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The Topology of Full and Weak Belief

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Abstract. We introduce a new topological semantics for belief logics in which the belief modality is interpreted as the interior of the closure of the interior operator. We show that the system **wKD45**, a weakened version of **KD45**, is sound and complete with respect to the class of all topological spaces. While generalizing the topological belief semantics proposed in [1,2] to all spaces, we model conditional beliefs and updates and give complete axiomatizations of the corresponding logics with respect to the class of all topological spaces.

Keywords: Topological models · Epistemic and doxastic logic · Updates · Conditional beliefs · (hereditarily) Extremally disconnected spaces

1 Introduction

Understanding the relation between knowledge and belief is an issue of central importance in formal epistemology. Especially after the birth of the knowledge-first epistemology in [36], the question of what exactly distinguishes an item of belief from an item of knowledge and how one can be defined in terms of the other has become even more pertinent. This problem has been tackled from two rather opposite perspectives in the literature. On the one hand, there has been proposals in the line of justified true belief account of knowledge (JTB), accepting the conceptual priority of belief over knowledge. According to this approach, one starts with a weak notion of belief (which is at least justified and true) and tries to reach knowledge by making the chosen notion of belief stronger in such a way that the defined notion of knowledge would no longer be subject to Gettier-type counterexamples [15]. Among this category, we can mention the conception of knowledge as *correctly justified belief*: *not only the content of belief has to*

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be true, but its justification has to be correct. This approach can be formalized via topologies under the *interior-based semantics* (see, e.g., Sect. 2.2). Other responses falling under the first category include the *defeasibility analysis of knowledge* [20,21], the *sensitivity account* [24], the *contextualist account* [12] and the *safety account* [29][1].

The second perspective, on the other hand, challenges the 'conceptual priority of belief over knowledge' [36] and reverts the relation by giving priority to knowledge. When knowledge has priority, other attitudes (e.g. beliefs) should be explainable or definable in terms of it. One of the few philosophers who has worked out a formal system that ties in with this second approach is Stalnaker. In [30], Stalnaker uses a relational semantics for knowledge based on *reflexive, transitive and directed* Kripke models. In his work, he analyses the relation between knowledge and belief and builds a combined modal system for these notions with the axioms extracted from this analysis. He intends to capture a strong notion of belief based on the conception of 'subjective certainty'

$$B\varphi \rightarrow BK\varphi$$

meaning that *believing implies believing that one knows* [30, p. 179]. Stalnaker refers to this concept as 'strong belief', but following our previous work [1,2] we prefer to call it *full belief.* In fact, the above axiom holds biconditionally in his system and belief therefore becomes *subjectively indistinguishable from knowledge: an agent (fully) believes φ iff she (fully) believes that she knows φ* [1,2]. Moreover, Stalnaker argues that the 'true' logic of knowledge is **S4.2** and that (full) belief can be defined as the *epistemic possibility of knowledge.* More precisely,

$$B\varphi = \neg K \neg K\varphi$$

meaning that *an agent believes φ iff she does't know that she does't know φ*[2]. He moreover states that his system embeds the logic of belief **KD45** when B is defined as $\langle K\rangle K$[3] (and K is an **S4.2** modality).

In [1,2] Stalnaker's semantics was generalized from a relational setting to a topological setting. In particular, a topological semantics was given for full belief extending the interior semantics for knowledge with a semantic clause for the belief modality via the *closure of the interior* operator and it was shown that the proposed semantics on extremally disconnected spaces constitutes the canonical (most general) semantics for Stalnaker's axiom. In this way, Stalnaker's formalization was generalized by making it independent from its relational semantics. [1,2] focused on the unimodal cases for knowledge and belief and proved that while the knowledge logic of extremally disconnected spaces under the interior-based semantics is indeed **S4.2**, its belief logic under the proposed topological semantics is **KD45**. In this paper (Sect. 3), we give a brief overview of the work

[1] For an overview of responses to the Gettier challenge and a detailed discussion, we refer the reader to [18,27].

[2] For a more detailed discussion on Stalnaker's approach, we refer the reader to [2].

[3] $\langle K\rangle$ denotes the dual of K, i.e., $\neg K\neg\varphi := \langle K\rangle\varphi$.

done in [1,2]. We refer to [1,2] for a more detailed discussion. This framework, however, comes with a problem when extended to a dynamic setting by adding update modalities in order to capture the action of learning (conditioning with) new 'hard' (true) information P, as also elaborated in [2]. Conditioning with new 'hard' (true) information P is commonly modelled by deleting the 'non-P' worlds from the initial model. Its natural topological analogue, as recognized in [5,6,38] (among others) and also applied in [2], is a topological update operator, using the restriction of the original topology to the subspace induced by the set P. In order for this interpretation to be successfully implemented, the subspace induced by the new information P should possess the same structural properties as the initial topology that renders the axioms of the underlying knowledge/belief system sound. More precisely, we demand the subspace induced by the new information P to be in the class of structures with respect to which the (static) knowledge/belief logics in questions are sound and complete. However, extremally disconnectedness is not a hereditary property. In other words, it is not guaranteed that an arbitrary subspace of a given extremally disconnected space is extremally disconnected. Therefore, the aforementioned topological interpretation of conditioning with true, hard information cannot be implemented on extremally disconnected spaces. In [2], we present a solution for this problem by modelling updates on the topological spaces whose every subspace is extremally disconnected, i.e., by modelling updates on hereditarily extremally disconnected spaces.

In this paper, we propose another solution for this problem via arbitrary topological spaces. More precisely, we do it by introducing a topological semantics for belief based on *all* topological spaces in terms of the *interior of the closure of the interior operator*. It is important that this semantics coincides with the topological belief semantics introduced in [1,2] on extremally disconnected space, thus, we here generalize the semantics proposed in [1,2] to all topological spaces. Further, while the complete logic of knowledge is actually **S4** (McKinsey and Tarski [23]), we show that the complete logic of belief is a weaker system than **KD45**, namely the logic **wKD45**. The latter result follows by translating **S4** fully and faithfully into **wKD45**. The restriction of this translation to **S4.2** coincides with Stannaker's embedding of **KD45** into **S4.2**. We also formalize a notion of conditional belief $B^{\varphi}\psi$ by *relativizing* the semantic clause for simple belief modality to the extension of the learnt formula φ. We moreover formalize updates $\langle !\varphi\rangle\psi$ again as a topological update operator using the *restriction* of the initial topology to its subspace induced by the new information φ and show that we no longer encounter the problem about updates risen in the case of extremally disconnected spaces: *updates on all topological spaces are 'well-behaved'.*

We note that the interior of the closure of the interior operator is also interesting from a purely topological point of view. This operation can be seen as a *regularization* of an open set. Geometrically this operation 'patches up cracks' of an open region (see Sect. 4.1 for more details on this as well as for an epistemic interpretation of this operation). Furthermore, from a purely syntactical point of view, part of our work can be seen as studying the $B := K\langle K\rangle K$-fragment of the system **S4** for K and providing a complete axiomatization for this modality (which is interpreted as belief (B) in this particular setting). Given that our work is inspired by Stalnaker's [30], one natural question to ask is why we are interested in the $K\langle K\rangle K$-fragment of **S4** rather than the $\langle K\rangle K$-fragment as a belief system. In fact, the latter approach, namely logics of belief as *epistemic possibility of knowledge*, stemming from knowledge modalities of different strength, has been of interest in recent years. Klein et al. [19] investigate this fragment when K is not positively introspective, more precisely, when K is of type **KT.2**. To the best of our knowledge, finding a complete axiomatization of the $\langle K\rangle K$-fragment of **S4** is still an open and interesting question from a proof theoretical perspective. However, we know that it is neither a normal modal logic nor does it include the (D)-axiom. It therefore does not form a 'good' logic of belief in this particular setting with highly idealized agents. The $K\langle K\rangle K$-fragment on the other hand is equivalent to the $\langle K\rangle K$-fragment when K is of **S4.2** type. Moreover, it is the only non-empty, positive modality that is normal in **S4** and not equivalent to the knowledge modality K (see e.g., [10, Ex. 3.14, p. 102]). Hence, it is the only alternative for Stalnaker's belief as *subjective certainty* that can satisfy most of the standard axioms of belief.

The paper is structured as follows. In Sect. 2 we introduce the topological preliminaries used in this paper and present the interior-based topological semantics as well as its connection to the standard Kripke semantics and to the topological interpretation of knowledge. Section 3 gives a brief overview of the previous related work. Sections 4 and 5 constitute the main parts of this paper: while the former presents a topological semantics for belief based on all topological spaces, the latter is concerned with the topological interpretation of conditional beliefs and updates. In Sect. 6 we conclude by giving a summary of our results and pointing out a number of directions for future research.

2 Background

2.1 Topological Preliminaries

In this section, we introduce the basic topological concepts that will be used throughout this paper. For more detailed discussion we refer the reader to [13,14].

A ***topological space*** is a pair (X, τ), where X is a non-empty set and τ is a family of subsets of X containing X and $\emptyset$ and is closed under finite intersections and arbitrary unions. The set X is called a ***space***. The subsets of X belonging to τ are called ***open sets*** (or ***opens***) in the space; the family τ of open subsets of X is called a ***topology*** on X. Complements of open sets are called ***closed sets***. An open set containing $x \in X$ is called an ***open neighbourhood*** of x. The ***interior*** $\mathrm{Int}(A)$ of a set $A \subseteq X$ is the largest open set contained in A whereas the ***closure*** $\mathrm{Cl}(A)$ of A is the least closed set containing A. It is easy to see that Cl is the De Morgan dual of Int (and vice versa) and can be written as $\mathrm{Cl}(A) = X\backslash\mathrm{Int}(X\backslash A)$. Moreover, the set of boundary points of a set $A \subseteq X$, denoted by $\mathrm{Bd}(A)$, is defined as $\mathrm{Bd}(A) = \mathrm{Cl}(A)\backslash\mathrm{Int}(A)$.

2.2 The Interior Semantics for Modal (Epistemic) Logic

In this section we provide the formal background for the aforementioned interior-based topological semantics for modal (epistemic) logic that originated in the work of McKinsey and Tarski [23]. Moreover, we present important completeness results concerning logics of knowledge **S4**, **S4.2** and **S4.3** based on the interior semantics, explain the connection between the interior and standard Kripke semantics, and focus on the topological (evidence-based) interpretation of knowledge.

Syntax. We consider the standard unimodal (epistemic) language $\mathcal{L}_K$ with a countable set of propositional letters Prop, Boolean operators $\neg$ and $\wedge$ and a modal operator K. Formulas of $\mathcal{L}_K$ are defined as usual by the following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi$$

where $p \in$ Prop. Abbreviations for the connectives $\vee$, $\to$, $\leftrightarrow$ are standard. Moreover, the existential modal operator $\langle K\rangle$ and $\bot$ are defined as $\langle K\rangle\varphi := \neg K\neg\varphi$ and $\bot := p \wedge \neg p$,

Semantics. Given a topological space (X, τ), we define a ***topological model*** (or simply a ***topo-model***) as $\mathcal{M} = (X, \tau, \nu)$ where $\nu : \text{Prop} \to \mathcal{P}(X)$ is a valuation function.

Definition 1. *Given a topo-model $\mathcal{M} = (X, \tau, \nu)$, we define the **interior semantics** for the language $\mathcal{L}_K$ recursively as:*

$$\begin{array}{lll}
\mathcal{M}, x \models p & \textit{iff} & x \in \nu(p) \\
\mathcal{M}, x \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, x \models \varphi \\
\mathcal{M}, x \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, x \models \varphi \textit{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models K\varphi & \textit{iff} & (\exists U \in \tau)(x \in U \wedge \forall y \in U,\ \mathcal{M}, y \models \varphi)
\end{array}$$

where $p \in$ Prop[4].

We let $[\![\varphi]\!]^{\mathcal{M}} = \{x \in X \mid \mathcal{M}, x \models \varphi\}$ denote the ***extension*** of a modal formula φ in a topo-model $\mathcal{M}$, i.e., the *extension* of a formula φ in a topo-model $\mathcal{M}$ is defined as the set of points in $\mathcal{M}$ satisfying φ. We skip the index when it is clear in which model we are working. It is now easy to see that $[\![K\varphi]\!] = \text{Int}([\![\varphi]\!])$ and $[\![\langle K\rangle\varphi]\!] = \text{Cl}([\![\varphi]\!])$. We use this extensional notation throughout the paper as it makes clear the fact that the modalities, K and $\langle K\rangle$, are interpreted in terms of specific and *natural* topological operators. More precisely, K and $\langle K\rangle$ are modelled as the *interior* and the *closure* operators, respectively.

We say that φ is true in a topo-model $\mathcal{M} = (X, \tau, \nu)$ if $[\![\varphi]\!]^{\mathcal{M}} = X$, and that φ is valid in (X, τ) if $[\![\varphi]\!]^{\mathcal{M}} = X$ for all topo-models $\mathcal{M}$ based on (X, τ), and finally we say that φ is valid in a class of topological spaces if φ is valid in every member of the class [33]. Soundness and completeness with respect to the interior semantics are defined as usual.

[4] Originally, McKinsey and Tarski [23] introduce the interior semantics for the basic modal language. Since we talk about this semantics in the context of *knowledge*, we use the basic *epistemic* language.

Theorem 1 ([23]). **S4** *is sound and complete with respect to the class of all topological spaces under the interior semantics.*

Topological interpretation of knowledge: open sets as pieces of evidences. One of the reasons as to why the interior operator is interpreted as knowledge is that the Kuratowski properties (see, e.g., [13,14]) of the interior operator amount to **S4** axioms written in topological terms. This implies that (as we can also read from Theorem 1), *topologically*, knowledge is *Truthful*

$$K\varphi \to \varphi,$$

Positively Introspective

$$K\varphi \to KK\varphi,$$

but *not necessarily* Negatively Introspective

$$\neg K\varphi \to K\neg K\varphi.$$

From a philosophical point of view, the principle of Negative Introspection is arguably the most controversial axiom regarding the characterization of knowledge. It leads to some undesirable consequences, such as Voorbraak's paradox (see e.g., [1,35]), and is rejected by some prominent people in the field such as Hintikka [17], Lenzen [22], Stalnaker [30] (among others).

Another argument in favour of *knowledge as the interior operator* conception is of a more 'semantic' nature: the interior semantics provides a deeper insight into the evidence-based interpretation of knowledge. We can interpret opens in a topological model as 'pieces of evidence' and, in particular, open neighborhoods of a state x as the pieces of *true* (*sound, correct*) evidence that are observable by the agent at state x. If an open set U is included in the extension of a proposition φ in a topo-model $\mathcal{M}$, i.e. if $U \subseteq [\![\varphi]\!]^{\mathcal{M}}$, we say that *the piece of evidence U entails* (*supports, justifies*) *the proposition* φ. Recall that, for any topo-model $\mathcal{M} = (X, \tau, \nu)$, any $x \in X$ and any $\varphi \in \mathcal{L}_K$, we have

$$x \in [\![K\varphi]\!]^{\mathcal{M}} \text{ iff } (\exists U \in \tau)(x \in U \wedge U \subseteq [\![\varphi]\!]^{\mathcal{M}}).$$

Thus, taking open sets as pieces of evidence and in fact open neighbourhoods of a point x as *true* pieces of evidence (that the agent can observe at x), we obtain the following evidence-based interpretation for knowledge: *the agent knows φ iff she has a true piece of evidence U that justifies φ*. In other words, *knowing φ* is the same as *having a correct justification for φ*. The necessary and sufficient conditions for one's belief to qualify as knowledge consist in it being not only truthful, but also in having a correct (evidential) justification. Therefore, the interior semantics implements the widespread intuitive response to Gettier's challenge: knowledge is *correctly* justified belief (rather than being simply true justified belief) [1,2].

Connection between Kripke frames and topological spaces. The interior semantics is closely related to the standard Kripke semantics of **S4** (and of its normal extensions): every reflexive and transitive Kripke frame corresponds to a special kind of (namely, Alexandroff) topological spaces.

Let us now fix some notation and terminology. We denote a ***Kripke frame*** by $\mathcal{F} = (X, R)$, a ***Kripke model*** by $M = (X, R, \nu)$ and $\|\varphi\|^M$ denotes the *extension* of a formula φ in a Kripke model $M = (X, R, \nu)$[5]. A topological space (X, τ) is called ***Alexandroff*** if τ is closed under arbitrary intersections, i.e., $\bigcap \mathcal{A} \in \tau$ for any $\mathcal{A} \subseteq \tau$. Equivalently, a topological space (X, τ) is Alexandroff iff every point in X has a least neighborhood. As mentioned, there is a one-to-one correspondence between reflexive and transitive Kripke frames and Alexandroff spaces. More precisely, given a reflexive and transitive Kripke frame $\mathcal{F} = (X, R)$, we can construct a topological space, indeed an Alexandroff space, $\mathcal{X} = (X, \tau_R)$ by defining τ_R to be the set of all upsets[6] of $\mathcal{F}$. Moreover, the evaluation of modal formulas in a reflexive and transitive Kripke model coincides with their evaluation in the corresponding (Alexandroff) topological space (see e.g., [26, p. 306]). As a result of this connection, the Kripke completeness of the normal extensions of **S4** implies topological completeness under the interior semantics (see, e.g., [33]).

Normal extensions of S4: the logics S4.2 and S4.3. There are two other knowledge systems, namely **S4.2** and **S4.3** , that are of particular interest for us. Both **S4.2** and **S4.3** are strengthenings of **S4** which are defined as

$$\begin{aligned}\mathbf{S4.2} &:= \mathbf{S4} + \langle K\rangle K\varphi \to K\langle K\rangle\varphi, \text{ and}\\ \mathbf{S4.3} &:= \mathbf{S4} + K(K\varphi \to \psi) \vee K(K\psi \to \varphi)\end{aligned}$$

where $\mathbf{L} + \varphi$ denotes the smallest logic containing $\mathbf{L}$ and φ.

We recall that a topological space (X, τ) is ***extremally disconnected*** if the closure of every open subset of X is open and it is ***hereditarily extremally disconnected*** if every subspace of (X, τ) is extremally disconnected. We here would like to remind that extremally disconnectedness is, in general, not a hereditary property[7].

Theorem 2 (cf. [33]). S4.2 *is sound and complete with respect to the class of extremally disconnected spaces under the interior semantics.*

Theorem 3 ([2,7]). S4.3 *is sound and complete with respect to the class of hereditarily extremally disconnected spaces under the interior semantics.*

We note that the completeness parts of Theorems 2 and 3 follow from the Kripke completeness of **S4.2** and **S4.3** (which is a direct consequence of the

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[5] The reader who is not familiar with the standard Kripke semantics is referred to [8,11] for an extensive introduction to the topic.

[6] A set $A \subseteq X$ is called an *upset* of (X, R) if for each $x, y \in X$, xRy and $x \in A$ imply $y \in A$.

[7] A topological property is said to be *hereditary* if for any topological space (X, τ) that has the property, every subspace of (X, τ) also has it [14, p. 68].

Sahlqvist theorem) and the fact that Alexandroff spaces corresponding to transitive, reflexive and directed Kripke frames (**S4.2**-frames) are extremally disconnected and Alexandroff spaces corresponding to reflexive and transitive Kripke frames with no branching to the right (**S4.3**-frames) are hereditarily extremally disconnected. The soundness with respect to the topological semantics, however, needs some argumentation. The detailed proofs can be found in [33, p. 253] and [7, Proposition 3.1]. The logical counterpart of the fact that extremally disconnected spaces (**S4.2**-spaces) are not closed under subspaces is that **S4.2** is not a subframe logic [10, Sect. 9.4]. The logical counterpart of the fact that hereditarily extremally disconnected spaces (**S4.3**-spaces) are extremally disconnected spaces closed under subspaces is that the subframe closure of **S4.2** is **S4.3**, see [37, Sect. 4.7]. For examples of extremally disconnected and hereditarily extremally disconnected spaces, we refer to [2,7,28].

3 The Topology of Full Belief: Overview of [1]

3.1 Stalnaker's Combined Logic of Knowledge and Belief

In his paper [30], Stalnaker focuses on the properties of knowledge and belief and the relation between the two and he approaches the problem of understanding the concrete relation between knowledge and belief from an unusual perspective. Unlike most research in the formal epistemology literature, he starts with a chosen notion of knowledge and weakens it to obtain belief. He bases his analysis on a conception of belief as 'subjective certainty': *from the point of the agent in question, her belief is subjectively indistinguishable from her knowledge* [1]. In this section, we briefly recall Stalnaker's proposal of the 'true' logic of knowledge and belief. Throughout this paper, following [1,2,25], we will refer to Stalnaker's notion as 'full belief'.

The *bimodal language* $\mathcal{L}_{KB}$ of *knowledge and (full) belief* is obtained by extending $\mathcal{L}_K$ by a belief modality B:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi.$$

We define the doxastic possibility modality $\langle B\rangle\varphi$ by $\neg B\neg\varphi$. We call Stalnaker's system, given in the following Table 1, **KB**.

We refer to [1,2,25] for a discussion on the axioms of **KB** and continue with some conclusions of philosophical importance derived by Stalnaker in [30] and stated in the following proposition:

Proposition 1 (Stalnaker [30]). *The following equivalence is provable in the system* **KB***:*

$$B\varphi \leftrightarrow \langle K\rangle K\varphi. \tag{1}$$

Moreover, the axioms (K) $B(\varphi \to \psi) \to (B\varphi \to B\psi)$*,* (D) $B\varphi \to \langle B\rangle\varphi$*,* (4) $B\varphi \to BB\varphi$*,* (5) $\neg B\varphi \to B\neg B\varphi$ *of the system* **KD45** *and the* (.2)*-axiom* $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$ *of the system* **S4.2** *are provable in* **KB***.*

Proposition 1 thus shows that full belief is definable in terms of knowledge as 'epistemic possibility of knowledge' via equivalence (1), the 'true' logic of belief is **KD45** and the 'true' logic of knowledge is **S4.2** (see [2] for the proof).

Table 1. Stalnaker's System **KB**

	Stalnaker's axioms	
(K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Knowledge is additive
(T)	$K\varphi \to \varphi$	Knowledge implies truth
(KK)	$K\varphi \to KK\varphi$	Positive introspection for K
(CB)	$B\varphi \to \neg B\neg\varphi$	Consistency of belief
(PI)	$B\varphi \to KB\varphi$	(Strong) positive introspection of B
(NI)	$\neg B\varphi \to K\neg B\varphi$	(Strong) negative introspection of B
(KB)	$K\varphi \to B\varphi$	Knowledge implies Belief
(FB)	$B\varphi \to BK\varphi$	Full Belief
	Inference rules	
(MP)	From φ and $\varphi \to \psi$ infer ψ	Modus Ponens
(K-Nec)	From φ infer $K\varphi$	Necessitation

3.2 The Topological Semantics of Full Belief

In [1,2,25], a topological semantics for full belief and knowledge is proposed by extending the interior semantics for knowledge with a semantic clause for belief. The belief modality B is interpreted as the *closure of the interior operator* on *extremally disconnected spaces*. Several topological soundness and completeness results for both bimodal and unimodal cases, in particular for **KB** and **KD45**, with respect to the proposed semantics are also proved. We now briefly overview the topological semantics for full belief introduced in [1,2,25] and state the completeness results. The proofs can be found in [2,25].

Definition 2 (Topological Semantics for Full Belief and Knowledge). *Given a topo-model $\mathcal{M} = (X, \tau, \nu)$, the semantics for the formulas in $\mathcal{L}_{KB}$ is defined for Boolean cases and $K\varphi$ the same way as in the interior semantics. The semantics for $B\varphi$ is defined as*

$$[\![B\varphi]\!]^{\mathcal{M}} = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})).$$

Truth and validity of a formula, soundness and completeness are defined the same way as in the interior semantics.

Proposition 2. *A topological space validates all the axioms and rules of Stalnaker's system* **KB** *(under the semantics given above) iff it is extremally disconnected.*

Theorem 4. *The sound and complete logic of knowledge and belief on extremally disconnected spaces is given by Stalnaker's system* **KB**.

Besides, as far as full belief is concerned, the above topological semantics constitutes the *most general extensional semantics* for Stalnaker's system **KB** [1,2,25]. Moreover, Stalnaker's combined logic of knowledge and belief yields the system **S4.2** as the unimodal logic of knowledge and the system **KD45** as the unimodal logic of belief (see Proposition 1). It has been already proven that **S4.2** is complete with respect to the class of extremally disconnected spaces under the interior semantics. This raises the question of topological soundness and completeness for **KD45** under the proposed semantics for belief in terms of the *closure and the interior operator*:

Theorem 5 **([1,2,25]). KD45** *is sound and complete with respect to the class of extremally disconnected spaces under the topological belief semantics.*

Theorem 5 therefore shows that the belief logic of extremally disconnected spaces is **KD45** when B is interpreted as the closure of the interior operator. These results on extremally disconnected spaces, however, encounter problems when extended to a dynamic setting by adding update modalities formalized as model restriction by means of subspaces.

Topological Semantics for Update Modalities. We now consider the language $\mathcal{L}_{!KB}$ obtained by adding to the language $\mathcal{L}_{KB}$ (existential) dynamic update modalities $\langle !\varphi\rangle\psi$ meaning that φ *is true and after the agent learns* φ, ψ *becomes true*. As also observed in [5,6,38], the topological analogue of updates corresponds to taking the restriction of a topology τ on X to a subset $P \subseteq X$, i.e., it corresponds to the restriction of the original topology to its subspace induced by the new, true information P.

Given a topological space (X, τ) and a non-empty set $P \subseteq X$, a space (P, τ_P) is called a ***subspace*** of (X, τ) where $\tau_P = \{U \cap P : U \in \tau\}$.

For a topo-model (X, τ, ν) and $\varphi \in \mathcal{L}_{!KB}$, we denote by $\mathcal{M}_\varphi$ the ***restricted model*** $\mathcal{M}_\varphi = ([\![\varphi]\!], \tau_{[\![\varphi]\!]}, \nu_{[\![\varphi]\!]})$ where $[\![\varphi]\!] = [\![\varphi]\!]^{\mathcal{M}}$ and $\nu_{[\![\varphi]\!]}(p) = \nu(p) \cap [\![\varphi]\!]$ for any $p \in \text{Prop}$. Then, the semantics for the dynamic language $\mathcal{L}_{!KB}$ is obtained by extending the semantics for $\mathcal{L}_{KB}$ with:

$$[\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} = [\![\psi]\!]^{\mathcal{M}_\varphi}.$$

To explain the problem: Given that the underlying *static* logic of knowledge and belief is the logic of extremally disconnected spaces (see e.g., Theorems 2, 4 and 5) and extremally disconnectedness is not inherited by arbitrary subspaces, we cannot guarantee that the restricted model induced by an arbitrary formula φ remains extremally disconnected. Under the topological belief semantics, both the (K)-axiom (also known as the axiom of *Normality*) $B(\varphi \wedge \psi) \leftrightarrow (B\varphi \wedge B\psi)$ and the (D)-axiom (also named as the *Consistency of Belief*) $B\varphi \rightarrow \langle B\rangle\varphi$ characterize extremally disconnected spaces [2,25]. Therefore, if the restricted model is not extremally disconnected, the agent comes to have inconsistent beliefs after an update with true information: the formula $B\varphi \wedge B\neg\varphi$ is satisfiable in a non-extremally disconnected topo-model. For an example illustrating this problem, we refer to [2, p. 21].

One possible solution for this problem is a further limitation on the class of spaces we work with: we can restrict our attention to *hereditarily extremally disconnected spaces*, thereby, we guarantee that no model restriction leads to inconsistent beliefs. As the logic of hereditarily extremally disconnected spaces under the interior semantics is **S4.3**, the underlying static logic, in this case, would consist in **S4.3** as the logic of knowledge but again **KD45** as the logic of belief. In [2], we examine this solution. In this paper, we present another solution which approaches the issue from the opposite direction: we propose to work with all topological spaces instead of working with a restricted class. This solution, unsurprisingly, leads to a weakening of the underlying static logic of knowledge and belief. As we already mentioned earlier, it is a classic result that the knowledge logic of all topological spaces is **S4** and here we will explore the (weak) belief logic of all topological spaces under the topological belief semantics.

4 The Topology of Weak Belief

4.1 Topological Semantics of Weak Belief

Recall that given an extremally disconnected space (X, τ), we have

$$\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$$

for any $A \subseteq X$. Hence, given a topo-model $\mathcal{M} = (X, \tau, \nu)$, the semantic clause for the belief modality can be written in the following equivalent forms

$$[\![B\varphi]\!]^{\mathcal{M}} \overset{(1)}{=} \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})) \overset{(2)}{=} \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})))$$

if (X, τ) is an extremally disconnected space. However, $\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ is not always the case for all topological spaces and all $A \subseteq X$; the equation demands the restriction to extremally disconnected spaces. Besides, if we evaluate B as the closure of the interior operator on *all* topological spaces, we obtain that neither the (K)-axiom nor the (D)-axiom is sound. Syntactically speaking, B defined as $\langle K \rangle K$ does not yield a 'good' logic of belief when K is an **S4**-type modality: $\langle K \rangle K$ is neither normal nor does satisfy the (D)-axiom. Moreover, purely **S4**-type knowledge could not have been what Stalnaker had in mind while considering B as $\langle K \rangle K$ since this would violate his principles (CB) and (PI). Moreover, given that K is interpreted as the interior operator on topological spaces, equation (1) makes the schema $B\varphi \leftrightarrow \langle K \rangle K\varphi$ and equation (2) makes the schema $B\varphi \leftrightarrow K\langle K \rangle K\varphi$ valid on all topological spaces. While $\mathbf{S4.2} \vdash \langle K \rangle K\varphi \leftrightarrow K\langle K \rangle K\varphi$, we have $\mathbf{S4} \nvdash \langle K \rangle K\varphi \leftrightarrow K\langle K \rangle K\varphi$ and B as $K\langle K \rangle K$ is the only alternative holding the property of being equivalent to $\langle K \rangle K$ in **S4.2** and being not equivalent to $\langle K \rangle K$ in **S4**. Moreover, $K\langle K \rangle K$ is the only non-empty and positive modality that is normal and is *not* equivalent to knowledge in **S4** [10, Ex. 3.14, p. 102]. Therefore, a notion of belief that works well on all topological spaces and coincides with Stalnaker's belief as subjective certainty on extremally disconnected spaces demands the alternative interpretation of belief in terms of the interior of the closure of the interior operator.

We thus concentrate on the latter equation: we interpret B as the *interior of the closure of the interior* operator on all topological spaces.

Semantics. Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model. The semantic clauses for the propositional variables and the Boolean connectives are the same as in the interior semantics. For the modal operator B, we put

$$[\![B\varphi]\!]^{\mathcal{M}} \quad = \quad \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})))$$

and the semantic clause for $\langle B \rangle$ is easily obtained as

$$[\![\langle B \rangle\varphi]\!]^{\mathcal{M}} \quad = \quad \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}}))).$$

Validity of a formula is defined as usual. We call this semantics **w**-*topological belief semantics* referring to the system **wKD45** for which we will prove soundness and completeness. This way we distinguish it from the *topological belief semantics* presented in Sect. 3.2 w.r.t. to which we proved the soundness and completeness of the system **KD45**.

Our new topological interpretation of belief also comes with intrinsic philosophical motivation that fits well with the topologically defined notions of *closeness* and *small/negligible sets*. To elaborate, it is well-known that the closure operator represents a topological conception of 'closeneess'. Intuitively speaking, we can read $x \in \mathrm{Cl}(A)$ as *x is very close to the set A*, i.e., *it cannot be sharply distinguished from the elements of A via an open set*. Therefore, recalling that K is interpreted as the interior modality, according to the semantics for (full) belief in terms of the closure and the interior operator introduced in Sect. 3.2, 'the agent fully believes φ at a state x iff she cannot sharply distinguish x from the worlds in which she has knowledge of φ' [2, p. 24]. Therefore, *full belief is very close to knowledge* when 'close' is interpreted topologically [2]. This interpretation in fact captures the notion of belief as *subjective certainty*. Our new interpretation of belief in terms of the interior of the closure of the interior operator $[\![B\varphi]\!] = \mathrm{Int}(\mathrm{Cl}([\![K\varphi]\!]))$, on the other hand, makes the connection between these two notions even stronger: the belief operator interpreted this way comes even *closer* to knowledge, yet does not coincide with it. According to the new interpretation of belief, the agent believes φ at a state x iff there exists an open neighbourhood U of x such that $U \subseteq \mathrm{Cl}([\![K\varphi]\!])$. This implies, since $[\![K\varphi]\!]$ is open, that $x \in U \subseteq [\![K\varphi]\!] \cup \mathrm{Bd}([\![K\varphi]\!])$, where $\mathrm{Bd}([\![K\varphi]\!])$ is the set of boundary points of $[\![K\varphi]\!]$. As $U \subseteq [\![K\varphi]\!] \cup \mathrm{Bd}[\![K\varphi]\!]$, the set $U \cap [\![\neg K\varphi]\!] = U \cap \mathrm{Bd}([\![K\varphi]\!])$ and it is possibly non-empty. Thus, it is still not guaranteed that the agent can distinguish the states in which she knows φ from the ones in which she does not. However,

$$U \cap [\![\neg K\varphi]\!] = U \cap \mathrm{Bd}([\![K\varphi]\!]) \subseteq \mathrm{Bd}([\![K\varphi]\!])$$

and, since $[\![K\varphi]\!] = \mathrm{Int}[\![\varphi]\!]$ is an open set, $\mathrm{Bd}([\![K\varphi]\!])$ is nowhere dense[8]. Therefore, $U \cap [\![\neg K\varphi]\!]$ is also nowhere dense. As nowhere dense sets constitute one of the topological notions of 'small, negligible sets' and $U \subseteq [\![K\varphi]\!] \cup \mathrm{Bd}([\![K\varphi]\!])$, we can

[8] A subset $A \subseteq X$ is called *nowhere dense* in (X, τ) if $\mathrm{Int}(\mathrm{Cl}(A)) = \emptyset$.

say that *the agent believes* φ *at* x *iff she can almost sharply distinguish* x *from the states in which she does not know* φ. The part of U that is consistent with $\neg K\varphi$ is topologically negligibly small. We therefore further argue that this is the "closest-to-knowledge" notion of belief that can be defined in terms of the topological tools at hand and that is not identical with the notion of knowledge taken as the primitive operator.

Topologically, our new belief operator behaves like a 'regularization' operator for the opens in a topology. Given a topological space (X, τ), we can define $B : \tau \to \tau$ such that $B(U) = \mathrm{Int}(\mathrm{Cl}(U))$. Therefore, B takes an open set and makes it *regular open*[9]. In fact, for any open set $U \in \tau$, the set $\mathrm{Int}(\mathrm{Cl}(U))$ is the smallest regular open such that $U \subseteq \mathrm{Int}(\mathrm{Cl}(U))$[10]. Therefore, this operator extends an open set in a minimal way by gluing its holes and cracks together. To illustrate, consider the natural topology on the real line $(\mathbb{R}, \tau)$ and let $P = [-2, 3) \cup (3, 5) \cup \{7\}$ (see Fig. 1).

Fig. 1. $(\mathbb{R}, \tau)$

We have $\mathrm{Int}(P) = (-2, 3) \cup (3, 5)$ and $\mathrm{Int}(P)$ is not regular open. However, $B(\mathrm{Int}(P)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(P))) = (-2, 5)$, which is the smallest regular open containing $\mathrm{Int}(P)$. Similarly, on the Euclidean plane, the belief operator patches up the cracks of an open set (see Fig. 2).

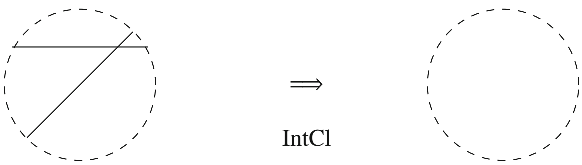

Fig. 2. From U to $\mathrm{Int}(\mathrm{Cl}(U))$

4.2 The Axiomatization of wKD45

We define the logic **wKD45** as

$$\mathbf{wKD45} = \mathbf{K} + (B\varphi \to \langle B\rangle\varphi) + (B\varphi \to BB\varphi) + (B\langle B\rangle B\varphi \to B\varphi)$$

[9] A subset $A \subseteq X$ of a topological space (X, τ) satisfying the condition $A = \mathrm{Int}(\mathrm{Cl}(A)$ is called *regular open* [14].

[10] In fact, for any $A \subseteq X$, the set $\mathrm{Int}(\mathrm{Cl}(A))$ is regular open, however, it is not always the case that $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$.

and call it *weak* **KD45**. This logic is weaker than **KD45** since it is obtained by replacing the (5)-axiom with the axiom $B\langle B\rangle B\varphi \to B\varphi$, and while $B\langle B\rangle B\varphi \to B\varphi$ is a theorem of **KD45**, the (5)-axiom is not a theorem of **wKD45**. More precisely, **KD45** $\vdash B\langle B\rangle B\varphi \to B\varphi$ but **wKD45** $\nvdash \langle B\rangle\varphi \to B\langle B\rangle\varphi$. We find it hard to give a direct and clear interpretation for this axiom as is given for the axiom of Negative Introspection, since it is too complex in the sense that it includes three consecutive modalities. However, we can interpret it on the basis of the axioms that we have already given an interpretation, in particular, based on the interpretation of Negative Introspection. It is easier to see the correspondence if we state the weak axiom in the following equivalent form:

$$\neg B\varphi \to \langle B\rangle B\neg B\varphi.$$

Recall that the principle of Negative Introspection says that *if an agent does not believe φ, then she believes that she does not believe φ*. On the other hand, taking the reading of Negative Introspection as the reference point, a direct doxastic reading for this axiom is *if the agent does not believe φ, then it is doxastically possible to her that she believes that she does not believe φ*. Therefore, in this section, we work with consistent, positively introspective yet not fully negatively introspective belief. This weakened system **wKD45** stands between **KD4** and **KD45**. While the latter is commonly used as the standard logic for belief, the former has also been studied as a belief system [16,31,32].

4.3 Soundness and Completeness of wKD45

In this section, we prove that **wKD45** for B is sound and complete with respect to the class of all topological spaces. Soundness proof can be presented in a standard way by checking the validity of the axioms and inference rules of **wKD45** with respect to the **w**-topological belief semantics. We leave this proof to the reader and argue for soundness in a different way. For completeness, we follow a technique which allows us to reduce the completeness problem of **wKD45** to the topological completeness of **S4** in the interior semantics. We do this so by defining a translation $(.)^{\circledast}$ from the doxastic language $\mathcal{L}_B$ to the epistemic language $\mathcal{L}_K$ such that for any $\varphi \in \mathcal{L}_B$, we obtain

$$\mathbf{S4} \vdash \varphi^{\circledast} \text{ iff } \mathbf{wKD45} \vdash \varphi.$$

Although we only need the direction from left-to-right for completeness, the other direction comes almost for free and we use this direction to argue for soundness. The above implication can be seen as the key intermediate result for the topological completeness proof of **wKD45**. In order to reach this result, we also make use of soundness and completeness of **S4** and **wKD45** in the standard Kripke semantics. Moreover, we believe that the full and faithful translation of **wKD45** into **S4** given by $(.)^{\circledast}$ is also of interest from a purely modal logical perspective. It implies that the $K\langle K\rangle K$-fragment of **S4** is **wKD45**. In the same way, a full and faithful translation of **KD45** into **S4.2** given by the $\langle K\rangle K$-modality implies that the $\langle K\rangle K$-fragment of **S4.2** is **KD45** [2]. To the best of our knowledge it is still an (interesting) open question how to axiomatize the (non-normal) modal logic **L** which is the $\langle K\rangle K$-fragment of **S4**.

Throughout this section, we use the notation $[\varphi]^{\mathcal{M}}$ for the extension of a formula $\varphi \in \mathcal{L}_K$ w.r.t. the *interior semantics* in order to make clear in which semantics we work. We reserve the notation $[\![\varphi]\!]^{\mathcal{M}}$ for the extensions of the formulas $\varphi \in \mathcal{L}_B$ w.r.t. the **w**-topological belief semantics. We skip the index when confusion is unlikely to occur.

Definition 3 (Translation $(.)^{\circledast} : \mathcal{L}_B \to \mathcal{L}_K$). *For any $\varphi \in \mathcal{L}_B$, the translation $(\varphi)^{\circledast}$ of φ into $\mathcal{L}_K$ is defined recursively as follows:*

1. $(p)^{\circledast} = p$, *where* $p \in \mathrm{Prop}$
2. $(\neg\varphi)^{\circledast} = \neg\varphi^{\circledast}$
3. $(\varphi \wedge \psi)^{\circledast} = \varphi^{\circledast} \wedge \psi^{\circledast}$
4. $(B\varphi)^{\circledast} = K\langle K\rangle K\varphi^{\circledast}$

Note that $(\langle B\rangle\varphi)^{\circledast} = \langle K\rangle K\langle K\rangle\varphi^{\circledast}$.

Proposition 3. *For any topo-model $\mathcal{M} = (X, \tau, \nu)$ and for any formula $\varphi \in \mathcal{L}_B$ we have*

$$[\![\varphi]\!]^{\mathcal{M}} = [\varphi^{\circledast}]^{\mathcal{M}}.$$

Proof. We prove the lemma by induction on the complexity of φ. The cases for the propositional variables and Booleans are straightforward. Now let $\varphi = B\psi$, then

$$\begin{aligned}
[\![\varphi]\!]^{\mathcal{M}} &= [\![B\psi]\!]^{\mathcal{M}} \\
&= \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\psi]\!]^{\mathcal{M}}))) && \text{(by the } \mathbf{w}\text{-topological belief semantics for } \mathcal{L}_B) \\
&= \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\psi^{\circledast}]^{\mathcal{M}}))) && \text{(by I.H.)} \\
&= [K\langle K\rangle K\psi^{\circledast}]^{\mathcal{M}} && \text{(by the interior semantics for } \mathcal{L}_K) \\
&= [(B\psi)^{\circledast}]^{\mathcal{M}} && \text{(by the translation } {}^{\circledast}) \\
&= [\varphi^{\circledast}]^{\mathcal{M}}.
\end{aligned}$$

We now recall some frame conditions concerning the relational completeness of the respective systems.

Let (X, R) be a transitive Kripke frame. Recall that a *cluster* is an equivalence class wrt the equivalence relation $\sim$ defined by $x \sim y$ if xRy and yRx for each $x, y \in X$. We denote the set of final clusters of (X, R) by $\mathfrak{C}_R$. A transitive Kripke frame (X, R) having at least one final cluster is called ***weakly cofinal*** if for each $x \in X$ there is a $C \in \mathfrak{C}_R$ such that for all $y \in C$ we have xRy. In fact, every finite reflexive and transitive frame is weakly cofinal. We call a weakly cofinal frame a ***weak brush*** if $X\setminus\bigcup\mathfrak{C}_R$ is an irreflexive anti-chain, i.e., for each $x, y \in X\setminus\bigcup\mathfrak{C}_R$ we have $\neg(xRy)$. A weak brush with a singleton $X\setminus\bigcup\mathfrak{C}_R$ is called a ***weak pin***[11]. By definition, every weak brush and every weak pin is transitive and also serial. A transitive frame (X, R) is called ***rooted***, if there is

[11] Brushes and pins are introduced in [25] and a similar terminology is used in this paper.

an $x \in X$, called a root, such that for each $y \in X$ with $x \neq y$ we have xRy. Finally, we say that a transitive frame (X, R) is of depth n if there is a chain of points $x_1 R x_2 R \ldots R x_n$ such that $\neg(x_{i+1} R x_i)$ for any $i \leq n$ and there is no chain of greater length satisfying this condition. It is hard to draw a generic picture of a weak brush, but the following figures illustrate weak pins and how a weak brush could look like (where top squares correspond to final clusters) (Figs. 3 and 4).

It is well-known that the (D)-axiom corresponds to seriality and the (4)-axiom corresponds to transitivity of a Kripke frame, under the standard Kripke interpretation (see, e.g., [8, Chap. 4]). It is not very hard to see that the contraposition equivalent $\langle B\rangle\varphi \to \langle B\rangle B\langle B\rangle\varphi$ of our new axiom $B\langle B\rangle B\varphi \to B\varphi$ is a Sahlqvist formula and the first order property corresponding to this axiom is

$$\forall x \forall y (xRy \ \Rightarrow \exists z (xRz \wedge \forall w (zRw \Rightarrow wRy))). \qquad \text{(wE)}$$

Therefore, a **wKD45** *frame* is a serial and transitive Kripke frame satisfying the above property (wE). We refer the reader to [8, Chap. 3.6] for a more detailed discussion on Sahlqvist formulas.

Let us recall that a point y in a reflexive and transitive Kripke frame (X, R) is called *quasi-maximal* if yRz for some $z \in X$ implies zRy.

Lemma 1. *A rooted Kripke frame $\mathcal{F} = (X, R)$ is a* **wKD45** *frame iff it is a cluster or it is a weak pin.*

Proof. The right-to-left direction is trivial. For the other direction, suppose $\mathcal{F} = (X, R)$ is a rooted **wKD45** frame that is not a cluster and assume $x \in X$ is the root. As $\mathcal{F}$ is serial, every quasi-maximal point of it is in a final cluster. Hence, for any $y \in X$, y is a quasi-maximal point iff there is a final cluster C of $\mathcal{F}$ such that $y \in C$, i.e. the set of quasi-maximal points of $\mathcal{F}$ is $\bigcup \mathfrak{C}_R$. Recall that a weak pin is a weakly cofinal frame with a singleton irreflexive $X \setminus \bigcup \mathfrak{C}_R$. We hence need to show that (1) x is an irreflexive point and (2) every successor of x is a quasi-maximal point. Since x is the root and $\mathcal{F} = (X, R)$ is not a cluster, there exists $y \in X$ such that xRy and $\neg(yRx)$.

For (1), suppose that we have xRx. Then, by (wE), there exists $z_0 \in X$ such that xRz_0 and for all $w \in X$ with $z_0 Rw$, we have wRx. Since R is serial, z_0 has at least one successor w, therefore, it is guarateed that there is at least one element $w \in X$ such that wRx. Since $wRxRy$ and R is transitive, we obtain

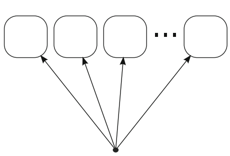

Fig. 3. Weak pin

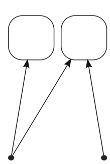

Fig. 4. An example of a weak brush

wRy implying, again by taransitivity, that $z_0 Ry$. Therefore, by (wE), we have yRx, contradicting $\neg(yRx)$. So x is irreflexive.

For (2), suppose there exists $y_0 \in X$ such that xRy_0 and y_0 is not a quasi-maximal element. This means that there is $t_0 \in X$ such that $y_0 Rt_0$ but $\neg(t_0 Ry_0)$. By (wE), xRy_0 implies that there exists $z_0 \in X$ such that xRz_0 and for all $w \in X$ with $z_0 Rw$, we have wRy_0. Similarly to the argument above, since it is guaranteed that z_0 has at least one successor w, R is transitive and $z_0 RwRy_0 Rt_0$ implies $z_0 Rt_0$. Therefore, again by (wE), $t_0 Ry_0$ contradicting our assumption. Thus, every successor of x is a quasi-maximal point. Finally, (1) and (2) together yield that (X, R) is a weak pin.

Lemma 2

1. *Each reflexive and transitive weakly cofinal frame is an* **S4***-frame. Moreover,* **S4** *is sound and complete w.r.t. the class of finite rooted reflexive and transitive weakly cofinal frames.*
2. *Each weak brush is a* **wKD45***-frame. Moreover,* **wKD45** *is sound and complete w.r.t. the class of finite weak brushes, indeed, w.r.t. the class of finite weak pins.*

Proof. (1) is well known, see e.g., [8,10]. For (2), we proved in Lemma 1 that the **wKD45**-frames are of finite depth. It is well known that every logic over **K4** that has finite depth is locally tabular and has the finite model property (e.g., [10, Theorem 12.21]). This implies that **wKD45** as well has the finite model property and thus it has the finite model property w.r.t. finite rooted **wKD45-**frames. Then by Lemma 1, we have that **wKD45** is in fact complete w.r.t. finite weak brushes and weak pins.

For any reflexive and transitive weakly cofinal frame (X, R) we define R_B on X by

$$xR_B y \text{ if } y \in \bigcup \mathfrak{C}_{R(x)}$$

for each $x, y \in X$, where $\bigcup \mathfrak{C}_{R(x)} = R(x) \cap \bigcup \mathfrak{C}_R$. In other words, $R_B(x) = \bigcup \mathfrak{C}_{R(x)}$ for each $x \in X$. Moreover, we have the following equivalence.

Lemma 3. *For any reflexive and transitive weakly cofinal frame (X, R) we have*

$$\bigcup \mathfrak{C}_{R_B} = \bigcup \mathfrak{C}_R.$$

Proof. Let (X, R) be a reflexive and transitive weakly cofinal frame and $x \in X$.

($\subseteq$) Suppose $x \in \bigcup \mathfrak{C}_{R_B}$ and $x \notin \bigcup \mathfrak{C}_R$. Then $x \in \bigcup \mathfrak{C}_{R_B}$ means that $x \in C$ for some $C \in \mathfrak{C}_{R_B}$. As C is a final cluster, there is no $y \in X$ such that $xR_B y$ and $\neg(yR_B x)$. On the other hand, since (X, R) is a weakly cofinal frame, there is a $C' \in \mathfrak{C}_R$ such that xRz for all $z \in C'$. Hence, $C' \subseteq \bigcup \mathfrak{C}_{R(x)}$. Thus, by the definition of R_B, we have $C' \subseteq R_B(x)$. However, as $x \notin \bigcup \mathfrak{C}_R$, we have that $\neg(zRx)$ and thus $\neg(zR_B x)$ for any $z \in C'$ contradicting $x \in C$ for a final cluster C of (X, R_B). In fact, there is a unique $C \in \mathfrak{C}_{R_B}$ such that $R_B(x) = C$ since C is a final cluster.

($\supseteq$) Suppose $x \in \bigcup \mathfrak{C}_R$. Then, there is a (unique) $C \in \mathfrak{C}_R$ such that $x \in C$ and in fact $R(x) = C$. Also suppose that $x \notin \bigcup \mathfrak{C}_{R_B}$. Hence, there is a $y_0 \in X$ such that xR_By_0 and $\neg(y_0R_Bx)$. Then, $y_0 \in \bigcup \mathfrak{C}_{R(x)}$ but $x \notin \bigcup \mathfrak{C}_{R(y_0)}$ by definition of R_B. By definition of R_B, we have that xR_By_0 implies xRy_0. Hence, as $y_0 \in R(x)$, we also have $R(y_0) = R(x) = C$. Thus, $\bigcup \mathfrak{C}_{R(y_0)} = \bigcup \mathfrak{C}_{R(x)}$. As R is reflexive, $x \in \bigcup \mathfrak{C}_{R(x)}$ and hence $x \in \bigcup \mathfrak{C}_{R(y_0)}$ contradicting $\neg(y_0R_Bx)$.

Lemma 4. *For any reflexive and transitive weakly cofinal Kripke model $\mathcal{M} = (X, R, \nu)$, any $\varphi \in \mathcal{L}_K$ and any $x \in X$, we have*

$$\bigcup \mathfrak{C}_{R(x)} \subseteq \|\varphi\|^{\mathcal{M}} \textit{ iff } x \in \|K\langle K\rangle K\varphi\|^{\mathcal{M}}.$$

Proof. Let $\mathcal{M} = (X, R, \nu)$ be a reflexive and transitive weakly cofinal model, $\varphi \in \mathcal{L}_K$ and $x \in X$.

($\Rightarrow$) Suppose $\bigcup \mathfrak{C}_{R(x)} \subseteq \|\varphi\|^{\mathcal{M}}$. Let $y \in X$ be such that xRy. As R is transitive and xRy, we have $R(y) \subseteq R(x)$ implying that $\bigcup \mathfrak{C}_{R(y)} \subseteq \bigcup \mathfrak{C}_{R(x)}$. Hence, by our assumption, $\bigcup \mathfrak{C}_{R(y)} \subseteq \|\varphi\|^{\mathcal{M}}$. Thus, there is a $C \in \mathfrak{C}_R$ such that $C \subseteq R(y)$ and $C \subseteq \|\varphi\|^{\mathcal{M}}$. Since for all $z \in C$, we have $R(z) = C$ and $C \subseteq \|\varphi\|^{\mathcal{M}}$, we obtain $C \subseteq \|K\varphi\|^{\mathcal{M}}$. As $C \subseteq R(y)$, we have $y \in \|\langle K\rangle K\varphi\|^{\mathcal{M}}$. Therefore, as y has been chosen arbitrarily from $R(x)$ we obtain $x \in \|K\langle K\rangle K\varphi\|^{\mathcal{M}}$.

($\Leftarrow$) Suppose $\bigcup \mathfrak{C}_{R(x)} \nsubseteq \|\varphi\|^{\mathcal{M}}$. This implies that there exists a $y \in \bigcup \mathfrak{C}_{R(x)}$ such that $y \notin \|\varphi\|^{\mathcal{M}}$. Now $y \in \bigcup \mathfrak{C}_{R(x)}$ implies that there is a $C \in \mathfrak{C}_R$ such that $R(y) = C$ and $R(y) \subseteq R(x)$. As zRy for all $z \in C$ and $y \notin \|\varphi\|^{\mathcal{M}}$, we have $z \notin \|K\varphi\|^{\mathcal{M}}$ for all $z \in C$. Then, $R(y) = C$ yields $y \notin \|\langle K\rangle K\varphi\|^{\mathcal{M}}$. Finally, since xRy, we obtain $x \notin \|K\langle K\rangle K\varphi\|^{\mathcal{M}}$.

Lemma 5. *For any reflexive and transitive weakly cofinal frame (X, R),*

1. *(X, R_B) is a weak brush.*
2. *For any valuation ν on X and for each formula $\varphi \in \mathcal{L}_B$ we have $\|\varphi^{\circledast}\|^{\mathcal{M}} = \|\varphi\|^{\mathcal{M}_B}$, where $\mathcal{M} = (X, R, \nu)$ and $\mathcal{M}_B = (X, R_B, \nu)$.*

Proof. Let (X, R) be a reflexive and transitive weakly cofinal frame.

1. – *Transitivity:* Let $x, y, z \in X$ such that xR_By and yR_Bz. This means that $y \in \bigcup \mathfrak{C}_{R(x)}$ and $z \in \bigcup \mathfrak{C}_{R(y)}$. As R is transitive and xRy we have $\bigcup \mathfrak{C}_{R(y)} \subseteq \bigcup \mathfrak{C}_{R(y)}$. Hence, $z \in \bigcup \mathfrak{C}_{R(x)}$, i.e., xR_Bz.
   – *Seriality:* Let $x \in X$. Since (X, R) is weakly cofinal, there is a $y \in \bigcup \mathfrak{C}_{R(x)}$, i.e., xR_By.
   – *Irreflexive, antichain:* Suppose there is an $x \in X \backslash \bigcup \mathfrak{C}_{R_B}$ such that xR_Bx. This implies, $x \in \bigcup \mathfrak{C}_{R(x)}$, thus, $x \in \bigcup \mathfrak{C}_R$. By Lemma 3, $x \in \bigcup \mathfrak{C}_{R_B}$ which contradicts our assumption. Moreover, suppose that $X \backslash \bigcup \mathfrak{C}_{R_B}$ is not an antichain, i.e., there are $x, y \in X \backslash \bigcup \mathfrak{C}_{R_B}$ such that either xR_By or yR_Bx. W.l.o.g., assume xR_By. Hence, by definition of R_B, we have $y \in \bigcup \mathfrak{C}_{R(x)}$. Thus, $y \in \bigcup \mathfrak{C}_R$ and, by Lemma 3, $y \in \bigcup \mathfrak{C}_{R_B}$ contradicting $y \in X \backslash \bigcup \mathfrak{C}_{R_B}$.

2. We prove this item by induction on the complexity of φ. Let $\mathcal{M} = (X, R, \nu)$ be a model on (X, R). The cases for $\varphi = \bot$, $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Now let $\varphi = B\psi$.
   ($\subseteq$) Let $x \in \|(B\psi)^{\circledast}\|^{\mathcal{M}} = \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}}$. Then, by Lemma 4, $\bigcup \mathfrak{C}_{R(x)} \subseteq \|\psi^{\circledast}\|^{\mathcal{M}}$. By I.H, we obtain $\bigcup \mathfrak{C}_{R(x)} \subseteq \|\psi\|^{\mathcal{M}_B}$. Since $\bigcup \mathfrak{C}_{R(x)} = R_B(x)$, we have $R_B(x) \subseteq \|\psi\|^{\mathcal{M}_B}$ implying that $x \in \|B\psi\|^{\mathcal{M}_B}$.
   ($\supseteq$) Let $x \in \|B\psi\|^{\mathcal{M}_B}$. Then, by the standard Kripke semantics, we have $R_B(x) \subseteq \|\psi\|^{\mathcal{M}_B}$. By I.H, we obtain $R_B(x) \subseteq \|\psi^{\circledast}\|^{\mathcal{M}}$. Since $\bigcup \mathfrak{C}_{R(x)} = R_B(x)$, we have $\bigcup \mathfrak{C}_{R(x)} \subseteq \|\psi^{\circledast}\|^{\mathcal{M}}$. Thus, by Lemma 4, $x \in \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}} = \|(B\psi)^{\circledast}\|^{\mathcal{M}}$.

Lemma 6. *For any weak brush* (X, R),

1. (X, R^+) *is a reflexive and transitive weakly cofinal frame.*
2. *For any valuation* ν *on* X *and for each formula* $\varphi \in \mathcal{L}_B$ *we have* $\|\varphi\|^{\mathcal{M}} = \|\varphi^{\circledast}\|^{\mathcal{M}^+}$, *where* $\mathcal{M} = (X, R, \nu)$ *and* $\mathcal{M}^+ = (X, R^+, \nu)$.

Proof. Let (X, R) be a serial weak brush.

1. Since R is transitive, R^+ is also transitive and it is reflexive by definition. Moreover, (X, R^+) is weakly cofinal since (X, R) is a weak brush.
2. We prove (2) by induction on the complexity of φ. Let $\mathcal{M} = (X, \tau, \nu)$ be a model on (X, R). The cases for $\varphi = \bot$, $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Let $\varphi = B\psi$.
   ($\subseteq$) Let $x \in \|B\psi\|^{\mathcal{M}}$. Then, by the standard Kripke semantics, we have $R(x) \subseteq \|\psi\|^{\mathcal{M}}$. Hence, by I.H., $R(x) \subseteq \|\psi^{\circledast}\|^{\mathcal{M}^+}$. Since (X, R) is a weak brush, $R(x) = \bigcup \mathfrak{C}_{R(x)} \subseteq \bigcup \mathfrak{C}_{R^+(x)}$. Hence, $x \in \bigcup \mathfrak{C}_{R^+(x)}$. Then, by Lemma 4, $x \in \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}^+}$.
   ($\supseteq$) Let $x \in \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}^+}$. Then, by Lemma 4, $\bigcup \mathfrak{C}_{R^+(x)} \subseteq \|\psi^{\circledast}\|^{\mathcal{M}^+}$. Thus, by I.H., $\bigcup \mathfrak{C}_{R^+(x)} \subseteq \|\psi\|^{\mathcal{M}}$. Then, as above, $R(x) \subseteq \|\psi\|^{\mathcal{M}}$ implying that $x \in \|B\psi\|^{\mathcal{M}}$.

Theorem 6. *For each formula* $\varphi \in \mathcal{L}_B$, $\mathbf{S4} \vdash \varphi^{\circledast}$ *iff* $\mathbf{wKD45} \vdash \varphi$.

Proof. Let $\varphi \in \mathcal{L}_B$.

($\Rightarrow$) Suppose $\mathbf{wKD45} \nvdash \varphi$. By Lemma 2(2), there exists a Kripke model $\mathcal{M} = (X, R, \nu)$, where (X, R) is a finite weak pin such that $\|\varphi\|^{\mathcal{M}} \neq X$. Then, by Lemma 6, $\mathcal{M}^+$ is a model based on the finite reflexive and transitive weakly cofinal frame (X, R^+) and $\|\varphi^{\circledast}\|^{\mathcal{M}^+} \neq X$. Hence, by Lemma 2(1), we have $\mathbf{S4} \nvdash \varphi^{\circledast}$.

($\Leftarrow$) Suppose $\mathbf{S4} \nvdash \varphi^{\circledast}$. By Lemma 2(1), there exists a Kripke model $\mathcal{M} = (X, R, \nu)$ where (X, R) is a finite reflexive and transitive weakly cofinal frame such that $\|\varphi^{\circledast}\|^{\mathcal{M}} \neq X$. Then, by Lemma 5, $\mathcal{M}_B$ is a model based on the (finite) weak brush (X, R_B) and $\|\varphi\|^{\mathcal{M}_B} \neq X$. Hence, by Lemma 2(2), we have $\mathbf{wKD45} \nvdash \varphi$.

Theorem 7. **wKD45** *is sound and complete w.r.t. the class of all topological spaces in the* **w***-topological belief semantics.*

Proof. As we noted in the beginning of this section, soundness can be proven directly. Another way of arguing for the topological soundness of **wKD45** is via Theorem 6: let $\varphi \in \mathcal{L}_B$ such that **wKD45** $\vdash \varphi$. Then, by Theorem 6, **S4** $\vdash \varphi^{\circledast}$. By the topological soundness of **S4** w.r.t. the class of all topological spaces in the interior semantics, we obtain that for any topological space (X, τ) we have $(X, \tau) \models \varphi^{\circledast}$. Then, by Proposition 3, we conclude that in the **w**-topological belief semantics $(X, \tau) \models \varphi$.

For completeness, let $\varphi \in \mathcal{L}_B$ be such that **wKD45** $\not\vdash \varphi$. By Theorem 6, **S4** $\not\vdash \varphi^{\circledast}$. Hence, by topological completeness of **S4** w.r.t. the class of all topological spaces in the interior semantics, there exists a topo-model $\mathcal{M} = (X, \tau, \nu)$ such that $[\varphi^{\circledast}]^{\mathcal{M}} \neq X$. Then, by Proposition 3, $[\![\varphi]\!]^{\mathcal{M}} \neq X$. Thus, we found a topological space (X, τ) which refutes φ in the **w**-topological belief semantics. Hence, **wKD45** is complete w.r.t. the class of all topological spaces in the **w**-topological belief semantics.

We point out that the above completeness proof crucially uses reasoning in Kripke frames rather than topology. However, as already mentioned earlier in the paper, topological (and geometrical) reading of our belief modality is key for its intuitive understanding as well as for viewing it as a Stalnaker-like belief operator.

5 The Topology of Static and Dynamic Belief Revision

5.1 Static Belief Revision: Conditional Beliefs

In this section, we explore the topological analogue of static conditioning by providing a topological semantics for conditional belief modalities based on arbitrary topological spaces[12]. We obtain the semantics for a conditional belief modality $B^{\varphi}\psi$ in a natural and standard way, as in [2], by relativizing the semantics for the simple belief modality to the extension of the learnt formula φ. Unlike model *restriction* in the case of updates, our conditional belief semantics does not lead to a change in the initial model. Conditional belief modalities intend to capture the *hypothetical* belief changes of an agent in case she would receive new information (see, e.g., [2] for a more detailed discussion on the topological interpretation of conditional beliefs).

Syntax and Semantics. We now consider the language $\mathcal{L}_{KCB}$ obtained by adding conditional belief modalities $B^{\varphi}\psi$ to $\mathcal{L}_{KB}$, where $B^{\varphi}\psi$ reads *if the agent would learn φ, then she would come to believe that ψ was the case before the learning* [4, p. 12].

[12] In [2], we propose topological semantics for conditional beliefs based on hereditarily extremally disconnected spaces.

For any subset P of a topological space (X, τ), we can generalize the belief modality B on the topo-models by relativizing the closure and the interior operators to the set P. More precisely, given a topological model $\mathcal{M} = (X, \tau, \nu)$, the additional semantic clause reads

$$[\![B^{\varphi}\psi]\!]^{\mathcal{M}} = \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \to \mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}} \cap \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \to [\![\psi]\!]^{\mathcal{M}})))$$

where $[\![\varphi]\!]^{\mathcal{M}} \to [\![\psi]\!]^{\mathcal{M}} := (X \backslash [\![\varphi]\!]^{\mathcal{M}}) \cup [\![\psi]\!]^{\mathcal{M}}$.

One possible justification for the above semantics of conditional belief is that it validates an equivalence that generalizes the one for belief in a natural way:

Proposition 4. *The following equivalence is valid in all topological spaces wrt the refined topological semantics for conditional beliefs and knowledge*

$$B^{\varphi}\psi \;\leftrightarrow\; K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi))).$$

This shows that, just like simple beliefs, conditional beliefs can be defined in terms of knowledge and this identity corresponds to the definition of the "conditional connective ⇒" in [9]. Moreover, as a corollary of Proposition 4, we obtain that the equivalences

$$B^{\top}\psi \overset{(1)}{\leftrightarrow} K(\top \to \langle K\rangle(\top \wedge K(\top \to \psi)) \overset{(2)}{\leftrightarrow} K\langle K\rangle K\psi \overset{(3)}{\leftrightarrow} B\psi$$

valid in all topological spaces, and thus our semantics for conditional beliefs and simple beliefs (in terms of the interior of the closure of the interior operator) are perfectly compatible with each other. Last but not least, we obtain the complete logic **KCB** of knowledge and conditional beliefs w.r.t. all topological spaces in the following way.

Theorem 8. *The logic* **KCB** *of knowledge and conditional beliefs is axiomatized completely by the system* **S4** *for the knowledge modality K together with the following equivalences:*

1. $B^{\varphi}\psi \leftrightarrow K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi)))$
2. $B\varphi \leftrightarrow B^{\top}\varphi$.

5.2 Dynamic Belief Revision: Updates on All Topological Spaces

In this section, we implement updates on arbitrary topological spaces and show that the problems occurred when we work with extremally disconnected spaces do not arise here: we in fact obtain a complete dynamic logic of knowledge and conditional beliefs with respect to the class of all topological spaces.

We now consider the language $\mathcal{L}_{!KCB}$ obtained by adding (existential) dynamic modalities $\langle !\varphi\rangle\psi$ to $\mathcal{L}_{KCB}$ and we model $\langle !\varphi\rangle\psi$ by means of subspaces exactly the same way as formalized in Sect. 3.2 , i.e., by using the restricted model $\mathcal{M}_{\varphi}$ with the semantic clause

$$[\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} = [\![\psi]\!]^{\mathcal{M}_{\varphi}}.$$

In this setting, however, as the underlying static logic **KCB** is the logic of all topological spaces, we implement updates on arbitrary topological spaces. Since the resulting restricted model $\mathcal{M}_\varphi$ is always based on a topological (sub)space and no additional property of the initial topology needs to be inherited by the corresponding subspace (unlike the case for extremally disconnected spaces), we do not face the problem of loosing some validities of the corresponding static system: all the axioms of **KCB** (and, in particular, of **S4** and **wKD45**) will still be valid in the restricted space. Moreover, we obtain a complete axiomatization of the dynamic logic of knowledge and conditional beliefs:

Theorem 9. *The complete and sound dynamic logic* **!KCB** *of knowledge and conditional beliefs with respect to the class of all topological spaces is obtained by adding the following reduction axioms to any complete axiomatization of the logic* **KCB***:*

1. $\langle !\varphi\rangle p \leftrightarrow (\varphi \wedge p)$
2. $\langle !\varphi\rangle \neg\psi \leftrightarrow (\varphi \wedge \neg\langle !\varphi\rangle\psi)$
3. $\langle !\varphi\rangle(\psi \wedge \theta) \leftrightarrow (\langle !\varphi\rangle\psi \wedge \langle !\varphi\rangle\theta)$
4. $\langle !\varphi\rangle K\psi \leftrightarrow (\varphi \wedge K(\varphi \rightarrow \langle !\varphi\rangle\psi))$
5. $\langle !\varphi\rangle B^\theta\psi \leftrightarrow (\varphi \wedge B^{\langle !\varphi\rangle\theta}\langle !\varphi\rangle\psi)$
6. $\langle !\varphi\rangle\langle !\varphi\rangle\theta \leftrightarrow \langle !\langle !\varphi\rangle\psi\rangle\theta$

Proof. Proof of this theorem follows, in a standard way, by the soundness of the reduction axioms with respect to all topological spaces. For proof details, we refer to [2, Theorem 12].

6 Conclusion and Future Work

In this paper, we proposed a new topological semantics for belief in terms of the *interior of the closure of the interior* operator which coincides with the one introduced in [1,2,25] on extremally disconnected spaces and diverges from it on arbitrary topological spaces. This new topological semantics for belief comes with significant advantages especially concerning static and dynamic belief revision (in particular, concerning conditional belief and update semantics) and a few disadvantages compared to the setting in [1,2].

In [1,2], we worked with the knowledge system **S4.2** and the standard belief system **KD45**, however, on a restricted class of topological spaces, namely on extremally disconnected spaces. Although the framework of [1,2] provides a solid ground for the static systems of knowledge and belief and the relation between the two, the topological semantics based on extremally disconnected spaces falls short of dealing with updates as shown in Sect. 3.2. In particular, in order to deal with updates one needs to further restrict the class of extremally disconnected spaces to hereditarily extremely disconnected spaces.

In this paper, we did not only provide a semantics for belief based on *all topological spaces* but we also showed that its natural extension to conditional beliefs and updates gave us a 'well-behaved' semantics. In other words, while extending the class of topo-models we could work within the context of knowledge and belief, we also resolved the problem about updates present in the previous setting. The price we had to pay for these results, however, was a weakening of the underlying static knowledge and belief logics: we weakened the knowledge logic **S4.2** to **S4** and the belief logic **KD45** to a slightly weaker one **wKD45**.

This paper can be seen as a continuation of the research program that we have been pursuing on a topological semantics for belief: in [1] we proposed a topological belief semantics based on extremally disconnected spaces and in [2] we investigated a topological belief semantics on hereditarily extremally disconnected spaces and further extended this setting with conditional beliefs and updates. The current work takes a broader perspective and examines belief, conditional beliefs and updates on arbitrary topological spaces.

In on-going work, we investigate a more natural axiomatization of the logic of knowledge and conditional beliefs **KCB** and its dynamic counterpart with respect to arbitrary topological spaces. Moreover, we also investigate the topological semantics for evidence and evidence-based justification in connection with topological interpretations of knowledge and belief in [3] and, following [34], we further explore the dynamics of evidence in a topological setting in the extended version of [3].

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