics as described by F. D. M. Haldane. [S0031-9007(98)06847-1]

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Soon after the discovery of the fractional quantum Hall effect, Laughlin proposed the many body wave functions that have ever since served as the theoretical basis for the understanding of the phenomenon. It was quickly established that the fundamental quasiparticles over the Laughlin states are unusual in a number of ways: in particular, they have fractional charge and fractional braiding statistics.

The prediction of fractionally charged excitations has recently been confirmed in a number of independent experiments based on shot noise [1] and on the physics of the Coulomb blockade [2]. The fractional braiding statistics of Laughlin quasiparticles are harder to detect in a direct way. Braiding particles in a two-dimensional system is a theoretical notion that does not have a direct experimental counterpart.

To explore experimental consequences of fractional *braiding* statistics, one may as a first step try to translate this notion into a related notion of fractional *exclusion statistics*, i.e., into some generalization of the Pauli exclusion principle [3,4]. The exclusion statistics lead to a generalization of the Fermi-Dirac distribution, and from this one derives predictions for observable quantities.

In recent work [5,6], we have carried out part of the above program for the quasiparticles over the  $\nu = \frac{1}{m}$  Laughlin state. In particular, we have linked the braiding statistics of these excitations to the exclusion statistics of the edge quasiparticles in the same system. The connection is established with the help of a conformal field theory (CFT) whose conformal blocks reproduce the wave functions for the quasiparticle excitations and whose finite size spectrum coincides with the spectrum of edge excitations. The braiding statistics of excitations over the Laughlin state are Abelian with statistics angle  $\theta = \frac{\pi}{m}$  for the charge  $\pm \frac{e}{m}$  excitations and  $\theta = m\pi$  for the excitations of charge  $\pm e$  [7]. In [5,6] we established that the exclusion statistics of edge electrons (of charge  $-e$ ) and edge quasiholes (of charge  $+\frac{e}{m}$ ) are given by fractional exclusion statistics in the sense of Haldane [3], with statistics parameter  $g = m$  and  $g = \frac{1}{m}$ , respectively, and no mutual statistics between the two. This result was

established by a method based on recursion relations for truncated conformal spectra [5].

For a quantum Hall state with fundamental charged edge excitations of charge  $q$  (measured in units of  $e$ ) and distribution function  $n(\epsilon)$  (generalizing the Fermi-Dirac distribution), the zero temperature Hall conductance is quickly shown to be

$$\sigma_H = q^2 n^{\max} \frac{e^2}{h}, \quad (1)$$

with  $n^{\max}$  the maximum (reached for  $\epsilon \rightarrow -\infty$ ) of  $n(\epsilon)$ . For Haldane  $g$ -ons one has  $n_g^{\max} = \frac{1}{g}$  and this gives the familiar result  $\sigma_H = \frac{1}{m} \frac{e^2}{h}$ , both for the edge electrons and the edge quasiholes over the  $\nu = \frac{1}{m}$  Laughlin state. In this quasiparticle formalism, the fact that the Hall conductance does not depend on temperature comes about as a consequence of the duality between the statistics  $g$  and  $\frac{1}{g}$  distribution functions [8].

In the present Letter, we extend the above reasoning to the case of a quantum Hall state (the so-called  $q = 2$  Pfaffian state) with excitations that obey non-Abelian braiding statistics. In this context, “non-Abelian” means that the wave function for a set of quasiparticle excitations at given positions has more than one component, and that the action of braiding (exchanging) two of those excitations is represented by a nondiagonal matrix acting on the multi-component wave function.

The issue addressed in this Letter is a fundamental one: we are exploring the consequences for thermodynamics and transport of the non-Abelian braiding statistics of a set of fundamental excitations in a condensed matter system. To our knowledge, a similar problem has not been addressed until now [9]. As a characteristic feature of “non-Abelian exclusion statistics,” we shall identify a nontrivial prefactor in the high-energy Boltzmann tail of a generalized distribution function.

After the experimental observation [10] of a quantum Hall effect at the even denominator filling fraction  $\nu = \frac{5}{2}$ , a number of candidate ground states with filling fraction  $\nu = \frac{1}{2}$  have been proposed. One of these is the so-called

$q = 2$  Pfaffian state [7,11] with wave function

$$\Psi_{\text{Pf}} = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i>j} (z_i - z_j)^2 \prod_i e^{-(|z_i|^2/4\ell^2)} \quad (2)$$

with  $\text{Pf} \left( \frac{1}{z_i - z_j} \right) = \mathcal{A} \left( \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \dots \right)$  denoting the antisymmetrized product over pairs of electrons and with  $\ell = \sqrt{\frac{\hbar c}{eB}}$  the magnetic length. Following [7], one may interpret this state as a conformal block of  $n$  chiral vertex operators in a  $c = \frac{3}{2}$  CFT. In the same manner, the wave functions for electron and quasihole excitations over the Pfaffian state are obtained from conformal blocks with the appropriate insertions of electron and quasihole operators. To write these operators, we represent the CFT as a product of a  $c = \frac{1}{2}$  (Ising) CFT with Majorana fermion  $\psi$  and spin operator  $\sigma$  and a  $c = 1$  Gaussian theory with scalar field  $\varphi$ . In terms of these fields, the electron operator  $G(z)$  (of charge  $-e$ ) and the quasihole operator  $\phi(z)$  (of charge  $+\frac{e}{4}$ ) take the following form:

$$G(z) \equiv (\psi e^{-i\sqrt{2}\varphi})(z), \quad \phi(z) \equiv (\sigma e^{i(1/2\sqrt{2})\varphi})(z). \quad (3)$$

The electron and quasihole operators play a double role in the theory. Via their conformal blocks they give bulk wave functions for excited states. At the same time, they act as spectrum generating fields in the edge CFT. In terms of statistics, the implications of the underlying CFT are again twofold. On the one hand, the braiding statistics in the CFT directly translate into braiding statistics of electron and quasihole (bulk) excitations. On the other hand, the exclusion statistics properties of the electron and quasihole operators (see below) dictate the thermodynamic (and, to some extent, transport) properties of the low-energy edge theories. For the Pfaffian state, the (non-Abelian) braiding statistics were studied in great detail in [7,12]. The study of the exclusion statistics aspects is the subject of this Letter.

The statistics of quasiparticles over the  $\nu = \frac{1}{m}$  Laughlin states are Abelian and can be inferred from the properties of two-particle states. As such, they can be studied by inspecting the short distance operator product expansions (OPE) of the electron and quasihole fields. The OPE's lead to Abelian braiding statistics, with angles  $m\pi$  and  $\frac{\pi}{m}$  for the electrons and quasiholes, respectively. The corresponding exclusion statistics are of the type proposed by Haldane, with statistics parameters  $g = m, g = \frac{1}{m}$ , respectively [6]. The OPE's for the electron and quasihole edge excitations over the the Pfaffian state have a more complicated structure, and it is no longer possible to derive the statistics from here. Indeed, the non-Abelian nature of the braiding statistics of the quasihole excitations becomes apparent only at the four-particle level.

In [12], it was argued that the wave-function specifying a collection of  $2n$  quasiholes (at given locations) over the Pfaffian state has  $2^{n-1}$  independent components. In the thermodynamic limit, this degeneracy amounts to

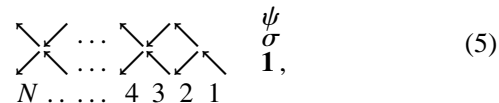
a degeneracy of  $\sqrt{2}$  per quasihole. This noninteger degeneracy factor, which ends up as a prefactor in the Boltzmann tail of the generalized distribution function, is a manifestation of the non-Abelian nature of the quasihole exclusion statistics.

To derive quantitative results for the exclusion statistics of the edge excitations over the Pfaffian state, we employ the method of truncated conformal spectra [5]. The idea of this method is to establish an explicit quasiparticle basis of the CFT chiral finite size spectrum, and from there derive the statistics. In [6], this procedure was carried out in great detail for the Laughlin states and for the charged sector for more general hierarchical quantum Hall states (Jain series).

The complication that is new for the case of non-Abelian quantum Hall states is the occurrence of so-called fusion channels in the specification of a multi- $\phi$  state. These have their origin in the following fusion rule for the spin field of the Ising CFT:

$$[\sigma] \cdot [\sigma] = [\mathbf{1}] + [\psi]. \quad (4)$$

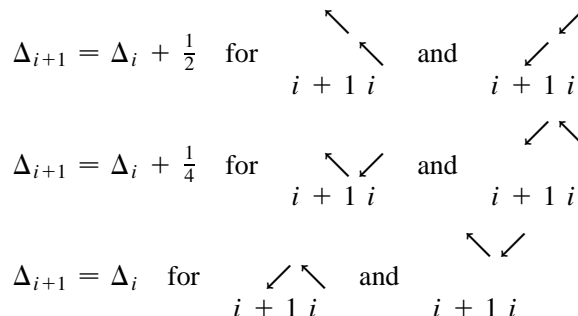
Each time we apply a  $\phi$  mode to a state containing an odd number of  $\phi$  quanta, we have to specify a choice for either the first or the second term on the right-hand side of (4). For a state containing  $N$  quasiholes, the total fusion path is encoded in a Bratteli diagram, which is an arrowpath on the following grid:



with the vertical direction indicating the Ising sectors  $\mathbf{1}, \sigma,$  and  $\psi$  and the horizontal index  $i = 1, \dots, N$  labeling the  $\phi$  modes. The starting point of an arrow path is the CFT vacuum  $|0\rangle$ . The multi- $\phi$  states for a given fusion path are [with  $\phi(z) = \sum_s \phi_{-s} z^{s-1/8}$ ]

$$\phi_{-1/8-\Delta_N-n_N} \dots \phi_{-1/8-\Delta_2-n_2} \phi_{-1/8-\Delta_1-n_1} |0\rangle, \quad (6)$$

with integers  $n_N \geq \dots n_2 \geq n_1 \geq 0$  and with increments  $\Delta_i$  determined by the Bratteli diagram according to



with  $\Delta_1 = 0$ . For example, the lowest energy four-quasihole states for the two possible fusion paths are

$$\begin{aligned} \phi_{-3/8} \phi_{-3/8} \phi_{-1/8} \phi_{-1/8} |0\rangle, \quad E = 1, \\ \phi_{-11/8} \phi_{-7/8} \phi_{-5/8} \phi_{-1/8} |0\rangle, \quad E = 3. \end{aligned} \quad (7)$$

In the construction of multielectron states the issue of fusion rules does not explicitly arise, but the details of the rules are different from what we had for the Laughlin states [6]. On the vacuum  $|0\rangle$ , we allow the following multielectron states [with  $G(z) = \sum_l G_{-l} z^{l-3/2}$ ]:

$$G_{-(1/2)-m_M} \dots G_{-(1/2)-m_2} G_{-\frac{1}{2}-m_1} |0\rangle \quad (8)$$

with integers  $m_i$  satisfying

$$m_i \geq \max(m_{i-1} + 1, m_{i-2} + 4), \quad (9)$$

with  $m_0 = -2$ ,  $m_{-1} = -3$ . For example, the lowest energy multi- $G$  states over  $|0\rangle$  are of the form

$$\dots G_{-13/2} G_{-11/2} G_{-5/2} G_{-3/2} |0\rangle, \quad (10)$$

with increments alternating between 1 and 3. It is quickly seen that the number of states allowed by these rules is larger than for the case with  $g = 2$ , where the construction rule is  $m_i \geq m_{i-1} + 2$ .

We are now ready to specify a complete set of independent multiparticle states. We start by constructing a tower of states over  $|0\rangle$  (which we shall write as  $|0, \mathbf{1}\rangle$  from now on) by acting independently with  $\phi$  and  $G$  modes according to the rules that we specified. We then introduce five additional reference states for which we repeat the procedure. The lowest  $\phi$  and  $G$  modes that are allowed on each of the reference states are specified in Table I.

The central claim of this Letter is that the union of all the quasiparticle states in all six sectors precisely forms a basis of the chiral Hilbert space of the CFT, and thereby (up to certain projections, see [6]) of physical edge theories. This is a mathematical result which, however, has a clear physical interpretation in terms of exclusion statistics (see below). The chiral spectrum of the CFT for the Pfaffian state is usually understood in terms of representations of a  $U(1)$  superaffine (Kac-Moody) current algebra. The new point of view that we advocate here is very different and directly refers to the charge carrying excitations in the system. Using MAPLE, we have checked that the character identities that are implied by our claim are satisfied to the first 25 nontrivial orders in  $q$ .

The absence of any interference between the construction rules for the  $\phi$  and the  $G$  quanta means that there is

TABLE I. Reference states for the quasiparticle construction of the chiral Hilbert space over the  $q = 2$  Pfaffian state. The creation operators  $\phi_{-s}$  and  $G_{-l}$  act on these states according to the exclusion rules specified in the text. The symbols  $\mathbf{1}$ ,  $\psi$ , and  $\sigma$  denote Ising sectors and the quantities  $\Delta_1$  and  $m_0, m_{-1}$  determine the lowest  $\phi$  and  $G$  modes that are allowed.

State	Charge	Energy	$\Delta_1$	$m_0, m_{-1}$
$ 0, \mathbf{1}\rangle$	0	0	0	-2, -3
$ 0, \psi\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$	-1, -4
$ -\frac{e}{4}, \sigma\rangle$	$-\frac{e}{4}$	$\frac{1}{8}$	$\frac{3}{4}$	-1, -3
$ -\frac{e}{2}, \mathbf{1}\rangle$	$-\frac{e}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	-1, -2
$ -\frac{e}{2}, \psi\rangle$	$-\frac{e}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	0, -3
$ -\frac{3e}{4}, \sigma\rangle$	$-\frac{3e}{4}$	$\frac{5}{8}$	$\frac{1}{2}$	0, -2

no mutual exclusion statistics between the two. This is a first sign of a (generalized) particle-hole duality between the two types of quasiparticles. We shall make this duality precise in what follows.

Having specified the quasiparticle basis we can implement the procedure of [5] and derive the thermodynamics. For the  $\phi$  modes, we define truncated and projected partition sums  $X_l(x, q)$  and  $Y_l(x, q)$ ,  $l = \frac{1}{8}, \frac{3}{8}, \dots$ , as expressions of the form

$$\text{trace}_{\leq l}(x^{N_\phi} q^E), \quad (11)$$

with  $N_\phi$  the number of  $\phi$  modes,  $E$  the dimensionless energy,  $x = e^{\beta\mu_{\text{qh}}}$  and  $q = e^{-\beta(2\pi/L\rho_0)}$  with  $\rho_0$  the density of states per unit length. The subscript  $\leq l$  indicates that the highest occupied mode  $\phi_{-s}$  is such that  $l - s$  is a nonnegative integer.  $X_l$  ( $Y_l$ ) is further restricted by requiring that the total charge be an even (odd) multiple of  $\frac{e}{4}$ . The rules specified above lead to the recursion relations

$$\begin{aligned} X_l &= X_{l-1} + xq^l(Y_l + Y_{l-1/2}), \\ Y_l &= Y_{l-1} + xq^l X_{l-1/4}. \end{aligned} \quad (12)$$

The initial conditions, which are sector dependent, follow from the information in Table I. The solution of (12) can be approximated as  $X_l \sim \prod_{i=1}^l \lambda_+(q^i x)$ , with  $\lambda_+(z_{\text{qh}})$  the largest real solution of the equation

$$(\lambda - 1)(\lambda^{1/2} - 1) = z_{\text{qh}}^2 \lambda^{5/4}. \quad (13)$$

This leads to a distribution function

$$n_{\text{qh}}(\epsilon) = [z_{\text{qh}} \partial_{z_{\text{qh}}} \log(\lambda_+)](z_{\text{qh}} = e^{\beta(\mu_{\text{qh}} - \epsilon)}) \quad (14)$$

for the quasihole excitations.

For the  $G$  quanta, the above rules are easily converted into a recursion relation for the quantity  $\Omega_l(y, q)$ , which is the trace of  $y^{N_G} q^E$ , with  $N_G$  the number of  $G$  quanta and  $y = e^{\beta\mu_e}$ , over all multi- $G$  states with modes  $\leq l - \frac{1}{2}$ . We find

$$\begin{aligned} \Omega_l &= \Omega_{l-1} + yq^{l-(1/2)}\Omega_{l-2} + y^2 q^{2l-2}\Omega_{l-4} \\ &\quad - y^3 q^{3l-(13/2)}\Omega_{l-6}. \end{aligned} \quad (15)$$

The corresponding distribution function is

$$n_e(\epsilon) = [z_e \partial_{z_e} \log(\mu_+)](z_e = e^{\beta(\mu_e - \epsilon)}), \quad (16)$$

with  $\mu_+(z_e)$  the largest real solution of

$$(\mu^4 - z_e^2)(\mu^2 - z_e) = \mu^5. \quad (17)$$

For temperature  $T = 0$ , the distribution functions  $n_{\text{qh}}(\epsilon)$  and  $n_e(\epsilon)$  are step-down functions with maximum  $n_{\text{qh}}^{\text{max}} = 8$  and  $n_e^{\text{max}} = \frac{1}{2}$ . At finite  $T$ , the asymptotic behavior for  $\epsilon \rightarrow \infty$  is  $n_{\text{qh}} \sim \sqrt{2} e^{-\beta\epsilon}$  and  $n_e \sim e^{-\beta\epsilon}$ . We stress that the prefactor  $\sqrt{2}$  is a direct manifestation of the non-Abelian nature of the exclusion statistics of the quasiholes. As an easy consequence of the above formulas, we establish the duality relation

$$2n_e(\epsilon) = 1 - \frac{1}{8} n_{\text{qh}}\left(-\frac{\epsilon}{4}\right), \quad (18)$$

provided  $\mu_{\text{qh}} = -\frac{1}{4}\mu_e$ . This result generalizes the duality for Haldane's fractional exclusion statistics [8]. The duality allows us to interpret the spectrum in different ways. One picture has quasiholes with energies ranging from  $-\infty$  to  $\infty$ , with the CFT vacuum and other reference states represented as filled seas of quasiholes. An alternative is a picture based on edge electrons, and then there is the "mixed" picture that we started from, with  $\phi$  and  $G$  quanta of positive energies only.

In the mixed picture, the total central charge  $c = \frac{3}{2}$  of the edge CFT is established as the sum of a contribution  $c_{\text{qh}}^+$  from the quasiholes and a piece  $c_e^+$  from the electrons. Using MAPLE, we evaluated

$$c_{\text{qh}}^+ = 1.059\dots, \quad c_e^+ = 0.440\dots \quad (19)$$

To compute the Hall conductance, we insert  $q_{\text{qh}} = \frac{1}{4}$  and  $n_{\text{qh}}^{\text{max}} = 8$  or  $q_e = -1$  and  $n_e^{\text{max}} = \frac{1}{2}$  into (1), giving  $\sigma_H = \frac{1}{2} \frac{e^2}{h}$  as expected. The fact that this result is independent of temperature is guaranteed by the duality relation (18). We refer to [6] for the details of this way of computing and for further applications of the quasiparticle formalism.

In conclusion, we have established an appropriate generalization of the Pauli principle for a situation where electrons and quasiholes obey non-Abelian braiding statistics. We have also demonstrated a particle-hole duality in this non-Abelian theory. In a future publication, we shall give a more extensive treatment of exclusion statistics in CFT's with non-Abelian braiding and give more examples. These will include the so-called parafermionic quantum Hall states, higher level Wess-Zumino-Witten theories, and a spinon representation for the  $\text{SO}(5)_1$  CFT, which is relevant for  $\text{SO}(5)$  invariant ladders models for correlated electrons.

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*Note added.*—The results presented here are easily generalized to the more general Pfaffian states with factor  $(z_i - z_j)^q$  in (2). Equations (13) and (17) become

$$\begin{aligned} (\lambda - 1)(\lambda^{1/2} - 1) &= z_{\text{qh}}^2 \lambda^{(3q-1)/(2q)} \\ (\mu^{2q} - z_e^2)(\mu^q - z_e) &= \mu^{3q-1}. \end{aligned} \quad (20)$$

and the duality relation reads

$$qn_e(\epsilon) = 1 - \frac{1}{4q} n_{\text{qh}} \left( -\frac{\epsilon}{2q} \right). \quad (21)$$

After submitting this manuscript, we learned that S. B. Isakov (private communication; to be published) has determined the exclusion statistics for *bulk* quasiholes over the general  $q$  Pfaffian state, and found results that are identical to those presented here for *edge* quasiholes.

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