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## Off-shell WZW models in extended superspace <sup>☆</sup>

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We investigate the off-shell  $N=2$  and  $N=4$  structure of the supersymmetric Wess–Zumino–Witten model. We present an off-shell formulation of the  $SU(2) \otimes U(1)$  supersymmetric WZW model in terms of  $N=2$  chiral and twisted chiral multiplets. We show that such a formulation is not possible for any other group and suggest a larger multiplet structure to cover these cases.

Off-shell formulations of WZW-models are known in  $N=0, 1$  superspace [1]. For example, the  $N=1$  action is

$$S = \int d^2z \nabla_+ \nabla_- [(g_{ij} + b_{ij}) \nabla_+ \phi^i \nabla_- \phi^j], \quad (1)$$

where  $\phi^i$  is an unconstrained scalar superfield that coordinatizes the group manifold,  $g_{ij}$  is the metric, and  $b_{ij}$  is the potential for the parallelizing torsion. It is known that any even dimensional group allows for an  $N=2$  super Kac–Moody symmetry, and a subset of these models have an  $N=4$  symmetry [2]. On dimensional grounds, it is clear that  $N=2, 4$  superspace actions are simply functions of the superfields without any derivatives; hence it is not evident how one can write  $g$  and  $b$  terms separately. For example, if one writes an action that depends on the most familiar  $N=2$  scalar multiplet, a chiral superfield, then one finds that  $g$  is necessarily Kähler and  $b=0$  [3]. For WZW models,  $g$  is *never* Kähler and  $b \neq 0$ , so chiral superfields are not enough. This is a new feature of extended supersymmetry: the dynamics is not determined entirely by the form of the action, but also by the kinematical nature of the superfields. A particular example of a variant (twisted) scalar multiplet was

introduced by Gates, Hull, and Roček [4]. In this letter we show that the  $SU(2) \otimes U(1)$  super WZW model can be formulated in  $N=2, 4$  superspace using a usual chiral and a twisted chiral superfield. We further show that all other WZW models require more exotic representations.

In  $N=2$  superspace, we work with complex left- and right-handed spinor derivatives  $D_{\pm}$  satisfying the algebra

$$\{D_{\pm}, \bar{D}_{\pm}\} = \partial_{\pm\pm}; \quad (2)$$

all other anticommutators vanish. Here  $\partial_{++} = \partial_z$ , etc. Chiral superfields obey

$$\bar{D}_{\pm} \Phi = 0, \quad D_{\pm} \bar{\Phi} = 0. \quad (3)$$

In contrast, twisted chiral superfields obey [4]

$$\bar{D}_+ A = 0, \quad D_- A = 0, \quad D_+ \bar{A} = 0, \quad \bar{D}_- \bar{A} = 0. \quad (4)$$

Both superfields can be reduced to  $N=1$  superfields as follows: We define real  $N=1$  spinor derivatives  $\nabla_{\pm} = D_{\pm} + \bar{D}_{\pm}$  and “extra” supersymmetry generators  $\tilde{Q}_{\pm} = i(\bar{D}_{\pm} - D_{\pm})$ . The resulting  $N=1$  superfields  $\phi, \lambda$  are unconstrained scalars with the following transformations under the extra supersymmetry:

$$\begin{aligned} \tilde{Q}_{\pm} \phi &= -i \nabla_{\pm} \phi, & \tilde{Q}_{\pm} \bar{\phi} &= +i \nabla_{\pm} \bar{\phi}, \\ \tilde{Q}_{+} \lambda &= -i \nabla_{+} \lambda, & \tilde{Q}_{-} \lambda &= +i \nabla_{-} \lambda, \end{aligned} \quad (5)$$

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$$\tilde{Q}_+ \bar{\lambda} = +i\nabla_+ \bar{\lambda}, \quad \tilde{Q}_- \bar{\lambda} = -i\nabla_- \bar{\lambda}. \quad (5 \text{ cont'd})$$

However, it is known that extra supersymmetries can be written in  $N=1$  superspace as [4]

$$\tilde{Q}_\pm \phi^i = J_{(\pm)}^i \nabla_\pm \phi^j. \quad (6)$$

Comparing (5) with (6), we can read off  $J_{(\pm)}$ , and find that they are both constant, distinct, commuting complex structures. This is in fact a general feature of complex structures on models constructed with only chiral and twisted chiral multiplets: the resulting left and right complex structures must commute [4]. We will see that such commuting structures exist on  $SU(2) \otimes U(1)$ , but not on other group manifolds.

A supersymmetric non-linear  $\sigma$ -model has  $N$  left- and right-handed supersymmetries when there exist two sets of  $N-1$  covariantly constant complex structures [5,2]. All the complex structures within each set anticommute, and the metric has to be hermitian with respect to all of them. When the connection has torsion, integrability requires the vanishing of the Nijenhuis tensors and the left-handed (right-handed) complex structures have to be covariantly constant with respect to the connection consisting of the metric connection plus (minus) the torsion ( $\Gamma_\pm = \{\} \pm T$ ).

In the case of supersymmetric WZW models, these conditions were completely solved in ref. [2]. A complex structure is in one-to-one correspondence with a Cartan decomposition of the Lie algebra. On the root space, the complex structure is diagonalized and has eigenvalue  $i$  or  $-i$ , when the root is positive or negative, respectively; the Cartan subalgebra is mapped to itself. The existence of a second complex structure, anticommuting with the first one, implies a third complex structure (the product of the first two), i.e.,  $N=3$  implies  $N=4$  supersymmetry. It turns out that  $N=4$  is only possible on a restricted set of group manifolds. These group manifolds are such that they can be written as a product of coset spaces which have the following structure. Given a group  $G$  with Lie algebra  $\mathfrak{g}$  and a Cartan decomposition, we consider the highest root  $\theta$ . Then  $E_{\pm\theta}$  and  $\theta \cdot H$  form an  $\mathfrak{su}(2)$  subalgebra, which we call  $\mathfrak{su}(2)_\theta$ . The remainder of the Cartan subalgebra together with all roots perpendicular to  $\theta$  form another subalgebra  $H_\perp$ . The coset space  $W = G/H_\perp \otimes SU(2)_\theta$  is a Wolf space [6].

An  $N=4$  group manifold can be decomposed as products of coset spaces of the form  $W \otimes SU(2)_\theta \otimes U(1)$ . The second complex structure acts within each of these coset spaces. The action on  $W$  is clear as it decomposes in doublets under  $SU(2)_\theta$ . The action on  $SU(2) \otimes U(1)$  is such that  $E_{\pm\theta}$  get mapped to the Cartan subalgebra and vice versa. more details are given in refs. [2,7].

We now analyze the case of  $SU(2) \otimes U(1)$  in detail. Following the discussion above, we have essentially unique candidates for  $J_{(\pm)}$ :

$$\begin{aligned} J_{(\pm)} E_+ &= i E_+, & J_{(\pm)} E_- &= -i E_-, \\ J_{(\pm)} (H_0 + i H_3) &= \pm i (H_0 + i H_3), \\ J_{(\pm)} (H_0 - i H_3) &= \mp i (H_0 - i H_3), \end{aligned} \quad (7)$$

where  $H_0$  generates  $U(1)$  transformation and  $E_\pm, H_3$  are the generators of  $SU(2)$ . The form is fixed by the condition that  $J_{(+)}$  and  $J_{(-)}$  commute. Eq. (6) implies analogous relations for the Lie algebra valued currents:

$$\begin{aligned} (g^{-1} \tilde{Q}_+ g)^a &= J_{(+)}^a{}_b (g^{-1} \nabla_+ g)^b, \\ (\tilde{Q}_- g g^{-1})^a &= J_{(-)}^a{}_b (\nabla_- g g^{-1})^b. \end{aligned} \quad (8)$$

Using the explicit form of  $J_{(\pm)}$  (7), and the relation to the  $N=2$  derivatives  $D = \frac{1}{2}(\nabla + i\tilde{Q})$ ,  $\bar{D} = \frac{1}{2}(\nabla - i\tilde{Q})$ , we can lift the relations (8) to  $N=2$  superspace. This leads to the following parametrization of  $g$  in terms of a chiral superfield  $\Phi$  and a twisted chiral superfield  $\mathcal{A}$ :

$$g = \frac{\exp(i\theta)}{\sqrt{\Phi\bar{\Phi} + \mathcal{A}\bar{\mathcal{A}}}} \begin{pmatrix} \mathbb{1} & \bar{\Phi} \\ -\Phi & \bar{\mathcal{A}} \end{pmatrix}, \quad (9)$$

where  $\theta = -\frac{1}{2} \ln(\Phi\bar{\Phi} + \mathcal{A}\bar{\mathcal{A}})$ . This gives an off-shell  $N=2$  formulation of the group  $SU(2) \otimes U(1)$ . In these coordinates, the metric on the group manifold is

$$ds^2 = \frac{d\Phi d\bar{\Phi} + d\mathcal{A} d\bar{\mathcal{A}}}{\Phi\bar{\Phi} + \mathcal{A}\bar{\mathcal{A}}}. \quad (10)$$

In ref. [4] it was shown that the metric can be expressed in terms of a potential function (analogous to a Kähler potential in the case without torsion):  $ds^2 = K_{\Phi\bar{\Phi}} d\Phi d\bar{\Phi} - K_{\mathcal{A}\bar{\mathcal{A}}} d\mathcal{A} d\bar{\mathcal{A}}$ . Here, we find

$$K = - \int^{\mathcal{A}\bar{\mathcal{A}}/\Phi\bar{\Phi}} \frac{dx}{x} \ln(1+x) + \ln \Phi \ln \bar{\Phi}. \quad (11)$$

This is the  $N=2$  superspace lagrangian. We can read off the torsion potential from  $K_{\Phi\bar{A}}$ , etc. (see ref. [4]).

As noted above,  $SU(2)\otimes U(1)$  actually admits  $N=4$  supersymmetry. In  $N=2$  superspace, the necessary condition for  $N=4$  supersymmetry is  $K_{\Phi\Phi} + K_{\bar{A}\bar{A}}=0$ , which is clearly satisfied in this case [4]. In refs. [4,8] the  $N=4$  superspace description is given; we briefly review it here. It is convenient to group the four real spinor derivatives of  $N=4$  superspace into a complex doublet  $D_{\pm}^a, \bar{D}_{a\pm}$ ; these satisfy the algebra

$$\{D_{\pm}^a, \bar{D}_{b\pm}\} = \delta^a_b \delta_{\pm\pm}, \tag{12}$$

with  $D_{\pm}^1 = (\bar{D}_{1\pm})^*$ , etc. We introduce a left and a right Riemann sphere with homogeneous coordinates  $u_a^{(\pm)}$ , which can be used to parametrize a maximal abelian subalgebra [4,8] (see also ref. [9]):

$$\mathcal{D}_{\pm} = u_a^{(\pm)} D_{\pm}^a, \quad \tilde{\mathcal{D}}_{\pm} = u^{(\pm)a} \bar{D}_{a\pm}, \tag{13}$$

where  $u^a = \epsilon^{ab} u_b$ .  $\mathcal{D}$  and  $\tilde{\mathcal{D}}$  map into each other under the real structure  $\sim$  defined by the composition of complex conjugation with the antipodal map on the spheres, which simply takes  $\bar{u}^a = (u_a)^*$  into  $u^a$ . Since the algebra (13) is abelian, we can consistently constrain an  $N=4$  superfield  $\eta$  by  $\mathcal{D}\eta = \tilde{\mathcal{D}}\eta = 0$ . To completely define the multiplet, we specify its analytic and reality properties:

$$\eta = u_a^+ u_b^- \eta^{ab}, \quad \bar{\eta} = \eta. \tag{14}$$

As explained in refs. [4,8], this implies that  $\eta$  has  $N=2$  components

$$\eta^{11} = \Phi, \quad \eta^{12} = \bar{A}, \quad \eta^{21} = -A, \quad \eta^{22} = \Phi, \tag{15}$$

with  $\Phi$  and  $A$  as in (3) and (4). The  $N=4$  superspace action can be taken to be

$$S = \int d^2z \int_C \frac{d\zeta^{(+)}}{2\pi i} \times \int_{C'} d\zeta^{(-)} D_+^1 D_-^1 \bar{D}_{1+} \bar{D}_{1-} f(\zeta^{(+)}, \zeta^{(-)}, \eta), \tag{16}$$

where  $\zeta^{(\pm)} = u_1^{(\pm)}/u_2^{(\pm)}$  are inhomogeneous coordinates on the Riemann spheres and the measure, written in terms of  $N=2$  spinor derivatives, is  $N=4$  invariant because  $\mathcal{D}, \tilde{\mathcal{D}}$  annihilate the Lagrangian  $f$ . (In inhomogeneous coordinates,  $\eta = \Phi + \zeta^{(-)} \bar{A} - \zeta^{(+)} A$

+  $\zeta^{(+)} \zeta^{(-)} \Phi$ .) The contours  $C$  and  $C'$  are to be chosen appropriately, and depend on the form of  $f$ .

In the case of the super WZW model on  $SU(2)\otimes U(1)$ ,

$$f = \frac{\ln \eta}{\zeta^{(+)} \zeta^{(-)}}, \tag{17}$$

and the contours appear difficult to determine. For

$$K_{\Phi\Phi} = \frac{1}{\Phi\bar{\Phi} + A\bar{A}} = - \int_C \frac{d\zeta^{(+)}}{2\pi i} \int_{C'} d\zeta^{(-)} \frac{1}{\eta^2}, \tag{18}$$

the contours are:  $C'$  is any open contour with endpoints  $a$  and  $b$ , and  $C$  is any contour enclosing the point  $\eta(\zeta^{(+)}, b) = 0$ :

$$\zeta^{(+)} = - \frac{\eta^{11} + b\eta^{12}}{\eta^{21} + b\eta^{22}} = \frac{\Phi + b\bar{A}}{A - b\Phi}. \tag{19}$$

(The contour  $C$  should *not* enclose the point  $\eta(\zeta^{(+)}, a) = 0$ .)

The existence of a fully off-shell formulation of the model has an important consequence: it is straightforward to deform the model while maintaining full  $N=4$  supersymmetry, and hence conformal invariance. An obvious deformed action is

$$f(\zeta^{(\pm)}, \eta) = \frac{1}{\zeta^{(+)} \zeta^{(-)}} \times \left( a \frac{\eta^2}{\zeta^{(+)} \zeta^{(-)}} + k \sum_i \ln(\eta - \eta_i) \right), \tag{20}$$

where  $a$  is a constant and the  $\eta_i$  depend only on  $\zeta^{(\pm)}$ , and hence introduce four more constants for each  $i$ . This CFT has recently been proposed as a stringy instanton solution [10].

The question which arises now is whether an off-shell formulation solely in terms of chiral and twisted chiral superfields is possible for any  $N=2$  WZW model. As we will argue now,  $SU(2)\otimes U(1)$  is a solitary case. We start from the observation, given before, that the left- and right-handed complex structures should commute if we insist on using only chiral and twisted chiral multiplets. This condition becomes

$$[J_{(+)}, gJ_{(-)}g^{-1}] = 0, \tag{21}$$

where  $J_{(+)}$  and  $J_{(-)}$  are the complex structures in the left and right invariant bases, respectively, and  $g$  is a group element in the adjoint representation. To sec-

ond order these conditions are equivalent to

$$[J_{(+)}, J_{(-)}] = 0, \quad [J_{(+)}, [T, J_{(-)}]] = 0, \quad (22)$$

where  $T$  is an element of the Lie algebra in the adjoint representation. We first diagonalize  $J_{(+)}$ . Then the first equation implies that  $J_{(-)}$  does not mix holomorphic with anti-holomorphic directions. The  $[J_{(+)}, [T_a, J_{(-)}]]_{\bar{\alpha}^{\beta}} = 0$  component of the second condition implies  $f_{\bar{\alpha}\bar{\alpha}^{\beta}} = 0$  where  $\bar{\alpha}$  is a negative root and  $\beta$  a positive root, and  $T_a$  is any basis vector for the Lie algebra. One sees that one can always find a positive root  $a$  such that this condition is violated except in the case that the group is  $[\text{SU}(2)]^m \otimes [\text{U}(1)]^n$ . Analyzing the next order relation  $[J_{(+)}, [T_a, [T_b, J_{(-)}]]] = 0$  in the same way further restricts the allowed groups to  $[\text{SU}(2) \otimes \text{U}(1)]^m \otimes [\text{U}(1)]^{2n}$ .

In ref. [8], new  $N=2$  and  $N=4$  superfields were introduced which make it possible to write superspace actions for models where  $[J_{(+)}, J_{(-)}] \neq 0$ . These superfields are semi-chiral, and arise naturally in the  $N=4$  context described above when one simply changes the analytic dependence of  $\eta$ . We speculate that super WZW models on other group manifolds can be described off shell with the aid of such superfields.

The explicit  $N=2$  superspace action has the form that admits a duality transformation [11,4,12]. After the transformation, all the superfields are chiral, and the manifold is therefore Kähler (the torsion vanishes). The manifold turns out to be a product manifold of a torus with the disk bearing the singular metric

$$ds^2 = \frac{du d\bar{u}}{1 - u\bar{u}}, \quad (23)$$

and a dilaton  $\ln(1 - u\bar{u})$  [12]. Conformal field theoretical aspects of this observation are being investigated [13]. Very recently, an analytic continuation of this metric has been interpreted as a two-dimensional black hole [14].

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