Appendix to Behavioral Learning Equilibria
Appendix on the New Keynesian Philips Curve with Infinite Horizon Learning
(not for publication)

Cars Hommes[a], Mei Zhu[b,a] *

[a] CeNDEF, School of Economics, University of Amsterdam

Roetersstraat 11, 1018 WB Amsterdam, Netherlands

[b] Institute for Advanced Research, Shanghai University of Finance and Economics,

and the Key Laboratory of Mathematical Economics(SUFE), Ministry of Education, Shanghai 200433, China

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* *E-mail addresses:* C.H.Hommes@uva.nl, M.Zhu@uva.nl

1 Microfoundation of the New Keynesian Philips Curve

In the paper *Behavioral Learning Equilibria*, Hommes and Zhu (2013), one of the applications is the New Keynesian Philips Curve (NKPC). In the paper the NKPC with Euler Equation (EE) learning is investigated. In this appendix we consider the NKPC with Infinite Horizon (IH) learning. We first derive the NKPC from microfoundations with monopolistic competition and staggered price setting (e.g. Woodford, 2003) and then introduce IH-learning following Preston (2005).

There is a continuum of firms indexed by $i \in [0, 1]$. Each firm produces a differentiated good, but they all use the same technology which uses labor as the only factor of production. The demand curve for the good produced by firm i is given by

$$Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\eta_t}, \tag{1.1}$$

where Y_t is the aggregator function defined as $Y_t = [\int_0^1 Y_t(i)^{(\eta_t-1)/\eta_t} di]^{\eta_t/(\eta_t-1)}$, P_t is the aggregate price level defined as $P_t = [\int_0^1 P_t(i)^{1-\eta_t} di]^{1/(1-\eta_t)}$, and η_t is the elasticity of substitution among goods which varies over time according to some stationary stochastic process.

Aggregate price dynamics

Following Calvo (1983) we assume that in every period only a fraction $(1-\omega)$ of firms are able to reset their prices, while a fraction ω keep their price unchanged. In such an environment the aggregate price dynamics are described by

$$P_t = (\omega P_{t-1}^{1-\eta_t} + (1-\omega)(P_t^*)^{1-\eta_t})^{\frac{1}{1-\eta_t}}, \tag{1.2}$$

where P_t^* is the price set in period t by firms reoptimizing their price in that period. Notice that, as shown below, all firms will set the same price since they face an identical problem.

Optimal price setting

We assume that firms have the same subjective beliefs, denoted by $\widetilde{E}_t$, and that each firm hires labor from the same integrated economy-wide labor market. Therefore, all firms face the same optimization problem and they will set the same price when reoptimizing.

A firm reoptimizing in period t will choose the price P_t^* to maximize expected discounted profits, which are given by

$$\max_{P_t^*} \widetilde{E}_t \sum_{s=0}^{\infty} \omega^s \Lambda_{t,t+s} \left(\frac{P_t^*}{P_{t+s}} - \Phi_{t+s} \right) \left(\frac{P_t^*}{P_{t+s}} \right)^{-\eta_{t+s}} Y_{t+s},$$

where $\Lambda_{t,t+s}$ is the stochastic discount factor and Φ_t are real marginal costs of production. The stochastic discount factor is defined as $\Lambda_{t,t+s} = \delta^s (Y_{t+s}/Y_t)^{-\sigma}$, where δ is the time discount factor.

The first-order condition associated with the problem above is given by

$$\frac{P_t^*}{P_t} = \frac{\widetilde{E}_t \sum_{s=0}^{\infty} \omega^s \Lambda_{t,t+s} \Phi_{t+s} \eta_{t+s} \left(\frac{P_{t+s}}{P_t} \right)^{\eta_{t+s}}}{\widetilde{E}_t \sum_{s=0}^{\infty} \omega^s \Lambda_{t,t+s} (\eta_{t+s} - 1) \left(\frac{P_{t+s}}{P_t} \right)^{\eta_{t+s}-1}}. \tag{1.3}$$

Define $Q_t^* = P_t^*/P_t$ and log-linearize Eq. (1.3) around a zero inflation steady state to get

$$\widehat{q}_t^* = \widetilde{E}_t \sum_{s=0}^{\infty} (\omega\delta)^s \left((1-\omega\delta)\widehat{\varphi}_{t+s} + \frac{1-\omega\delta}{1-\eta} \widehat{\eta}_{t+s} + \omega\delta\widehat{\pi}_{t+s+1} \right), \tag{1.4}$$

where "hatted" lower case letters denote log-deviations from steady state, η is the mean of the stochastic process $\{\eta_t\}$, and $\widehat{\pi}_t$ is the inflation rate.

Log-linearizing Eq. (1.2) around the zero inflation steady state we get

$$\widehat{q}_t^* = \frac{\omega}{1-\omega} \widehat{\pi}_t. \tag{1.5}$$

Combining Eqs. (1.4) and (1.5), and dropping hats for notational convenience, we get

$$\pi_t = \widetilde{E}_t \sum_{s=0}^{\infty} (\omega\delta)^s (\delta(1-\omega)\pi_{t+s+1} + \gamma\varphi_{t+s} + \lambda\eta_{t+s}), \tag{1.6}$$

where $\gamma = \frac{(1-\omega\delta)(1-\omega)}{\omega}$ and $\lambda = \frac{\gamma}{1-\eta}$ are functions of the structural parameters. Here $\widetilde{E}_t$ are *subjective expectations* and inflation depends on infinite horizon subjective expectations over a discounted sum of future inflation, marginal costs and a noise term.

Under rational expectations, the infinite horizon setup of the NKPC can be reduced to a temporary equilibrium equation with only a one-period ahead inflation forecast:

$$\pi_t = \delta \widetilde{E}_t \pi_{t+1} + \gamma\varphi_t + u_t, \tag{1.7}$$

where $u_t = \lambda\eta_t$. Eq. (1.7) is derived by leading (1.6) one period, taking (rational) conditional expectations and using the law of iterated expectations for conditional expectations, i.e. $E_t E_{t+1}(\cdot) \equiv E_t(\cdot)$; see e.g. Preston, 2005 and Massaro, 2013).

The NKPC under Euler equation (EE) learning is obtained by substituting the adaptive learning rule into the temporary equilibrium aggregate equations (1.7). Euler equations for the current period then determine the behavioral rule of boundedly rational agents describing current decisions as functions of subjective one-period ahead expectations; see Evans and Honkapohja (2006) for a more detailed discussion.

2 The NKPC under infinite horizon learning

Preston (2005) studies the NKPC under Infinite Horizon (IH) learning, where individual decision rules depend on infinite horizon forecasts. Here, we derive the IH version of SAC-learing in the NKPC. As shown below, this leads to a somewhat different system than under EE-learning. Nevertheless, we will show that the IH-learning dynamics is similar to EE-learning and, e.g., also exhibits co-existence of multiple BLE.

Consider the infinite horizon NKPC in (1.6) and assume that agents use AR(1) forecasting rules for inflation and marginal costs. The s-period ahead subjective forecast for the endogenous variable, inflation, is given as

$$\widetilde{E}_t \pi_{t+s+1} = \alpha + \beta^{2+s}(\pi_{t-1} - \alpha). \tag{2.1}$$

We assume that agents know that marginal costs (output gap) follows an exogenous AR(1) process, $\varphi_t = a + \rho\varphi_{t-1} + \varepsilon_t$. Assuming that agents observe φ_t and know the parameters of the exogenous process[1], the s-period ahead subjective forecast for marginal costs becomes

$$\widetilde{E}_t \varphi_{t+s} = \bar{a} + \rho^s(\varphi_t - \bar{a}), \tag{2.2}$$

where $\bar{a} = a/(1-\rho)$ is the long run average. Substituting these s-period ahead subjective

[1]If agents do not observe φ_t, but just forecast based on the observed data up to $t-1$, the s-period ahead subjective forecast for marginal costs becomes $\widetilde{E}_t \varphi_{t+s} = \bar{a} + \rho^{s+1}(\varphi_{t-1} - \bar{a})$. Following the same steps as below, it can be shown that inflation π_t satisfies $\pi_t = \bar{\delta}(\alpha + h(\beta)\beta^2(\pi_{t-1} - \alpha)) + \bar{\gamma}(\bar{a} + h(\rho)\rho(\varphi_{t-1} - \bar{a})) + u_t$, slightly different from (2.4). The first-order autocorrelation coefficient $F(\beta)$ then satisfies $F(\beta) = \frac{\bar{\gamma}^2 h(\rho)^2\rho^2(\bar{\delta}h(\beta)\beta^2+\rho)+\bar{\delta}h(\beta)\beta^2(1-\rho^2)(1-\bar{\delta}h(\beta)\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{\bar{\gamma}^2 h(\rho)^2\rho^2(\bar{\delta}h(\beta)\beta^2\rho+1)+(1-\rho^2)(1-\bar{\delta}h(\beta)\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}$, slightly different from (2.5). However, the following results on behavioral learning equilibria (existence and possibility of multiple equilibria) as well as the stability of the SAC-learning dynamics still hold.

forecasts in the infinite horizon NKPC (1.6) we obtain

$$
\begin{aligned}
\pi_t &= \delta(1-\omega)\sum_{s=0}^{\infty}(\omega\delta)^s\widetilde{E}_t\pi_{t+s+1} + \gamma\sum_{s=0}^{\infty}(\omega\delta)^s\widetilde{E}_t\varphi_{t+s} + \lambda\sum_{s=0}^{\infty}(\omega\delta)^s\widetilde{E}_t\eta_{t+s} \\
&= \delta(1-\omega)\sum_{s=0}^{\infty}(\omega\delta)^s(\alpha+\beta^{2+s}(\pi_{t-1}-\alpha)) + \gamma\sum_{s=0}^{\infty}(\omega\delta)^s(\bar{a}+\rho^s(\varphi_t-\bar{a})) + \lambda\eta_t \\
&= \delta(1-\omega)(\frac{\alpha}{1-\omega\delta} + \frac{\beta^2}{1-\omega\delta\beta}(\pi_{t-1}-\alpha)) + \gamma(\frac{\bar{a}}{1-\omega\delta} + \frac{1}{1-\omega\delta\rho}(\varphi_t-\bar{a})) + \lambda\eta_t \\
&= \delta\frac{1-\omega}{1-\omega\delta}(\alpha + \frac{1-\omega\delta}{1-\omega\delta\beta}\beta^2(\pi_{t-1}-\alpha)) + \frac{\gamma}{1-\omega\delta}(\bar{a} + \frac{1-\omega\delta}{1-\omega\delta\rho}(\varphi_t-\bar{a})) + \lambda\eta_t.
\end{aligned}
\tag{2.3}
$$

We rewrite as

$$
\begin{aligned}
\pi_t &= \bar{\delta}(\alpha + h(\beta)\beta^2(\pi_{t-1}-\alpha)) + \bar{\gamma}(\bar{a} + h(\rho)(\varphi_t-\bar{a})) + \lambda\eta_t, \\
&=: \bar{\delta}(\alpha + h(\beta)\beta^2(\pi_{t-1}-\alpha)) + \bar{\gamma}(\bar{a} + h(\rho)(\varphi_t-\bar{a})) + u_t,
\end{aligned}
\tag{2.4}
$$

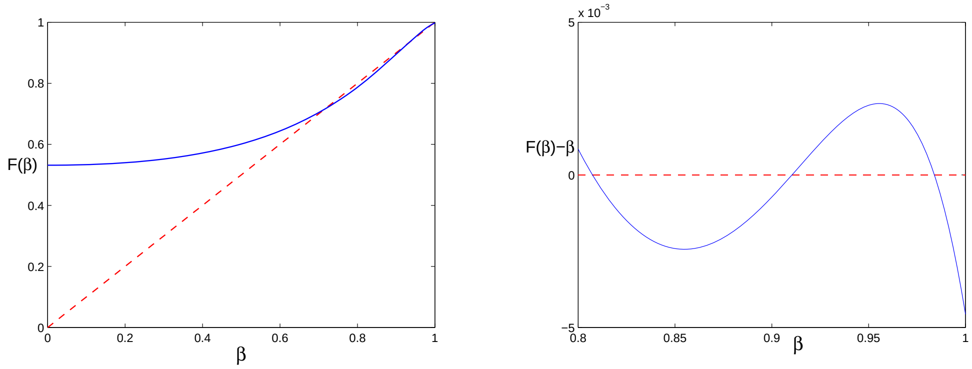
where $\bar{\delta} = \delta(1-\omega)/(1-\omega\delta) \in (0,1)$, $\bar{\gamma} = \gamma/(1-\omega\delta) = (1-\omega)/\omega > 0$ and $h(x) = (1-\omega\delta)/(1-\omega\delta x) \in (0,1)$, $0 \le x \le 1$. Hence, using an AR(1) forecasting rule in an infinite horizon NKPC, we obtain a (somewhat) different ALM than Eq. (1.7) under EE learning. The EE and IH learning are similar however, and one would therefore expect similar results for the behavioral learning equilibria as well as the SAC-learning dynamics. We prove below that under IH learning at least one SCEE exists, that multiple SCEE may co-exist and that the stability conditions for SAC-learning are similar.

Following the same method as in Appendix C in Hommes and Zhu (2013) the first-order autocorrelation coefficient $F(\beta)$ for the ALM (2.4) is given by

$$
\begin{aligned}
F(\beta) &= \bar{\delta}h(\beta)\beta^2 + \frac{\bar{\gamma}^2h(\rho)^2\rho(1-\bar{\delta}^2h(\beta)^2\beta^4)}{\bar{\gamma}^2h(\rho)^2(\bar{\delta}h(\beta)\beta^2\rho+1) + (1-\rho^2)(1-\bar{\delta}h(\beta)\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} \\
&= \frac{\bar{\gamma}^2h(\rho)^2(\bar{\delta}h(\beta)\beta^2+\rho) + \bar{\delta}h(\beta)\beta^2(1-\rho^2)(1-\bar{\delta}h(\beta)\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{\bar{\gamma}^2h(\rho)^2(\bar{\delta}h(\beta)\beta^2\rho+1) + (1-\rho^2)(1-\bar{\delta}h(\beta)\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}.
\end{aligned}
\tag{2.5}
$$

Note that $h(\beta)\beta^2|_{\beta=0} = 0,\ h(\beta)\beta^2|_{\beta=1} = 1$. Thus

$$
F(0) = \frac{\bar{\gamma}^2h(\rho)^2\rho}{\bar{\gamma}^2h(\rho)^2 + (1-\rho^2)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} > 0,
$$



(a) (b)

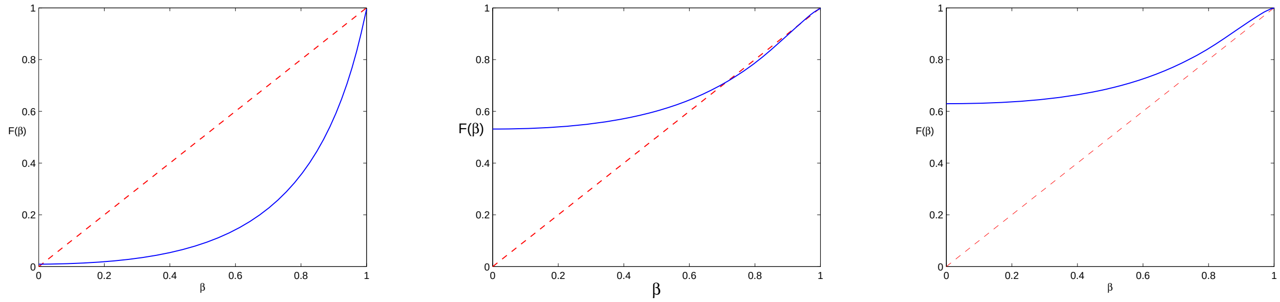
Figure 1: Three co-existing BLE. Two stable BLE $\beta_1^* = 0.8074$ and $\beta_3^* = 0.984$, separated by an unstable BLE $\beta_2^* = 0.9104$. Parameters: $\rho = 0.91, \sigma_u = 0.003162, \sigma_\varepsilon = 0.01, \omega = 0.813, \delta = 0.99$.

and

$$
\begin{aligned}
F(1) &= \frac{\bar{\gamma}^2 h(\rho)^2(\bar{\delta}+\rho) + \bar{\delta}(1-\rho^2)(1-\bar{\delta}\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{\bar{\gamma}^2 h(\rho)^2(\bar{\delta}\rho+1) + (1-\rho^2)(1-\bar{\delta}\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} - 1 \\
&= \frac{-\bar{\gamma}^2 h(\rho)^2(1-\bar{\delta})(1-\rho) - (1-\bar{\delta})(1-\rho^2)(1-\bar{\delta}\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{\bar{\gamma}^2 h(\rho)^2(\bar{\delta}\rho+1) + (1-\rho^2)(1-\bar{\delta}\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} \\
&< 0.
\end{aligned}
$$

Therefore, there exists at least one β^* such that $F(\beta^*) = \beta^*$. Moreover, following the proof in Appendix F, we obtain the same property as in Proposition 4, that the SAC-learning dynamics is (locally) stable if $F'(\beta^*) < 1$.

The following numerical simulation shows that multiple equilibria may co-exist. Figure 1 illustrates the co-existence of three BLE. In order to see the three BLE more clearly, in Figure 1b we plot the graph of $F(\beta) - \beta$ for $\beta \in [0.8, 1]$. Also, as in the case of EE learning, as ρ or δ increases, $F(\beta)$ tends to move upwards and hence under IH learning the system may move from a unique low persistence BLE, to three co-exisiting BLE and then to a unique high persistence BLE as ρ or δ increases, as illustrated in Figure 2. In this sense, in the NKPC the dynamics under IH-learning are similar to the dynamics under EE-learning.



(a) $\rho = 0.5$ (b) $\rho = 0.91$ (c) $\rho = 0.92$

Figure 2: BLE β^*, for different values of ρ. (a) unique low persistence BLE $\beta^* = 0.0352$ for $\rho = 0.5$; (b) three co-existing BLE for $\rho = 0.91$ (as in Figure 1); (c) unique high persistence BLE $\beta^* = 0.9911$ for $\rho = 0.92$. Other parameters: $\sigma_u = 0.003162, \sigma_\varepsilon = 0.01, \omega = 0.813, \delta = 0.99$.

References

[1] Calvo, G., 1983. Staggered contracts in a utility-maximizing framework, *Journal of Monetary Economics*, 12, 383–398.

[2] Evans, G.W., Honkapohja, S., 2006. Monetary policy, expectations and commitment, *Scandinavian Journal of Economics* 108, 15-38.

[3] Massaro, D., 2013. Heterogeneous expectations in monetary DSGE models, *Journal of Economic Dynamics & Control*, 37(3), 680-692.

[4] Preston, B., 2005. Learning About Monetary Policy Rules When Long-Horizon Expectations Matter, *International Journal of Central Banking* 1(2): 81-126.

[5] Woodford, M., 2003. Interest and Prices: Foundations of a Theory of Monetary Policy, Princeton University Press, Princeton, NJ.