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Numerical bifurcation analysis of distance-dependent on-center off-surround shunting neural networks

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Abstract. On-center off-surround shunting neural networks are often applied as models for content-addressable memory (CAM), the equilibria being the stored memories. One important demand of biological plausible CAMs is that they function under a broad range of parameters, since several parameters vary due to postnatal maturation or learning. ELLIAS, COHEN and GROSSBERG have put much effort into showing the stability properties of several configurations of on-center off-surround shunting neural networks. In this article we present numerical bifurcation analysis of distance-dependent on-center off-surround shunting neural networks with fixed external input. We varied four parameters that may be subject to postnatal maturation: the range of both excitatory and inhibitory connections and the strength of both inhibitory and excitatory connections. These analyses show that fold bifurcations occur in the equilibrium behavior of the network by variation of all four parameters. The most important result is that the number of activation peaks in the equilibrium behavior varies from one to many if the range of inhibitory connections is decreased. Moreover, under a broad range of the parameters the stability of the network is maintained. The examined network is implemented in an ART network, Exact ART, where it functions as the classification layer F2. The stability of the ART network with the F2-field in different dynamic regimes is maintained and the behavior is functional in Exact ART. Through a bifurcation the learning behavior of Exact ART may even change from forming local representations to forming distributed representations.

1 Introduction

On-center off-surround shunting neural networks (SNN) can function as content-addressable memory (CAM). In a CAM a pattern is transformed and stored by competitive neural populations. The equilibria are the stored memories. An extensively examined property of CAMs is the stability of equilibrium behavior under a wide range of parameters that may vary slowly, but in an unpredictable way, during the maturation or learning process. Asymptotic stability of the equilibria under a wide range of parameter values is considered as necessary for fault tolerance and robustness. GROSSBERG (1973), ELLIAS and GROSSBERG (1975), and COHEN and GROSSBERG (1983) have proven this to be the case in on-center off-surround SNNs, although, with specific parameter and input configurations, persistent stable oscillations may also occur. COHEN (1988, 1990) proves the latter results by showing the occurrence of Hopf bifurcations in on-center off-surround SNNs with symmetric negative feedback connections and positive self-excitatory connections only.

Apart from examining the stability of equilibria, as in stability analysis, bifurcation analysis bears also on the nature of change in equilibrium behavior due to variation of parameters. Bifurcations are abrupt changes in an equilibrium state of a system of differential (or difference) equations at critical values of control parameters (GUCKENHEIMER and HOLMES 1986). Once bifurcations are found in a neural network within plausible regions of parameter values, the main questions are: (i) Is the functionality of the network maintained in each dynamic regime? (ii) Is the functionality of the network increased, so that the bifurcation is related to 'learning' or 'development' (see also WANG and BLUME 1995). In neural networks, the control parameters of bifurcations can be parameters that change slowly during the learning process, such as adaptable weights (as in PINEDA 1988; MCFADDEN et al. 1993; WANG and BLUM 1995) or the power of inputs (as in SCHUSTER and WAGNER 1990), but also parameters that define the structural properties of the network. With the latter we mean parameters in the

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Supplementary material: Colored versions of figures 2b and 4 deposited in electronic form and can be obtained from <http://science.springer.de/biolcyb/biolcyb.htm>

learning equations (as in Van der Maas et al. 1990) or parameters that define the architecture of the network (as in Borisjuk and Kirillov 1992; Erdi et al. 1993).

Development or maturation is one of the main sources of the latter kind of changes. SNNs with on-center and off-surround connections are found in several neural structures as the hippocampus, the neocortex, and the cerebellum (Eccles et al. 1967; Anderson et al. 1969; Stefanis 1969; Kandall et al. 1991). During postnatal maturation several structural properties of these networks change. The main properties of cortical networks that change are the length of axons and dendrites, the number of spines, the number of synaptic connections, myelination of axons, the number of glial cells, and the vascular tissue (Purves 1994). There is, for example, no evidence for postnatal growth of the number of cells (Huttenlocher 1993).

In the present article our main question is whether bifurcations in the SNN occur between two dynamic regimes of the SNN that implement a functionally operating network. The scope of this article is limited to the most simple bifurcations, which occur due to variation of one parameter: codimension-one bifurcations. To detect bifurcations with higher codimensions, which will be a future objective, the method of bifurcation analysis used will need to be extended. However, the study of codimension-one bifurcations already provides an interesting answer to our main question. The present article shows the occurrence of codimension-one bifurcations in the distance-dependent on-center off-surround SNN with fixed external input under variation of the four parameters that define the shunting connections in the network. Furthermore, we show that the equilibrium states that occur in different dynamic regimes of the SNN are stable. To examine the functionality of the SNN in different dynamic regimes we place the SNN in an Adaptive Resonance Theory network (ART: Grossberg 1980). In ART, the classification of input patterns takes place in the category representation field F2, which is implemented by a SNN with winner-takes-all dynamics (Grossberg 1980). In the existing implementations of ART – ART1 (Carpenter and Grossberg 1987a) and ART2 (Carpenter and Grossberg 1987b) – F2 is approximated by the equilibrium behavior of the SNN only. However, dynamic implementations of F2 also exist: ARC (Ryan and Winter 1987) and Exact ART (Raijmakers and Molenaar 1994, 1996). We study the classification behavior of Exact ART with an F2-field in different dynamic regimes that occur with different values of the parameters that define the shunting connections of the network.

In the following section we introduce the on-center off-surround SNN and describe its basic properties. Section 3 presents the numerical bifurcation analysis of the SNN: the applied techniques and the results. Additionally, for the ranges of parameters examined we present a qualitative description of network equilibria, related to the nature of category representations formed by Exact ART. Section 4 describes the learning behavior on a simple classification task of Exact ART with the F2-field in several dynamic regimes.

2 The on-center off-surround shunting neural network

Grossberg (1973) introduced several strongly related models of on-center off-surround SNNs. Such a network consists of mutually connected neurons (also called units) with activities that obey the nerve-cell-membrane equation (Hodgkin and Huxley 1952). The activities of units j are described by:

$$\frac{dx_j}{dt} = -Ax_j + (B - x_j) \left(\sum_{k=1}^M C_{kj} f(x_k) + I_j \right) - (x_j + D) \sum_{k \neq j}^M E_{kj} f(x_k) \quad (1)$$

where $j = 1, 2, \dots, M$, x_j is the potential or short-term memory activity of the j th cell (population) or unit in the SNN. Term $-Ax_j$ describes the passive decay of the activity. The second term describes a multiplicative (i.e., shunting) effect of external input, I_j , and excitatory feedback signals, $C_{kj}f(x_k)$, on activity. As in the nerve-cell-membrane equation, if $x_j(0) \leq B$ then $x_j(t) \leq B$ for $t \geq 0$, which means that B limits activities x_j from above. Parameters C_{kj} define the lateral excitatory connections within the SNN, both their range and their strength. The third term describes the shunting effect of inhibitory feedback signals. If $x_j(0) \geq D$ the $x_j(t) \geq D$ for $t \geq 0$, which means that D limits activities x_j from below. Parameters E_{kj} define both the range and the strength of the lateral inhibitory connections within the SNN. Parameters C_{kj} and E_{kj} are defined by:

$$C_{kj} = \hat{C} \exp(-\sigma_e^{-2}|k - j|^2),$$

$$\hat{C} \text{ is set such that } \sum_{k=1}^M C_{kj} = d_e \quad (2a)$$

$$E_{kj} = \hat{E} \exp(-\sigma_i^{-2}|k - j|^2),$$

$$\hat{E} \text{ is set such that } \sum_{k \neq j}^M E_{kj} = d_i, E_{jj} = 0 \quad (2b)$$

C_{kj} and E_{kj} are Gaussian distributed along the distance between units j and units k . σ_e and σ_i determine the range across which lateral connections exist. Parameters d_e and d_i determine the strength of lateral connections. These parameters can be interpreted as properties of cortical networks that change during postnatal development: σ_e and σ_i represent the range of axons and dendrites; the density of synapses and spines are represented by parameters d_e and d_i . The other properties of cortical networks that change during postnatal development, as described in Sect. 1, have no equivalent in (1). In the SNN that we will examine, 25 units defined by (1) and (2) are connected in a ring such that the distance between units 1 and 25 is equal to 1. Figure 1 shows a schematic view of the model.

The SNN activities, x_j , have many equilibrium states which depend, among other factors, on the initial activity values, $x_j(0)$. If we study the change of equilibria we can not examine all the coexisting equilibria, because they are too many. Therefore, the numerical bifurcation analysis of the M -dimensional SNN defined by (1) concerns only

interneurons. Ellias and Grossberg (1975) show peak splitting of activity patterns if a rectangular input is made broader. Peak splitting means that two activity peaks arise after presenting one rectangular input. Ellias and Grossberg conclude that this result generally depends on the quotient of the range of excitatory and inhibitory connections. It appears that in the lumped on-center off-surround SNN that we analyze, a comparable phenomenon occurs. The number of clusters of active units above threshold displays these changes. Evidently, this description of an equilibrium state depends on the number of units, M , in the SNN. However, the purpose of the qualitative description is to discriminate different kinds of equilibria of a system with a fixed number of units.

The numerical bifurcation analysis concentrates on fold-bifurcation and Hopf-bifurcation points. These bifurcation points are characterized by special conditions of the linearization matrix. The linearization matrix is the Jacobian matrix of an equilibrium state of the system. Consider the system of first-order differential equations with (1) as a special case:

$$\frac{dx_j}{dt} = F_j(x_1, \dots, x_j, \dots, x_M), \quad j = 1, \dots, M \quad (3)$$

Equilibrium states are defined by

$$F_j(x_1, \dots, x_j, \dots, x_M) = 0, \quad \forall j$$

The behavior of solutions near $\bar{x}$, where $\bar{x} = (\bar{x}_1, \dots, \bar{x}_j, \dots, \bar{x}_M)^T$ is an equilibrium state, can be characterized by linearizing (3) at $\bar{x}$. The resulting linear system is

$$\dot{\xi} = DF(\bar{x})\xi, \quad \xi \in \mathbb{R}^M$$

where

$$DF = \begin{bmatrix} \frac{\partial F_j}{\partial x_k} \end{bmatrix} \quad (4)$$

is the Jacobian matrix of the first partial derivatives of the function $F = (F_1, \dots, F_j, \dots, F_M)^T$ and

$$x = \bar{x} + \xi, \quad |\xi| \ll 1$$

For (1), element DF_{jk} of the Jacobian matrix are defined by (5a) and (5b):

$$DF_{jk} = (B - \bar{x}_j)(2C_{kj}\bar{x}_k) - \bar{x}_j(2E_{kj}\bar{x}_k), \quad j \neq k \quad (5a)$$

$$DF_{jj} = -A - \left(\sum_{k=1}^M C_{kj}\bar{x}_k^2 + I_j \right) + (B - \bar{x}_j)(2C_{jj}\bar{x}_j) - \left(\sum_{k \neq j}^M E_{kj}\bar{x}_k^2 \right) \quad (5b)$$

where $\bar{x}_j$ and $\bar{x}_k$ are vector elements of the equilibrium state $\bar{x}$ of system (1).

Fold-bifurcation points occur at those parameter values for which one eigenvalue of the linearization matrix of the equilibrium state becomes zero. Hopf-bifurcation points are characterized by a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real part. This means that we can first detect the bifurcation points with zero real part and then, depend-

ing on the imaginary part, we determine whether the point is a fold or a Hopf bifurcation.

To detect bifurcation points, we make a grid of two parameters that we pass through by varying the parameters in two nested loops. At the beginning of each step we set x_j , all $j \leq M$, to zero and we integrate the system defined by (1) until equilibrium is reached, according to the first and second derivatives of variables x_j , or until a maximum time ($T_{\max} = 1000$) is exceeded. LSODAR is used as integration method (Hindmarsh 1983). Then, we calculate the maximum eigenvalue of the linearization matrix. If a local maximum appears in this series of maximum eigenvalues, we search for a zero point by means of a one-dimensional search method: Golden Section Search (Press et al. 1989). A zero eigenvalue indicates a bifurcation point. An eigenvalue, λ , is considered as zero iff $|\lambda| < \sqrt{\varepsilon}$, ε being the error of the integration process. We pass through the grid twice, horizontally and vertically, to find bifurcations in the direction of both parameters. The stability of equilibria depends on the eigenvalues of the Jacobian matrix. In all eigenvalues are negative, the equilibrium is stable. Otherwise the equilibrium is unstable. However, the fact that we stop the integration process when both the first derivative and the second derivative of all state variables are zero almost guarantees that we only terminate the integration at stable equilibria. For possible limit cycles, the integration process will stop if the maximum integration time is exceeded.

The application of a one-dimensional search procedure implies that only codimension-one bifurcations can be detected. More complex search procedures are necessary to detect bifurcations in a higher-dimensional parameter space.

3.2 Results

The parameter regions that we examined are $\sigma_i \times \sigma_e$ ($[0.6, 10.5] \times [0.1, 2.06]$) and $d_i \times d_e$ ($[0.12, 23.88] \times [0.1, 0.99]$). The results are based on a fixed input vector. However, it appeared from repeated simulations with different random input vectors that the overall picture of the qualitative description is a general result. This will be discussed more extensively in the next section.

3.2.1 Qualitative description of equilibria. Figure 2b shows a qualitative description of the equilibrium states of (1) in the $\sigma_e \times \sigma_i$ area ($d_i = 24.0$, $d_e = 1.0$). The colors denote both N_u and N_e for each combination of σ_e and σ_i , after the manner of Fig. 2a. It appears that only isolated activity peaks, instead of clusters of activity peaks, are found in the equilibrium behavior (i.e., $N_u = N_e$). Consequently, only plain-colored areas, instead of striped areas, occur (cf. Fig. 4). The most striking result is that by decreasing the range of inhibitory connections, i.e., decreasing σ_i , the number of active units increases, starting from 0 or 1, depending on the value of σ_e , to 12. This result appears to be general for different random input vectors as can be seen in Table 1a. Table 1 shows the frequency of particular numbers of active units and clusters of units (N_u and N_e) resulting from simulations with

Fig. 2. A qualitative description of the equilibria of (1), with $x_j(0) = 0.0$ for all $j \leq M$, in the $\sigma_i \times \sigma_e$ parameter area. The colors reflect N_u (the number of active units) and N_c (the number of clusters of active units) of each combination of σ_i and σ_e in the format of **a**. All areas are plain-colored since the number of active units equals the number of clusters. The white squares indicate fold-bifurcation points, which are

found in either the σ_i direction or the σ_e direction. Apart from the Golden Section Search, the step of σ_i is equal to 0.1 and the step of σ_e is equal to 0.02. Values of fixed parameters are given in the text ($d_e = 1.0$, $d_i = 24$) (Colored version of Fig. 2b has been deposited in electronic form and can be obtained from <http://science.springer.de/biolcyb/biolcyb.htm>)

Fig. 3. Activity patterns for different values of σ_i , ranging from 0.7 in **a** to 10.0 in **k** ($d_e = 1$, $d_i = 24$, $\sigma_e = 1.02$). In general, the number of active units and active clusters, N_u and N_c , decrease with increasing σ_i : i.e. 12, 6, 5, 4, 3, 3, 2, 3, 2, 1, and **h** being an exception

ten different input vectors. Figure 3 shows examples of equilibrium activity patterns, $\bar{x}$, with different numbers of active units. These figures are generated from a section of Fig. 2 at $\sigma_e = 1.02$. Figure 3a–k show that the decrease in the number of peaks is not monotonic, which can also be seen from Fig. 2b. Many changes of equilibrium patterns due to variation of σ_i and σ_e consist of either the appearance or the disappearance of peaks in the activity of units. Figure 3g, h and i show, however, an exception in that one existing peak disappears at the same values of σ_i at which two new peaks arise. Another striking fact, which is shown by Fig. 3e and f, is that the equilibrium state configuration can change while the number of active units remains the same. All these phenomena are also found with different input vectors.

The variation of σ_e causes fewer changes in the equilibrium state of the SNN than does σ_i , but its effect is considerable. If the range of excitatory connections is too broad none of the activities rises above threshold. In Exact ART zero active units means that no learning takes

Table 1. Frequencies of N_u and N_c resulting from ten random input vectors

a

$N_{u,c}$	σ_i				
	0.6	1.6	2.6	3.6	10.6
0					2
1					8
2				6	
3			2	4	
4		3	8		
5		7			
10	2				
11	4				
12	4				

$d_e = 1, d_i = 24, \sigma_e = 1.02$

b

$N_{u,c}$	σ_e		
	0.1	1.1	2.1
0		2	10
1	8	8	
2	2		

$d_e = 1, d_i = 24, \sigma_i = 8.5$

c

$N_{u,c}$	d_i		
	12	18	24
1	4	8	10
2	6	2	

$\sigma_e = 0.6, \sigma_i = 9, d_e = 0.97$

d

N_u	d_i			
	0.12	0.84	1.56	2.28
6				8
7				2
8			4	
9			4	
11			2	
15		2		
16		2		
17		4		
20		1		
25	10	1		

$\sigma_e = 0.6, \sigma_i = 9, d_e = 0.97$

e

N_c	d_i			
	0.12	0.84	1.56	2.28
0	10	1		
2		2	7	9
3		7	3	1

$\sigma_e = 0.6, \sigma_i = 9, d_e = 0.97$

f

$N_{u,c}$	d_e			
	0.01	0.41	0.81	1.01
0	10	2		
1		8	9	6
2			1	4

$\sigma_e = 0.6, \sigma_i = 9, d_i = 16$

place (see Sect. 4). Hence, this kind of equilibrium will not induce the formation of a category representation in Exact ART. This rather large area is a general result and does not depend on the specific input vector, as Table 1b shows. In general we can conclude that the area with a winner-takes-all dynamics, i.e. one peak, is limited and that there exists a large area within which no selection of units takes place. However, if σ_e is below 1, activity patterns with various numbers of active units occur. In Sect. 4 we will examine the learning behavior of Exact ART with an F2-field in these different dynamic areas.

The colors in Fig. 4 represent a qualitative description of equilibrium states in the $d_i \times d_e$ area ($\sigma_i = 9, \sigma_e = 0.6$), in the same manner as Fig. 2. The ranges of d_i and d_e are chosen such that the maximum excitatory feedback and the maximum inhibitory feedback of one unit to another (C_{jk} and E_{jk}) equal 1. With large values of d_i , that is with strong inhibitory feedback, only isolated activity peaks occur as we saw in the $\sigma_i \times \sigma_i$ area examined (see also Table 1c). With smaller values of d_i broader peaks also occur³ (see also Table 1d and e). In Fig. 4 this is reflected by striped areas ($N_u \neq N_c$). Very small values of d_i result in uniform activity ($N_c = 0, N_u = 25$; blue-red stripes). Increasing d_i increases the number of clusters by decreasing the number of activated units until only isolated activity peaks are found. The area with clusters of active units instead of isolated activity peaks ($N_u \neq N_c$) is rather small in terms of the parameter range. For large values of d_i , an approximately linear combination of d_i and d_e determines where 0, 1 or 2 units become active. Figure 5 shows some equilibrium states of (1) for different values of d_i ($d_e = 0.97$).

Summarizing, we can say that again the parameter range where (1) obeys a winner-takes-all dynamics is limited and that within a large parameter area, the activation of all units stays under threshold (see also Table 1f). Clustered peaks also occur, but only exist for a rather small range of d_i . Simulations with different random input vectors show equivalent results.

3.2.2 Bifurcations and changes of equilibria. In Figs. 2b and 4, the white squares reflect bifurcation points, which all appear to be fold-bifurcation points. That is, none of the zero points of the highest eigenvalue of the Jacobian matrix had an imaginary part. Moreover, only in the

³ In contrast to isolated activity peaks, where clusters of units are active, θ_v determines the precise boundaries between different equilibrium states.

Fig. 4. The fold-bifurcation points of the equilibrium states of a 25-dimensional SNN defined by (1) with $x_j(0) = 0.0$ for all $j \leq M$, in the $d_e \times d_i$ parameter area. In the *striped areas* the *upper color* denotes the number of clusters of activated units, the *lowest color* denotes the number of activated units (as indicated in Fig. 2a). The *white squares* indicate fold-bifurcation points, which are found in either the d_i direc-

tion or the d_e direction. The step of d_i is equal to 0.24 and the step of d_e is equal to 0.02. Values of fixed parameters are given in the text ($\sigma_e = 0.6$, $\sigma_i = 9.0$) (Colored version of Fig. 4 has been deposited in electronic form and can be obtained from <http://science.springer.de/biolcyb/biolcyb.htm>)

Fig. 5. Activity patterns with different values of d_i , ranging from 0.12 in a to 10.2 in g ($d_e = 0.97$, $\sigma_e = 0.6$, $\sigma_i = 9$)

vicinity of bifurcation points was the maximum integration time reached (the latter is known as ‘critical slowing down’ in catastrophe theory: Gilmore 1981). Hence, no Hopf-bifurcation points and no stable limit cycles were found. Most of the fold-bifurcation points coincide with a transition between two different kinds of equilibrium states, that is a change of N_u or N_c . And, vice versa, many changes of N_u or N_c coincide with fold-bifurcations. However, changes of N_u or N_c also occur where we do not find a fold bifurcation. One such change occurs with σ_i in $[3.2, 3.3]$ ($\sigma_e = 1.02$; see Fig. 3g, h). An explanation for this is illustrated in Fig. 6.

Figure 6 shows the course of the equilibrium value of x_{10} resulting from different starting values, $x_j(0)$, $j \leq M$. The black squares show the equilibria of x_{10} calculated with $x_j(0) = 0.0$, for all $j \leq M$, at each value of σ_i (as in

Fig. 6. Three courses of equilibrium values of x_{10} defined by (1) at different values of σ_i , as a result of different starting values, $x_j(0)$. The first calculation regime is equivalent to that of Fig. 3: for each value of σ_i the equilibrium state of the SNN is calculated with $x_j(0) = 0.0$, for all $j \leq M$ (equilibrium values of x_{10} are drawn as *black squares*). In the second calculation regime (the *black line*) the calculation started with $\sigma_i = 3.1$ and $x_j(0) = 0$, for all $j \leq M$. After reaching equilibrium, at every turn σ_i is increased (for $\sigma_i > 3.1$) or decreased (for $\sigma_i < 3.1$) by 0.1 and the state is again adjusted until equilibrium. The third calculation regime (the *gray line*) is equivalent to the former calculation regime except that here the calculation started with $\sigma_i = 3.5$ and $x_j(0) = 0$, for all $j \leq M$. This figure shows that for $3.2 < \sigma_i < 3.3$ two equilibria coexist, and that neither of the two equilibria disappears. Consequently, no bifurcation point is found at $3.2 < \sigma_i < 3.3$. Parameter values are as in Fig. 3, and $\sigma_e = 1.02$

Figs. 2b and 3). The black line shows equilibria of x_{10} resulting from other starting values: at $\sigma_i = 3.1 \times x_j(0) = 0$ for all $j \leq M$. At other values of σ_i , $x_j(0)$ is assigned its equilibrium value of the preceding σ_i (for $\sigma_i > 3.1$) or succeeding σ_i (for $\sigma_i < 3.1$). The third regime (the gray line) is equivalent to the former calculation regime, but here the calculation started at $\sigma_i = 3.5$, with

$x_j(0) = 0$ for all $j \leq M$. Fold bifurcations only occur if an equilibrium suddenly appears or disappears. It appears, however, that (1) with σ_i between 3.0 and 3.5 has at least two equilibria (but probably more), which both remain stable and which coexist. Hence, neither of them meets a bifurcation between $\sigma_i = 3.2$ and $\sigma_i = 3.3$, but the zero state $[x_j(t) = 0, \text{ for all } j \leq M]$ goes from the basin of attraction of one equilibrium to the other. Hence, the changing basins of attraction cause the observed change in N_c and N_u .

The reverse situation also occurs: fold-bifurcation points are found that do not change N_c and N_u . In such a case, activity peaks appear and disappear simultaneously (Fig. 3e, f). According to Fig. 4, fold bifurcations may also occur when clusters of active units become narrower but do not change in number (only N_u , but not N_c , decreases). For example, the transitions between Fig. 5b and c, Fig. 5c and d, and Fig. 5d and 5e are marked by fold-bifurcation points.

4 Exact ART under various dynamic regimes

One of the applications of on-center off-surround SNNs is a CAM. ART networks contain a CAM as the category representation field, F2 (Grossberg 1980). As we briefly discussed above, the setting of the F2-field in ART constrains the configurations of the SNN that we have subjected to numerical bifurcation analysis. The most common implementations of ART, i.e., ART1, ART2, and ART3 (Carpenter and Grossberg, 1987a, b, 1990, respectively), are such that minimal computational time is needed. For that reason, several aspects of the model are implemented by their equilibrium behavior only. Exact ART (Raijmakers and Molenaar 1996) is a complete implementation of an ART network by a system of differential equations. This means, among other things, that the F2-field is implemented by a system of differential equations which comprises (1). In the present section we briefly explain Exact ART and summarize which constraints appear from the application of the SNN in Exact ART. As we showed previously, the most common equilibrium states of the SNN are one or more isolated activity peaks. The number of activity peaks can be manipulated by means of σ_i . Hence, we examine the classification behavior of Exact ART with different values of σ_i .

4.1 Exact ART

Exact ART (Raijmakers and Molenaar 1996) is an implementation of ART and bears a strong resemblance to ART2 (Carpenter and Grossberg 1987b). The main difference is that, in contrast to ART 2, Exact ART is completely defined by a system of ordinary differential equations. (This counts also for the implementation of Ryan and Winter 1987.) Furthermore, the category representation field F2 is implemented by a gated dipole field (GDF), which is based on Grossberg (1980). Calculation of activities and weights takes place by integrating this system of differential equations, including the weight

Fig. 7. An outline of Exact ART. The model consists of five major parts: F0, F1, F2, the adaptive filter, and the orienting subsystem. Each filled black circle denotes a unit of a layer of equivalent units. Each white circle denotes a single unique unit. Striped circles denote nonspecific interactions: excitatory for the dark gray circles and inhibitory for the light gray circles. Black arrows denote specific interactions: excitatory interactions for continuous lines and inhibitory interactions for dashed lines. Gray arrows denote nonspecific inhibitory arousal. Filled rectangles and half ellipses denote adaptive weights. Note that the model is designed after the second version of ART2 (Carpenter and Grossberg 1987b), and bears a strong resemblance to it. The mathematical definition of the model is given in Appendix A

equations, simultaneously during the presentation of external input. Appendix A shows the complete system of differential equations. A complete description of the model is given in Raijmakers and Molenaar (1996). As is ART2, Exact ART is built up of several subsystems: F0 is the input transformation field, F1 combines bottom-up input and top-down expectations, F2 is the category representation field, bottom-up and top-down weights (LTM) act as the adaptive filter, and an orienting subsystem measures the match between bottom-up input and top-down expectations and resets both F1 and F2 in case a mismatch between F1 and F2 occurs. The functions of these different parts in learning stable recognition codes and the functional interactions of the different parts are equivalent to those of ART2. Figure 7 shows an outline of the model.

4.2 The shunting neural network in exact ART

In Exact ART, categories are represented by the F2-field. The F2-field consists of different kind of units which form a GDF (see Appendix A). The y_5 -units of F2 constitute a SNN as described by (1). In most implementations of

ART the F2-field is supposed to follow a winner-takes-all dynamics: the activity of the unit with the highest input becomes maximal, and the activities of the other units decrease to zero (winner-takes-all dynamics imply that $N_e = N_u = 1$). This is established in (1) if the positive feedback signals are self-excitatory only, and the inhibitory feedback signals are uniform, without self-inhibitory feedback. In terms of (2a) and (2b) this means that $\sigma_e \ll 1$, $\sigma_i \gg M$, $d_e = 1$, and $d_i = M - 1$. In Exact ART the $y5_j$ -units of F2 are implemented by (6) [see also (A11) in Appendix A]:

$$\frac{dy5_j}{dt} = -Ay5_j + (B - y5_j) \left(\sum_{k=1}^M C_{kj}f(y5_k) + y3_j + e \sum_{k=1}^N \sqrt{z_{kj}p_k} \right) - y5_j \left(\sum_{k \neq j}^M E_{kj}f(y5_k + y4_j) \right) \quad (6)$$

Equation (6) is almost equivalent to (1), apart from two terms: the excitatory $y3_j$ and the inhibitory $y4_j$. As can be seen in Fig. 7 and Appendix A, $y4_j$ is only active if the system is being reset. Hence, the inhibitory feedback in (1) is equal to the inhibitory feedback in (6). In (1), the excitatory feedback is as follows:

$$\sum_{k=1}^M C_{kj}f(y5_k), f(y5_k) = y5_k^2 \quad (7)$$

In (6), $y3_j$ is an additional self-excitatory feedback, which is not equal to zero. After substituting equilibrium values of $y3_j$ and $y1_j$ in (6) [see (A9) and (A7) in Appendix A], the excitatory feedback in (6) is as follows:

$$\sum_{k=1}^M C_{kj}(y5_k), h(y5_k) = y5_k^2 \quad \text{for } k \neq j$$

$$h(y5_k) = y5_k^2 + (z1_k y5_k)/C_{kj} \quad \text{for } k = j \quad (8)$$

$z1_j$ is adaptive, which means that $z1_j$ decreases from γ to $\beta\gamma/(\beta + 1 - \Gamma)$ if $y5_j \geq \Gamma$ (parameters β , γ , and Γ belong to the gated dipole and are described in Appendix B). Like $f(x)$ in (7), $h(x)$ in (8) is a faster-than-linear function. An important finding of Grossberg (1973) concerns the effect of the signal function $f(x_j)$ on the transformation of initial values $x_j(0)$ by systems that are strongly related to (1) and $C_{kj} = 1$ for $k = j$, $C_{kj} = 0$ for $k \neq j$, $E_{kj} = 1$ for all $k \neq j$. Grossberg shows that if $f(x_k)$ is linear the SNN preserves the initial activity pattern $x_k(0)$ and normalizes the total activity. If $f(x_k)$ grows slower than linearly, i.e., $x_k^{-1}f(x_k)$ decreases monotonically, all values $x_k(t)$ for $t \rightarrow \infty$ are equal. If $f(x_k)$ grows faster than linearly, that is, $x_k^{-1}f(x_k)$ increases monotonically, the SNN performs a winner-takes-all: unit j with maximal initial value $x_j(0)$ wins the competition. Hence, we expect that the difference between (1) and (6) does not affect the qualitative description of the equilibrium behavior. This should appear from the simulation results presented below.

In Exact ART starting values of $y5_j$ are usually equal to zero, which we used as a constraint in the bifurcation analysis. This comes from the orienting subsystem, which affects Exact ART during the presentation of a new input as follows. If a new input pattern is presented, the orienting subsystem will probably detect a mismatch between

the previously established F2 equilibrium and the new input. Consequently, the orienting subsystem inhibits $y5_j$ -units strongly by means of A_E [via $y4_j$ -units: see (A7) and (A9) in Appendix A]. If the activities of units within both F1 and F2 are zero (which is their lower bound), A_E decreases to zero such that a new classification takes place. Hence, the starting values of activities $y5_j$ are usually equal to zero.

In the numerical bifurcation analysis of (1) that we performed the external input is fixed on a random vector, which is a typical input vector of F2 with initial bottom-up weights, $z_{ij}(0)$. In (6), the external input consists of the sum of $e\sqrt{(z_{ij}p_i)}$ for all $i \leq N$. $z_{ij}(0)$ has a random value between 0 and 0.4, $N = 10$, and, in equilibrium, the sum of all p_i , $i \leq N$, is equal to 1, $e = 0.01$.

For the application of the SNN in Exact ART, we are interested in the change in those aspects of the behavior of the SNN that are important with respect to Exact ART. One of the important aspects of an ART model is learning. Learning, that is adapting weights z_{ij} and z_{ji} , takes place only if unit $y5_j > \theta_y$ [see (A20) and (A21) in Appendix A]. For that reason we classified the behavior of the SNN (in Figs. 2 and 4) by counting the number of units that are activated above θ_y (i.e., N_u). As we showed in Sect. 3.2.1, the number of F2-units that become active simultaneously in Exact ART can be manipulated by parameter σ_i . In the next section we will examine the classification behavior of Exact ART with an F2-field in several dynamic regimes: with 2, 4, 5, and 6 simultaneously activated units.

4.3 Classification behavior of Exact ART

An ART network performs an unsupervised classification of input patterns (but can be part of a supervised network ARTMAP). During the classification process, prototypes of the classes of patterns formed are preserved in the adaptive weights. We describe the performance of an ART network with three measures. First, it is necessary for ART in learning a stimulus set, that the match of each input pattern with its representation is below the vigilance parameter ρ (i.e., the mismatch R should be below $1 - \rho$: see Appendix A). Second, after learning, the representation on F2 should be stable such that each input pattern is always represented by the same F2 code. These two criteria, which are both used to study ART2, indicate whether the network learns a classification task at all. A third measure characterizes the kind of representation of input patterns that is formed during the learning process: local or distributed. In ART1 and ART2, due to the winner-takes-all dynamics in F2, one category is represented by the weights of one F2-unit. As a consequence, representations are local and distinct. In contrast to the other ART networks, ART3 can also form distributed code (Carpenter and Grossberg 1990). However, the F2-field of ART3 is not built out of a SNN. In ART3 the distributed representation is established as follows. Some input patterns are represented by more than one F2-unit and some F2-units become active during the presentation of input patterns from different categories. To measure whether representations are

Fig. 8. Input patterns A, B, C, and D, which are classified by Exact ART. The sequence of presentation is A B C A D, which is represented for many trials

Fig. 9. F2 activities during presentation of input patterns A, B, C, A, and D at successive trials. Parameters of the F2-field are the following: $(\sigma_e, \sigma_i, d_e, d_i) = (0.4, 7.5, 1, 24)$. The sequence of presentation of input is A B C A D, which is repeated for 15 trials. Each figure shows the $y5_j$

value at the end of each trial, associated with a particular input. The horizontal axis of each figure denotes the trial (1–15), the vertical axis denotes the unit j (1–25). The hue represents the activity of $y5_j$ -units (black means $y5_j = 0$, white means $y5_j = 1$)

Fig. 10. F2-activities during presentation of input patterns at successive trials with the following parameters: $(\sigma_e, \sigma_i, d_e, d_i) = (0.4, 3.0, 1, 24)$. For further explanation see Fig. 9

distributed, we count the number of units that become active as part of different representations of input patterns.

The classification task we use to test Exact ART with different dynamic regimes of F2 is the same task as Carpenter and Grossberg (1987b) used to test the stability of learned representations in ART2. The task consists of four different input patterns that are presented in a fixed order: A, B, C, A, and D (Fig. 8). This sequence is presented repeatedly for many trials. Each input presentation persists for 500 time steps. Other parameter values of Exact ART that are constant during the simulations, are given in Appendix B. As is shown in Rajmakers and Molenaar (1996), Exact ART learns the simple classification task in a stable manner. With a vigilance parameter of 0.7, which is the match criterion, the network with a winner-takes-all dynamics in F2 forms four different categories for the four different input patterns. To induce simultaneous activity of more than one unit in F2, we vary σ_i . In the following paragraphs the results of the simulations with increasing number of active F2-units are discussed.

4.3.1 Two simultaneously active F2-units. The simulations with two simultaneously active F2-units show that at the end of each trial the match is perfect ($R = 0 < 1 - \rho$). Moreover, the representations on F2 of all four input patterns stabilize. Figure 9 shows the F2 representation of each input at 15 successive trials. It appears that F2-units restrict their activity to one input pattern, which makes the representation of input patterns local and distinct.

4.3.2 Three or four simultaneously active F2-units. The same simulation procedure as described above is carried out with an F2-field with the following parameters: $(\sigma_e, \sigma_i, d_e, d_i) = (0.4, 3.0, 1, 24)$. According to the numerical bifurcation analysis, the equilibrium of the F2-field should show three simultaneously active units. Figure 10 shows that for the input patterns used the number of active units is either three or four. The equilibrium patterns of F2 that represent the input patterns stabilize rather quickly: after the fourth trial no changes occur. Moreover, the match is perfect for each input pattern ($R = 0 < 1 - \rho$). None of the F2-units is active for more than one input pattern; hence the input representation is not genuinely distributed.

4.3.3 Four or five simultaneously active F2-units. Again, the same simulation is performed with parameter σ_i decreased such that more F2-units become active simultaneously: $(\sigma_e, \sigma_i, d_e, d_i) = (0.4, 2.2, 1, 24)$. According to Fig. 2, the number of active units should be four. Figure 11 shows that in the F2-field the number of active units is four or five. As with higher values of σ_i the representations of all input patterns stabilize and the match is perfect for each input pattern. The overlap between input representations is again zero.

4.3.4 Six or seven peaks of activity. If we decrease σ_i to 1.3, six active units in F2 are expected. Figure 12 shows that six or seven units become active simultaneously. With this number of simultaneously active units, the total number of units in F2 is too small to form four totally distinct representations. Increasing the number of input

Fig. 11. F2-activities during presentation of input patterns at successive trials with the following parameters: $(\sigma_e, \sigma_i, d_e, d_i) = (0.4, 2.2, 1, 24)$. For further explanation see Fig. 9

Fig. 12. F2-activities during presentation of input patterns at successive trials with the following parameters: $(\sigma_e, \sigma_i, d_e, d_i) = (0.4, 1.3, 1, 24)$. For further explanation see Fig. 9. Note that, in contrast to the simula-

tions with larger values of σ_i , representations of input patterns C and D have some F2-units but not all, i.e. five, in common

patterns, instead of increasing the number of simultaneously active units, can have the same effect.⁴ As expected, overlap between input representations is now found: representations of input patterns C and D share five out of seven F2-units. As can be verified in Fig. 8, of all pairs, input patterns C and D are most alike. Hence, the network formed a distributed code where it is forced to do so. The representations are again stable and the match between input and learned representation is sufficient for each input pattern. Although the match is sufficient, it is no longer perfect: The match is 0.01, 0.02, 0.27, and 0.19 for inputs. A, B, C, and D respectively. Also in ART3 the match is not perfect if distributed representations are formed. In contrast to an F2-field with a winner-takes-all dynamics, unique representations can now be formed for each input pattern while at the same time similarities between patterns are detected.

5 Conclusion

Numerical bifurcation analysis of a distance-dependent on-center off-surround shunting neural network (SNN) with fixed external input shows that fold-bifurcation points occur by variation of both the range and the strength of lateral connections. The equilibrium states of the SNN in different dynamic regimes are distinguished by the number of units that become active simulta-

neously. It is mainly parameter σ_i , which represents the range of inhibitory connections, that has a strong influence on the number of active units. The SNN is, among other neural networks, applied to Adaptive Resonance Theory (Grossberg 1980) as a winner-takes-all network. However, only within a limited range of parameters σ_i , σ_e , d_i , and d_e does the SNN obey a winner-takes-all dynamics. Since bifurcations are found in the dynamics of the SNN, two questions arise. First, is the functionality of the SNN maintained in various dynamic regimes? Is the functionality of the SNN increased, so that the bifurcation is related to learning or development?

We studied the functionality of Exact ART, which is a complete implementation of an ART network, under variation of σ_i . In Exact ART the SNN functions as a category representation field F2. It appears that the representation of input patterns stabilizes with two, four, five, and six simultaneously active units in F2. The mismatch between input and learned categories is sufficiently low in all cases. From this we can conclude that the F2-field of ART is robust. The answer to the first question is thus affirmative: the functionality of the F2-field in Exact ART is maintained in different dynamic regimes of the SNN.

The most striking result is, however, that with these lower values of σ_i the representation of input patterns appears to be distributed, since some F2-units are part of the representation of different categories. This means that with gradual variation of a structural parameter of the F2-field, the ART network acquires (or loses) the possibility of generating a distributed code of learned input patterns. This ability appears (or disappears) suddenly, since the dynamic transition of a SNN from one to more simultaneous active units due to variation of σ_i is a fold bifurcation. From this we can conclude that the functionality of the F2 field is changed due to bifurcations in the SNN. The on-center off-surround connectivity structure of the SNN examined here is found in several neural areas such as the neocortex. Among other variables, the range of inhibitory connections is known to vary in these neural areas during development. However, the interpretation of the above results in developmental terms is

⁴ A simulation with Exact ART with $\sigma_i = 2.2$ (other parameters remain equal) and six input patterns results in a representation on F2 such that each input pattern is represented by four or three active F2-nodes. All six input representations are unique; six nodes are shared by two input representations. If $\sigma_i = 3.0$ and ten input patterns are presented, each input pattern is represented by four or three active F2-nodes. Three activity patterns on F2 represent two input patterns each; the other activity patterns are unique. Three nodes are shared by different input representations. If $\sigma_i = 7.5$ and 16 input patterns are presented, each input pattern is represented by two active F2-nodes. Four activity patterns on F2 represent two or three input patterns each; the other activity patterns are unique. Five nodes are shared by different input representations.

not evident, because the change in the range of lateral connections during development is not monotonic and differs between areas. This means that the second question can only be answered partly: the functionality of the SNN changes under variation of σ_i , but the direction of change is not monotonic during development. Most important, however, is that the examined application of the SNN maintains its functionality under variation of parameters that might change during learning or developing. The latter was shown for ART neural networks.

Appendix A. Exact ART: system of ordinary differential equations

F0: Input transformation

$$\frac{1}{\phi} \frac{dx_0_i}{dt} = -Ax_0_i + (B - x_0_i)I_i - x_0_i \sum_{k \neq i}^N I_k \quad (A1)$$

$$\begin{aligned} \frac{1}{\phi} \frac{du_0_i}{dt} = & -Au_0_i + (B - u_0_i)f(x_0_i, \theta) - u_0_i \\ & \times \left(\sum_{k \neq i}^N f(x_0_k, \theta) + cf(A_E, \theta) \right) \end{aligned} \quad (A2)$$

F1: Combination of bottom-up input and top-down expectation

$$\begin{aligned} \frac{dx_i}{dt} = & -Ax_i + (B - x_i)(u_0_i + au_i) \\ & - x_i \left(\sum_{k \neq i}^N u_0_k + a \sum_{k \neq i}^N u_k + cf(A_E, \theta) \right) \end{aligned} \quad (A3)$$

$$\begin{aligned} \frac{du_i}{dt} = & -Au_i + (B - u_i)(x_i + bq_i) \\ & - u_i \left(\sum_{k \neq i}^N x_k + b \sum_{k \neq i}^N q_k + cf(A_E, \theta) \right) \end{aligned} \quad (A4)$$

$$\frac{dq_i}{dt} = -Aq_i + (B - q_i)p_i - q_i \left(\sum_{k \neq i}^N p_k + cf(A_E, \theta) \right) \quad (A5)$$

$$\frac{dp_i}{dt} = -p_i + d \sum_{j=1}^M y_j z_{ji} + u_i - p_i cf(A_E, \theta) \quad (A6)$$

F2: Gated dipole field

$$\frac{dy_1_j}{dt} = -y_1_j + y_5_j + A_E \quad (A7)$$

$$\frac{dy_2_j}{dt} = -y_2_j + [y_6_j]^+ + A_E \quad (A8)$$

$$\frac{dy_3_j}{dt} = -y_3_j + z_1_j y_1_j \quad (A9)$$

$$\frac{dy_4_j}{dt} = -y_4_j + z_2_j y_2_j \quad (A10)$$

$$\begin{aligned} \frac{dy_5_j}{dt} = & -Ay_5_j + (B - y_5_j) \\ & \times \left(\sum_{k=1}^M C_{kj} f(y_5_k) + y_3_j + e \sum_{k=1}^N \sqrt{z_{kj} p_k} \right) \\ & - y_5_j \left(\sum_{k \neq j}^M E_{kj} f(y_5_k) + y_4_j \right) \end{aligned} \quad (A11)$$

$$\frac{dy_6_j}{dt} = -y_6_j + (y_4_j - y_3_j) \quad (A12)$$

$$\frac{1}{\varepsilon} \frac{dz_1_j}{dt} = \beta(\gamma - z_1_j) - \delta[y_1_j - \Gamma]^+ z_1_j \quad (A13)$$

$$\frac{1}{\varepsilon} \frac{dz_2_j}{dt} = \beta(\gamma - z_2_j) - \delta[y_2_j - \Gamma]^+ z_2_j \quad (A14)$$

$$f(x) = x^2 \quad (A15)$$

Orienting Subsystem: matching and nonspecific arousal

$$\frac{dr_i}{dt} = -r_i + \frac{1}{2} |u_0_i - q_i| \quad (A16)$$

$$\frac{1}{\eta} \frac{dA_E}{dt} = -A_E + g(a_e(c_1 A_E - c_2 A_1 - \theta_e + S)) \quad (A17)$$

$$\frac{1}{\eta} \frac{dA_1}{dt} = -A_1 + g(a_i(c_3 A_E - c_4 A_1 - \theta_i)) \quad (A18)$$

$$\begin{aligned} \frac{1}{\tau} \frac{dS}{dt} = & -AS + (B - S) \left[\sum_{i=1}^N r_i - (1 - \rho) \right]^+ \\ & - \omega S \left[(1 - \rho) - \sum_{i=1}^N r_i \right]^+ \end{aligned} \quad (A19)$$

LTM: Connections between F1 and F2

$$\frac{1}{\alpha} \frac{dz_{ji}}{dt} = df(y_j, \theta_y)(f(p_i, \theta) - z_{ji}) \quad (A20)$$

$$\frac{1}{\alpha} \frac{dz_{ij}}{dt} = df(y_j, \theta_y)(f(p_i, \theta) - z_{ij}) \quad (A21)$$

Nonlinear signal functions

$$f(x, \theta) = \begin{cases} x & \text{if } x > \theta \\ 0 & \text{if } x \leq \theta \end{cases} \quad (A22)$$

$$g(x) = \frac{1}{1 + \exp(-x)} \quad (A23)$$

The notation used in this appendix and the text is as follows. Vector elements are denoted by italic letters with an index; i is the index of F1- and F0-units, j is the index of F2-units ($x_0_i, \dots, p_i, y_1_j, y_6_j, z_1_j, z_2_j, r_i, z_{ij}, z_{ji}$). N is the number of units within the F1-field and the F0-field. M is the number of F2-units, i.e., gated dipoles. Default parameter values are given in Appendix B.

Appendix B. Default parameter values of exact ART

F0 and F1 parameters

Number of F0-units and F1-units (N)	10
Decay (A)	0.001
Upper bound of activation (B)	1.0
Positive connection from q to $u(b)$	10.0
Positive connection from u to $x(a)$	10.0
Rapidity of F0 equations (ϕ)	100.0

F2 parameters

Number of F2-units (M)	25
Range of excitatory connections (σ_e)	0.01
Range of inhibitory connections (σ_i)	500.0
Total strength of excitatory connections (d_e)	1.0
Total strength of inhibitory connections (d_i)	24.0
Strength of input to F2 (e)	0.01
Strength of input to F1 (d)	0.5
Increase of transmitter (β)	0.1
Decrease of transmitter (δ)	5.0
Threshold of transmitter (Γ)	0.1
Upper bound of transmitter (γ)	0.5
Rapidity of dipole-transmitter equations (ε)	0.001

LTM parameters

Maximum value of initial weights (z_0)	0.4
Threshold in LTM equations (θ_y)	0.35
Rapidity of LTM ($F1 \leftrightarrow F2$) (α)	0.005

OSS parameters

Vigilance (ρ)	0.7
Rapidity of OSS equations (η)	100.0
Rapidity of S equation (τ)	0.1
Decrease of S (ω)	1.0

General parameters

Threshold in signal function (θ)	0.1
Strength of arousal (c)	10.0

Integration parameters

Time at start of numerical integration	0.0
Time at end of numerical integration	500.0
Time step of numerical integration	25.0
Tolerance of numerical integrator	0.0001

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