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DISPERSIVE OPTICAL BISTABILITY IN A NONIDEAL FABRY-PEROT CAVITY

I. STABILITY ANALYSIS OF THE MAXWELL-BLOCH EQUATIONS

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Abstract

A stability analysis is performed for optical bistability in a Fabry-Pérot cavity with mirrors of arbitrary transmission coefficient. The mixed absorptive and dispersive régime is covered. In order to describe the system we use the Maxwell-Bloch equations formulated in terms of slowly varying envelopes. Standing-wave effects are completely taken into account by refraining from a truncation of the harmonic expansions for the polarization and the inversion density. We represent the solutions of the linearized Bloch hierarchy in terms of Chebyshev polynomials depending on the stationary electric field envelopes. In this way, we reduce the stability problem to a four-dimensional set of linear differential equations. Together with a couple of boundary conditions these equations govern the spatial behaviour of the deviations of the forward and the backward electric field envelopes. Our final stability problem becomes much simpler in the uniform-field limit and in the adiabatic limit. If we choose the stationary backward electric field equal to zero we recover results that were derived earlier for the case of a ring cavity.

1 Introduction

The Maxwell-Bloch theory is widely recognized as being an appropriate starting-point for the investigation of cooperative phenomena in nonlinear optical systems. Indeed, over the last decade a huge amount of articles has been published in which the dynamical behaviour of both passive and active media is studied on the basis of the Maxwell-Bloch equations; numerous reviews have appeared as well [1]–[5]. The reason for the frequent use of the Maxwell-Bloch formalism is that on the one hand it does not exclude analytical treatment from the outset, while on the other hand it is capable of providing reliable predictions on the coherent interaction between a laser beam and a nonlinear medium.

One of the central topics in Maxwell-Bloch theory is the study of instabilities in a passive medium which consists of two-level particles and which is enclosed in an optical cavity with feedback. Besides bistability the laser output of such a system can exhibit a spontaneous self-pulsing behaviour. This feature makes a passive medium with feedback a most interesting object of theoretical study.

Up to now the majority of treatments on passive media has focused on the case of a ring cavity [1]. In this arrangement the feedback of the laser beam is external so as to avoid standing-wave effects in the nonlinear medium. The extensive work for the Maxwell-Bloch equations describing a nonlinear ring cavity has furnished us with a good understanding of instabilities in passive media. Especially in the single-mode area important progress has been made: a satisfactory agreement with experimental data has been achieved [6].

In a common experimental setup the optical cavity is not of the ring but rather of the Fabry-Pérot variety [7]. For such a cavity the feedback of the laser beam is internal. This means that theoretical results derived for the case of a ring cavity cannot be applied because of the presence of standing-wave effects. Recently, we showed [8] that these standing-wave effects must be taken into account systematically. If the mirrors of the cavity are nonideal, it is not allowed to employ methods such as the uniform-field approximation [9] or a single-mode type approximation [10].

If one wishes to cover experimental situations the incorporation of standing-wave effects is not the only point of interest. Another prerequisite for making contact with experiments is the inclusion of detuning. Both the atomic frequency and the frequency of the central cavity mode should be chosen different from the frequency of the laser beam. Altogether, we can say that there is a demand for an analysis of the Maxwell-Bloch theory that describes the dynamics of a dispersive medium enclosed in a nonideal Fabry-Pérot cavity. Up to now such an analysis has not been performed.

In the present treatment we adopt as a starting-point the plane-wave Maxwell-Bloch equations for a passive medium in a Fabry-Pérot cavity. Both atomic detuning and cavity detuning are present, so that the mixed absorptive and dispersive régime is described. Standing-wave effects in the cavity will be taken into account completely. Furthermore, we shall refrain from making any approximations on the spatial dependence of the fields. After a discussion of the steady-state solutions of our equations a linear stability analysis will be performed. Previous expertise acquired for the purely absorptive case [11] will be to our advantage. Nevertheless, the derivation we give is self-contained.

In this work our aim is to reduce the stability problem of the Maxwell-Bloch equations to a mathematical problem which may be solved numerically. We shall not only discuss the general theory, but turn our attention to some special cases as well. In particular, we shall consider the uniform-field limit and the adiabatic limit. These limits will be of help in the numerical analysis the results of which will be presented in a companion paper [12].

2 Stationary analysis

For a gas of homogeneously broadened two-level particles contained in a Fabry-Pérot cavity the Maxwell-Bloch equations can be written in the following form

$$\frac{1}{c}\frac{\partial E_{\rm F}}{\partial t} + \frac{\partial E_{\rm F}}{\partial z} = g P_{{\rm F},1} \quad , \tag{2.1}$$

$$\frac{1}{c}\frac{\partial E_{\rm B}}{\partial t} - \frac{\partial E_{\rm B}}{\partial z} = g P_{{\rm B},1} \quad , \tag{2.2}$$

$$\frac{\partial P_{\mathrm{F},m}}{\partial t} = -\gamma_\perp(1+i\Delta)P_{\mathrm{F},m} + \frac{\mu}{\hbar}(E_\mathrm{F}D_{m-1} + E_\mathrm{B}D_m) \ , \tag{2.3}$$

$$\frac{\partial P_{\mathrm{B},m}}{\partial t} = -\gamma_\perp(1+i\Delta)P_{\mathrm{B},m} + \frac{\mu}{\hbar}(E_\mathrm{F}D_m{}^* + E_\mathrm{B}D_{m-1}{}^*) \ , \tag{2.4}$$

$$\frac{\partial D_0}{\partial t} = -\gamma_{||}(D_0+1) - \frac{2\mu}{\hbar}(E_\mathrm{F}P_{\mathrm{F},1}{}^* + E_\mathrm{B}P_{\mathrm{B},1}{}^* + E_\mathrm{F}{}^*P_{\mathrm{F},1} + E_\mathrm{B}{}^*P_{\mathrm{B},1}) \ , \tag{2.5}$$

$$\frac{\partial D_m}{\partial t} = -\gamma_{||}D_m - \frac{2\mu}{\hbar}(E_\mathrm{F}P_{\mathrm{B},m}{}^* + E_\mathrm{B}P_{\mathrm{B},m+1}{}^* + E_\mathrm{F}{}^*P_{\mathrm{F},m+1} + E_\mathrm{B}{}^*P_{\mathrm{F},m}) \ , \tag{2.6}$$

with m=1,2,3,... . In the above hierarchy of equations only envelopes of fields figure, namely $\{P_{\mathrm{F},m}, P_{\mathrm{B},m}\}_{m=1}^{\infty}$ for the polarization, $\{D_m\}_{m=0}^{\infty}$ for the inversion and $E_\mathrm{F}, E_\mathrm{B}$ for the electric fields. This has been achieved by carrying out a Fleck expansion [13] for the polarization density, the inversion density and the electric fields. The expansion of the electric fields has been truncated [14]. The advantage of formulating the Maxwell-Bloch equations in terms of envelopes is that these fields are slowly varying in space (z) and time (t).

In (2.1)–(2.2) the coupling constant g is given by $\frac{1}{2}\mu nk$, where μ is the modulus of the dipole moment of the constituent particles, k the modulus of the wave vector of the coherent external field, and n the particle density. The transverse and the longitudinal damping coefficients of the medium are denoted by $\gamma_\perp$ and $\gamma_{||}$, respectively. As usual, the former coefficient is employed to measure the difference between the atomic frequency ω_a and the laser frequency ω=ck; hence, the dimensionless atomic detuning parameter is equal to Δ=$(\omega_\mathrm{a}$–$\omega)/\gamma_\perp$.

The Fabry-Pérot configuration imposes a set of boundary conditions on the envelopes of the electric fields. These conditions read

$$E_\mathrm{F}(0,t) = R^{1/2}E_\mathrm{B}(0,t) + T^{1/2}E_\mathrm{I}(t) \ , \tag{2.7}$$

$$E_\mathrm{B}(L,t) = R^{1/2}E_\mathrm{F}(L,t)\,\mathrm{e}^{i\theta} \ , \tag{2.8}$$

$$E_\mathrm{T}(t) = T^{1/2}E_\mathrm{F}(L,t) \ . \tag{2.9}$$

Here E_I and E_T denote the envelopes of the incident electric field and the transmitted electric field, respectively. The mirrors of the cavity have a reflection coefficient R and a transmission coefficient T=1–R. In general, the frequency of the coherent field does not fit to the cavity. In the boundary condition (2.8) this is reflected by the presence of the cavity detuning parameter θ=$2L(\omega$–$\omega_n)/c$. The symbol ω_n stands for a cavity frequency; the difference between adjacent cavity frequencies is given by the free spectral range $\pi c/L$, with L the length of the cavity. If we choose both the cavity detuning and the atomic detuning equal to zero in (2.1)–(2.9) we find the set of equations which describes the purely absorptive case and which we analyzed earlier [11].

If we introduce scaled quantities

$$f = \bar{\mu}E_\mathrm{F} \quad , \quad b = \bar{\mu}E_\mathrm{B} \ , \tag{2.10}$$

$$y = 2\bar{\mu}T^{-1/2}E_\mathrm{I} \quad , \quad x = 2\bar{\mu}T^{-1/2}E_\mathrm{T} \ , \tag{2.11}$$

$$P_{\mathrm{F},m}{}' = (\gamma_\perp/\gamma_{||})^{1/2}\,P_{\mathrm{F},m} \quad , \quad P_{\mathrm{B},m}{}' = (\gamma_\perp/\gamma_{||})^{1/2}\,P_{\mathrm{B},m} \ , \tag{2.12}$$

with m=1,2,3,... and $\bar{\mu} = \mu/[\hbar(\gamma_\perp\gamma_{||})^{1/2}]$, the coefficients in the equations (2.1)–(2.9) get a simpler form. As a dimensionless spatial variable we introduce ζ=z/L. The coupling coefficient in the Maxwell equations (2.1)–(2.2) then becomes proportional to the cooperation parameter

$$C = \frac{\mu^2 n\omega L}{2\hbar c\gamma_\perp T} \quad . \tag{2.13}$$

For the case of optical bistability, considered in this paper, this parameter is always positive. As a last preparatory step we truncate the Bloch hierarchy

$$P_{\mathrm{F},m}{}' = P_{\mathrm{B},m}{}' = D_m = 0 \quad , \qquad\qquad m > N \quad , \tag{2.14}$$

where the truncation parameter N is a positive integer. For the following this step implies that we shall not need to operate with infinite-dimensional matrices.

To investigate the steady-state behaviour of the Maxwell-Bloch hierarchy we choose all time derivatives equal to zero. In this way the Bloch hierarchy (2.3)–(2.6) reduces to a set of algebraic equations from which all polarization envelopes can readily be eliminated. What then results is a set of equations for the inversion envelopes $\{D_m\}$ in which the electric fields f and b play the role of coefficients. This set can be solved along the same lines as for the absorptive case [11]. We find

$$D_m = (-1)^m D_0 \exp[im(\arg f - \arg b)] \, \frac{C_{N-m}}{C_N} \quad , \tag{2.15}$$

$$-D_0{}^{-1} = 1 + 4|f'|^2 + 4|b'|^2 - 8|f'||b'| \mathrm{Re}\left(\frac{C_{N-1}}{C_N}\right) \quad , \tag{2.16}$$

$$\begin{aligned} C_m = {} & \frac{4|f'|^2 + 4|b'|^2 - 4i\Delta(|f'|^2 - |b'|^2)}{1 + 4|f'|^2 + 4|b'|^2} T_m(u) \\ & + \frac{1 + 4i\Delta(|f'|^2 - |b'|^2)}{1 + 4|f'|^2 + 4|b'|^2} U_m(u) \quad , \end{aligned} \tag{2.17}$$

with m=1,2,3,...,N and with the abbreviations $u=(1+4|f'|^2+4|b'|^2)/(8|f'||b'|)$ and $f'=f(1+\Delta^2)^{-1/2}$, $b'=b(1+\Delta^2)^{-1/2}$. For an arbitrary field v we have adopted the notation $v=|v| \exp(i\arg v)$. The symbols T_m and U_m stand for the Chebyshev polynomials of the first and the second kind, respectively. With the help of a recursion relation for the Chebyshev form [15], given by

$$C_{m+2} = 2uC_{m+1} - C_m \tag{2.18}$$

for m=0,1,2,...,N–2, it can easily be verified that the above expressions indeed satisfy the hierarchy for the inversion envelopes.

Having solved the Bloch hierarchy we now want to remove the restriction of the truncation parameter being finite and take the limit $N \rightarrow \infty$. To that end, the following representation [15] for the Chebyshev form C_m is most useful

$$\begin{aligned} C_m = {} & \left[\frac{4W + 1 + 4i\Delta(|f'|^2 - |b'|^2)}{8W}\right]\left[\frac{1 + 4|f'|^2 + 4|b'|^2 + 4W}{8|f'||b'|}\right]^m \\ & + \left[\frac{4W - 1 - 4i\Delta(|f'|^2 - |b'|^2)}{8W}\right]\left[\frac{1 + 4|f'|^2 + 4|b'|^2 - 4W}{8|f'||b'|}\right]^m \quad , \end{aligned} \tag{2.19}$$

with

$$4W = [1 + 8(|f'|^2 + |b'|^2) + 16(|f'|^2 - |b'|^2)^2]^{1/2} \quad , \tag{2.20}$$

and m=0,1,2,...,N. Clearly, for large values of m the first contribution at the right-hand side of (2.19) exceeds the second one, implying that the limiting behaviour of the inversion envelopes is

$$D_m{}^{(\infty)} = -\frac{1}{4W}\left(\frac{-8f'b'^*}{1+4|f'|^2+4|b'|^2+4W}\right)^m \quad , \tag{2.21}$$

with m=0,1,2,... .

Upon substitution of the result (2.21) in the Maxwell equations (2.1)–(2.2) and separation of these into equations for the moduli and for the arguments of the fields, we arrive at

$$\frac{\mathrm{d}|f'|}{\mathrm{d}\zeta} = -\frac{C'T}{32W|f'|}(4W+4|f'|^2-4|b'|^2-1) \quad , \tag{2.22}$$

$$\frac{\mathrm{d}|b'|}{\mathrm{d}\zeta} = \frac{C'T}{32W|b'|}(4W+4|b'|^2-4|f'|^2-1) \quad , \tag{2.23}$$

$$\frac{\mathrm{d}\arg f'}{\mathrm{d}\zeta} = -\frac{\Delta}{|f'|}\frac{\mathrm{d}|f'|}{\mathrm{d}\zeta} \quad , \tag{2.24}$$

$$\frac{\mathrm{d}\arg b'}{\mathrm{d}\zeta} = -\frac{\Delta}{|b'|}\frac{\mathrm{d}|b'|}{\mathrm{d}\zeta} \quad , \tag{2.25}$$

with C'=$C/(1+\Delta^2)$. These differential equations are subject to a set of boundary conditions which can be obtained from (2.7)–(2.9). The equations (2.22)–(2.23) have been discussed already for the absorptive case [11], so that we do not need to derive their solutions here. The integration of (2.24)–(2.25) is straightforward and yields with the help of the boundary conditions

$$\arg f(\zeta) = \arg x + \Delta\log\left|\frac{x}{2f(\zeta)}\right| \quad , \tag{2.26}$$

$$\arg b(\zeta) = \arg x + \theta + \Delta\log\left|\frac{R^{1/2}x}{2b(\zeta)}\right| \quad . \tag{2.27}$$

From the boundary condition (2.7) we see that the steady-state curve follows from

$$\begin{aligned}\tfrac{1}{4}T^2|y|^2 &= |f(0)|^2+R|b(0)|^2 \\ &\quad -2R^{1/2}|f(0)||b(0)|\cos\left[\theta+\Delta\log\left|\frac{R^{1/2}f(0)}{b(0)}\right|\right] \quad .\end{aligned} \tag{2.28}$$

The quantities $|f'(\zeta)|$, $|b'(\zeta)|$ can be found by performing the transformations $f\to|f'|$, $b\to|b'|$, $x\to|x'|$ and $C\to C'$ in the results for the absorptive case, with the definition x'=$x(1+\Delta^2)^{-1/2}$. Hence, they are determined by the identities

$$\xi = \tfrac{1}{4} - K + [\psi^2+\tfrac{1}{8}-\tfrac{1}{2}K]^{1/2} \quad , \tag{2.29}$$

$$[\psi+(\psi^2+\tfrac{1}{8}-\tfrac{1}{2}K)^{1/2}]\exp(4\psi) = (K-\tfrac{1}{4}+\tfrac{1}{2}|x'|^2)\exp[2C'T(1-\zeta)+T|x'|^2] \quad , \tag{2.30}$$

where we abbreviated

$$\xi = |f'|^2+|b'|^2 \quad , \qquad \psi = |f'|^2-|b'|^2 \quad , \tag{2.31}$$

and where K follows from

$$4K = -|x'|^2(1+R) + [T^2|x'|^4 + 2|x'|^2(1+R) + 1]^{1/2} \quad . \tag{2.32}$$

From the above treatment it turns out that for the stationary Maxwell-Bloch theory the extension of results from the purely absorptive case to the mixed absorptive and dispersive case is a simple exercise. Indeed, for the stationary case extensive analyses of the dispersive Maxwell-Bloch theory have already been carried out [16]. As compared to the purely absorptive case the steady-state equations (2.28)–(2.32) are richer because they predict that several bistable ranges can exist for a given set of parameters C, T, Δ and θ.

3 Stability analysis

In the previous section we solved the stationary Maxwell-Bloch hierarchy. We now wish to test the stability of our stationary solutions against small perturbations. Our tool will be the usual linear stability analysis, that is to say, for every field $v(\zeta,t)$ we substitute the expression $v^{\mathrm{st}}(\zeta)+\delta\, v(\zeta,t)$ in the original equations (2.1)–(2.6) and we only keep contributions which are linear in the deviations $\delta\, v(\zeta,t)$. Next, we model the time behaviour of the deviations by

$$\begin{aligned} \delta v(\zeta,t) &= \delta v(\zeta)\mathrm{e}^{\lambda t} \quad , & (3.1)\\ \delta v^*(\zeta,t) &= \delta v^{\mathrm{c}}(\zeta)\mathrm{e}^{\lambda t} \quad . & (3.2)\end{aligned}$$

The eigenvalue λ is complex and determines the stability of the stationary solution.

If we follow the above recipe for the Maxwell-Bloch hierarchy we find that the eigenvalue λ is determined by a four-dimensional set of differential equations with boundary conditions. The differential equations read

$$\begin{aligned} \frac{\mathrm{d}\delta f}{\mathrm{d}\zeta} &= -\tilde{\lambda}\delta f + CT\delta P_{\mathrm{F},1}{}' \quad , & (3.3)\\ \frac{\mathrm{d}\delta b}{\mathrm{d}\zeta} &= \tilde{\lambda}\delta b - CT\delta P_{\mathrm{B},1}{}' \quad , & (3.4)\\ \frac{\mathrm{d}\delta f^{\mathrm{c}}}{\mathrm{d}\zeta} &= -\tilde{\lambda}\delta f^{\mathrm{c}} + CT\delta P_{\mathrm{F},1}{}'^{\mathrm{c}} \quad , & (3.5)\\ \frac{\mathrm{d}\delta b^{\mathrm{c}}}{\mathrm{d}\zeta} &= \tilde{\lambda}\delta b^{\mathrm{c}} - CT\delta P_{\mathrm{B},1}{}'^{\mathrm{c}} \quad . & (3.6)\end{aligned}$$

We have introduced the scaled eigenvalue $\tilde{\lambda}=\lambda L/c$. The polarization deviations in the Maxwell equations (3.3)–(3.6) are determined by the linearized Bloch hierarchy, which reads

$$\begin{aligned} (\lambda_\perp + i\Delta)\delta P_{\mathrm{F},m}{}' &= f\delta D_{m-1} + D_{m-1}\delta f + b\delta D_m + D_m\delta b \quad , & (3.7)\\ (\lambda_\perp + i\Delta)\delta P_{\mathrm{B},m}{}' &= f\delta D_m{}^{\mathrm{c}} + D_m{}^*\delta f + b\delta D_{m-1}{}^{\mathrm{c}} + D_{m-1}{}^*\delta b \quad , & (3.8)\\ -\tfrac{1}{2}\lambda_{||}\delta D_0 &= f^*\delta P_{\mathrm{F},1}{}' + P_{\mathrm{F},1}{}'\delta f^{\mathrm{c}} + b^*\delta P_{\mathrm{B},1}{}' + P_{\mathrm{B},1}{}'\delta b^{\mathrm{c}} & \\ &\quad + f\delta P_{\mathrm{F},1}{}'^{\mathrm{c}} + P_{\mathrm{F},1}{}'^*\delta f + b\delta P_{\mathrm{B},1}{}'^{\mathrm{c}} + P_{\mathrm{B},1}{}'^*\delta b \quad , & (3.9)\\ -\tfrac{1}{2}\lambda_{||}\delta D_m &= f^*\delta P_{\mathrm{F},m+1}{}' + P_{\mathrm{F},m+1}{}'\delta f^{\mathrm{c}} + b^*\delta P_{\mathrm{F},m}{}' + P_{\mathrm{F},m}{}'\delta b^{\mathrm{c}} & \\ &\quad + f\delta P_{\mathrm{B},m}{}'^{\mathrm{c}} + P_{\mathrm{B},m}{}'^*\delta f + b\delta P_{\mathrm{B},m+1}{}'^{\mathrm{c}} + P_{\mathrm{B},m+1}{}'^*\delta b \quad , & (3.10)\end{aligned}$$

with m=1,2,3,... and with the definition $\lambda_i=1+\gamma_i{}^{-1}\lambda$ for $i=\perp,\|$. The coefficients in front of the deviations at the right-hand sides of (3.7)–(3.10) are the stationary envelopes discussed in the previous section. The deviations δf and δb must satisfy the following boundary conditions

$$\begin{aligned} \delta f(0) &= R^{1/2}\delta b(0) \ , & (3.11)\\ \left(\begin{array}{c}\delta f(1)\\ \delta b(1)\end{array}\right) &= \frac{1}{2}\delta x\left(\begin{array}{c}1\\ R^{1/2}\mathrm{e}^{i\theta}\end{array}\right) \ . & (3.12)\end{aligned}$$

The Bloch equations and the boundary conditions for the conjugate fields $\delta v^{\mathrm{c}}(\zeta)$ follow from (3.7)–(3.12) if we interchange δv^{c} and δv, change the sign of the detuning parameters Δ and θ and take the complex conjugate of all stationary envelopes. Notice that we have $\delta D_0=\delta D_0{}^{\mathrm{c}}$.

To solve the Bloch hierarchy we impose the truncation condition (2.14) both for the stationary fields and the deviations. Subsequently, we eliminate all polarization envelopes, as we did in the stationary case. Then (3.9) gets the form

$$\begin{aligned} &(1+\Delta_\lambda{}^2)\lambda_{\mathrm{p}}[(1+4|f'|_\lambda^2+4|b'|_\lambda^2)\delta D_0+4f_\lambda'^{\times}b_\lambda'\delta D_1+4f_\lambda' b_\lambda'^{\times}\delta D_1{}^{\mathrm{c}}] = \\ &-2(1+\lambda_\perp)(1-i\Delta_\lambda)(1-i\Delta)^{-1}[(f^*D_0+b^*D_1{}^*)\delta f+(f^*D_1+b^*D_0)\delta b] \\ &-2(1+\lambda_\perp)(1+i\Delta_\lambda)(1+i\Delta)^{-1}[(fD_0+bD_1)\delta f^{\mathrm{c}}+(fD_1{}^*+bD_0)\delta b^{\mathrm{c}}] \ , \qquad (3.13)\end{aligned}$$

with $\lambda_{\mathrm{p}}=\lambda_\perp\lambda_\|$, $\Delta_\lambda=\Delta/\lambda_\perp$. Furthermore we introduced the notations

$$|f'|_\lambda=\frac{|f|}{\lambda_{\mathrm{p}}^{1/2}(1+\Delta_\lambda{}^2)^{1/2}} \quad , \quad |b'|_\lambda=\frac{|b|}{\lambda_{\mathrm{p}}^{1/2}(1+\Delta_\lambda{}^2)^{1/2}} \qquad (3.14)$$

and $v_\lambda'\equiv|v'|_\lambda\exp(i\arg v)$, $v_\lambda'^{\times}\equiv|v'|_\lambda\exp(-i\arg v)$ with $v=f,b$.

Elimination of the polarizations from equation (3.10) and its conjugate gives rise to two decoupled sets of dimension N, one for the deviations $\{\delta D_m\}_{m=1}^N$ and another for the deviations $\{\delta D_m{}^{\mathrm{c}}\}_{m=1}^N$. In these sets the deviations δf, δb, δf^{c}, δb^{c} and δD_0 figure at the right-hand sides of the equality signs. Both sets can be transformed into each other via the recipe mentioned above, so that we only need to solve the deviation δD_1 from the first set in terms of δf, δb, δf^{c}, δb^{c} and δD_0, and subsequently apply the recipe in order to obtain $\delta D_1{}^{\mathrm{c}}$. With the help of the relation (3.13) the deviation δD_0 is found then as a function of the deviations δf, δb, δf^{c} and δb^{c}. Once the zeroth-order and the two first-order deviations of the inversion are known we can eliminate the polarization deviations in the Maxwell equations (3.3)–(3.6).

The execution of the above program involves rather technical manipulations which we transfer to the Appendix. Here we give the final result, i.e. the linearized Maxwell equations written in terms of the deviations of the electric field. For the sake of a systematic presentation we transform the original deviations (3.1)–(3.2) into amplitude and phase deviations, which we shall denote by $\Delta^{(+)}v(\zeta)$ and $\Delta^{(-)}v(\zeta)$, respectively. The transformation in question is

$$\Delta^{(\pm)}v(\zeta)=\frac{1}{2\eta_\pm}\left[\mathrm{e}^{-i\arg v_{\mathrm{st}}}\delta v(\zeta)\pm\mathrm{e}^{i\arg v_{\mathrm{st}}}\delta v^{\mathrm{c}}(\zeta)\right] \ , \qquad (3.15)$$

with $\eta_+=1$ and $\eta_-=i$. If the truncation parameter N becomes infinitely large the stability problem attains the form

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\begin{pmatrix}\Delta^{(+)}f\\ \Delta^{(+)}b\\ \Delta^{(-)}f\\ \Delta^{(-)}b\end{pmatrix}=\begin{pmatrix}H_{11} & H_{12} & H_{13} & H_{14}\\ H_{21} & H_{22} & H_{23} & H_{24}\\ H_{31} & H_{32} & H_{33} & H_{34}\\ H_{41} & H_{42} & H_{43} & H_{44}\end{pmatrix}\begin{pmatrix}\Delta^{(+)}f\\ \Delta^{(+)}b\\ \Delta^{(-)}f\\ \Delta^{(-)}b\end{pmatrix}\quad , \tag{3.16}$$

with

$$\begin{aligned} H_{11} &= -\tilde{\lambda}-\frac{CT{\lambda_\perp}^{-1}}{4(1+{\Delta_\lambda}^2)W}\\ &\quad -\frac{C'T(1+{\lambda_\perp}^{-1})}{16|f'|^2}\left[G+\frac{4\Delta_1|f'|^2}{(\lambda_\mathrm{d}-1)W}+\frac{4\Delta_1|f'|^2_\lambda}{({\lambda_\mathrm{d}}^{-1}-1)W_\lambda}\right] \quad , \end{aligned}\tag{3.17}$$

$$\begin{aligned} H_{12} &= \frac{2CT{\lambda_\perp}^{-1}|f'||b'|}{(1+{\Delta_\lambda}^2)W(1+4|f'|^2+4|b'|^2+4W)}\\ &\quad +\frac{C'T(1+{\lambda_\perp}^{-1})}{16|f'||b'|}\left[2\Delta_1+G+\frac{\Delta_1(1+4|f'|^2+4|b'|^2)}{2(\lambda_\mathrm{d}-1)W}\right.\\ &\quad \left.+\frac{\Delta_1(1+4|f'|^2_\lambda+4|b'|^2_\lambda)}{2({\lambda_\mathrm{d}}^{-1}-1)W_\lambda}\right] \quad , \end{aligned}\tag{3.18}$$

$$\begin{aligned} H_{13} &= \frac{C'T\Delta}{32|f'|^2W}(4W+4|f'|^2-4|b'|^2-1)-\frac{CT\Delta_\lambda{\lambda_\perp}^{-1}}{4(1+{\Delta_\lambda}^2)W}\\ &\quad +\frac{C'T\Delta(1+{\lambda_\perp}^{-1})}{16|f'|^2}\left[G+\frac{4\Delta_2|f'|^2}{(\lambda_\mathrm{d}-1)W}+\frac{4\Delta_2|f'|^2_\lambda}{({\lambda_\mathrm{d}}^{-1}-1)W_\lambda}\right] \quad , \end{aligned}\tag{3.19}$$

$$\begin{aligned} H_{14} &= \frac{2CT\Delta_\lambda{\lambda_\perp}^{-1}|f'||b'|}{(1+{\Delta_\lambda}^2)W(1+4|f'|^2+4|b'|^2+4W)}\\ &\quad -\frac{C'T\Delta(1+{\lambda_\perp}^{-1})}{16|f'||b'|}\left[2\Delta_2+G+\frac{\Delta_2(1+4|f'|^2+4|b'|^2)}{2(\lambda_\mathrm{d}-1)W}\right.\\ &\quad \left.+\frac{\Delta_2(1+4|f'|^2_\lambda+4|b'|^2_\lambda)}{2({\lambda_\mathrm{d}}^{-1}-1)W_\lambda}\right] \quad . \end{aligned}\tag{3.20}$$

We introduced the abbreviations $\lambda_\mathrm{d}=\lambda_\mathrm{p}(1+{\Delta_\lambda}^2)(1+\Delta^2)^{-1}$,

$$\Delta_1=\frac{1+\Delta\Delta_\lambda}{1+{\Delta_\lambda}^2}\quad , \quad \Delta_2=\frac{1-\lambda_\perp^{-1}}{1+{\Delta_\lambda}^2} \tag{3.21}$$

and

$$G=-1+\frac{4W}{1-\lambda_\mathrm{d}}+\frac{4W_\lambda}{1-{\lambda_\mathrm{d}}^{-1}}\quad . \tag{3.22}$$

Here the quantity W_λ is obtained from the root W by performing the substitution $|f'|\to|f'|_\lambda$, $|b'|\to|b'|_\lambda$. Hence, the root W_λ is complex; the prescription

$$\mathrm{Re}\frac{W_\lambda}{1+4|f'|^2_\lambda+4|b'|^2_\lambda}>0 \tag{3.23}$$

determines its sign.

In (3.17)–(3.20) we have only given four matrix elements H_{ij}. The matrix elements H_{33}, H_{34} are obtained from the expressions for H_{11}, H_{12} by changing the sign of G and making the substitution $\Delta_1 \to \Delta_1 - 1$. Likewise, we get H_{31}, H_{32} from $-H_{13}$, $-H_{14}$ upon changing the sign of G and making the replacement $\Delta_2 \to \Delta_2 - 1$. The remaining eight matrix elements follow from the symmetry relations

$$H_{ij}(f,b) = -H_{mn}(b,f) \quad , \tag{3.24}$$

with m, n following from i, j by the interchanges $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. The matrix elements H_{ij} satisfy the identities

$$\begin{aligned} \Delta H_{11} + H_{13} + H_{31} - \Delta H_{33} &= 0 \quad , \\ \Delta H_{12} + H_{14} + H_{32} - \Delta H_{34} &= 0 \quad . \end{aligned} \tag{3.25}$$

Two other identities of this type are found by using the symmetry relations (3.24).

In terms of amplitude and phase deviations the boundary conditions (3.11)–(3.12) read

$$\begin{pmatrix} \Delta^{(\pm)} f(1) \\ \Delta^{(\pm)} b(1) \end{pmatrix} = \tfrac{1}{2} \Delta^{(\pm)} x \begin{pmatrix} 1 \\ R^{1/2} \end{pmatrix} \quad , \tag{3.26}$$

$$\begin{pmatrix} \Delta^{(+)} f(0) \\ \Delta^{(-)} f(0) \end{pmatrix} = R^{1/2} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \Delta^{(+)} b(0) \\ \Delta^{(-)} b(0) \end{pmatrix} \quad , \tag{3.27}$$

with $\alpha = \arg f(0) - \arg b(0)$ determined by (2.26)–(2.27).

The expressions for the matrix elements H_{ij} can be verified by performing an analysis of the linearized Maxwell-Bloch equations for the complete polarization and inversion fields, without using the Fleck expansion. For the absorptive uniform-field case this method has been employed in ref. [9].

Other checks can be carried out by considering special cases. If we set the detuning parameters Δ and θ equal to zero the set (3.16) decouples into two sets of dimension two, which have been derived in an earlier paper [11]. Since the boundary condition (3.27) then decouples as well, it follows that the complete stability problem for the purely absorptive case consists of separate amplitude and phase problems.

A second special case is obtained by putting the stationary backward field b equal to zero. Then eight matrix elements H_{ij} vanish, namely those with i=2,4, j=1,3 and their counterparts (3.24), so that the deviations of the forward electric field and of the backward electric field decouple. The set for the former deviations reads

$$H_{11} = -\tilde{\lambda} + \frac{C'T(4|f'|^2_\lambda - \lambda_\| {\lambda_\mathrm{d}}^{-1})}{(1+4|f'|^2)(1+4|f'|^2_\lambda)} \quad , \tag{3.28}$$

$$H_{13} = \frac{C'T\Delta(1 - {\lambda_\perp}^{-2})}{(1+{\Delta_\lambda}^2)(1+4|f'|^2)(1+4|f'|^2_\lambda)} \quad , \tag{3.29}$$

$$H_{33} = -\tilde{\lambda} - \frac{C'T\lambda_\perp^{-1}(4|f'|^2_\lambda + \lambda_\mathrm{p} {\lambda_\mathrm{d}}^{-1})}{(1+4|f'|^2)(1+4|f'|^2_\lambda)} \quad , \tag{3.30}$$

$$H_{31} = -\frac{C'T\Delta(1 + {\lambda_\perp}^{-1})[4(1+{\Delta_\lambda}^2)|f'|^2_\lambda + 1 - {\lambda_\perp}^{-1}]}{(1+{\Delta_\lambda}^2)(1+4|f'|^2)(1+4|f'|^2_\lambda)} \quad . \tag{3.31}$$

The right-hand sides are in accordance with results that have been derived before for the case of a unidirectional ring cavity [17].

In this section we have reduced the stability problem for the plane-wave Maxwell-Bloch equations describing dispersive optical bistability in a nonideal Fabry-Pérot cavity to a boundary-value problem for a four-dimensional set of linear differential equations. The latter problem can be solved numerically along similar lines as for the purely absorptive case [8]. We shall present our numerical results in an accompanying paper [12]. In the following we will turn our attention to two special limits, namely the adiabatic limit and the uniform-field limit. The investigation of these limits will help us in analyzing the general stability problem.

4 Limiting cases

If the medium response times $\gamma_\perp^{-1}$ and $\gamma_{||}^{-1}$ in the Bloch equations (2.3)–(2.6) are small with respect to the cavity round-trip time, we may adiabatically eliminate the time derivatives of the polarization and the inversion envelopes. For the linear stability analysis, as presented in the previous section, the so-called adiabatic limit implies that the quantities $\lambda_\perp$, $\lambda_{||}$ and $\lambda_{\rm d}$ tend to unity, at least if one assumes that the eigenvalue λ remains finite. So one may expand the root W_λ as

$$\lambda_{\rm d} W_\lambda = W[1 + \frac{1}{16W^2}(1 + 4|f'|^2 + 4|b'|^2)(\lambda_{\rm d} - 1)] \quad , \tag{4.1}$$

where the modulus of $\lambda_{\rm d}$–1 is small. On insertion of this result in the expressions for the matrix elements H_{ij} we find that in the adiabatic régime these become

$$\begin{aligned}
H_{11} &= -\tilde{\lambda} - \frac{C'T}{32|f'|^2W}[1 - 4|f'|^2 + 4|b'|^2 - 4W + \frac{|f'|^2}{W^2}(1 + 4|f'|^2 + 4|b'|^2)] \quad , &(4.2)\\
H_{12} &= \frac{C'T|f'||b'|}{4W^3} \quad , &(4.3)\\
H_{31} &= \frac{C'T\Delta}{16|f'|^2W}[1 - 4|f'|^2 + 4|b'|^2 - 4W + \frac{|f'|^2}{2W^2}(1 + 4|f'|^2 + 4|b'|^2)] \quad , &(4.4)\\
H_{33} &= -\tilde{\lambda} + \frac{C'T}{32|f'|^2W}[1 - 4|f'|^2 + 4|b'|^2 - 4W] \quad , &(4.5)
\end{aligned}$$

with H_{32}=–ΔH_{12} and H_{13}=H_{14}=H_{34}=0. The other matrix elements can be obtained from the symmetry relations (3.24).

The vanishing of several matrix elements enables us, upon using the identities (3.25), to reduce the four-dimensional set of differential equations (3.16) to two sets of dimension two. This is achieved by introducing new phase deviations $\tilde{\Delta}^{(-)}v$ via

$$\tilde{\Delta}^{(-)}v(\zeta) = \frac{1}{2i}\left[(1 + i\Delta){\rm e}^{-i\arg v_{\rm st}}\delta v(\zeta) - (1 - i\Delta){\rm e}^{i\arg v_{\rm st}}\delta v^{\rm c}\right] \quad . \tag{4.6}$$

The transformed matrix H now possesses a block structure such that the transformed phase deviations do not couple with the amplitude deviations. Consequently, the integration of (3.16) can be done for the amplitude and the phase deviations (4.6) separately. In particular, the integration of the phase differential equations can be carried out along the same lines as for the absorptive case. The boundary condition (3.27) does not decouple after the transformation (4.6), so that the stability problem as a whole does not factorize in the adiabatic limit.

The adiabatic limit is not the only interesting special case. The uniform-field limit is worth a study as well. It consists [1] in putting the transmission coefficient T equal to zero for a fixed finite value of the cooperation parameter C and of the ratio $\delta{=}\theta/T$. The actual computation of the eigenvalue λ in the uniform-field limit does not differ much from that carried out for the absorptive case [11]. As usual, we put $\tilde{\lambda}{=}\tilde{\lambda}^{(0)}{+}\, T\tilde{\lambda}^{(1)}{+}\cdots$. In lowest order of T we find a twofold degenerate solution $\tilde{\lambda}^{(0)}{=}\pi n i$, with n an integer. This solution just gives the frequencies of the cavity modes. To calculate the first-order eigenvalue $\tilde{\lambda}^{(1)}$ one must distinguish between the cases $n{=}0$ and $n{\neq}0$. The result of the tedious but straightforward calculation is

$$\tilde{\lambda}_n{}^{(1)} = \tfrac{1}{2}\alpha \pm \tfrac{1}{2}(\alpha^2 - 4\beta)^{1/2} \quad . \tag{4.7}$$

The quantities α and β are certain functions of the parameters of our model. For the off-resonant case $n{\neq}0$ their explicit form is fixed by the relations

$$\begin{aligned} \alpha &= C'A^{(\mathrm{uf})} + C'B^{(\mathrm{uf})} - 1 \ , && (4.8)\\ \beta &= (C'A^{(\mathrm{uf})} - \tfrac{1}{2})(C'B^{(\mathrm{uf})} - \tfrac{1}{2}) \\ &\quad -(C'C^{(\mathrm{uf})} - \tfrac{1}{2}\delta - 2\Delta h)(C'D^{(\mathrm{uf})} + \tfrac{1}{2}\delta + 2\Delta h) \ . && (4.9) \end{aligned}$$

Here $A^{(\mathrm{uf})}$ follows from $(H_{11}{+}\tilde{\lambda})/(C'T)$ by inserting $|f'|{=}|b'|{=}|x'|/2$; as a result we find

$$\begin{aligned} A^{(\mathrm{uf})} &= \frac{(1+\lambda_\perp^{-1})(1+\Delta^2)}{4|x|^2}\left(1+\frac{U^{-1}}{\lambda_\mathrm{d}-1}+\frac{U_\lambda^{-1}}{\lambda_\mathrm{d}{}^{-1}-1}\right) - \frac{\lambda_\perp^{-1}(1+\Delta^2)U^{-1}}{1+\Delta_\lambda^2} \\ &- \frac{(1+\lambda_\perp^{-1})(\Delta_1-1)}{\lambda_\mathrm{d}-1}(U^{-1}-U_\lambda^{-1}) \ , \end{aligned} \tag{4.10}$$

where we introduced $U=(1{+}4|x'|^2)^{1/2}$ and $U_\lambda=(1{+}4\lambda_\mathrm{d}{}^{-1}|x'|^2)^{1/2}$ with $\mathrm{Re}\,U_\lambda>0$.. For λ we must substitute its zeroth-order value $i\pi nc/L$. In a similar way we get $B^{(\mathrm{uf})}$ from $(H_{33}{+}\tilde{\lambda})/(C'T)$, $C^{(\mathrm{uf})}$ from $H_{13}/(C'T)$ and $D^{(\mathrm{uf})}$ from $H_{31}/(C'T)$. Finally, we abbreviated $h{=}C'/\{1{+}4|x'|^2{+}[1{+}4|x'|^2]^{1/2}\}$.

It is instructive to set the detuning parameters Δ and δ equal to zero in (4.7). In that case the argument of the root at the right-hand side can be written as a square and the two solutions for λ reduce to those we discussed earlier [11], [18] for the absorptive case.

In order to make a further comparison with results obtained in the literature we also studied the single-mode limit. It is found by taking the uniform-field limit for fixed ratios $a_i{=}L\gamma_i/cT$, with $i{=}\perp,\ \|$, and putting $\tilde{\lambda}{=}\tilde{\lambda}^{(1)}T$. If we start from (3.28)–(3.31) for the ring cavity and use the well-known boundary conditions for this arrangement we end up with a fifth-order polynomial equation for $\tilde{\lambda}^{(1)}$, which is consistent with results derived elsewhere [6]. For the Fabry-Pérot cavity we can do single-mode theory as well, but here the resulting equation for the eigenvalue is not of a polynomial form. It still contains the single-mode counterpart of the root W_λ.

Appendix. Solution of the linearized Bloch hierarchy

In section 3 we remarked that if all polarization envelopes are eliminated from the linearized Bloch hierarchy two N-dimensional sets of equations are generated, a first for the deviations $\{\delta D_m\}_{m=1}^N$ and a second for the deviations $\{\delta D_m{}^{\mathrm{c}}\}_{m=1}^N$. The first set contains the deviation δD_0 at the right-hand side and the second one the deviation $\delta D_0{}^{\mathrm{c}}$. Therefore, both sets are coupled to each other by the relation (3.13). They transform into each other via a well-defined prescription given in section 3. Because of this property we limit our attention to the first set only.

If we write the N equations for the deviations $\{\delta D_m\}_{m=1}^N$ in a matrix form it appears that the matrix follows from that occurring in the set of equations for the static inversion envelopes D_m with m=1,2,3,...,N by making the replacements

$$f' \to f'_\lambda \ \ , b' \to b'_\lambda \ \ , f'^* \to f'^\times_\lambda \ \ , b'^* \to b'^\times_\lambda \ \ , \Delta \to \Delta_\lambda \ \ . \tag{A.1}$$

We can exploit this convenient property in a similar manner as for the absorptive case [11]. In this way we find that the deviation δD_1 is equal to

$$\delta D_1 = \frac{-1}{4|f||b|}\sum_{m=1}^N \left(\frac{-f^* b}{|f||b|}\right)^{m-1} \frac{C_{\lambda,N-m}}{C_{\lambda,N}} F_m \ \ , \tag{A.2}$$

with $C_{\lambda,m}$=$C_m(u_\lambda)$ and u_λ to be obtained from u via the prescription (A.1). The quantities F_m make up the N-dimensional vector figuring at the right-hand side of the matrix equation that leads to (A.2). They are defined as

$$\begin{aligned} F_m \ &= \ 4 f b^* \delta D_0 \delta_{m,1} \\ &+2(1+\lambda_\perp)\Lambda^-[(f^* D_m + b^* D_{m-1})\delta f + (f^* D_{m+1} + b^* D_m)\delta b] \\ &+2(1+\lambda_\perp)\Lambda^+[(f D_m + b D_{m+1})\delta f^{\mathrm{c}} + (f D_{m-1} + b D_m)\delta b^{\mathrm{c}}] \ \ , \end{aligned} \tag{A.3}$$

with m=1,2,3,...,N–1,

$$\begin{aligned} F_N \ &= \ 2\Lambda^- \left[\lambda_\perp(1+i\Delta_\lambda) f^* D_N + (1+\lambda_\perp) b^* D_{N-1}\right]\delta f \\ &+2(1-i\Delta_\lambda) b^* D_N \delta b + 2(1+i\Delta_\lambda) f D_N \delta f^{\mathrm{c}} \\ &+2\Lambda^+ \left[(1+\lambda_\perp) f D_{N-1} + \lambda_\perp(1-i\Delta_\lambda) b D_N\right]\delta b^{\mathrm{c}} \ \ . \end{aligned} \tag{A.4}$$

Here we abbreviated $\Lambda^\pm$=$(1\pm i\Delta_\lambda)/(1\pm i\Delta)$.

If we substitute the expressions (A.3) for F_m, with the result (2.15) for the stationary inversion fields inserted, into (A.2) we arrive at

$$\begin{aligned} \delta D_1 \ &= \ -\frac{f b^*}{|f||b|}\frac{C_{\lambda,N-1}}{C_{\lambda,N}}\delta D_0 + \frac{(1+\lambda_\perp)D_0}{2|f||b|}\left\{\Lambda^- b^*\left(\frac{|f|}{|b|}S_0 - S_+\right)\delta f\right. \\ &+\Lambda^-\frac{f b^{*2}}{|b|^2}\left(\frac{|b|}{|f|}S_0 - S_-\right)\delta b + \Lambda^+\frac{f^2 b^*}{|f|^2}\left(\frac{|f|}{|b|}S_0 - S_-\right)\delta f^{\mathrm{c}} \\ &\left.+\Lambda^+ f\left(\frac{|b|}{|f|}S_0 - S_+\right)\delta b^{\mathrm{c}}\right\} + \frac{(-1)^N}{4|f||b|}\left(\frac{f^* b}{|f||b|}\right)^{N-1}\frac{F_N}{C_{\lambda,N}} \ \ , \end{aligned} \tag{A.5}$$

where we defined

$$S_0 = \sum_{m=1}^{N-1}\frac{C_m C_{\lambda,m}}{C_N C_{\lambda,N}} \ \ , \tag{A.6}$$

$$S_\pm = \sum_{m=1}^{N-1} \frac{C_{m\pm1}C_{\lambda,m}}{C_N C_{\lambda,N}} \quad . \tag{A.7}$$

We now focus on the evaluation of the sums S.

Let us start by deriving two closely analogous relations for the sum and the difference of S_+ and S_-. These follow from (A.7) upon performing a shift of indices such that the summand becomes either symmetric or antisymmetric under the interchange of u and u_λ. One finds

$$S_+ \pm S_- = \sum_{m=1}^{N-1} \frac{C_m C_{\lambda,m-1} \pm C_{m-1}C_{\lambda,m}}{C_N C_{\lambda,N}} + \frac{C_N C_{\lambda,N-1} - C_1 C_{\lambda,0}}{C_N C_{\lambda,N}} \quad . \tag{A.8}$$

Two other independent identities for $S_+ \pm S_-$ are obtained from (A.6) and (A.7) by employing the recurrence relation (2.18). In fact, using this relation for C_m one immediately gets

$$u^{-1}(S_+ + S_-) = 2S_0 \quad . \tag{A.9}$$

The right-hand side is symmetric under the interchange $u \leftrightarrow u_\lambda$. On the other hand, starting again from (A.7), using the recurrence relation for $C_{\lambda,m}$ and shifting the indices in a suitable way one arrives at an identity for the difference $S_+ - S_-$

$$\begin{aligned} 2u_\lambda(S_+ - S_-) \quad = \quad & \sum_{m=1}^{N-1} \frac{C_{m+1}C_{\lambda,m-1} - C_{m-1}C_{\lambda,m+1}}{C_N C_{\lambda,N}} \\ & + \frac{C_N C_{\lambda,N} + C_{N-1}C_{\lambda,N-1} - C_1 C_{\lambda,1} - C_0 C_{\lambda,0}}{C_N C_{\lambda,N}} \quad , \end{aligned} \tag{A.10}$$

where the summand at the right-hand side is antisymmetric under the interchange of u and u_λ. We have now obtained a set of four relations, two for $S_+ + S_-$ and two for $S_+ - S_-$. The sums occurring in these relations have a definite symmetry character under the interchange of u and u_λ. We can use that symmetry to eliminate these sums altogether. In this way we arrive at four linear equations for four unknowns, namely $S_+ \pm S_-$ and the analogous expressions with u and u_λ interchanged. Solution of these equations yields

$$S_+ + S_- \quad = \quad \frac{1 + 4|f'|^2 + 4|b'|^2}{(\lambda_\mathrm{d} - 1)C_N C_{\lambda,N}}(C_{N-1}C_{\lambda,N} - C_N C_{\lambda,N-1} + C_1 C_{\lambda,0} - C_0 C_{\lambda,1}) \quad , \tag{A.11}$$

$$\begin{aligned} S_+ - S_- \quad = \quad & \frac{8|f'||b'|}{(\lambda_\mathrm{d} - 1)C_N C_{\lambda,N}}(C_N C_{\lambda,N} + C_{N-1}C_{\lambda,N-1} - C_1 C_{\lambda,1} - C_0 C_{\lambda,0}) \\ - \quad & \frac{1 + 4|f'|^2 + 4|b'|^2}{(\lambda_\mathrm{d} - 1)C_N C_{\lambda,N}}(C_{N-1}C_{\lambda,N} + C_N C_{\lambda,N-1} - C_0 C_{\lambda,1} - C_1 C_{\lambda,0}) \quad . \end{aligned} \tag{A.12}$$

From these expressions and (A.9) one can derive simple expressions for S_+, S_- and S_0.

For large values of the truncation parameter N the limiting behaviour of the Chebyshev forms C_m was already discussed in section 2. Using the same representation (2.19) for the transformed Chebyshev forms $C_{\lambda,m}$ and the fact that $|C_{\lambda,N}| \to \infty$ for N tending to infinity, we see that

$$\lim_{N\to\infty} \frac{C_{\lambda,N-1}}{C_{\lambda,N}} = \frac{8|f'|_\lambda |b'|_\lambda}{1 + 4|f'|^2_\lambda + 4|b'|^2_\lambda + 4W_\lambda} \quad , \tag{A.13}$$

under the condition that (3.23) is satisfied. Employing (A.13) and the limiting behaviour for the stationary inversion envelopes, given in section 2, we can evaluate the right-hand side of (A.5) for $N\to\infty$. In this limit the contribution that contains the quantity F_N vanishes. The corresponding result for the deviation $\delta D_1{}^{\mathrm{c}}$ is found if we interchange according to $\delta v^{\mathrm{c}}\leftrightarrow\delta v$, use the fact that $\delta D_0=\delta D_0{}^{\mathrm{c}}$, change the sign of the detuning parameter Δ and take the complex conjugate of all stationary envelopes.

We now return to equation (3.13) and insert the results for the first-order deviations of the inversion. After some algebra the deviation δD_0 attains the final form

$$\begin{aligned}\delta D_0 \quad = \quad & \frac{(1+\lambda_\perp)}{16(1-\lambda_{\mathrm{d}})|f|^2}\left(\frac{1-4\psi}{W}-\frac{1-4\psi_\lambda}{W_\lambda}\right)(\Lambda^- f^*\delta f+\Lambda^+ f\delta f^{\mathrm{c}}) \\ & +\frac{(1+\lambda_\perp)}{16(1-\lambda_{\mathrm{d}})|b|^2}\left(\frac{1+4\psi}{W}-\frac{1+4\psi_\lambda}{W_\lambda}\right)(\Lambda^- b^*\delta b+\Lambda^+ b\delta b^{\mathrm{c}}) \quad , \qquad \text{(A.14)}\end{aligned}$$

where we used the definitions (2.31). If we return to the expressions for the first-order deviations of the inversion and use the above results we end up with

$$\begin{aligned}\delta D_1 \quad = \quad & \frac{(1+\lambda_\perp)\Lambda^- b^*}{16(1-\lambda_{\mathrm{d}})|b|^2}\left[\frac{1+4\psi}{W}-\frac{1+4\psi_\lambda}{W_\lambda}\right]\delta f \\ & +\frac{(1+\lambda_\perp)\Lambda^- f b^{*2}}{16(1-\lambda_{\mathrm{d}})|f||b|^3}\left[\frac{1-\lambda_{\mathrm{d}}}{|f'||b'|}-\frac{1+4\xi+4|f'|^2(1+4\psi)}{4|f'||b'|W}\right. \\ & \left.+\frac{1+4\xi_\lambda+4|f'|_\lambda^2(1+4\psi_\lambda)}{4|f'|_\lambda|b'|_\lambda W_\lambda}\right]\delta b \\ & +\frac{(1+\lambda_\perp)\Lambda^+ f^2 b^*}{16(1-\lambda_{\mathrm{d}})|f|^3|b|}\left[\frac{1-\lambda_{\mathrm{d}}}{|f'||b'|}-\frac{1+4\xi+4|b'|^2(1-4\psi)}{4|f'||b'|W}\right. \\ & \left.+\frac{1+4\xi_\lambda+4|b'|_\lambda^2(1-4\psi_\lambda)}{4|f'|_\lambda|b'|_\lambda W_\lambda}\right]\delta f^{\mathrm{c}} \\ & +\frac{(1+\lambda_\perp)\Lambda^+ f}{16(1-\lambda_{\mathrm{d}})|f|^2}\left[\frac{1-4\psi}{W}-\frac{1-4\psi_\lambda}{W_\lambda}\right]\delta b^{\mathrm{c}} \quad . \qquad \text{(A.15)}\end{aligned}$$

The deviation δD_1^{c} follows from this result by using the recipe mentioned above.

We have now calculated the zeroth-order and first-order inversion deviations from the linearized Bloch hierarchy. With the help of the identities (A.14) and (A.15) we can eliminate the polarization deviations in the linearized Maxwell equations (3.3)–(3.6). This yields a set of differential equations of the type (3.16) for the deviations of the electrical field δf, δb, δf^{c}, δb^{c}. Finally, if we transform according to (3.15) and use the results (2.24)–(2.25) from stationary theory the matrix elements of the four-dimensional set of equations indeed turn out to be given by the expressions of section 3.

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