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# Nonlinear Yang–Mills theories

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We construct Yang–Mills type gauge theories for quadratically nonlinear algebras in any number of spacetime dimensions. They contain gauge fields  $h_\mu^A$  and coadjoint scalars  $t_A$ . We derive conditions under which these theories are equivalent to ordinary Yang–Mills theories at the classical level by field redefinitions. The example of a nonlinear extension of  $SU(1, 1)$  is worked out in detail. The quantum theory is briefly discussed.

## 1. Introduction

Nonlinear Lie algebras are a generalization of ordinary Lie algebras which contain squares, and possibly higher order products, of the generators on the right-hand side of the defining brackets without violating the Jacobi identities. Quantum groups are an example of nonlinear algebras with an infinite set of products. We shall consider below algebras with at most squares. The Jacobi identities restrict the possible quadratically nonlinear algebras severely, and reveal that they are always an extension of ordinary (linear) Lie algebras if the brackets are Poisson brackets. For quantum brackets this is not always the case [1]. We shall only consider Poisson brackets in this letter.

In two previous articles we have begun to construct a general gauge formalism for quadratically nonlinear algebras [2,3], and this letter is intended to be the completion of this program, in particular of ref. [3]. We shall study whether invariant gauge actions in general exist. For the case of the  $W_3$  algebra (to be defined below), matter couplings have already been constructed in refs. [4,2], and we shall not discuss matter couplings in this note.

It is tempting to speculate on applications to supersymmetry because the spectrum of a nonlinear supersymmetric algebra need not contain a boson for every fermion, or vice versa. As we shall see, the formalism

automatically introduces a scalar  $t_A$  for every gauge field  $h_\mu^A$ . We are not sure about the possible physical meaning of these scalars but one might think of standard model physics. In the matter coupling of the  $W_3$  algebra we found that the scalars  $t_A$  were fields  $t_{++}(x^+, x^-)$  and  $w_{+++}(x^+, x^-)$ , and that the  $t_{++}$  had to be replaced by the stress tensor of matter in the final transformation rules. This suggests that, at least in some models, the  $t_A$  are auxiliary fields of a gauge action which can be eliminated after coupling the gauge sector to matter and then become matter currents. With these four- and two-dimensional motivations in mind we decided to study the construction of gauge actions for general quadratically nonlinear algebras.

Before working out general results, we present two examples of nonlinear algebras. The  $W_3$  algebra [5] is a quadratically nonlinear infinite dimensional algebra which is a spin 3 extension of the Virasoro algebra in two dimensions. Its Poisson brackets are given by [6]

$$\begin{aligned} [L_m, L_n] &= (-m+n)L_{m+n}, \\ [L_m, W_n] &= (-2m+n)W_{m+n}, \\ [W_m, W_n] &= \sum_{k=-\infty}^{\infty} L_{m+n-k}L_k. \end{aligned} \quad (1)$$

An example of a finite dimensional quadratically nonlinear algebra is the following extension of  $SU(1, 1)$ :

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$$[K_{\pm}, H] = \pm K_{\pm}, \quad [K_+, K_-] = H + \alpha H^2, \quad (2)$$

where  $\alpha$  is a real constant. One may check that in both examples the Jacobi identities are indeed satisfied; in fact, in (2) any function of  $H$  is allowed instead of the term  $\alpha H^2$ .

Both examples display a coset structure, namely, one can divide the generators into generators of a subalgebra  $H_i$  ( $L_m$  and  $H$ ) and coset generators  $K_{\alpha}$  ( $W_m$  and  $K_{\pm}$ ) such that nonlinearities appear only in the coset-coset brackets and are proportional to  $H_i H_j$ . We restrict our attention to such algebras in general. They are formally defined by

$$[\hat{T}_A, \hat{T}_B] = \hat{T}_C f_{AB}^C + \hat{T}_D \hat{T}_D V_{AB}^{CD},$$

$$\{\hat{T}_A\} = \{H_i, K_{\alpha}\}; \quad \text{only } V_{\alpha\beta}^{\dot{ij}} \neq 0. \quad (3)$$

The Jacobi identities require that  $f_{AB}^C$  satisfy the ordinary Jacobi identities for Lie algebras, while  $V$  must satisfy

$$V_{[AB}^{DE} f_{C]D}^E + V_{[AB}^{DF} f_{C]D}^E = V_{D[A}^{EF} f_{BC]}^D. \quad (4)$$

Due to the coset structure, contractions between two  $V$  symbols vanish, which simplifies the analysis a great deal, and we may without loss of generality assume that  $V_{\alpha\beta}^{\dot{ij}}$  is symmetric in  $(ij)$ , even at the quantum level. An analysis of the BRST charge for algebras of this type, both at the classical and at the quantum level, was given in ref. [1].

We begin by associating to every generator  $\hat{T}_A$  a gauge field  $h_{\mu}^A$  and local gauge parameter  $\epsilon^A$  and define gauge transformations of the gauge fields by

$$\delta(\epsilon) h_{\mu}^A = D_{\mu} \epsilon^A \equiv \partial_{\mu} \epsilon^A + \hat{f}_{BC}^A h_{\mu}^C \epsilon^B,$$

$$\hat{f}_{BC}^A = f_{BC}^A + t_D V_{BC}^{DA}. \quad (5)$$

A new feature is the appearance of scalars  $t_D$ . The algebra can be rewritten as  $[\hat{T}_A, \hat{T}_B] = \hat{T}_C \hat{f}_{AB}^C$  with  $\hat{f}_{AB}^C = f_{AB}^C + \hat{T}_D V_{AB}^{DC}$ , but since we cannot admit generators in  $\delta h_{\mu}^A$ , we cannot use  $\hat{f}_{BC}^A$  in  $\delta h_{\mu}^A$  and we are forced to introduce the scalars  $t_A$ . We choose their scale such that the  $tV$  term in (5) has unit coefficient. The scalars  $t_A$  transform by definition as follows:

$$\delta(\epsilon) t_A = t_C f_{AB}^C \epsilon^B + \frac{1}{2} t_C t_D V_{AB}^{DC} \epsilon^B. \quad (6)$$

Due to the factor  $\frac{1}{2}$ , their covariant derivatives transform as tensors in the coadjoint representation,

$$\delta(D_{\mu} t_A) = (D_{\mu} t_C) \hat{f}_{AB}^C \epsilon^B, \quad (7)$$

where  $D_{\mu} t_A = \partial_{\mu} t_A - t_C (f_{AB}^C + \frac{1}{2} t_D V_{AB}^{DC}) h_{\mu}^B$ .

One may prove the following results, using heavily and heavenly the Jacobi identities,

$$(i) \quad [\delta(\epsilon_1), \delta(\epsilon_2)] h_{\mu}^A$$

$$= \delta(\epsilon_3) h_{\mu}^A - (D_{\mu} T_D) V_{BC}^{DA} \epsilon_1^C \epsilon_2^B,$$

$$[\delta(\epsilon_1), \delta(\epsilon_2)] t_A = \delta(\epsilon_3) t_A,$$

$$\epsilon_3^A = \hat{f}_{BC}^A \epsilon_1^C \epsilon_2^B,$$

$$(ii) \quad [D_{\mu}, D_{\nu}] t_A = \delta(\epsilon = -R_{\mu\nu}) t_A,$$

$$R_{\mu\nu}^A = \partial_{\mu} h_{\nu}^A - \partial_{\nu} h_{\mu}^A + \hat{f}_{BC}^A h_{\mu}^C h_{\nu}^B,$$

$$(iii) \quad \delta(\epsilon) R_{\mu\nu}^A$$

$$= \hat{f}_{BC}^A R_{\mu\nu}^C \epsilon^B + (D_{\mu} t_D) V_{BC}^{DA} h_{\nu}^C \epsilon^B - \mu \leftrightarrow \nu),$$

$$D_{[\mu} R_{\nu\rho]}^A = 0 \quad (\text{Bianchi identity!}). \quad (8)$$

The gauge commutator has the same composite parameter  $\epsilon_3$  on  $t_A$  as on  $h_{\mu}^A$  provided the coefficient in the last term of (6) is  $\frac{1}{2}$ . This is thus a confirmation of the correctness of our definition in (6). (In ref. [2] we chose a factor unity, but in ref. [3] we changed this to  $\frac{1}{2}$ . This change has no effect on the final results for  $W_3$  gravity in ref. [2], because there we replaced the fields  $t_A$  by matter currents whose transformation rules were determined by the chain rule.)

If the  $D_{\mu} t_A$  terms in (8) would have been absent, the gauge algebra would have been closed, and curvatures would have transformed homogeneously into themselves. In refs. [2,3], where we applied this scheme to  $W_3$  gravity, we imposed  $D_{\mu} t_A = 0$  as a constraint (which implied that part of the theory was on-shell, and had to be moved off-shell in a next step), but below we follow a different approach, namely we further modify the rules in (5) by admitting  $D_{\mu} t_A$  terms in  $\delta h_{\mu}^A$ . This raises the hope that we can obtain a closed gauge algebra without constraints, as in ordinary Yang-Mills theory. We shall then construct actions of the form  $\mathcal{L} = \gamma_{AB} R_{\mu\nu}^A R_{\mu\nu}^B + \dots$  where  $\gamma_{AB}$  is (an extension of) the Killing metric. To focus our ideas, we shall first, in the next section, use the Noether method to construct an action through first order in  $\alpha$  for nonlinear  $SU(1, 1)$ . Then we shall consider general algebras in section 3, where dynamical and kinematical aspects are analyzed. Our conclusions and conjectures are summarized in section 4.

**2. Noether results for nonlinear SU(1, 1)**

From (2) we read off  $f_{+-}^0 = f_{+0}^+ = f_{-0}^- = 1$ ,  $V_{+-}^{00} = \alpha$ , and this yields the results in table 1. We consider the following action:

$$\begin{aligned} \mathcal{L} = & aR_{\mu\nu}^+ R_{\mu\nu}^- + bR_{\mu\nu}^0 R_{\mu\nu}^0 \\ & + c(D_\mu t_+) (D_\mu t_-) + d(D_\mu t_0)^2 \\ & + (At_+ R_{\mu\nu}^+ + Bt_- R_{\mu\nu}^-) R_{\mu\nu}^0 \\ & + Ct_0 R_{\mu\nu}^+ R_{\mu\nu}^- + Dt_0 (R_{\mu\nu}^0)^2 \\ & + \sum_{a,b,c=\pm,0} \alpha_{bc}^a (D_\mu t_a) R_{\mu\nu}^b h_\nu^c + \dots \end{aligned} \quad (9)$$

Since one can always complete curls of  $h_\mu^a$  to  $R_{\mu\nu}^a$ , and gradients of  $t_a$  to  $D_\mu t_a$ , this action is the most general Yang–Mills type action as long as we allow further polynomials in  $R_{\mu\nu}^a$ ,  $D_\mu t_a$ ,  $h_\mu^a$  and  $t_a$ . These are indicated by the dots. The need for the last term in (9) becomes immediately clear if one realizes that  $\delta R$  contains a term  $\alpha(Dt)h\epsilon$  so that the variation of the  $a, b$  terms can only be canceled by using  $\delta(Dt) = (Dt)\epsilon$  and  $\delta R = \tilde{f}R\epsilon$  in the  $\alpha$ -terms. It follows that the  $\alpha$ -terms are of order  $\alpha$ . In all terms

Table 1

Definitions and transformation rules for the gauge theory based on the nonlinear extension (2) of SU(1, 1). In the final formulation the  $\delta h^a$  rules are modified as in (11), with  $W_c^{ab}$  to first order in  $t_a$  given in (12).

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$$\begin{aligned} R_{\mu\nu}^+ &= \partial_\mu h_\nu^+ - \partial_\nu h_\mu^+ + h_\nu^+ h_\mu^0 - h_\nu^0 h_\mu^+ \\ R_{\mu\nu}^- &= \partial_\mu h_\nu^- - \partial_\nu h_\mu^- - h_\nu^- h_\mu^0 + h_\nu^0 h_\mu^- \\ R_{\mu\nu}^0 &= \partial_\mu h_\nu^0 - \partial_\nu h_\mu^0 + (1 + \alpha t_0) (h_\nu^+ h_\mu^- - h_\nu^- h_\mu^+) \\ D_\mu t_+ &= \partial_\mu t_+ - t_+ h_\mu^0 - (1 + \frac{1}{2}\alpha t_0) t_0 h_\mu^- \\ D_\mu t_- &= \partial_\mu t_- + t_- h_\mu^0 + (1 + \frac{1}{2}\alpha t_0) t_0 h_\mu^+ \\ D_\mu t_0 &= \partial_\mu t_0 + t_+ h_\mu^+ - t_- h_\mu^- \\ \delta h_\mu^+ &= \partial_\mu \epsilon^+ + (h_\mu^0 \epsilon^+ - h_\mu^+ \epsilon^0) \\ \delta h_\mu^- &= \partial_\mu \epsilon^- - (h_\mu^0 \epsilon^- - h_\mu^- \epsilon^0) \\ \delta h_\mu^0 &= \partial_\mu \epsilon^0 + (1 + \alpha t_0) (h_\mu^- \epsilon^+ - h_\mu^+ \epsilon^-) \\ \delta t_+ &= t_+ \epsilon^0 + (1 + \frac{1}{2}\alpha t_0) t_0 \epsilon^- \\ \delta t_- &= -t_- \epsilon^0 - (1 + \frac{1}{2}\alpha t_0) t_0 \epsilon^+ \\ \delta t_0 &= -t_+ \epsilon^+ + t_- \epsilon^- \\ \delta R_{\mu\nu}^+ &= \epsilon^+ R_{\mu\nu}^0 - \epsilon^0 R_{\mu\nu}^+ \\ \delta R_{\mu\nu}^- &= -\epsilon^- R_{\mu\nu}^0 + \epsilon^0 R_{\mu\nu}^- \\ \delta R_{\mu\nu}^0 &= (1 + \alpha t_0) (\epsilon^+ R_{\mu\nu}^- - \epsilon^- R_{\mu\nu}^+) + \alpha (D_\mu t_0) (\epsilon^+ h_\nu^- - \epsilon^- h_\nu^+) \\ &\quad - \alpha (D_\nu t_0) (\epsilon^+ h_\mu^- - h_\mu^+ \epsilon^-) \\ \delta (D_\mu t_+) &= (D_\mu t_0) \epsilon^- - (1 + \alpha t_0) + (D_\mu t_+) \epsilon^0 \\ \delta (D_\mu t_-) &= -(D_\mu t_0) \epsilon^+ + (1 + \alpha t_0) - (D_\mu t_-) \epsilon^0 \\ \delta (D_\mu t_0) &= -(D_\mu t_+) \epsilon^+ + (D_\mu t_-) \epsilon^- \end{aligned}$$


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the + and - signs balance each other, as in  $\alpha_{+0}^+(D_\mu t_+) R_{\mu\nu}^+ h_\nu^0$ .

We shall work order by order in  $\alpha$ , or, equivalently, order by order in the number of  $t_a$  fields. To lowest order we find invariance of the action, putting  $b=1$  for convenience, provided

$$a = -2, \quad c + 2d = 0. \quad (10)$$

However, varying  $h_\nu^c$  in the last term of (9) yields a variation of the form  $\alpha(Dt)R(D\epsilon)$  which, upon partial integration, produces the lowest order  $h$  field equation  $(D_\mu R_{\mu\nu})$  times  $\alpha(Dt)\epsilon$ , and a term  $\alpha RRt\epsilon$ . The latter can be canceled against variations of the  $ABCD$  terms, but the former can only be canceled by adding a term  $\alpha(Dt)\epsilon$  to  $\delta h$ . This brings us quite generally to define the gauge transformation of the gauge fields  $h_\mu^A$  by

$$\delta h_\mu^A = \partial_\mu \epsilon^A + \tilde{f}_{BC}^A h_\mu^C \epsilon^B + (D_\mu t_C) W_B^{CA} \epsilon^B, \quad (11)$$

where  $W_B^{CA}$  are functions of  $t_a$  to be studied further below. Using (11) in our toy model (9), we find that all variations of the form  $R(Dt)h\epsilon$ ,  $Rhh\epsilon$  and  $RRt\epsilon$  cancel provided

$$\begin{aligned} \alpha_{bc}^a &= \frac{1}{2} W_c^{ad} \gamma_{db}, \quad \gamma_{00} = -\gamma_{+-} = 2, \\ A &= \frac{1}{2} q_1, \quad B = \frac{1}{2} q_2, \quad C = \frac{1}{2} (q_1 + q_2), \quad D = -\frac{2}{3} \alpha, \\ W_0^{+-} &= W_0^{-+} = -W_0^{00} = \frac{1}{3} \alpha, \\ W_+^{0+} &= -\frac{1}{4} q_2, \quad W_-^{0-} = -\frac{1}{4} q_1, \\ W_+^{+0} &= \frac{1}{4} q_1 + \frac{1}{3} \alpha, \quad W_-^{-0} = \frac{1}{4} q_2 + \frac{1}{3} \alpha. \end{aligned} \quad (12)$$

At order  $\alpha$  there are also variations of the form  $t(Dt)(Dt)\epsilon$  which one could try to cancel by adding  $t(Dt)(Dt)$  terms to the action, but since the  $c, d$  terms do not communicate with the rest of the model to this order in  $\alpha$ , it is plausible that the  $c, d$  terms form the beginning of a separate invariant. We shall therefore put from now on  $c=d=0$ .

Rather than go on to the next order in  $\alpha$ , we shall now turn to a general analysis and leave the Noether method which has given us (11).

**3. General dynamics and kinematics**

The appearance of the  $(Dt)\epsilon$  term in  $\delta h_\mu^A$  suggests to consider field redefinitions of ordinary Yang–Mills fields  $H_\mu^A$  of the form  $H_\mu^A = \lambda_B^A(t) h_\mu^B$  and redefini-

tions of the gauge parameter  $\eta^A$  of ordinary Yang–Mills theory of the form  $\eta^A = \zeta_B^A(t)\epsilon^B$ . These redefinitions will indeed produce  $(Dt)\epsilon$  terms in  $\delta h_\mu^A$  as in (11). We shall find the conditions under which ordinary Yang–Mills theory exactly produces (11) and an invariant action as in (9). Of course we are interested in the cases where this equivalence by field redefinition is not possible. Note, however, that quantizing a classically equivalent nonlinear gauge theory by the standard rules will lead to a quantum theory which is inequivalent to ordinary Yang–Mills theory because, from a path-integral point of view, one drops the jacobian.

From the Yang–Mills transformation law  $\delta H_\mu^A = \partial_\mu \eta^A + f_{BC}^A H_\mu^C \eta^B$  we find, upon making the substitutions mentioned above, a rule for  $\delta h_\mu^A$  which agrees with (11) provided the following two conditions are met:

$$\lambda_B^A \tilde{f}_{PQ}^B = f_{BC}^A \lambda_P^B \lambda_Q^C + (t_R f_{(1/2)PS}^R \partial^S \lambda_Q^A - P \leftrightarrow Q), \tag{13}$$

$$W_C^{AB} = (\lambda^{-1})_D^B \partial^A \lambda_C^D. \tag{14}$$

The symbol  $f_{(1/2)BC}^A$  lies halfway between  $\tilde{f}_{BC}^A$  and  $f_{BC}^A$ , namely  $f_{(1/2)BC}^A = f_{BC}^A + \frac{1}{2} T_D V_{BC}^D A$ , and  $\delta t_A = t_C f_{(1/2)AB}^C \epsilon^B$ . Clearly at  $t=0$ , where  $\lambda_B^A = \delta_B^A$ , (13) is identically satisfied. Under the substitution  $H_\mu^A = \lambda_B^A h_\mu^B$ , the standard Yang–Mills curvature  $R_{\mu\nu}^A(H)$  goes over into  $\lambda_B^A R_{\mu\nu}^{B,\text{cov}}(h)$  where

$$R_{\mu\nu}^{A,\text{cov}}(h) = R_{\mu\nu}^A(h) + (D_\mu t_C W_D^{CB} h_\nu^B - \mu \leftrightarrow \nu). \tag{15}$$

An invariant action of the form in (9) is then at once obtained by making this substitution in the standard Yang–Mills action  $\frac{1}{2} \gamma_{AB} R_{\mu\nu}^A(H) R_{\mu\nu}^B(H)$  and reads

$$\mathcal{L} = \frac{1}{2} \gamma_{AB} \lambda_C^A \lambda_D^B R_{\mu\nu}^{C,\text{cov}} R_{\mu\nu}^{D,\text{cov}}. \tag{16}$$

At this point we return to the Noether results in section 2 to investigate whether they form the beginnings of such a redefinition. From (12) and (14) to first order in  $\alpha$ , we immediately find  $\lambda$  to first order in  $t$ . (For example,  $W_0^{+-} = \frac{1}{2}\alpha$  implies  $\lambda_0^- = \frac{1}{2}\alpha t_+ + O(t^2)$ .) These  $\lambda$ 's satisfy the master equation in (13) to linear order in  $t$ , and reproduce via (16) the correct  $A, B, C, D$  in (12). However, putting  $A=+, P=-, Q=0$  in (13) one finds, using  $\lambda_\pm^+ \lambda_\pm^+ \sim t_\pm^2$ , that  $q_1 = q_2 = -\frac{4}{3}\alpha$ . The Noether method

did not give this restriction. If we would have allowed further additive redefinitions  $h_\mu^{\pm} = h_\mu^{\pm} + q^{(\pm)} D_\mu t_\mp$  and  $h_\mu^0 = h_\mu^0 + q^{(0)} D_\mu t_0$ , the rule for  $\delta h_\mu^A$  is again of the form (11) to this order in  $t$ , except that one should replace  $q_1$  by  $q_1 - 4(q^{(0)} + q^{(-)})$  and  $q_2$  by  $q_2 + 4(q^{(0)} + q^{(+)})$ . This explains the presence of the two free parameters in the Noether results. If we would have extended the master equation by taking these additive redefinitions into account the above mentioned restriction on  $q_1$  and  $q_2$  would have disappeared. This strongly suggests that completing the Noether analysis for the  $SU(1, 1)$  model is equivalent to solving the master equation in (13). In general there are  $D^2$  unknown functions  $\lambda_B^A$  of  $T_C$ , whereas the master equation contains  $\frac{1}{2} D^2 (D-1)$  equations, where  $D$  is the number of generators. In our example of  $SU(1, 1)$  these numbers match and indeed we found (the beginnings of) a solution, but in the general case the situation is unclear.

We now switch gears and turn to kinematical aspects of general nonlinear gauge theories of the type in (3). We find the following results:

(i) Requiring closure of the gauge algebra

$$[\delta(\epsilon_1), \delta(\epsilon_2)](h_\mu^A \text{ or } t_A) = \delta(\epsilon_3)(h_\mu^A \text{ or } t_A) \tag{17}$$

with  $\epsilon_3$  given in (8) and  $\delta(\epsilon)h_\mu^A$  given in (11), we find as necessary and sufficient condition

$$\begin{aligned} & (-\frac{1}{2} V_{PQ}^{AB} - \frac{1}{2} W_C^{BA} \tilde{f}_{PQ}^C + \tilde{f}_{PC}^A W_Q^{BC} + \tilde{f}_{CQ}^B W_P^{CA} \\ & - t_R f_{(1/2)CD}^R W_P^{CA} W_Q^{BD} + t_R f_{(1/2)CQ}^R \partial^C W_P^{BA}) \\ & \times (D_\mu t_B) (\epsilon_1^P \epsilon_2^Q - \epsilon_2^P \epsilon_1^Q) = 0. \end{aligned} \tag{18}$$

If  $W$  is pure gauge as in (14), this equation follows from differentiating the master equation in (13) with respect to  $t_A$ .

(ii) Assuming closure of the gauge algebra as in (i), we can go one step further by considering the covariant extension of the nonlinear curvature  $R_{\mu\nu}^A$  in (8). The extra term  $(Dt)\epsilon$  in  $\delta h$  determines this covariant curvature and yields the same result as in (15). Requiring that this  $R_{\mu\nu}^{A,\text{cov}}$  transforms only into itself leads to a further constraint on  $W_C^{AB}$ ,

$$\partial^A W_Q^{BP} + W_R^{AP} W_Q^{BR} - A \leftrightarrow B = 0. \tag{19}$$

This states that  $W$  is pure gauge, as in (14). The covariant curvature transforms then as follows:

$$\delta R_{\mu\nu}^{A,\text{cov}} = (\tilde{f}_{BC}^A - T_D f_{(1/2)EC}^D W_B^{EA}) R_{\mu\nu}^{C,\text{cov}} \epsilon^B. \quad (20)$$

We conclude that the two requirements of a closed gauge algebra and a homogeneous transformation law for the covariantized curvatures, bring one back to ordinary Yang–Mills theory (on an unusual basis).

#### 4. Conclusions

We have completed our gauge formalism for general nonlinear algebras of the type in (3). The final transformation rules of the gauge fields are given in (11) and of the coadjoint scalars in (6).

If one has a solution for  $\lambda_B^A$  in the master equation in (13), one has an invariant action; this theory is then obtained from ordinary Yang–Mills theory by a redefinition of the gauge fields. The combined kinematical requirements of closure of the gauge algebra and homogeneity of the transformation rule of the covariantized curvatures in (19) leads to the same conclusions.

One could consider more general redefinitions of ordinary Yang–Mills gauge fields  $H_\mu^A = \lambda_B^A h_\mu^B + (D_\mu T_C) U^{CA}$ , where  $U^{CA}$  is a new function of  $t_D$ . Requiring that  $\delta h_\mu^A$  be again of the form (11) leads to a modified master equation, and  $W$  will in general no longer be pure gauge as in (14). In our  $SU(1, 1)$  model this freedom corresponded to two free parameters  $q_1$  and  $q_2$ . One could also study whether there exist scalars  $s_A$  in ordinary Yang–Mills theory which produce our  $\delta t_A$  law by the field redefinition  $s_A = \mu_B^A(t) t_B$ , using, of course, that  $\eta^A = \lambda_B^A \epsilon^B$ . Or one might try to find analogous redefinitions at the level of the generators which produce (3) from an ordinary Lie algebra. Finally, one could consider actions of the kind  $F\tilde{F}$ , or Chern–Simons action for nonlinear algebras.

For infinite dimensional nonlinear algebras such as  $W_3$ , we expect the classical theory to be inequivalent to ordinary Yang–Mills theory. This would mean that for such algebras no solution of the master equation exists.

It is interesting to study these new gauge theories at the quantum level. At the one-loop level the  $t$ -tadpoles vanish for massless  $t$  fields. The  $\hbar$  two-point function is then transversal, as we have explicitly checked and also proved in the usual way from Ward identities. When  $t$  becomes massive, tadpoles no longer vanish. A Higgs effect is then generated by the new vertices in the nonlinear theory rather than by the double-well potential in the usual theory. Further renormalizability, unitarity and anomaly aspects are under study.

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