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QUANTUM W_3 GRAVITY IN THE CHIRAL GAUGE

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We analyze the quantum theory of W_3 gravity in the chiral gauge, coupled to scalar fields or to W_3 matter. We evaluate the effective action through 3-loop order, using both Feynman diagrams and OPE methods. We exhibit two distinct Ward identities for the W_3 symmetry, in which new terms appear due to the essential non-linearities in the theory. The relation with KPZ's program for chiral gravity and its extension to W_3 gravity are briefly discussed.

1. Introduction

Over the years there have been attempts to generalize theories of classical and quantum gravity (in arbitrary dimension D) to higher-spin extensions that include gauge fields of spin greater than or equal to three. Only recently it has become clear that in two dimensions such extensions – both at the classical and the quantum level – can successfully be made. The appropriate gauge algebras turn out to be so-called W-algebras, which are higher spin extensions of the Virasoro algebra; hence the name W-gravities. The prototype of these theories in the spin-3 extension of 2D gravity which is based on Zamolodchikov's W_3 algebra [1], called W_3 gravity, which is the subject of this paper.

A good understanding of the classical theory of W_3 gravity has been achieved recently. A first result was the formulation of Hull [2] of W_3 gravity in the chiral gauge coupled to scalar matter fields. In refs. [3, 4] this result was extended to a fully covariant formulation with vielbein-type gauge fields $e_\mu^\pm$ and $B_\mu^{\pm\pm}$ for the spin-2 and spin-3 gauge fields, respectively.

In this paper we focus on quantum W_3 gravity in the chiral gauge. Our first goal is to reach a proper understanding of this theory in the framework of perturbative quantum field theory; in particular we would like to understand the anomaly structure and to obtain exact results for the effective action. Such results can then be compared to results in the conformal gauge, and, eventually, be extended to covariant results. Later on one would like to compare this continuum formulation

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with predictions based on matrix models and on topological field theory. A particular challenge is to try to connect results based on lagrangian field theory with the group-theoretical analysis of ref. [5] and with considerations based on quantum W_3 geometry [6]. With these developments ahead, we now focus on the lagrangian formulation of chiral W_3 gravity.

Let us first recall some facts about pure gravity in the chiral gauge, with gravitational field h_{++} . One defines the so-called induced action $\Gamma[h]$, which is obtained by first coupling pure gravity to matter fields ϕ , giving rise to a classical action $S[h, \phi]$, and subsequently integrating out the matter fields

$$e^{\Gamma[h]} = \int \mathcal{D}\phi e^{S[h, \phi]}. \quad (1.1)$$

The induced action $\Gamma[h]$ is then taken as the starting point for the formulation of quantum gravity. The contents of the latter are summarized by the generating functional $W[t]$, defined as

$$e^{W[t]} = \int \mathcal{D}h e^{\Gamma[h] + (1/\pi) \int h t}. \quad (1.2)$$

The expression for $\Gamma[h]$, which is derived from the celebrated covariant form $\int R(1/\square)R$, is well known; in sect. 2 we will review its derivation in the chiral gauge as a prelude to our later computations in W_3 gravity. A strong result of KPZ [7], as interpreted in ref. [8], is an exact expression for the functional $W[t]$,

$$W[t] = \frac{1}{2} k_c W_L[z_c t], \quad (1.3)$$

where k_c and z_c depend as follows on the central charge c (which is a measure for the “size” of the matter system ϕ , for example the number n of scalar fields, and which enters the induced action $\Gamma[h]$ as a multiplicative factor) in the region $c < 1$

$$c - 13 = 6 \left(k + 2 + \frac{1}{k+2} \right),$$

$$z = \frac{2}{k+2}. \quad (1.4)$$

The functional $W_L(u)$ is defined through the following differential equation

$$(\partial_-^3 + 2u\partial_- + \partial_- u) \frac{\delta W_L}{\delta u} = (1/\pi) \partial_+ u, \quad (1.5)$$

which in the field theory has the interpretation of an anomalous Ward identity, see eq. (2.31).

The full result (1.3) for $W[t]$ can be understood in a loop expansion of the path integral (1.2), which corresponds to an expansion in $1/c$. The lowest-order (saddle point) contribution is given by

$$W_{\text{saddle}}[t] = \frac{c}{12} W_L \left[\frac{12}{c} t \right], \quad (1.6)$$

showing that the functional $W_L[t]$ is essentially given by the Legendre transform of the induced action $\Gamma[h]$. The one-loop, i.e. order $1/c$, corrections to the saddle-point result, which were computed by Polyakov (unpublished) and are reproduced in ref. [8], agree with the exact result (1.3).

In our analysis of W_3 gravity we follow steps similar to those above and eventually hope to achieve a result analogous to (1.3). Partial results in this direction were obtained in refs. [9, 10]. However, these papers did not properly treat the non-linear terms in the W_3 current algebra, which make out the essential complication of W_3 gravity with respect to ordinary gravity and which contribute if all orders in $1/c$ are considered. Some results in the conformal gauge were obtained in refs. [11, 12].

A classical action for spin-2 and spin-3 gravitons h_{++} and B_{+++} coupled to n real scalar fields φ^i , $i = 1, 2, \dots, n$, was given in ref. [2] (see formula (3.1)). Quantizing the φ^i fields (keeping h and B as external fields) leads to the effective action $\Gamma[h, B; \varphi_{\text{cl}}^i]$. Putting $\varphi_{\text{cl}}^i = 0$ in there leads to the induced action $\Gamma[h, B]$, which can again be defined directly as in (1.1). A striking difference with the case of pure gravity is that now $\Gamma[h, B]$ receives contributions from diagrams with arbitrarily many loops, as compared to the pure gravity case where only 1-loop diagrams contribute. As a consequence, the induced action for W_3 gravity is no longer proportional to c (or n) but it can be expanded in a large c (or n) expansion and contains then also terms proportional to $c^0, c^{-1}, \dots$ (or $n^0, n^{-1}, \dots$).

Only for $n = 2$ the n scalar theory has exact W_3 at the quantum level [2, 3]. We also define an induced action $\Gamma_{W_3}[h, B]$, which is formally obtained by integrating out (as in (1.1)) a matter system of central charge c with exact W_3 symmetry also at the quantum level. Although an explicit lagrangian for such matter systems is not known, one can still extract the expression for Γ_{W_3} by using operator product expansion (OPE) methods. Indicating the effective action obtained from the field theory in (3.1) by $\Gamma_\varphi[h, B; \varphi_{\text{cl}}^i]$, we have $\Gamma_\varphi[h, B] = \Gamma_{W_3}[h, B]$ (up to a scale factor in the definition of B , see eq. (3.18)) if $n = c = 2$. In sect. 3 we will find that explicit differences between Γ_φ and Γ_{W_3} start appearing at the 3-loop level if $n = c \neq 2$.

The main issue is now to try to establish anomalous Ward identities for the actions $\Gamma_\varphi[h, B; \varphi_{\text{cl}}^i]$, $\Gamma_\varphi[h, B]$ and $\Gamma_{W_3}[h, B]$, to all orders in perturbation theory.

This will turn out to be much more involved than in the case of pure gravity because the spin-3 gauge symmetry, called λ symmetry, is non-linear (as is dictated by the W_3 algebra). We will establish two alternative (anomalous) Ward identities for this λ symmetry.

The first Ward identity for Γ_φ is essentially the standard Ward identity which one can write in a general quantum field theory. However, in this case there is a technical complication which has been considered only recently in ref. [13], and which is due to the fact that under λ transformations the *external* field h_{++} transforms into a product of *quantum* fields φ^i . The correct Ward identity under this circumstance, which was proved in ref. [13], is given in eq. (4.7). It contains a composite operator on the l.h.s., which implies that it does not simply describe the response of the effective action under a gauge transformation. In sect. 4 we will propose an exact (all-order) result for the local quantity Δ_λ in the r.h.s. of the Ward identity. This Δ_λ is related to the λ anomaly $\text{An}[\lambda]$, but the two are not the same because of the presence of the composite operator.

In order to study the true ϵ and λ anomalies $\text{An}[\epsilon]$ and $\text{An}[\lambda]$, we will then turn to the Wess–Zumino consistency conditions for consistent anomalies. Starting from certain transformation rules for the fields h_{++} and B_{+++} , containing already some suitably chosen terms of order $\hbar$, we will find a solution for $\text{An}[\epsilon]$ and $\text{An}[\lambda]$ to all-loop order, which is valid under the assumption of certain local transformation rules, given in (5.9) and (5.15), for the effective currents T^{eff} and W^{eff} . With that result, it would be possible to remove all anomalies except the lowest ones (which are independent of quantum fields) by redefinitions of the transformation rules. We will argue that this proposal for the λ anomaly is correct through 2-loop order. However, explicit 3-loop computations show further (non-local) λ anomalies, implying that the transformation rules of the currents are different from the simple local form in (5.9) and (5.15).

It is possible to write an exact (but formal) expression for the response of the effective action under local λ transformations, i.e. for the λ anomaly $\text{An}[\lambda]$. It involves the (non-local) expression R^{eff} (in the n scalar theory) or Λ^{eff} (in the W_3 theory). By working out these expressions in perturbation theory, we will recover the results obtained from the Wess–Zumino conditions, and the extra 3-loop terms mentioned above. These results can then be cast into the form of functional equations for the induced actions Γ_φ (without external φ lines) and Γ_{W_3} . These equations, given in eqs. (5.21) and (5.22), involve the third- and fifth-order Gelfand–Dickey operators

$$\begin{aligned} D_1 &= \partial_-^3 + 2u\partial_- + u', \\ D_2 &= \partial_-^5 + 10u\partial_-^3 + 15u'\partial_-^2 + 9u''\partial_- + 2u''' + 16u^2\partial_- + 16uu', \end{aligned} \quad (1.7)$$

where the primes denote ∂_- . However, we should stress that they can not be

written as *local* relations among h , B and the effective currents T^{eff} and W^{eff} represented by u and v . This property is lost due to the presence of the 3-loop terms mentioned above.

One may hope that the locality of the functional equations, which, as we just mentioned, fails to hold for Γ_φ and Γ_{W_3} and hence for the saddle point approximations to $W_\varphi[u, v]$ and $W_{W_3}[u, v]$, will be restored if one includes loop corrections in the path-integral expressions for the functionals $W[u, v]$. The complete perturbative evaluation of $W_\varphi[u, v]$ and $W_{W_3}[u, v]$ involves two independent loop expansions, with matter loops due to the path integral over φ and gauge fields loops due to the integral and h and B . It is therefore much more intricate than the similar evaluation in pure gravity. We intend to come back to this issue in a future publication [14].

An alternative way to quantize chiral W_3 gravity is through the Batalin–Vilkovisky (BV) quantization scheme. For a pure W_3 theory with central charge $c = 100$, the relevant BRST charge is anomaly-free at the quantum level, as was shown in refs. [15, 16]. One finds that for other values of c and for the n scalar theory, there are BRST anomalies related to the fact that h and B are propagating fields in the full quantum W_3 theory. We intend to discuss this issue in a separate publication. In the approach that we follow here (compare with ref. [8]) one does not introduce ghosts in the path integral over h and B (as in (1.2)), simply because the effective action does not have exact gauge invariance due to anomalies. The anomalies provide kinetic terms for the gauge fields in $\Gamma[h, B]$, so that propagators are well defined. One does not need to split off the gauge volume from the path integral so that ghosts never come into the picture.

This paper is organized as follows. In the next section we review the treatment of pure 2D gravity as a prelude to our computations in W_3 gravity. In sect. 3 we present explicit Feynman diagram computations in W_3 gravity, which we compare with results obtained by the OPE method. These results include explicit expressions for the basic 3-loop diagrams in the induced action for W_3 gravity. In the sects. 4 and 5 we discuss the various Ward identities, which we already mentioned above. Sect. 6 contains an outlook on further developments.

2. Pure gravity

As a preliminary, we work through the case of (euclidean) chiral gravity [17] with action

$$S[h, \varphi^i] = (1/\pi) \int \left(-\frac{1}{2} \partial_+ \varphi^i \partial_- \varphi^i + \frac{1}{2} h_{++} \partial_- \varphi^i \partial_- \varphi^i + J_i \varphi^i \right) d^2z, \quad (2.1)$$

where $i = 1, 2, \dots, n$ and ∂_- and ∂_+ stand for ∂_z and $\partial_{\bar{z}}$, respectively. The

generating functional for connected graphs, $W[h_{++}, J^i]$, is given by

$$\begin{aligned} \exp W[h_{++}, J^i] / \hbar &= \sum_{m=0}^{\infty} \frac{1}{m!} \left[\frac{\pi \hbar}{2} h_{++} \left(\partial_- \frac{\delta}{\delta J^i} \right) \left(\partial_- \frac{\delta}{\delta J^i} \right) \right]^m \\ &\times \exp \frac{1}{2\pi \hbar} \int \left(\frac{1}{\partial_+} J^i \right) \left(\frac{1}{\partial_-} J^i \right) d^2 z. \end{aligned} \quad (2.2)$$

For the dressed propagator, that is the sum of all tree graphs with a φ -line and $0, 1, 2, \dots$ h -vertices, one finds

$$W^{(\text{tree})}[h_{++}, J^i] = -\frac{1}{2\pi} \int J^i \frac{1}{\partial_+} \frac{1}{1 - h_{++} \frac{\partial_-}{\partial_+}} \frac{1}{\partial_-} J^i d^2 z. \quad (2.3)$$

The dressed propagator is clearly proportional to the inverse of the φ -field equation since

$$\frac{1}{\partial_+} \frac{1}{1 - h_{++} \frac{\partial_-}{\partial_+}} \frac{1}{\partial_-} = \frac{1}{\partial_- \left(1 - h_{++} \frac{\partial_-}{\partial_+} \right) \partial_+} = [\partial_- \partial_+ - \partial_- (h_{++} \partial_-)]^{-1} \equiv \square^{-1}. \quad (2.4)$$

Under the ϵ -transformations

$$\begin{aligned} \delta_\epsilon \varphi^i &= \epsilon_+ \partial_- \varphi^i, & \delta_\epsilon J^i &= \partial_- (\epsilon_+ J^i), \\ \delta_\epsilon h_{++} &= \partial_+ \epsilon_+ - h_{++} \partial_- \epsilon_+ + \epsilon_+ \partial_- h_{++}, \end{aligned} \quad (2.5)$$

the action in equation (2.1) is invariant, and hence $W^{(\text{tree})}[h_{++}, J^i]$ should also be ϵ invariant. To prove this, note that $\square S$ is a scalar density if S is a scalar ($\delta_\epsilon S = \epsilon_+ \partial_- S$). Hence

$$\delta_\epsilon \square = \partial_- \epsilon_+ \square - \square \epsilon_+ \partial_-, \quad (2.6)$$

as one may also verify by direct evaluation, and with this result the ϵ -invariance of $W^{(\text{tree})}$ follows easily.

To obtain the generator $\Gamma[h_{++}; \varphi_{\text{cl}}^i]$ of one particle irreducible (1PI) graphs, one should make a Legendre transformation

$$\Gamma[h_{++}; \varphi_{\text{cl}}^i] = W[h_{++}, J^i] - (1/\pi) \int J^i \varphi_{\text{cl}}^i d^2z, \quad (2.7)$$

where the “classical field” φ_{cl}^i is given by $\pi \delta W / \delta J^i = -\square^{-1} J^i$. As expected, the tree result ($\hbar = 0$) is just the classical action

$$\Gamma^{(\text{tree})}[h_{++}; \varphi_{\text{cl}}^i] = (1/2\pi) \int \varphi_{\text{cl}}^i \square \varphi_{\text{cl}}^i d^2z. \quad (2.8)$$

Turning to loops, we first consider the two-point function. Using eq. (2.2), and putting $\hbar = 1$, one finds

$$\text{---}\bigcirc\text{---} = n \int \left[\frac{1}{2} h_{++}(z) \left\{ \frac{\partial_-}{\partial_+} \delta^2(z-w) \right\}^2 \frac{1}{2} h_{++}(w) \right] d^2z d^2w. \quad (2.9)$$

To make sense of the product of distributions, i.e. to specify our regularization procedure, we use the representation of the delta function in terms of $\ln(z-w)^2 = \ln(\bar{z}-\bar{w}) + \ln(z-w)$,

$$\pi \frac{1}{\partial_+} \delta^2(z-w) = \frac{1}{z-w}. \quad (2.10)$$

This relation holds only in euclidean space, and may be proved by partially integrating $\int \theta(z\bar{z}-\epsilon) \partial_+ f d^2z$. In Minkowski space-time one may use analytic regularization [18]; dimensional regularization might also work, although it is less obvious how this should be applied to a chiral theory. Repeated differentiation of (2.10) with respect to z yields the result

$$\pi \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial_-^{n-1}}{\partial_+} \delta^2(z-w) = \frac{1}{(z-w)^n}. \quad (2.11)$$

Using this relation twice leads to the identity

$$\left[\frac{\partial_-}{\partial_+} \delta^2(z-w) \right]^2 = -\frac{1}{6\pi} \frac{\partial_-^3}{\partial_+} \delta^2(z-w). \quad (2.12)$$

Since we may partially integrate with $1/\partial_+$, but note that the Leibniz rule does not

hold:

$$\frac{1}{\partial_+}(AB) \neq A \frac{1}{\partial_+} B + B \frac{1}{\partial_+} A,$$

we find

$$\text{tadpole} = \frac{-n}{24\pi} \int h_{++} \frac{\partial_-^3}{\partial_+} h_{++} d^2z. \quad (2.13)$$

For the tadpole diagram we obtain

$$\text{tadpole} = \frac{n}{2} \int h_{++}(z) \left[\frac{\partial_-}{\partial_+} \delta^2(z-w) \right] \delta^2(z-w) d^2z d^2w. \quad (2.14)$$

Putting $\delta^2(z-w) = \partial_+(z-w)^{-1}/\pi$ and $(\partial_-/\partial_+)\delta^2(z-w) = -(1/\pi)(z-w)^{-2}$, one obtains from eq. (2.11)

$$\frac{1}{3\pi^2} \partial_+ \frac{1}{(z-w)^3} = \frac{\partial_-^2}{6\pi} \delta^2(z-w). \quad (2.15)$$

This then shows that

$$\int \left[\frac{\partial_-}{\partial_+} \delta^2(z-w) \right] \delta^2(z-w) d^2w = 0, \quad (2.16)$$

so that the tadpole diagram vanishes

$$\text{tadpole} = 0. \quad (2.17)$$

Turning to a more complicated graph, we find

$$\begin{aligned} \text{graph} &= \frac{n}{3!} \int \left[h_{++}(z_1) \frac{\partial_-}{\partial_+} \delta^2(z_1-z_2) \right] \left[h_{++}(z_2) \frac{\partial_-}{\partial_+} \delta^2(z_2-z_3) \right] \\ &\quad \times \left[h_{++}(z_3) \frac{\partial_-}{\partial_+} \delta^2(z_3-z_1) \right] d^2z_1 d^2z_2 d^2z_3. \end{aligned} \quad (2.18)$$

Partial integration to liberate one delta function of derivatives, for example the first one, leads to the expression

$$\left[h_{++}(z_3) \frac{\partial_-}{\partial_+} \delta^2(z_3-z_2) \right] h_{++}(z_2) \frac{\partial_-}{\partial_+} \left[h_{++}(z_2) \frac{\partial_-}{\partial_+} \delta^2(z_2-z_3) \right], \quad (2.19)$$

which is cumbersome to deal with. A simple trick solves this problem: writing $h_{++} = (\partial_+/\partial_+)h_{++}$, we partially integrate the ∂_+ in the numerator. In this way we find

$$\begin{aligned} \text{Diagram} &= \frac{n}{36\pi} \int \left[h_{++} \left(\frac{\partial_-^2}{\partial_+} h_{++} \right) - \left(\frac{1}{\partial_+} h_{++} \right) (\partial_-^2 h_{++}) \right] \frac{\partial_-^2}{\partial_+} h_{++} d^2z \\ &= \frac{n}{24\pi} \int h_{++} \left[\frac{\partial_-^2}{\partial_+} h_{++} \right]^2 d^2z. \end{aligned} \quad (2.20)$$

For obtaining the last line we used once again the trick of replacing $\partial_-^2 h_{++}$ by $\partial_+ (\partial_-^2/\partial_+) h_{++}$.

For the sum of all 2, 3, 4, ... point one-loop graphs one expects a result proportional to $\sqrt{g} R \square^{-1} \sqrt{g} R$, with $\square = \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu$. In the chiral gauge, where only h_{++} is non-trivial, one has

$$\sqrt{g} R = R = \partial_-^2 h_{++}. \quad (2.21)$$

Indeed, the complete one-loop result reads

$$\Gamma^{(1\text{-loop})}[h_{++}] = \frac{-n}{24\pi} \int \partial_-^2 h_{++} \frac{1}{\partial_+} \frac{1}{1 - h_{++} \frac{\partial_-}{\partial_+}} \frac{1}{\partial_-} \partial_-^2 h_{++} d^2z. \quad (2.22)$$

This result contains eqs. (2.13) and (2.20). In fact, since there are only graphs with one loop, this is the complete quantum part of the effective action. In W_3 gravity, where vertices cubic in quantum fields are present, $\Gamma^{(1\text{-loop})}$ will also depend on φ_{cl}^i and the complete effective action will receive contributions from diagrams with arbitrary numbers of loops.

The effective action is not invariant under local ϵ transformations. The response under $\delta_\epsilon h_{++}$ as in (2.5) is most easily computed using eq. (2.6). One finds

$$\delta_\epsilon \Gamma = \frac{-n}{12\pi} \int h_{++} \partial_-^3 \epsilon_+ d^2z. \quad (2.23)$$

This is thus the ϵ -anomaly. The ϵ -transformations are a combination of the Einstein (= general coordinate), local Lorentz and Weyl (= local scale) transformations of the covariant theory, and the ϵ -anomaly should be viewed as a manifestation of the Weyl anomaly. This is clearly illustrated by the transformation

law of $\sqrt{g}R = \partial_-^2 h_{++}$,

$$\delta_\epsilon \sqrt{g}R = \partial_-(\epsilon_+ R) + \square(\partial_- \epsilon_+). \quad (2.24)$$

The effective stress tensor of this theory, T_{--}^{eff} , is defined as the variation of the effective action with respect to the field h_{++} . Normalizing it to $T_{--}^{\text{eff}} = -\frac{1}{2}\partial_- \varphi^i \partial_- \varphi^i + \dots$, we have

$$T_{--}^{\text{eff}} = -\pi \frac{\delta}{\delta h_{++}} \Gamma = -\frac{1}{2} \partial_- \varphi^i \partial_- \varphi^i + \frac{n}{12} \frac{1}{\partial_+ - h_{++} \partial_- - 2(\partial_- h_{++})} \partial_-^3 h_{++}. \quad (2.25)$$

One may prove this either by direct computation or by rewriting eq. (2.23) as in eq. (2.31).

Before we turn to W_3 gravity, let us mention that the above results are easily reproduced in a formalism based on operator product expansions (OPE's). Focusing on diagrams with no external φ 's, we can write

$$\begin{aligned} \exp \Gamma^{(1\text{-loop})}[h_{++}] &= \left\langle \exp \left\{ - (1/\pi) \int d^2 z h_{++} T_{--}^\varphi \right\} \right\rangle \\ &= \sum_N \frac{(-1)^N}{N!} \langle \Xi(1) \dots \Xi(N) \rangle, \end{aligned} \quad (2.26)$$

where $T_{--}^\varphi = -\frac{1}{2} \partial_- \varphi^i \partial_- \varphi^i$, $\Xi(j) = (1/\pi) \int d^2 z_j h_{++} T_{--}^\varphi$ and the brackets denote the expectation value with respect to φ^i . The N -point functions for T_{--}^φ are easily computed by using the basic OPE $\varphi^i(z_1) \varphi^j(z_2) = -\delta^{ij} \log(z_1 - z_2) + \dots$ and

$$T_{--}^\varphi(z_1) T_{--}^\varphi(z_2) = \frac{n/2}{(z_1 - z_2)^4} + \frac{2T_{--}^\varphi(z_2)}{(z_1 - z_2)^2} + \frac{\partial_- T_{--}^\varphi(z_2)}{(z_1 - z_2)} + \dots \quad (2.27)$$

For example, for the 3-point function one finds

$$\begin{aligned} &\langle T_{--}^\varphi(z_1) T_{--}^\varphi(z_2) T_{--}^\varphi(z_3) \rangle \\ &= n \left[\frac{1}{(z_1 - z_2)^2 (z_2 - z_3)^4} - \frac{2}{(z_1 - z_2)(z_2 - z_3)^5} + \dots + (2 \leftrightarrow 3) \right], \end{aligned} \quad (2.28)$$

which can be used to rederive eq. (2.20) by applying eq. (2.11). (The terms which are non-singular in one of the differences $(z_i - z_j)$, denoted by $\dots$, do not contribute since, when hit by a ∂_+ , they give rise to expressions like $z \delta^2(z, \bar{z})$ which vanish.)

An advantage of the OPE approach is that it allows one to derive anomalous conservation laws such as (2.23) without explicitly computing the effective action. Substituting the variation $\delta_\epsilon h_{++}$ given in (2.5) into (2.26) leads to

$$\begin{aligned}
 \delta_\epsilon \exp \Gamma &= \sum_N \frac{(-1)^N}{(N-1)!} \frac{1}{\pi} \int d^2 z_1 (\partial_+ \epsilon_+ + \epsilon_+ \partial_- h_{++} - \partial_- \epsilon_+ h_{++})(z_1) \\
 &\quad \times \langle T_{--}(z_1) \Xi(2) \dots \Xi(N) \rangle \\
 &= \sum_N \frac{(-1)^N}{(N-2)!} \frac{1}{\pi^2} \int d^2 z_1 d^2 z_2 (\partial_+ \epsilon_+ + \epsilon_+ \partial_- h_{++} - h_{++} \partial_- \epsilon_+)(z_1) h_{++}(z_2) \\
 &\quad \times \left\{ \frac{n/2}{(z_1 - z_2)^4} \langle \Xi(3) \dots \Xi(N) \rangle \right. \\
 &\quad \left. + \left[\frac{2}{(z_1 - z_2)^2} + \frac{1}{(z_1 - z_2)} \partial_-^{(2)} \right] \langle T_{--}(z_2) \Xi(3) \dots \Xi(N) \rangle \right\}. \quad (2.29)
 \end{aligned}$$

The terms in $\delta_\epsilon h_{++}$ that are linear in h_{++} precisely cancel the contributions from the $\partial_+ \epsilon$ terms with the second- and first-order poles, leaving us with

$$\begin{aligned}
 \delta_\epsilon \exp \Gamma &= \frac{1}{\pi^2} \int d^2 z_1 d^2 z_2 \partial_+ \epsilon_+(z_1) h_{++}(z_2) \frac{n/2}{(z_1 - z_2)^4} \exp \Gamma \\
 &= \left[-\frac{n}{12\pi} \int d^2 z_1 (\partial_-^3 \epsilon_+) h_{++} \right] \exp \Gamma \quad (2.30)
 \end{aligned}$$

as in eq. (2.23). This result can be rephrased in the form of the following functional equation for the induced action $\Gamma[h]$:

$$\left[\partial_+ - h_{++} \partial_- - 2(\partial_- h_{++}) \right] \frac{\delta \Gamma}{\delta h} = -\frac{n}{12\pi} \partial_-^3 h_{++}, \quad (2.31)$$

which can be shown, using (1.2) and (1.6), to be equivalent to eq. (1.5). In the case of W_3 gravity derivations such as the above will be more cumbersome due to the fact that in that case the operator product algebra is non-linear.

3. W_3 gravity: the effective action

3.1. ELEMENTARY DIAGRAMS

After our review of pure gravity, we now address chiral quantum W_3 gravity. We shall treat h_{++} and B_{+++} as external fields and work without ghosts. The action now reads [2]

$$S[h_{++}, B_{+++}, \varphi^i] = (1/\pi) \int \left[-\frac{1}{2} \partial_+ \varphi^i \partial_- \varphi^i + \frac{1}{2} h_{++} \partial_- \varphi^i \partial_- \varphi^i + \frac{1}{3} B_{+++} d^{ijk} \partial_- \varphi^i \partial_- \varphi^j \partial_- \varphi^k + J^i \varphi^i \right] d^2z. \quad (3.1)$$

This action is invariant under the following ϵ (gravity) and λ (W_3 gravity) gauge transformations:

$$\begin{aligned} \delta_{\epsilon, \lambda} h_{++} &= \partial_+ \epsilon_+ - h_{++} \partial_- \epsilon_+ + \epsilon_+ \partial_- h_{++} + (\lambda_{++} \partial_- B_{+++} - B_{+++} \partial_- \lambda_{++}) \partial_- \varphi^i \partial_- \varphi^i, \\ \delta_{\epsilon, \lambda} B_{+++} &= \epsilon_+ \partial_- B_{+++} - 2 B_{+++} \partial_- \epsilon_+ + \partial_+ \lambda_{++} - h_{++} \partial_- \lambda_{++} + 2 \lambda_{++} \partial_- h_{++}, \\ \delta_{\epsilon, \lambda} \varphi^i &= \epsilon_+ \partial_- \varphi^i + \lambda_{++} d^{ijk} \partial_- \varphi^j \partial_- \varphi^k, \\ \delta_{\epsilon, \lambda} J^i &= \partial_- [\epsilon_+ J^i + \lambda_{++} d^{ijk} \partial_- \varphi^j J^k]. \end{aligned} \quad (3.2)$$

The d -symbols d^{ijk} are completely symmetric, traceless, and satisfy ([2], see also ref. [19])

$$d^{i(jk} d^{lm)i} = \delta^{(jk} \delta^{lm)}. \quad (3.3)$$

Some useful identities involving d -symbols, which can be derived from these defining properties, are given in table 1.

The generating functional $W[h_{++}, B_{+++}, J^i]$ is again written as in (2.2), and we can compute 1PI loop graphs that contribute to the effective action $\Gamma_\varphi[h, B; \varphi_{cl}^i]$. For example, the simplest new graph without external φ_{cl} lines is the following diagram, which was first discussed by Matsuo in ref. [9]:

$$\begin{aligned} \text{Diagram} &= \frac{1}{3} \pi d^{ijk} d^{ijk} \int B_{+++}(z) \left[\frac{\partial_-}{\partial_+} \delta^2(z-w) \right]^3 B_{+++}(w) d^2z d^2w \\ &= \frac{\frac{1}{2} n(n+2)}{360 \pi} \int B_{+++} \frac{\partial_-^5}{\partial_+} B_{+++} d^2z. \end{aligned} \quad (3.4)$$

TABLE 1
Identities for the d -symbols d^{ijk} . Lines indicate contracted indices. For example: the second identity reads $d^{ijk}d^{ijk} = \frac{1}{2}n(n+2)$

$= \frac{1}{2}(n+2)\delta^{ij}$
$= \frac{1}{2}n(n+2)$
$= -\frac{1}{4}(n-2)d^{ijk}$
$= -\frac{1}{8}(n-2)n(n+2)$
$= \frac{1}{4}n(n+2)^2$

From now on we shall denote h_{++} by h , B_{+++} by B etc. and omit the symbols d^2z .

The next graph we consider has an extra h -vertex. Using the $B = (\partial_+/\partial_+)B$ trick, one finds after some tedious algebra

$$\begin{aligned}
 \text{Diagram: a circle with a wavy line on the left and a vertical line on the right, with a small 'x' above the circle.} &= \frac{\frac{1}{2}n(n+2)}{360\pi} \int \left[\left(h \frac{\partial_-}{\partial_+} B - 2\partial_- h \frac{1}{\partial_+} B \right) \left(\frac{\partial_-^5}{\partial_+} B \right) \right. \\
 &\quad \left. + \left(2B \frac{\partial_-^3}{\partial_+} B - 3\partial_- B \frac{\partial_-^2}{\partial_+} B + 3\partial_-^2 B \frac{\partial_-}{\partial_+} B - 2\partial_-^3 B \frac{1}{\partial_+} B \right) \left(\frac{\partial_-^3}{\partial_+} h \right) \right]. \quad (3.5)
 \end{aligned}$$

This result is clearly proportional to the lowest-order effective field equations of h and B . An alternative way of writing it (which one obtains by writing h as $(\partial_+/\partial_+)h$ and partially integrating) is

$$\text{Diagram: a circle with a wavy line on the left and a vertical line on the right, with a small 'x' above the circle.} = \frac{\frac{1}{2}n(n+2)}{360\pi} \int \left[4 \left(\frac{\partial_-}{\partial_+} h \right) B - 2 \left(\frac{1}{\partial_+} h \right) \partial_- B \right] \left(\frac{\partial_-^5}{\partial_+} B \right). \quad (3.6)$$

Let us now take a first look at the anomalies in chiral W_3 gravity. The terms in Γ_φ with only external h -lines leads to the ϵ -anomaly (2.23) which we already discussed for pure gravity. We will now consider the ϵ variation of the 2-loop diagrams with two external B -lines. To lowest order in h (i.e., variations of the

form (ϵBB) we have two contributions: one from substituting $\delta_\epsilon B \sim \epsilon B$ into eq. (3.4) and a second from the $\delta_\epsilon h = \partial_+ \epsilon$ term in eq. (3.5) or eq. (3.6). Using the expressions (3.4) and (3.6), one easily checks that the sum cancels

$$(3.7)$$

Hence, in this example, ϵ -invariance is unbroken to lowest order in h . We will later show that the sum of *all* higher-loop diagrams with arbitrary numbers of external B - and h -lines is invariant under ϵ transformations, or, stated differently, that the full ϵ anomaly in W_3 gravity is given by the 1-loop result (2.23).

Turning to λ -invariance, we find first of all that the leading variation $\delta_\lambda B = \partial_+ \lambda$ in (3.4) leads to the following leading term of the λ -anomaly

$$\delta_\lambda \Gamma_\varphi = \frac{n(n+2)}{360\pi} \int B \partial_-^5 \lambda, \quad (3.8)$$

which is the W_3 analogue of the ϵ -anomaly (2.23). To first order in h we find

$$= \frac{n(n+2)}{360\pi} \int (2B \partial_-^3 \lambda - 3 \partial_- B \partial_-^2 \lambda + 3 \partial_-^2 B \partial_- \lambda - 2 \partial_-^3 B \lambda) \left(\frac{\partial_-^3}{\partial_+} h \right). \quad (3.9)$$

Since the result is proportional to the 1-loop h field equation, we can remove this anomaly by adding the following new term $\delta_\lambda^{(1)} h$, of order $\hbar$, to the δ_λ law for h ,

$$\delta_\lambda^{(1)} h = \frac{(n+2)}{30} (2B \partial_-^3 \lambda - 3 \partial_- B \partial_-^2 \lambda + 3 \partial_-^2 B \partial_- \lambda - 2 \partial_-^3 B \lambda). \quad (3.10)$$

and using this variation in the 1-loop graph in eq. (2.13).

Before analyzing these results from a more profound point of view, we calculate a few more graphs, and again study their ϵ and λ properties.

At the 1-loop level, graphs with two external $B\varphi_{cl}$ pairs and any number of h -fields can be obtained from the effective action of pure gravity in eq. (2.22) by

replacing h by $H^{ij} \equiv h\delta^{ij} + 2Bd^{ijk}\partial_- \varphi_{\text{cl}}^k$. In particular,

$$\text{Diagram 1} = -\frac{(n+2)}{12\pi} \int \left[(B\partial_- \varphi_{\text{cl}}^i) \frac{\partial_-^3}{\partial_+} (B\partial_- \varphi_{\text{cl}}^i) \right], \quad (3.11)$$

$$\text{Diagram 2} = \frac{(n+2)}{12\pi} \int \left[h \left(\frac{\partial_-^2}{\partial_+} (B\partial_- \varphi_{\text{cl}}^i) \right)^2 + 2 \left(\frac{\partial_-^2}{\partial_+} h \right) (B\partial_- \varphi_{\text{cl}}^i) \frac{\partial_-^2}{\partial_+} (B\partial_- \varphi_{\text{cl}}^i) \right]. \quad (3.12)$$

Invariance under ϵ -symmetry to lowest order in h can explicitly be checked: varying B and φ_{cl} in the first graph, and using $\delta h = \partial_+ \epsilon_+$ in the second graph, one finds that all terms cancel. Again, we can deduce an alternative expression for the diagram in eq. (3.12) by replacing ϵ in the ϵ -variation of eq. (3.11) by $-(1/\partial_+)h$.

Let us now turn to the λ -variations of the graphs (3.11) and (3.12). Varying B into $\partial_+ \lambda$ in the first graph leads to a non-local result of the form $(\lambda B \varphi \varphi)$. With our present δ_λ rules there are no further variations of this form. However, since we expect that 1-loop anomalies are given by *local* expressions, we anticipate further 1-loop modifications of our transformation rules. In particular, one observes that a 1-loop nonlocal extra term in the δ_λ rule for φ_{cl}^i can make the above anomaly local.

The systematics of the correction terms in the transformation rules that lead to *local* results for all anomalies can be analyzed with standard quantum field theory methods. We refer to sect. 4 for a detailed discussion of this issue.

3.2. OPE METHODS; FURTHER DIAGRAMS THROUGH 3-LOOP

In order to evaluate some further diagrams without external φ_{cl} lines, we turn to the ‘‘OPE method’’, which we already explained in the context of pure gravity. In particular, we will obtain the lowest 3-loop diagrams (with four external B -lines) explicitly, both for the theory with n scalar fields φ^i and for the theory with a pure W_3 matter system of central charge c .

Both the induced actions $\Gamma_\varphi[h, B]$ and $\Gamma_{W_3}[h, B]$ can be written as

$$\begin{aligned} e^{\Gamma[h, B]} &= \langle e^{-(1/\pi) \int h_{++} T_{--} - (1/\pi) \int B_{+++} W_{--}} \rangle \\ &= \sum_{N \geq 0} \frac{(-1)^N}{N!} \langle \Xi(1) \dots \Xi(N) \rangle, \end{aligned} \quad (3.13)$$

where $\Xi(j)$ is now defined as $\Xi(j) = (1/\pi) \int d^2 z_j (h_{++} T_{--} + B_{+++} W_{--})$.

For the n scalar theory, the currents that go into the above expression are $T_{--} = T_{-+} = -\frac{1}{2}\partial_- \varphi^i \partial_- \varphi^i$ and $W_{---} = W_{-+} = -\frac{1}{3}d^{ijk}\partial_- \varphi^i \partial_- \varphi^j \partial_- \varphi^k$. The TT and TW operator products are the standard Virasoro and spin-3 OPE's; the WW operator product reads [2]

$$W^\varphi(z)W^\varphi(w) = -(n+2) \left\{ \frac{n/3}{(z-w)^6} + \frac{2T^\varphi(w)}{(z-w)^4} + \frac{\partial_- T^\varphi(w)}{(z-w)^3} \right. \\ \left. + \frac{1}{(z-w)^2} \left[R^\varphi(w) + \frac{3}{10}\partial_-^2 T^\varphi(w) \right] \right. \\ \left. + \frac{1}{(z-w)} \left[\frac{1}{2}\partial_- R^\varphi(w) + \frac{1}{15}\partial_-^3 T^\varphi(w) \right] \right\}, \quad (3.14)$$

where $R^\varphi(w)$ is given by

$$R^\varphi(w) = \frac{4}{n+2}(T^\varphi T^\varphi)(w) - \frac{1}{2} \frac{n-2}{n+2} (\partial_-^3 \varphi^i \partial_- \varphi^i)(w) - \frac{3}{10}\partial_-^2 T^\varphi(w). \quad (3.15)$$

The terminology "effective action" for the above path-integral is justified by the circumstance that only 1PI graphs contribute to $\Gamma[h, B]$.

In the theory with pure W_3 matter of central charge c the basic OPE's are, by definition, given by the standard W_3 algebra. The WW OPE reads

$$W(z)W(w) = \left\{ \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial_- T(w)}{(z-w)^3} \right. \\ \left. + \frac{1}{(z-w)^2} \left[2\beta\Lambda(w) + \frac{3}{10}\partial_-^2 T(w) \right] \right. \\ \left. + \frac{1}{(z-w)} \left[\beta\partial_- \Lambda(w) + \frac{1}{15}\partial_-^3 T(w) \right] \right\}, \quad (3.16)$$

where $\beta = 16/(22 + 5c)$ and $\Lambda(w)$ is given by

$$\Lambda(w) = (TT)(w) - \frac{3}{10}\partial_-^2 T(w). \quad (3.17)$$

In all these expressions ordinary brackets denote standard normal ordering. Apart from an overall factor $(-n+2)$, the expressions (3.14) and (3.16) agree (with the

identifications $T^\varphi \leftrightarrow T$ and $n \leftrightarrow c$) up to terms proportional to

$$R^\varphi - 2\beta\Lambda = \frac{-12}{22+5c} \frac{n-2}{n+2} \left((TT) + \frac{n+2}{8} \partial_-^2 T + \frac{22+5n}{24} \partial_-^3 \varphi^i \partial_- \varphi^i \right), \quad (3.18)$$

which is a primary field of conformal dimension 4 [2, 3]. As mentioned before, the difference vanishes if $n = c = 2$.

For diagrams without external φ_{cl}^i lines, the number of loops is simply proportional to the number of external B -lines (which is always even): N -loop diagrams have $2(N-1)$ external B -lines (and, of course, an arbitrary number of h -lines). One immediate consequence of the fact that the difference $(R^\varphi - 2\beta\Lambda)(w)$ is primary, is that the induced actions Γ_φ and Γ_{W_3} with $n = c$ are simply proportional through 2-loop order,

$$\Gamma_\varphi^{\leq 2\text{-loop}}[h, \sqrt{(-n+2)} B] = \Gamma_{W_3}^{\leq 2\text{-loop}}[h, B]. \quad (3.19)$$

The reason for this is that the N -point function of one primary field $\phi(z_1)$ with $N-1$ fields $T(z_2), \dots, T(z_N)$ vanishes. Below we will explicitly show that, starting from 3-loop, the effective actions of the two theories differ essentially if $n = c \neq 2$.

Before we come to that, let us reconsider one of the diagrams that we computed in sect. 2. According to eq. (3.13), the expression for the 2-loop diagram in eqs. (3.5) and (3.6) can be written as

$$-(1/2\pi^3) \int d^2 z_1 d^2 z_2 d^2 z_3 h(z_1) B(z_2) B(z_3) \langle T^\varphi(z_1) W^\varphi(z_2) W^\varphi(z_3) \rangle. \quad (3.20)$$

The 3-point function in (3.20) can be written in different ways, which, however, are all equivalent when interpreted as distributions. The simplest expression is

$$\begin{aligned} & \langle T^\varphi(z_1) W^\varphi(z_2) W^\varphi(z_3) \rangle \\ &= -n(n+2) \left[\frac{1}{(z_1 - z_2)^2 (z_2 - z_3)^6} - \frac{2}{(z_1 - z_2)(z_2 - z_3)^7} + (2 \leftrightarrow 3) \right], \end{aligned} \quad (3.21)$$

leading to the following expression for the diagram in question

$$\text{diagram} = \frac{n(n+2)}{360\pi} \int \left[3 \left(\frac{\partial_-}{\partial_+} h \right) \left(B \frac{\partial_-^5}{\partial_+} B \right) + \left(\frac{1}{\partial_+} h \right) \left(B \frac{\partial_-^6}{\partial_+} B \right) \right]. \quad (3.22)$$

This expression is equal to (3.6) which we derived by Feynman diagram methods, thus confirming the equivalence of both methods of computation.

With the OPE method the computation of the basic 3-loop diagrams reduces to the evaluation of the 4-point functions of the currents $W(w)$, which can be done straightforwardly. The final result can be expressed in terms of two basic structures which are given by

$$\begin{aligned}
 [\text{I}] &= \int \left(2B \frac{\partial_-^3}{\partial_+} B - 3\partial_- B \frac{\partial_-^2}{\partial_+} B + 3\partial_-^2 B \frac{\partial_-}{\partial_+} B - 2\partial_-^3 B \frac{1}{\partial_+} B \right) \\
 &\quad \times \frac{1}{\partial_+} \left(2B \frac{\partial_-^6}{\partial_+} B + 3\partial_- B \frac{\partial_-^5}{\partial_+} B \right), \\
 [\text{II}] &= \int \left(B \frac{\partial_-}{\partial_+} B - \partial_- B \frac{1}{\partial_+} B \right) \\
 &\quad \times \frac{1}{\partial_+} \left(B \frac{\partial_-^8}{\partial_+} B + 6\partial_- B \frac{\partial_-^7}{\partial_+} B + 14\partial_-^2 B \frac{\partial_-^6}{\partial_+} B + 14\partial_-^3 B \frac{\partial_-^5}{\partial_+} B \right). \quad (3.23)
 \end{aligned}$$

The lowest-order 3-loop contributions to Γ_φ and Γ_{W_3} (i.e. the terms with four B -factors and no h -factors) are found to be

$$\Gamma_\varphi^{(3)} = -\frac{n(n+2)^2}{60 \cdot 6!} \frac{1}{\pi} [\text{I}] - \frac{n(n+2)}{5 \cdot 8!} [8(n+2) - 5(n-2)] \frac{1}{\pi} [\text{II}], \quad (3.24)$$

$$\Gamma_{W_3}^{(3)} = -\frac{c}{60 \cdot 6!} \frac{1}{\pi} [\text{I}] - \frac{2\beta c}{5 \cdot 7!} \frac{1}{\pi} [\text{II}]. \quad (3.25)$$

In the case of the n scalar theory, the above is the full result, which turns out to be the sum of two Feynman diagrams. Since the n -dependence of both diagrams is different (compare with table 1), the contributions of each separate diagram are easily identified:

$$\begin{aligned}
 \text{Diagram 1} &= -\frac{n(n+2)^2}{60 \cdot 7!} \frac{1}{\pi} (7[\text{I}] + 12[\text{II}]), \\
 \text{Diagram 2} &= \frac{(n-2)n(n+2)}{8!} \frac{1}{\pi} [\text{II}]. \quad (3.26)
 \end{aligned}$$

These results explicitly show that the simple relation (3.19) between the induced actions of the n scalar theory in (3.24) and the pure W_3 theory in (3.25) breaks down at 3-loop level if $n = c \neq 2$.

3.3. GRAVITATIONAL COVARIANTIZATIONS

In general relativity one knows how to covariantize expressions involving derivatives, since a tensor calculus is available. In chiral W_3 gravity, the ϵ transformations are a combination of Einstein, local Lorentz and Weyl symmetries, and if there are indeed no other ϵ anomalies beyond the lowest order, it should be possible to develop an “ ϵ calculus” which automatically adds the h fields to the effective action, once the latter is known without h fields. Such a tensor calculus we now present. It is based on the Polyakov variable f [17], which parametrizes the gravitational field h_{++} according to

$$h_{++} = \frac{\partial_+ f}{\partial_- f}. \quad (3.27)$$

In order to deal as much as possible with covariant objects, we introduce two operators which “covariantize” the derivatives ∂_+ and ∂_- with respect to ϵ transformations

$$\nabla_+ \equiv \partial_+ - h_{++} \partial_-, \quad \delta = \frac{1}{\partial_- f} \partial_-. \quad (3.28)$$

We now derive a set of elementary lemmas for these operators.

Lemma 3.1. The operators ∇_+ and δ commute

$$[\nabla_+, \delta] = 0. \quad (3.29)$$

Lemma 3.2. If S is an ϵ scalar, then also $\nabla_+ S$ and δS are ϵ scalars.

Lemma 3.3. $\delta f = 1$ and $\nabla_+ f = 0$.

Lemma 3.4. f is an ϵ scalar. Indeed, $\delta_\epsilon f = \epsilon_+ \partial_- f$ reproduces the correct $\delta_\epsilon h_{++}$.

Lemma 3.5. Partial integration of ∇_+ and δ is allowed if one uses $\partial_- f$ as measure. This means that if S_1 and S_2 are ϵ scalars, then

$$\int (\partial_- f) (\nabla_+ S_1) S_2 d^2 z = - \int (\partial_- f) S_1 (\nabla_+ S_2) d^2 z, \quad (3.30)$$

and idem for δ .

As a first application, we rewrite the effective action of pure gravity in terms of f and read off its ϵ variation from the result. Using

$$\log \partial_- f = \frac{1}{\partial_+} (\partial_+ \log \partial_- f) = \frac{1}{\partial_+} (\partial_- h_{++} + h_{++} \partial_- \log \partial_- f), \quad (3.31)$$

we find that $\nabla_+ \log \partial_- f = \partial_- h_{++}$ and $\square^{-1} \partial_-^2 h_{++} = \log \partial_- f$, so that

$$\int R \frac{1}{\square} R d^2 z = \int \partial_-^2 h_{++} \log \partial_- f d^2 z = \int \partial_+ f \delta \delta \partial_- f d^2 z. \quad (3.32)$$

By using the chain rule, one may express the effective stress tensor in terms of f ,

$$T_{--}^{\text{eff}} = -\pi \frac{\delta}{\delta h_{++}} \left[\frac{-n}{24\pi} \int R \frac{1}{\square} R d^2 z \right] = \frac{n}{12} \left[\frac{\partial_-^3 f}{\partial_- f} - \frac{3}{2} \left(\frac{\partial_-^2 f}{\partial_- f} \right)^2 \right], \quad (3.33)$$

which is in agreement with eq. (2.25). Using this expression one easily proves that the ϵ variation of T_{--}^{eff} is given by

$$\delta_\epsilon T_{--}^{\text{eff}} = \epsilon_+ \partial_- T_{--}^{\text{eff}} + 2(\partial_- \epsilon_+) T_{--}^{\text{eff}} + \frac{1}{12} n \partial_-^3 \epsilon_+. \quad (3.34)$$

These results, first discussed in ref. [17], clearly show how the use of the variable f allows one to exhibit the dependence of the effective actions and currents on h explicitly and to all orders in h .

As a new result we covariantize the Matsuo term in the effective action, which is given in eq. (3.4). We begin by expressing B_{+++} in terms of the gravitational ϵ scalar f , and a new ϵ scalar f_3 , which is the W_3 -counterpart of f ,

$$B_{+++} = \frac{\nabla_+ f_3}{(\partial_- f)^2}. \quad (3.35)$$

The reader may verify that with this parametrization B_{+++} transforms correctly under ϵ variations. In fact, one needs not introduce f_3 for our purposes, because it will be sufficient to use that the combination $B_{+++}(\partial_- f)^2$ is an ϵ scalar. However, we already introduce f_3 , since it can later be used to set up a “ λ calculus” for the spin-3 symmetry*.

Since f_3 is an ϵ scalar, we can write down the following action, which, according to our lemmas, will be ϵ invariant:

$$I_2 = \frac{\frac{1}{2}n(n+2)}{360\pi} \int (\partial_- f)(\nabla_+ f_3) \delta^5 f_3 d^2 z. \quad (3.36)$$

* Results of a λ calculus based on f, f_3 will be presented elsewhere [30]. Here we only mention that the assignments $\delta_\lambda f_3 = \lambda(\partial_- f)^2$ and $\delta_\lambda f = 0$ lead to the correct $\delta_\lambda B$ rule, as in (3.2), and the linearized $\delta_\lambda h$ rule $\delta_\lambda h = 0$.

It can be rewritten in the form

$$I_2 = \frac{\frac{1}{2}n(n+2)}{360\pi} \int \partial_- Y \nabla_+ Y d^2z, \quad (3.37)$$

with the scalar field Y given by

$$Y = \delta\delta f_3 = \frac{1}{\nabla_+} \delta\delta [B(\partial_- f)^2] \\ = \frac{1}{\nabla_+} \left[\partial_- + \left(\frac{\partial_-}{\partial_+} - \frac{1}{1-h\frac{\partial_-}{\partial_+}} \partial_- h \right) \right] \left[\partial_- B + 2B \left(\frac{\partial_-}{\partial_+} - \frac{1}{1-h\frac{\partial_-}{\partial_+}} \partial_- h \right) \right], \quad (3.38)$$

where we used that $\partial_-^2 f / \partial_- f = \partial_- \square^{-1} \partial_-^2 h$. (This result corrects ref. [20], where $I_2 \sim \int \partial_-^3 B \square^{-1} \partial_-^3 B$ was proposed.) The first two orders in h of the action (3.36) correctly reproduce the results (3.4) and (3.5)–(3.6), which were obtained by direct computation. We now claim that (3.36) gives the full 2-loop contribution (to all orders in h) of the induced action of the n scalar theory. This claim will be proved in sects. 4 and 5, where we will show that the sum of all diagrams beyond 1-loop has exact ϵ invariance.

As a second application, we give the fully covariantized terms with two B lines and two φ_{cl} lines in the effective action of the n scalar theory

$$= -\frac{(n+2)}{12\pi} \int \left[\partial_- \left(\partial_- + \frac{\partial_-}{\partial_+} - \frac{1}{1-h\frac{\partial_-}{\partial_+}} \partial_- h \right) (B\partial_- \varphi_{\text{cl}}^i) \right] \\ \times \frac{1}{\partial_+} \frac{1}{1-h\frac{\partial_-}{\partial_+}} \left[\left(\partial_- + \frac{\partial_-}{\partial_+} - \frac{1}{1-h\frac{\partial_-}{\partial_+}} \partial_- h \right) (B\partial_- \varphi_{\text{cl}}^i) \right], \quad (3.39)$$

in agreement with the lowest terms in the h expansion given in eqs. (3.11), (3.12).

4. First Ward identity for the λ symmetry

4.1. FORMULATION OF THE IDENTITY

In sect. 3, we already briefly discussed (anomalous) Ward identities in W_3 gravity. It will turn out that there are two different ways to generalize the ϵ Ward identity of pure gravity to the λ symmetry of W_3 gravity. In the first one, which we discuss in this section, the λ anomaly will be expressed in terms of a local quantity $\Delta^{(\lambda)}$. The second identity will be discussed in sect. 5.

Before we formulate the first λ Ward identity, we briefly recall some results from quantum Yang–Mills theory, where one proves the renormalizability of the theory by BRST methods [21, 22]. The effective action Γ satisfies then the well-known “renormalization equation”

$$\delta\Gamma/\delta\phi^i\delta K_i\backslash\delta\Gamma = \Delta \cdot \Gamma, \quad (4.1)$$

where ϕ^i are the “classical fields”, i.e. the Yang–Mills fields, matter fields, ghosts and antighosts. The fields K_i are sources for the BRST transformation laws which have opposite statistics to ϕ^i , and $\Delta \cdot \Gamma$ denotes all 1PI graphs with precisely one insertion of the *local* vertex Δ . (Because the K_i are fermionic in our case, we must be careful about ordering; we indicate left and right differentiation by the symbols “ $\backslash$ ” and “ $/$ ”, respectively.) The fact that Δ is local is a consequence of the celebrated Lam theorem [23]. At the tree level, eq. (4.1) states that the quantum action is BRST invariant. At higher order in $\hbar$, all transformation laws are modified by corrections of order $\hbar$ and higher. If one has a term $K_i\delta_{\text{BRST}}\phi^i$ in the quantum action, the correct quantum law for ϕ^i is given by

$$\delta\phi^i(\text{quantum}) = \delta K^i\backslash\delta\Gamma = (\delta_{\text{BRST}}\phi^i \cdot \Gamma). \quad (4.2)$$

Combining eqs. (4.1) and (4.2) leads to the following BRST Ward identity

$$\delta\Gamma/\delta\phi^i(\delta_{\text{BRST}}\phi^i \cdot \Gamma) = \Delta \cdot \Gamma. \quad (4.3)$$

In our treatment of chiral W_3 gravity, h_{++} and B_{+++} are *external* gauge fields, which have not yet been quantized. This means of course that we do not add ghosts and that we consider gauge invariance rather than BRST invariance. Still, we can derive a Ward identity for effective action Γ_φ which is closely analogous to (4.3). It is derived by changing the integration variable φ^i in the path integral for $Z(h, B, J^i)$ into a gauge-transformed variable. The jacobian, if nonzero, leads to the gauge anomaly. Making a Legendre transformation from J^i to the classical fields φ_{cl}^i , and defining the effective action as usual one may prove the following

Ward identity^{*}

$$\delta\Gamma_\varphi/\delta\varphi_{\text{cl}}^i(\delta\varphi_{\text{cl}}^i \cdot \Gamma_\varphi) + (\delta S/\delta h \delta h) \cdot \Gamma_\varphi + (\delta S/\delta B \delta B) \cdot \Gamma_\varphi = \Delta \cdot \Gamma_\varphi. \quad (4.4)$$

In this formula, $\delta\varphi_{\text{cl}}^i$, δh and δB denote the tree level ϵ or λ variations given in (3.2). The symbol $X \cdot \Gamma$ denotes all 1PI graphs with precisely one insertion of the local operator X . Δ is a *local* expression, which parametrizes the (ϵ or λ) anomalies in the theory. For a proof of this identity see ref. [13].

The Ward identity for ϵ symmetry is as naively expected since the ϵ laws are linear in, or independent of, quantum fields. It reads

$$\delta\Gamma_\varphi/\delta\varphi_{\text{cl}}^i \delta_\epsilon \varphi_{\text{cl}}^i + \delta\Gamma_\varphi/\delta h \delta_\epsilon h + \delta\Gamma_\varphi/\delta B \delta_\epsilon B = \Delta_\epsilon \cdot \Gamma_\varphi. \quad (4.5)$$

In particular, there are no quantum corrections to $\delta_\epsilon \varphi_{\text{cl}}^i$.

The λ Ward identity contains, however, in addition to the unusual term in eq. (4.5), also quantum corrections to $\delta\varphi_{\text{cl}}^i$ since $\delta\varphi_{\text{cl}}^i$ is non-linear in quantum fields. This situation is analogous to Yang–Mills theory. Since $\delta_\lambda B = \partial_+ \lambda - h \partial_- \lambda + 2 \partial_- h \lambda$ is independent of quantum fields, we can write the third term in the λ Ward identity as $\delta\Gamma_\varphi/\delta B (\delta B \cdot \Gamma_\varphi)$, which is also of the form one expects from Yang–Mills theory. However, δh contains quantum fields, and instead of $\delta\Gamma_\varphi/\delta h (\delta h \cdot \Gamma_\varphi)$ one finds in the λ Ward identity a term

$$(\delta S/\delta h \delta h) \cdot \Gamma_\varphi = (2/\pi) \int (B \partial_- \lambda - \lambda \partial_- B) (T^{\varphi}_{--} T^{\varphi}_{--}) \cdot \Gamma_\varphi, \quad (4.6)$$

which contains, in general, “cross contractions”, i.e. does not factorize into $(T^{\varphi}_{--} \cdot \Gamma) (T^{\varphi}_{--} \cdot \Gamma_\varphi)$. The necessity of this term will be confirmed by the fact that it indeed leads to *local* Δ ’s. The complete Ward identity for λ symmetry reads now

$$\delta\Gamma_\varphi/\delta\varphi_{\text{cl}}^i (\delta_\lambda \varphi_{\text{cl}}^i \cdot \Gamma) + (2/\pi) \int (\lambda \partial_- B - B \partial_- \lambda) (T^{\varphi} T^{\varphi}) \cdot \Gamma_\varphi + \delta\Gamma_\varphi/\delta B \delta_\lambda B = \Delta_\lambda \cdot \Gamma_\varphi. \quad (4.7)$$

In this formulation, the 1-loop ϵ anomaly is given by

$$\Delta_\epsilon^{(1)} = -\frac{n}{12\pi} \int h \partial_-^3 \epsilon. \quad (4.8)$$

In subsect. 4.3 we shall show that there are no further ϵ anomalies beyond this lowest order one. The 2-loop λ -anomaly coming from the Matsuo diagram (3.4) is

^{*} We are indebted to Mr. Bastianelli for proposing and deriving this Ward identity.

given by

$$\Delta_\lambda^{(2)} = \frac{n(n+2)}{360\pi} \int B \partial_-^5 \lambda. \quad (4.9)$$

We now come back to the 1-loop λ anomalies with two external scalars, which we abandoned at the end of subsect. 3.1. We find a new contribution from the term $\delta\Gamma_\varphi/\delta\varphi_{\text{cl}}^i(\delta_\lambda\varphi_{\text{cl}}^i \cdot \Gamma_\varphi)$ if we take for $\delta\Gamma/\delta\varphi_{\text{cl}}^i$ the lowest order field equation $\partial_+\partial_-\varphi_{\text{cl}}^i$. Graphically

$$\text{Diagram 1} + \frac{1}{\partial_+\partial_-\phi} \times \text{Diagram 2} = \Delta_\lambda^{(1)} \quad (4.10)$$

Diagram 1: A circle with two external lines labeled B and $\partial B \sim \partial_+\lambda$.
Diagram 2: A circle with two external lines labeled λ and B .

As a confirmation of the correctness of the Ward identity (4.7), we find a *local* result for the 1-loop anomaly $\Delta_\lambda^{(1)}$. It is given by

$$\Delta_\lambda^{(1)} = \frac{(n+2)}{6\pi} \int (\lambda \partial_- \varphi_{\text{cl}}^i) \partial_-^3 (B \partial_- \varphi_{\text{cl}}^i). \quad (4.11)$$

The existence of this anomaly was first found in refs. [2, 24].

The reader may at this moment expect the worst, and anticipate similar anomalies in 1-loop and higher-loop graphs with further external $B\varphi$ pairs and/or further external h fields. However, we now claim that, remarkably, no further anomalous terms arise. In other words, we claim that the terms that we already encountered, i.e.

$$\Delta_\epsilon^{(1)}, \quad \Delta_\lambda^{(1)}, \quad \Delta_\lambda^{(2)} \quad (4.12)$$

actually represent *all* anomalies in the theory! In subsect. 4.2 we will illustrate this claim by explicit computations involving diagrams with up to three loops. After that, in subsect. 4.3, we will prove this claim for the special case where we only consider diagrams with no external φ_{cl} lines.

4.2. EXPLICIT CHECKS THROUGH 3-LOOP

Let us first consider a 1-loop 1PI graph with three instead of two external $B\varphi_{\text{cl}}$ pairs. It yields

$$\text{Diagram 3} = -\frac{(n-2)}{12\pi} \int d^{ijk} (B \partial_- \varphi_{\text{cl}}^i) \left[\frac{\partial_-^2}{\partial_+} (B \partial_- \varphi_{\text{cl}}^j) \right] \left[\frac{\partial_-^2}{\partial_+} (B \partial_- \varphi_{\text{cl}}^k) \right], \quad (4.13)$$

Diagram 3: A circle with three external lines.

where we used the expression for $\text{tr}(d^i d^j d^k)$ from table 1. The Ward identity with external lines $(\lambda B^2 \varphi_{\text{cl}}^3)$ gets contributions from the following terms

$$\frac{\delta \Gamma_\varphi^{(1)}}{\delta \varphi_{\text{cl}}^i} \delta \varphi_{\text{cl}}^{i(0)} + \frac{\delta \Gamma_\varphi^{(0)}}{\delta \varphi_{\text{cl}}^i} \delta^{(1)} \varphi_{\text{cl}}^i + \frac{2}{\pi} (\lambda \partial_- B - B \partial_- \lambda) (T^\varphi T^\varphi \cdot \Gamma_\varphi)^{(1)} + \frac{\delta \Gamma_\varphi^{(1)}}{\delta B} \partial_+ \lambda \quad (4.14)$$

(where the superscripts denote the number of loops) corresponding to the following diagrams

$$\begin{aligned} & \text{Diagram 1} + \left(\text{Diagram 2} + \text{Diagram 3} \right) \\ & + \text{Diagram 4} + \text{Diagram 5} = 0 \end{aligned} \quad (4.15)$$

Note in particular the fourth diagram, coming from a cross-contraction in $(T^\varphi T^\varphi \cdot \Gamma_\varphi)$. All diagrams have different n dependence, and the fact that the sum is local is a confirmation of the need for the $(T^\varphi T^\varphi \cdot \Gamma_\varphi)$ terms in the identity (4.7). Without this term the sum would not even have been local. In fact, the sum of all terms vanishes and this confirms that there are no corrections to the three terms in (4.12) at this order, in agreement with our claim in subsect. 4.1.

One might have anticipated that only the graphs with 2 external B lines and no h lines give an anomaly, since their λ variation contains only one non-local operator ∂_+^{-1} , which can be cancelled by a term proportional to $\partial_+ \partial_- \varphi_{\text{cl}}^i$. Graphs with more ∂_+^{-1} factors cannot be made local by graphs which contain only one factor $\partial_+ \partial_- \varphi_{\text{cl}}^i$. We explicitly checked that the Ward identity (4.7) with only the anomalies (4.12) is also satisfied if one considers the 1-loop terms with structure $(\lambda h B \varphi_{\text{cl}}^2)$. (One of the contributions comes from the diagram (3.12), with $\delta_\lambda B = \partial_+ \lambda$.)

At the 2-loop level the (λB) terms in the Ward identity lead to the anomaly $\Delta_\lambda^{(2)}$ (see eq. (4.9)). Let us consider the $(\lambda h B)$ terms in the Ward identity at 2-loop. On the l.h.s. of eq. (4.7) only the $\delta_\lambda B$ terms contribute; we already discussed their contribution in eq. (3.9). The $(T^\varphi T^\varphi \cdot \Gamma_\varphi)$ terms do not contribute in this case. This means that in the Ward identity the terms (3.9) are *not* canceled on the l.h.s. of the equation. (An alternative λ Ward identity, where this cancellation does take place, will be discussed in sect. 5.) One should now realize that the r.h.s. of the Ward identity (4.7) contains the term $\Delta_\lambda^{(1)} \cdot \Gamma_\varphi$, which does contribute at this level.

Explicit computation of the relevant 1-loop diagram leads to

$$\begin{aligned} \text{Diagram} &= \frac{n(n+2)}{360\pi} \int \left(\frac{\partial_-^3}{\partial_+} h \right) \\ &\times (2B\partial_-^3\lambda - 3\partial_-B\partial_-^2\lambda + 3\partial_-^2B\partial_-\lambda - 2\partial_-^3B\lambda), \end{aligned} \quad (4.16)$$

where we have denoted the vertex due to $\Delta_\lambda^{(1)}$ by a cross. Comparison with eq. (3.9) shows that the Ward identity is satisfied at this level.

Let us finally consider the identity (4.7) at 3-loop level, where we consider terms with structure (λB^3) . In subsect. 3.2 we discussed the two Feynman diagrams with four external B lines that contribute to the 3-loop effective action. In the Ward identity, these diagrams contribute in the sector $\delta I_\varphi^{(3)}/\delta B \delta B$ through the variation $\delta_\lambda B = \partial_+\lambda$. The OPE formalism shows that the full variation w.r.t. B of these expressions is found by taking four times the variation at the second position in each of the structures in (3.23). One finds

$$\begin{aligned} \text{Diagram 1} &= \frac{n(n+2)^2}{15 \cdot 7!} \frac{1}{\pi} \int \left(7P_{\lambda B}^3 \frac{1}{\partial_+} Q_{BB}^6 + 12P_{\lambda B}^1 \frac{1}{\partial_+} Q_{BB}^8 \right), \\ \text{Diagram 2} &= - \frac{(n-2)n(n+2)}{2 \cdot 7!} \frac{1}{\pi} \int P_{\lambda B}^1 \frac{1}{\partial_+} Q_{BB}^8, \end{aligned} \quad (4.17)$$

where we introduced the expressions

$$\begin{aligned} P_{XY}^1 &= X\partial_-Y - \partial_-XY, \\ P_{XY}^3 &= 2X\partial_-^3Y - 3\partial_-X\partial_-^2Y + 3\partial_-^2X\partial_-Y - 2\partial_-^3XY, \\ Q_{XY}^6 &= 2X\frac{\partial_-^6}{\partial_+}Y + 3\partial_-X\frac{\partial_-^5}{\partial_+}Y, \\ Q_{XY}^8 &= X\frac{\partial_-^8}{\partial_+}Y + 6\partial_-X\frac{\partial_-^7}{\partial_+}Y + 14\partial_-^2X\frac{\partial_-^6}{\partial_+}Y + 14\partial_-^3X\frac{\partial_-^5}{\partial_+}Y. \end{aligned} \quad (4.18)$$

They have the following properties:

$$\partial_-^7P_{XY}^1 = Q_{X,\partial_+Y}^8 - Q_{Y,\partial_+X}^8, \quad \partial_-^3P_{XY}^3 = Q_{X,\partial_+Y}^6 - Q_{Y,\partial_+X}^6. \quad (4.19)$$

In the h -sector, there is one contribution to the Ward identity at this level. It is due to a cross contraction in $(T^\varphi T^\varphi) \cdot \Gamma_\varphi$, namely twice the fields $\varphi^i \varphi^j$ coupled to the vertex $B(\partial_- \varphi)^3$. The resulting expression is

$$\begin{array}{c} \lambda \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \\ B \end{array} \times \begin{array}{c} B \\ \diagup \\ \text{---} \\ \diagdown \\ B \end{array} = -\frac{2n(n+2)}{7!} \frac{1}{\pi} \int P_{\lambda B}^1 \frac{1}{\partial_+} Q_{BB}^8. \quad (4.20)$$

At the r.h.s of the Ward identity there is again the occurrence of $\Delta_\lambda^{(1)}$ through $\Delta_\lambda^{(1)} \cdot \Gamma_\varphi$. One finds the following result:

$$\begin{array}{c} \lambda \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \\ B \end{array} \begin{array}{c} B \\ \diagup \\ \text{---} \\ \diagdown \\ B \end{array} = \frac{n(n+2)^2}{140 \cdot 5!} \frac{1}{\pi} \int P_{\lambda B}^1 \frac{1}{\partial_+} Q_{BB}^8 + \frac{n(n+2)^2}{15 \cdot 6!} \frac{1}{\pi} \int P_{\lambda B}^3 \frac{1}{\partial_+} Q_{BB}^6. \quad (4.21)$$

One easily checks that the contributions (4.17) and (4.20) on the l.h.s of the Ward identity correctly add up to (4.21), which shows that there is no new contribution to Δ_λ at this order. One might have expected this result, as it seems impossible to find linear combinations of diagrams with three non-local operators ∂_+^{-1} which are local.

Cursory contemplation of the task to compute 4-loop diagrams produces an irresistible urge to give an all loop proof that there are no higher-order ϵ anomalies and λ anomalies. To this problem we now turn.

4.3. ALL-LOOP RESULT FOR THE ANOMALIES IN $\Gamma_\varphi[h, B]$

Before we come to the anomalies we make a few definitions. As in sect. 2 we define T^{eff} as $-\pi(\delta/\delta h)\Gamma[h, B]$ and W^{eff} as $-\pi(\delta/\delta B)\Gamma[h, B]$. Note that they can be represented as

$$T^{\text{eff}}(z, \bar{z}) = \sum_N \frac{(-1)^N}{N!} \langle T(z, \bar{z}) \Xi(1) \dots \Xi(N) \rangle_{\varphi/Z}, \quad (4.22)$$

with a similar expression for W^{eff} and with $Z = \exp \Gamma$ as in eq. (3.13). We now define, R^{eff} (in the n scalar theory, see eq. (3.15)) and Λ^{eff} (in the W_3 theory, see

eq. (3.17)) by analogous expressions. (In the Feynman diagram language X^{eff} is of course represented as $X \cdot \Gamma_\varphi$.) In this section we focus on the Ward identities where the classical field φ_{cl}^i is set equal to zero.

Let us first consider the ϵ symmetry. By using the TT and TW OPE's we derive (compare with the derivation of (2.29))

$$\delta_\epsilon \Gamma_\varphi = \frac{\delta \Gamma_\varphi}{\delta B} \delta_\epsilon B + \frac{\delta \Gamma_\varphi}{\delta h} \delta_\epsilon h = -\frac{n}{12\pi} \int h \partial_-^3 \epsilon, \quad (4.23)$$

confirming our claim that the 1-loop ϵ anomaly is actually the full result. This result can be rephrased as

$$[\partial_+ - h \partial_- - 2(\partial_- h)] T^{\text{eff}} - [2B \partial_- + 3(\partial_- B)] W^{\text{eff}} = \frac{1}{12} n \partial_-^3 h. \quad (4.24)$$

A similar result (with $n \rightarrow c$) holds for the W_3 theory.

We now repeat the analysis for the λ symmetry. Our strategy is to first evaluate the consequences of varying the B fields. In the result we shall then recognize the various terms that make up the Ward identity. We have

$$\begin{aligned} \frac{\delta \Gamma_\varphi}{\delta B} \delta_\lambda B &= \frac{n(n+2)}{360\pi} \int B \partial_-^5 \lambda \\ &+ \frac{n+2}{30\pi} \int (2B \partial_-^3 \lambda - 3\partial_- B \partial_-^2 \lambda + 3\partial_-^2 B \partial_- \lambda - 2\partial_-^3 B \lambda) T^{\text{eff}} \\ &+ \frac{n+2}{2\pi} \int (B \partial_- \lambda - \partial_- B \lambda) R^{\text{eff}} \end{aligned} \quad (4.25)$$

for the n scalar theory.

At this point one should be careful about the normal ordering in the term $(T^\varphi T^\varphi)$ in R^φ . In the defining relation (3.15) it was understood that this ordering was done w.r.t. the modes of T^φ (as is standard in conformal field theory). However, in order to be able to compare with the Ward identity (4.7) we wish to do the ordering with respect to the modes of φ , using the relation (see formulas (A.15) and (A.10) of ref. [26])

$$(T^\varphi T^\varphi)_T = (T^\varphi T^\varphi)_\varphi - \frac{1}{2} \partial_-^3 \varphi^i \partial_- \varphi^i. \quad (4.26)$$

This then leads to

$$\begin{aligned}
 & \partial \Gamma_\varphi / \partial B \delta_\lambda B + (2/\pi) \int (\lambda \partial_- B - B \partial_- \lambda) ((T^\varphi T^\varphi)_\varphi \cdot \Gamma_\varphi) \\
 &= \Delta_\lambda^{(2)} \cdot \Gamma_\varphi + \frac{n+2}{30\pi} \int (2B \partial^3_- \lambda - 3\partial_- B \partial^2_- \lambda + 3\partial^2_- B \partial_- \lambda - 2\partial^3_- B \lambda) T^\varphi \cdot \Gamma_\varphi \\
 & \quad + \int (B \partial_- \lambda - \partial_- B \lambda) \left(-\frac{n+2}{4\pi} \partial^3_- \varphi^i \partial_- \varphi^i - \frac{3(n+2)}{30\pi} \partial^2_- T^\varphi \right) \cdot \Gamma_\varphi \\
 &= \Delta_\lambda^{(2)} \cdot \Gamma_\varphi + \frac{(n+2)}{6\pi} \int [(\lambda \partial_- \varphi^i) \partial^3_- (B \partial_- \varphi^i)] \cdot \Gamma_\varphi \\
 &= \Delta_\lambda^{(1)} \cdot \Gamma_\varphi + \Delta_\lambda^{(2)} \cdot \Gamma_\varphi. \tag{4.27}
 \end{aligned}$$

This proves that the 1-loop and 2-loop anomalies for the λ symmetry constitute the full anomaly in the Ward identity (4.7) with $\varphi_{\text{cl}}^i = 0$.

5. Second λ Ward identity and functional equations for the effective action

5.1. WESS-ZUMINO CONDITIONS FOR THE CONSISTENT λ ANOMALY

In sect. 4 we discussed the Ward identities (4.5) and (4.7) and evaluated the local expressions Δ_ϵ and Δ_λ which parametrize the anomalies. Unfortunately, the λ Ward identity (4.7) cannot directly be reformulated as a differential equation involving only h , B , T^{eff} and W^{eff} (such as eq. (4.26) which expresses the anomalous ϵ Ward identity (4.25)). The obstruction to this is the presence of the factor R^{eff} in eq. (4.27) (or A^{eff} in the W_3 theory), which cannot readily be expressed in terms of h , B , T^{eff} and W^{eff} .

If a second equation of the type (4.26) (but corresponding to λ rather than to ϵ symmetry) could be established, the two together would uniquely characterize the effective action $\Gamma_\varphi[h, B]$ (or $\Gamma_{W_3}[h, B]$) and could perhaps lead to a closed expression for Γ_φ in terms of h and B . For this reason we will now discuss an attempt to formulate an alternative λ Ward identity and the problems that go with it.

A sufficient result in order to establish a λ analogue of eq. (4.26) would be a relation of the form

$$\frac{\delta \Gamma}{\delta h} \delta_\lambda h + \frac{\delta \Gamma}{\delta B} \delta_\lambda B = \text{const.} \int B \partial^5_- \lambda, \tag{5.1}$$

where $\delta_\lambda h$ and $\delta_\lambda B$ are variations that include corrections of order $\hbar$ and higher, but can still be expressed as *local* expressions in h , B , T^{eff} and W^{eff} . We already saw an example of such a correction term, namely $\delta_\lambda^{(1)} h$ in (3.10), which was designed precisely to avoid extra terms on the r.h.s. of eq. (5.1).

In order to investigate the possibility of a relation like (5.1), we shall proceed as follows. We shall first fix a set of variation rules for h and B under ϵ and λ symmetry, namely the rules given in (3.2) for $\delta_\epsilon h$, $\delta_\epsilon B$ and $\delta_\lambda B$, and the rule (3.10) for the λ variation of h . We then consider the δ_ϵ and δ_λ variations of the effective action Γ_φ under these rules, which we shall denote by $\text{An}[\epsilon]$ and $\text{An}[\lambda]$, respectively. These anomalies satisfy integrability conditions, which are often called Wess–Zumino (WZ) conditions [25]. We already know that

$$\text{An}[\epsilon] = -\frac{n}{12\pi} \int h \partial_-^3 \epsilon, \quad \text{An}[\lambda] = \frac{n(n+2)}{360\pi} \int B \partial_-^5 \lambda + \int \lambda \text{An}_\lambda, \quad (5.2)$$

(compare with eqs. (4.25), (3.8)), where the function $\text{An}_\lambda(h, B)$ is to be determined. The idea is that we can use the WZ conditions to determine An_λ , or at least to find restrictions on it. Once $\text{An}[\lambda]$ is determined, one can try to absorb all terms in it beyond the minimal one into the transformation rules $\delta_\lambda h$ and $\delta_\lambda B$, such as to establish a relation like (5.1).

We begin with the ϵ – ϵ condition. From the commutator of two local ϵ transformations on h and B we deduce that

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta(\epsilon' = \epsilon_2 \partial_- \epsilon_1 - \epsilon_1 \partial_- \epsilon_2). \quad (5.3)$$

Hence

$$\delta(\epsilon_1) \text{An}[\epsilon_2] - \delta(\epsilon_2) \text{An}[\epsilon_1] = \text{An}[\epsilon'], \quad (5.4)$$

which is indeed satisfied by $\text{An}[\epsilon]$ in eq. (5.2).

For the ϵ – λ condition we should first evaluate the commutator $[\delta(\epsilon), \delta(\lambda)]$. On B one finds

$$[\delta(\epsilon), \delta(\lambda)] B = \delta(\lambda' = -\epsilon \partial_- \lambda + 2\lambda \partial_- \epsilon) B. \quad (5.5)$$

On h , however, one finds an extra term

$$[\delta(\epsilon), \delta(\lambda)] h = \delta(\lambda') h + \frac{8(n+2)}{15} (\lambda \partial_- B - B \partial_- \lambda) \partial_-^3 \epsilon. \quad (5.6)$$

The $\epsilon - \lambda$ condition becomes

$$\delta(\epsilon) \text{An}[\lambda] - \delta(\lambda) \text{An}[\epsilon] = \text{An}[\lambda'] - \frac{8(n+2)}{15\pi} \int (\lambda \partial_- B - B \partial_- \lambda) T^{\text{eff}} \partial_-^3 \epsilon, \quad (5.7)$$

where we used that $-\pi \delta \Gamma / \delta h = T^{\text{eff}}$. This equation reduces to the following equation for the function An_λ

$$\int [\lambda \delta(\epsilon) \text{An}_\lambda - 2\lambda \partial_- \epsilon \text{An}_\lambda + \partial_- \lambda \epsilon \text{An}_\lambda] = - \frac{8(n+2)}{15\pi} \int (\lambda \partial_- B - B \partial_- \lambda) T^{\text{eff}} \partial_-^3 \epsilon, \quad (5.8)$$

which states that An_λ transforms under ϵ transformations as a (-3) tensor with an anomalous term on the r.h.s. We now remark that if we *assume* that the δ_ϵ transformation rule of T^{eff} is given by

$$\delta_\epsilon T^{\text{eff}} = \epsilon \partial_- T^{\text{eff}} + 2(\partial_- \epsilon) T^{\text{eff}} + \frac{1}{12} n \partial_-^3 \epsilon \quad (5.9)$$

(which we know to be true in lowest order, see eq. (3.34)), then the following expression is an exact solution to eq. (5.8):

$$\int \lambda \text{An}_\lambda = - \frac{16(n+2)}{5\pi n} \int (\lambda \partial_- B - B \partial_- \lambda) [T^{\text{eff}}]^2. \quad (5.10)$$

In order to show this, it is useful to note that the r.h.s. is ϵ invariant except for the ϵ anomaly in $\delta_\epsilon T^{\text{eff}}$.

Turning to the $\lambda - \lambda$ consistency condition, we first evaluate the commutator of two local λ transformations on h and B . The result is

$$\begin{aligned} [\delta(\lambda_1), \delta(\lambda_2)] B &= \delta(\epsilon') B, \\ [\delta(\lambda_1), \delta(\lambda_2)] h &= \delta(\epsilon') h + \frac{8(n+2)}{15} (\lambda_1 \partial_- \lambda_2 - \lambda_2 \partial_- \lambda_1) \partial_-^3 h, \end{aligned} \quad (5.11)$$

where ϵ' is given by

$$\epsilon' = \frac{n+2}{30} (2\lambda_1 \partial_-^3 \lambda_2 - 3\partial_- \lambda_1 \partial_-^2 \lambda_2 + 3\partial_-^2 \lambda_1 \partial_- \lambda_2 - 2\partial_-^3 \lambda_1 \lambda_2). \quad (5.12)$$

The $\lambda - \lambda$ WZ condition becomes

$$\delta(\lambda_1) \text{An}[\lambda_2] - \delta(\lambda_2) \text{An}[\lambda_1] = \text{An}[\epsilon'] - \frac{8(n+2)}{15\pi} \int (\lambda_1 \partial_- \lambda_2 - \lambda_2 \partial_- \lambda_1) T^{\text{eff}} \partial_-^3 h. \quad (5.13)$$

It reduces to the following equation for the function An_λ

$$\int \lambda_2 \delta(\lambda_1) \text{An}_\lambda - \int \lambda_1 \delta(\lambda_2) \text{An}_\lambda = - \frac{8(n+2)}{15\pi} \int (\lambda_1 \partial_- \lambda_2 - \lambda_2 \partial_- \lambda_1) T^{\text{eff}} \partial_-^3 h. \quad (5.14)$$

Let us see if the proposed expression (5.10) satisfies this condition. Inserting (5.10) into the l.h.s of (5.14), and using the result (4.26), one finds that the identity reduces to

$$\delta_\lambda T^{\text{eff}} = 3 \partial_- \lambda W^{\text{eff}} + 2 \lambda \partial_- W^{\text{eff}}. \quad (5.15)$$

In lowest order, this equation can be checked by explicitly writing expressions for T^{eff} and W^{eff} .

Thus we see that the expression (5.10) is an exact solution of all WZ consistency conditions if we *assume* the validity of eqs. (5.9) and (5.15). These conditions (up to the anomalous term in (5.9)) were found previously in our analysis of classical W_3 gravity in ref. [4], which is based on a classical limit of the quantum W_3 algebra.

We now claim that (5.2) and (5.10) are the correct λ anomaly through 2-loop order (i.e. to first order in B but to arbitrary high order in $\hbar$), but that they are *not* the full result if one considers 3-loop order and higher.

As for the first claim: we explicitly checked that the conditions (5.9) and (5.10) are satisfied to high enough order as to guarantee the correctness of (5.10) as a solution of the WZ conditions through 2-loop order. We also checked that the leading term of An_λ (which has the structure $(\lambda B \hbar^2)$) is correctly reproduced by adding the following three contributions: (i) the variation under $\delta B = \partial_+ \lambda$ of the $(B^2 \hbar^2)$ term in the effective action, which can be extracted from the all order result (3.37) and which corresponds to the Matsuo diagram (3.4) with two h insertions, (ii) the variation under $\delta B \sim \hbar \lambda$ of the Matsuo diagram with one h insertion, given in (3.6), and (iii) the variation under $\delta_\lambda^{(1)} \hbar$ of the 3- h diagram (2.20).

The fact that (5.10) is not the full λ anomaly becomes apparent if we consider the λ variations of the 3-loop (B^4) terms in the effective action, which we gave in (4.17). One should add to that result the variation gotten by taking $\delta_\lambda^{(1)} \hbar$ in

diagram (3.5), which precisely cancels the $P_{\lambda B}^3(1/\partial_+)Q_{BB}^6$ term in (4.17). Hence, the full result for the (λB^3) terms in the λ anomaly is given by

$$-\frac{n(n+2)}{10 \cdot 7!} [8(n+2) - 5(n-2)] \frac{1}{\pi} \int P_{\lambda B}^1 \frac{1}{\partial_+} Q_{BB}^8. \quad (5.16)$$

This result clearly shows that the full λ anomaly is *not* simply given by (5.10). If we exclude the possibility that the extra terms in An_λ are homogeneous solutions to the WZ conditions, we should conclude that the assumptions (5.9) and (5.15) are not both valid in our theory.

We finally remark that expression (5.16) contains non-localities which cannot all be absorbed into the expressions T^{eff} and W^{eff} . This means that our effort to obtain a result of the form (5.1) breaks down at the 3-loop level.

5.2. FUNCTIONAL EQUATIONS FOR THE EFFECTIVE ACTION

In this subsection we reformulate the result of subsect. 5.1, and extend them to the W_3 theory. We will find that in the latter case the non-localities that we found in (5.16) are suppressed by a factor $1/c$ if we consider the large- c limit.

First of all, we give the analogue of formula (4.25) for the W_3 theory, which reads

$$\begin{aligned} \frac{\delta \Gamma_{W_3}}{\delta B} \delta_\lambda B = & -\frac{c}{360\pi} \int B \partial_-^5 \lambda - \frac{1}{30\pi} \int (2B \partial_-^3 \lambda - 3\partial_- B \partial_-^2 \lambda + 3\partial_-^2 B \partial_- \lambda - 2\partial_-^3 B \lambda) T^{\text{eff}} \\ & - \frac{\beta}{\pi} \int (B \partial_- \lambda - \partial_- B \lambda) A^{\text{eff}}. \end{aligned} \quad (5.17)$$

The first few terms in R^{eff} and A^{eff} can be worked out explicitly by using OPE relations. For example, if we focus on the terms in A^{eff} that are independent of B , we can derive from the OPE between $\Lambda(z)$ and $T(w)$ that

$$(\partial_+ - h\partial_- - 4(\partial_- h)) A^{\text{eff}}(h) = \frac{22+5c}{30} T^{\text{eff}}(h) \partial_-^3 h. \quad (5.18)$$

Precisely the same differential equation is satisfied by $((22+5c)/5c)[T^{\text{eff}}(h)]^2$ (compare with eq. (2.31)), and this leads us to identify the two results. (Strictly speaking the identification is up to a homogeneous solution of eq. (5.18), which we

do not expect to contribute here.) We found the following expansions:

$$\begin{aligned}
 R^{\text{eff}} &= \frac{32}{5n} [T^{\text{eff}}(h)]^2 - \frac{n}{5 \cdot 7!} [(8(n+2) - 5(n-2))] \frac{1}{\partial_+} Q_{BB}^8 \\
 &\quad + O(B^2 h^{\geq 1}, B^4, \dots), \\
 A^{\text{eff}} &= \frac{16}{5\beta c} [T^{\text{eff}}(h)]^2 - \frac{8c}{5 \cdot 7!} \frac{1}{\partial_+} Q_{BB}^8 + O(B^2 h^{\geq 1}, B^4, \dots), \quad (5.19)
 \end{aligned}$$

with Q_{BB}^8 defined as in (4.18). Note that the result for R^{eff} confirms the validity of (5.10) through 2-loop order.

We can now insert these expansions into eqs. (4.25) and (5.17) and extract differential equations satisfied by T^{eff} and W^{eff} . In doing so, we switch to variables h, b, u and v , where the new variables are defined as: $b = \sqrt{-(n+2)} B$ (n scalar theory) or $b = B$ (W_3 theory), $u = (12/n)T^{\text{eff}}$ or $u = (12/c)T^{\text{eff}}$, and $v = (360/n\sqrt{-(n+2)})W^{\text{eff}}$ or $v = (360/c)W^{\text{eff}}$.

For completeness we first express the relation (4.24) for the anomalous ϵ Ward identity in terms of the new variables

$$\partial_+ u = D_1 h + \frac{1}{15} [3v\partial_- + 2(\partial_- v)] b. \quad (5.20)$$

The other relations translate into

$$\begin{aligned}
 \partial_+ v &= [3v\partial_- + (\partial_- v)] h + D_2 b \\
 &\quad + \frac{1}{140} \left(8 - 5 \frac{n-2}{n+2} \right) \left(2\partial_- b \frac{1}{\partial_+} Q_{bb}^8 + b \frac{\partial_-}{\partial_+} Q_{bb}^8 \right) + O(b^3 h^{\geq 1}, b^5, \dots) \quad (5.21)
 \end{aligned}$$

for the n scalar theory, and

$$\begin{aligned}
 \partial_+ v &= [3v\partial_- + (\partial_- v)] h + D_2 b + \frac{4\beta}{35} \left(2\partial_- b \frac{1}{\partial_+} Q_{bb}^8 + b \frac{\partial_-}{\partial_+} Q_{bb}^8 \right) \\
 &\quad + O(b^3 h^{\geq 1}, b^5, \dots) \quad (5.22)
 \end{aligned}$$

for the pure W_3 theory. In here D_1 and D_2 are the Gelfand–Dickey operators given in eqs. (1.7).

For the W_3 theory we remark that the non-local terms in the functional equation (5.22) are proportional to β , which in the large- c limit is first order in $1/c$. In the introduction we mentioned that the induced action $\Gamma[h, b]$ corresponds (via a Legendre transformation) to the saddlepoint approximation of the generating functional $W[u, v]$. The loop corrections to the saddlepoint approximation will be of order $1/c$ and will therefore interfere with the non-local terms in the above

functional equations. One might expect that this works out in such a way that the full generating functional is essentially (up to finite renormalizations as in (1.3)) given by the Legendre transform of a quantity satisfying the above functional equations *without any order $1/c$ non-local terms*. At that point one would expect to make contact with a KPZ type approach, which could presumably yield the all-order result for the renormalization constants (compare with the proposals made in refs. [9, 10]). We leave this issue for further study.

6. Outlook

In sect. 5 we introduced the idea of a $1/c$ expansion for the induced action $\Gamma_{W_3}[h, b]$ of W_3 gravity coupled to a W_3 matter system of central charge c . We expect that the lowest order term in this expansion, which we will call the *classical* induced action, satisfies the Ward identities (5.20) and (5.22) without the subleading non-local terms:

$$\partial_+ u = D_1 h + \frac{1}{15} [3v\partial_- + 2(\partial_- v)] b, \quad \partial_+ v = [3v\partial_- + (\partial_- v)] h + D_2 b. \quad (6.1)$$

It has been observed in the literature [27, 28] that these identities, which are related to the so-called Boussinesq hierarchy of integrable differential equations, can be derived from the Ward identities of the $sl(3)$ Wess–Zumino–Witten (WZW) theory of the procedure of hamiltonian reduction. This result generalizes the result for pure gravity, where the exact Ward identity (2.31), which is related to the KdV hierarchy, can be derived from the Ward identity for the $sl(2)$ WZW theory.

We would like to point out that the relation, in general, of the “classical” W_n Ward identities with the $sl(n)$ WZW theories can be exploited to derive explicit expressions for the classical induced action of W_n gravity (compare with ref. [29]). We will here briefly indicate how this can be done in the case of pure gravity. Details and the extension to W_n will be given in a separate paper with H. Ooguri [30].

The induced action $\Gamma[A]$ for an external gauge field $A_a(z, \bar{z})$ coupled to WZW matter fields can be expanded according to

$$\begin{aligned} \Gamma[A] &= \text{diagram 1} + \text{diagram 2} + \dots \\ &= \frac{k}{4\pi} \int d^2z \left[\partial_- A^a \frac{1}{\partial_- \partial_+} \partial_- A_a + \frac{2}{3} \left(\frac{1}{\partial_+} A^a \right) A^b \left(\frac{\partial_-}{\partial_+} A^c \right) f_{abc} + O[A^4] \right], \quad (6.2) \end{aligned}$$

where f_{abc} are the structure constants of the underlying Lie algebra. In the case of $\mathfrak{sl}(2)$ we have three gauge fields A^+ , A^0 and A^- . Imposing the constraints

$$\frac{\delta W}{\delta A^0} = 0, \quad \frac{\delta W}{\delta A^+} = \frac{k}{2\pi} \quad (6.3)$$

and calling $A^- = h$, one finds that

$$A^+ = -\partial_-^2 h - h \left(\frac{\partial_-^3}{\partial_+} h \right) + \dots, \quad A^0 = \partial_- h, \quad A^- = h. \quad (6.4)$$

Using these constraints, one can reduce the action (6.2) and finds that it reproduces the leading terms of the induced action (2.22) for pure gravity, with the identification $c = k/6$. In ref. [30] we will extend this result to all orders in h and present the generalization to W_n .

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