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Subvarieties of the variety of meadows

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Abstract

Meadows—commutative rings equipped with a total inversion operation—can be axiomatized by purely equational means. We study subvarieties of the variety of meadows obtained by extending the equational theory and expanding the signature.
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1 Introduction

In [14], von Neumann introduced his regular rings during the study of von Neumann algebras and continuous geometry. A *von Neumann regular ring* is an algebra $\langle R, +, \cdot, -, 0, 1\rangle$ satisfying the axioms (1)–(5), (7), (8), (11) of Table 1 and 2. In general, y in (11) is not uniquely determined by x. However, if the underlying ring is commutative then the weak inverse is unique. *Meadows* were introduced in [4] as commutative rings equipped with a total inversion operation. A meadow is an algebra $\langle R, +, \cdot, -, {}^{-1}, 0, 1\rangle$ satisfying the axioms (1)–(10) in Table 1. The paradigmatic infinite meadows are the rational, real and complex numbers each equipped with a total multiplicative inversion operation where $0^{-1} = 0$. Every commutative von Neumann regular ring can be obtained as a reduct of a meadow, and conversely, every meadow can be obtained from a commutative von Neumann regular ring by defining the inversion operation. Thus, although the signatures of meadows and commutative von Neumann regular rings differ, they form the same category of objects and natural morphisms.

From the perspective of universal algebra, the situation is quite different. If K is a class of algebras of the same signature, then K is a *variety* if K is closed under subalgebras, homomorphic images and direct products of nonempty families. One of the most celebrated theorems of Birkhoff says that the *equational* classes of algebras—algebras axiomatized by identities—are precisely the varieties. Hence, the class of meadows is a variety. In contrast, the class of von Neumann regular rings is merely an *elementary* class—a class axiomatized by a set of first-order formulas—since it is not closed under subalgebras: e.g. the rational numbers $\mathbb{Q}$ form a von Neumann regular ring but its subalgebra $\mathbb{Z}$ of integers does not have weak inverses (except for 0 and 1). In [10], the concept of an *existence variety* is introduced which applies to (commutative) von Neumann regular rings. Existence varieties require only

$$(x+y)+z = x+(y+z) \tag{1}$$
$$x+y = y+x \tag{2}$$
$$x+0 = x \tag{3}$$
$$x+(-x) = 0 \tag{4}$$
$$(x\cdot y)\cdot z = x\cdot(y\cdot z) \tag{5}$$
$$x\cdot y = y\cdot x \tag{6}$$
$$1\cdot x = x \tag{7}$$
$$x\cdot(y+z) = x\cdot y + x\cdot z \tag{8}$$
$$(x^{-1})^{-1} = x \tag{9}$$
$$x\cdot(x\cdot x^{-1}) = x \tag{10}$$

Table 1: The set Md of axioms for meadows

$$\forall x \exists y\ x = x\cdot y\cdot x \tag{11}$$

Table 2: The axiom for the weak inverse

the closure of a class of algebras under elementary substructures, homomorphic images and direct products but also enjoy a Birkhoff-style theorem.

A challenging question is whether there exist finite complete axiomatizations for the meadows of the rational, real and complex numbers. [2] gives an affirmative answer in the case of the expansion of the meadow of the real numbers with a sign function. In this paper we try to attack the problem for the meadow of the rational numbers by studying the structure of the subvarieties of meadows. We presume a general familiarity with universal algebra and initial algebra semantics. Some references to various aspects of this subject are [7, 8, 9, 12, 15]. In Section 2, we determine the simple meadows—a special kind of subdirectly irreducible algebras having no nontrivial congruence relations. It turns out that the simple meadows are exactly the building blocks of meadows. In Section 3, we study subvarieties of meadows obtained by extending the theory. In Section 4, we expand meadows with additional signature. In special cases we can give an upper bound of the number of equations needed to axiomatize finitely based theories. In particular, we consider the expansion of meadows with a sign function. In Theorem 8 we prove that the sign function has an equational specification in the signature of the meadows. Section 5 ends the paper with open questions.

2 The cancellation meadows

In the sequel, we shall write $\mathfrak{Alg}(T)$ for the class of algebras axiomatized by the theory T. Thus e.g. $\mathfrak{Alg}(\mathsf{Md})$ denotes the variety of meadows. Moreover, if $\mathcal{F}$ is a field, we shall write $\mathcal{F}_0$ for the expansion of $\mathcal{F}$ where the inversion operation is completed by $0^{-1} = 0$. The meadow $\mathcal{F}_0$ is said to be a *zero-totalized expansion* field. Thus $\mathbb{Q}_0$ denotes the zero-totalized expansion field of the rational numbers.

The 'meaning' of an equational class V is often taken as the *initial algebra* of that class, i.e. the algebra $\mathcal{I}_V \in V$ which has the special property that for every algebra $\mathcal{A} \in V$ there exists a unique homomorphism $h : \mathcal{I}_V \to \mathcal{A}$. The initial algebra always exists, is unique up to isomorphism, and can be constructed from the closed term algebra by dividing out over provable equality. In [5], it is shown that the *initial meadow*—the initial algebra in $\mathfrak{Alg}(\mathsf{Md})$—can also be represented as the minimal subalgebra of the direct product of the zero-totalized expansions of the finite prime fields, i.e. finite meadows of the form $(\mathbb{Z}/p\mathbb{Z})_0$ with p a prime number. Dually, an algebra $\mathcal{A}$ is *final* in V if for every $\mathcal{B} \in V$, there is a unique homomorphism $h : \mathcal{B} \to \mathcal{A}$. The *final meadow*—the final algebra in $\mathfrak{Alg}(\mathsf{Md})$—is the trivial, one-element algebra.

A special class of meadows are the so-called *cancellation* meadows—meadows that satisfy the *Inverse Law*

$$x \neq 0 \longrightarrow x \cdot x^{-1} = 1. \quad (\mathsf{IL})$$

Cancellation meadows and zero-totalized expansion fields form the same class $\mathfrak{Alg}(\mathsf{Md}+\mathsf{IL})$—a class that is not closed under products: e.g. $(\mathbb{Z}/2\mathbb{Z})_0 \times (\mathbb{Z}/3\mathbb{Z})_0 \notin \mathfrak{Alg}(\mathsf{Md}+\mathsf{IL})$ since $\langle 0,1\rangle \neq \langle 0,0\rangle$ but $\langle 0,1\rangle \cdot \langle 0,1\rangle^{-1} = \langle 0,1\rangle \neq \langle 1,1\rangle$. The class of cancellation meadows is therefore not a (sub)variety of meadows. In particular, cancellation meadows cannot be axiomatized by purely equational means.

An algebra $\mathcal{A}$ is a *subdirect product* of an indexed family $(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}$ is a subalgebra of the direct product $\prod_{i\in I} \mathcal{A}_i$ and $\pi_i(\mathcal{A}) = \mathcal{A}_i$, for every $i \in I$. In [5], it is proved that every meadow is (isomorphic with) a subdirect product of cancellation meadows. This is a special instance of Birkhoff's Subdirect Decomposition Theorem. So cancellation meadows form the building blocks of meadows. It follows that the smallest variety of meadows containing all cancellation meadows is the entire class of meadows.

Theorem 1. *Let V be a variety with $\mathfrak{Alg}(\mathsf{Md}+\mathsf{IL}) \subseteq V \subseteq \mathfrak{Alg}(\mathsf{Md})$. Then $V = \mathfrak{Alg}(\mathsf{Md})$.*

Proof: V contains every meadow, since every meadow is a subdirect product of cancellation meadows and V is closed under products and subalgebras. □

Definition 1. *An algebra $\mathcal{A}$ is* simple *if the only congruences on its carrier set A are the diagonal $\{\langle a,a\rangle \mid a \in A\}$ and the all relation $A \times A$.*

It is not hard to see that an algebra is simple if and only if it has no proper homomorphic images. Some textbooks require that a simple algebra be nontrivial. For our development we find the discussion smoother by admitting the one-element meadow.

Theorem 2. $\mathfrak{Alg}(\mathsf{Md}+\mathsf{IL})$ *is the class of simple meadows.*

Proof: Let $\mathcal{A}$ be a cancellation meadow. Suppose the congruence Θ on its carrier A is neither the diagonal nor the all relation. Then we can pick $a_0, a_1, a_2, a_3 \in A$ such that $a_0 \neq a_1$, $\langle a_0, a_1\rangle \in \Theta$ and $\langle a_2, a_3\rangle \notin \Theta$. Since Θ is a congruence we have $\langle a_0 \cdot (a_2 - a_3), a_1 \cdot (a_2 - a_3)\rangle \in \Theta$ and hence $\langle a_0 \cdot a_2 - a_1 \cdot a_2, a_0 \cdot a_3 - a_1 \cdot a_3\rangle \in \Theta$. Therefore $\langle a_2 \cdot (a_0 - a_1), a_3 \cdot (a_0 - a_1)\rangle \in \Theta$ and thus $\langle a_2 \cdot (a_0 - a_1) \cdot (a_0 - a_1)^{-1}, a_3(a_0 - a_1) \cdot (a_0 - a_1)^{-1}\rangle \in \Theta$. So $\langle a_2, a_3\rangle \in \Theta$ by (IL).This yields a contradiction. Therefore Θ must be either the diagonal or the all relation, i.e. $\mathcal{A}$ is simple.

Conversely, let $\mathcal{A}$ be simple meadow. If $\mathcal{A}$ is final, then it is a cancellation meadow. Otherwise, $\mathcal{A}$ is a nontrivial subdirect product of cancellation meadows. Thus there is a projection π with $\pi(\mathcal{A})$ a nontrivial cancellation meadow. Now, since $ker(\pi)$ is a congruence on the carrier A of $\mathcal{A}$ and π is surjective, $ker(\pi) = \{\langle a, a'\rangle \in A \times A \mid \pi(a) = \pi(a')\}$ must be the diagonal. Hence $\mathcal{A} \cong \pi(\mathcal{A})$. Hence $\mathcal{A}$ is a cancellation meadow. $\square$

Corollary 1. *The minimal nontrivial simple meadows are precisely the zero-totalized expansions of prime fields.* $\square$

Corollary 2. *Every variety of meadows having a nontrivial member contains a nontrivial cancellation meadow.*

Proof: By a theorem of Magari [11], if a variety V has a nontrivial member, then V contains a nontrivial simple algebra. $\square$

3 Subvarieties of meadows

Subvarieties of $\mathfrak{Alg}(\mathsf{Md})$ arise when equations are added to Md.

Definition 2. *For $n \in \mathbb{N}^+$, we define the numeral $\underline{n}$ by $\underline{1} = 1$ and $\underline{n+1} = \underline{n} + 1$, and for $R \subseteq \mathbb{N}^+$, we put $\mathsf{Inv}_R = \{\underline{n} \cdot \underline{n}^{-1} = 1 \mid n \in R\}$.*

It is easy to see that

$$\mathfrak{Alg}(\mathsf{Md}) \models \underline{n+m} = \underline{n} + \underline{m} \text{ and } \mathfrak{Alg}(\mathsf{Md}) \models \underline{n \cdot m} = \underline{n} \cdot \underline{m} \qquad (\dagger)$$

for all $n, m \in \mathbb{N}^+$. In the sequel, we let P be the set of primes.

Theorem 3.

1. *For all $P \subseteq R \subseteq \mathbb{N}^+$, $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_R) = \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_{\mathbb{N}^+})$.*
2. *$\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$ is the largest subvariety of $\mathfrak{Alg}(\mathsf{Md})$ with initial algebra $\mathbb{Q}_0$.*
3. *If $R \subset P$, then $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P) \subset \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_R)$ and $I_{\mathfrak{Alg}(\mathsf{Md}+\mathsf{Inv}_R)} \ncong \mathbb{Q}_0$.*

Proof:

1. This follows from ($\dagger$) and the fact that $\mathfrak{Alg}(\mathsf{Md}) \models (x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.
2. In [4], it is proved that $\mathbb{Q}_0$ is the initial algebra of $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_{\mathbb{N}^+})$. Hence, since $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P) = \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_{\mathbb{N}^+})$, $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$ is a subvariety of $\mathfrak{Alg}(\mathsf{Md})$ with initial algebra $\mathbb{Q}_0$. Moreover, if V is a subvariety of $\mathfrak{Alg}(\mathsf{Md})$ with initial algebra $\mathbb{Q}_0$, then every algebra $\mathcal{A} \in V$ must satisfy Inv_P. Thus $V \subseteq \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$.

3. Let $R \subset P$. Then clearly $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P) \subseteq \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_R)$. Pick a prime $p \notin R$. Then $(\mathbb{Z}/p\mathbb{Z})_0 \in \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_R)$ but $p \cdot p^{-1} = 0$ in $(\mathbb{Z}/p\mathbb{Z})_0$. Hence $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P) \subset \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_R)$ and $\mathbb{Q}_0$ cannot be initial in $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_R)$. □

Zero-totalized finite expansion fields do not satisfy Inv_P; therefore $\mathfrak{Alg}(\mathsf{Md} + \mathsf{IL}) \not\subseteq \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$. Conversely, not every meadow in $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$ is a cancellation meadow.

Theorem 4. *Let $R \subseteq P$. Then*

1. $\mathfrak{Alg}(\mathsf{Md} + \mathsf{IL}) \subseteq \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_R)$ *if and only if* $R = \emptyset$, *and*
2. $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_R) \not\subseteq \mathfrak{Alg}(\mathsf{Md} + \mathsf{IL})$

Proof:

1. This follows from Theorem 1.
2. Choose a new constant symbol a. For $k \in \mathbb{N}^+$ let

$$E_k = \{a \neq 0\} \cup \{\underline{n} \neq 0 \mid 0 < n < k\} \cup \{\underline{n} \cdot \underline{n}^{-1} = 1 \mid 0 < n < k\} \cup \{\underline{2} \cdot a = 0\} \cup \mathsf{Md}.$$

Moreover, choose a prime $p \neq 2$ exceeding k and interpret a in $\mathbb{Z}/2p\mathbb{Z}$ by p. Since $2p$ is squarefree, $\mathbb{Z}/2p\mathbb{Z}$ is meadow (see [6]). Thus $\mathbb{Z}/2p\mathbb{Z} \models E_k$. It follows that E_k is consistent and therefore $E = \bigcup_{k=1}^{\infty} E_k$ is consistent by the compactness theorem. Let M be a model for E. Then

(a) M is a meadow, since $\mathsf{Md} \subseteq E$,

(b) M satisfies Inv_R,

(c) M is not a cancellation meadow: $M \models \underline{2} \neq 0$ but $M \models \underline{2} \cdot \underline{2}^{-1} \neq 1$, for otherwise

$$a = 1 \cdot a = \underline{2} \cdot \underline{2}^{-1} \cdot a = \underline{2}^{-1} \cdot \underline{2} \cdot a = 2 \cdot 0 = 0.$$

A frequently asked question of universal algebra is whether or not the identities valid in a class of algebras are *finitely based*, i.e. are the consequence of a finite number of identities. Below we give a negative answer in the case Inv_P.

Theorem 5. $\mathsf{Md} + Inv_P$ *has no finite basis.*

Proof: Suppose $\mathsf{Md} + Inv_P$ has a finite base. Then by the compactness theorem for equational logic, there must exist a finite set $R \subseteq P$ such that $\mathsf{Md} + Inv_R$ is a base for $\mathsf{Md} + Inv_P$. In order to obtain a contradiction, we choose a prime $p \in P$ larger then any prime in R. Then $(\mathbb{Z}/p\mathbb{Z})_0 \in \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_R)$ but $(\mathbb{Z}/p\mathbb{Z})_0 \not\models \underline{p} \cdot \underline{p}^{-1} = 1$. □

It follows that $\mathbb{Q}_0$ cannot have a finite complete axiomatization which also holds in $\mathbb{C}_0$—the zero-totalized expansion field of the complex numbers. This is proved in Corollary 3 below.

Theorem 6. *Let s, t be meadow terms. Then*

$$\mathbb{C}_0 \models s = t \quad \textit{iff} \quad \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P) \models s = t.$$

Proof: ($\Leftarrow$): This follows from the fact that $\mathbb{C}_0$ is a cancellation meadow.
($\Rightarrow$): Suppose $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P) \not\models s = t$. By Corollary 5.2, there exists a meadow $M \in \mathfrak{Alg}(\mathsf{Md}+\mathsf{IL}+\mathsf{Inv}_P)$ with $M \not\models s = t$. Observe that M is a cancellation meadow of characteristic 0. Let $\hat{M}$ be the algebraic closure of M. Since $s = t$ is a universal proposition, we have $\hat{M} \not\models s = t$. In [2] it is proved that every meadow equation has a first-order representation over the signature $\{0, 1, +, \cdot, -\}$ of fields. In particular, there exists a quantifier-free first-order formula $\phi(s,t)$ such that

$$\mathsf{Md} + \mathsf{IL} \vdash s = t \leftrightarrow \phi(s,t).$$

Thus

$$\hat{M} \models \exists \vec{x}\ \neg\phi(s,t)$$

where $\exists \vec{x}\ \neg\phi(s,t)$ is the existential closure of $\phi(s,t)$. Since M has characteristic 0, $\hat{\mathbb{Q}}_0$ can be embedded in $\hat{M}$. Moreover, since algebraically closed fields are model-complete, every embedding between algebraically closed fields is elementary (see e.g. [13]). We therefore have $\hat{\mathbb{Q}}_0 \models \exists \vec{x}\ \neg\phi(s,t)$ and hence $\mathbb{C}_0 \models \exists \vec{x}\ \neg\phi(s,t)$. It follows that $\mathbb{C}_0 \not\models s = t$. □

Corollary 3. *Let $V \subseteq \mathfrak{Alg}(\mathsf{Md})$ be a finitely based variety. Then*

$$\mathcal{I}_V \cong \mathbb{Q}_0 \Longrightarrow \mathbb{C}_0 \notin V.$$

Proof: Put $V = \mathfrak{Alg}(\mathsf{Md} + E)$ for some finite set of equations E. Suppose $\mathbb{C}_0 \models E$. Then $\mathsf{Md}+\mathsf{Inv}_P \vdash E$ by the previous theorem, and hence by equational compactness, $\mathsf{Md}+\mathsf{Inv}_R \vdash E$ for some finite set $R \subset \mathbb{N}^+$. It follows that $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$ is finitely based, a contradiction with Theorem 5. □

4 Expansions of meadows

In this section we consider expansions of meadows with new function symbols. If Σ is a set of new function symbols and E is a set of identities, we shall write $\mathfrak{Alg}_\Sigma(\mathsf{Md} + E)$ for the variety of Σ-expansion meadows that satisfy E; likewise, we shall write $\mathfrak{Alg}_\Sigma(\mathsf{Md} + \mathsf{IL} + E)$ for the Σ-expansions of cancellation meadows that satisfy E. Thus if $\Sigma = \emptyset = E$ we have the variety of meadows and the class of cancellation meadows. Moreover, if e is an identity we shall write $\mathfrak{Alg}_\Sigma(\mathsf{Md} + E) \models e$ ($\mathfrak{Alg}_\Sigma(\mathsf{Md} + \mathsf{IL} + E) \models e$) if all meadows in $\mathfrak{Alg}_\Sigma(\mathsf{Md} + E)$ ($\mathfrak{Alg}_\Sigma(\mathsf{Md} + \mathsf{IL} + E)$) satisfy e. Finally, if $\mathcal{A}$ is a Σ-expansion meadow we shall denote the reduct—the underlying meadow—by $\mathcal{A}\restriction_{\mathsf{Md}}$.

The fact that every meadow is a subdirect product of cancellation meadows can be generalized to varieties of expansions.

Definition 3. *Let $\mathcal{A}$ be a meadow with carrier A and let $(A_i)_{i\in I}$ be a family of sets such that $A \subseteq \prod_{i\in I} A_i$. $f : A^n \to A$ is* defined pointwise *if*

$$f(\langle a_1^i\rangle_{i\in I}, \ldots, \langle a_n^i\rangle_{i\in I}) = \langle \pi_i(f(Z_{a_1^i}, \ldots, Z_{a_n^i}))\rangle_{i\in I}.$$

Here we denote by $Z_{a_m^i}$ the element of $\prod_{i\in I} A_i$ that is 0 in all coordinates except i, where it is a_m^i.

Observe that interpretations of function symbols in direct products are defined pointwise. A typical expansion variety with a function symbol that cannot be defined pointwise is the following. Consider the expansion with a single binary function symbol **eq** and the equations $\mathbf{eq}(x,x) = 1$ and $\mathbf{eq}(\underline{i},\underline{j}) = 0$ for $0 \leq i \neq j \leq 5$ which define equality in $\mathbb{Z}/6\mathbb{Z}$. One can interpret **eq** in $\mathcal{A} = (\mathbb{Z}/2\mathbb{Z})_0 \times (\mathbb{Z}/3\mathbb{Z})_0$ by

$$\mathbf{eq}^{\mathcal{A}}(\langle i,i'\rangle,\langle j,j'\rangle) = \begin{cases} \langle 1,1\rangle & \text{if } i=j \text{ and } i'=j' \ , \\ \langle 0,0\rangle & \text{otherwise.} \end{cases}$$

However, $\mathbf{eq}^{\mathcal{A}}$ has no pointwise definition: for example

$$\mathbf{eq}^{\mathcal{A}}(\langle 1,0\rangle,\langle 2,0\rangle) = \langle 0,0\rangle \neq \langle 0,1\rangle = \langle \pi_1(\mathbf{eq}^{\mathcal{A}}(\langle 1,0\rangle,\langle 2,0\rangle)), \pi_2(\mathbf{eq}(\langle 0,0\rangle,\langle 0,0\rangle)\rangle.$$

Proposition 1. *Let Σ be a set of new function symbols and let E be a set of identities. Then $\mathcal{A} \in \mathfrak{Alg}_\Sigma(\mathsf{Md}+E)$ is a subdirect product of a family $(\mathcal{A}_i)_{i\in I}$ with $\mathcal{A}_i \in \mathfrak{Alg}_\Sigma(\mathsf{Md}+\mathsf{IL}+E)$ for every $i \in I$ if and only if every $f \in \Sigma$ can be defined pointwise in $\mathcal{A}$.*

Proof: If $\mathcal{A}$ is a subdirect product then clearly every $f \in \Sigma$ has a pointwise definition in $\mathcal{A}$. Conversely, suppose $\mathcal{A} \in \mathfrak{Alg}_\Sigma(\mathsf{Md}+E)$ has pointwise definitions for every $f \in \Sigma$. Pick a family $(\mathcal{A}_i)_{i\in I}$ of cancellation meadows such that $\mathcal{A}\restriction_{\mathsf{Md}}$ is a subdirect product of $(\mathcal{A}_i)_{i\in I}$. Since π_i is a surjective homomorphism from $\mathcal{A}\restriction_{\mathsf{Md}}$ to $\mathcal{A}_i$, we can expand every $\mathcal{A}_i$ to a Σ-cancellation meadow by stipulating

$$f^{\mathcal{A}_i}(a_1^i,\ldots,a_n^i) = \pi_i(f^{\mathcal{A}}(Z_{a_1^i},\ldots,Z_{a_n^i}))$$

for every $f \in \Sigma$. Then

$$f^{\mathcal{A}}(\langle a_1^i\rangle_{i\in I},\ldots,\langle a_n^i\rangle_{i\in I}) = \langle \pi_i(f^{\mathcal{A}}(Z_{a_1^i},\ldots,Z_{a_n^i}))\rangle_{i\in I} = \langle f^{\mathcal{A}_i}(a_1^i,\ldots,a_n^i)\rangle_{i\in I}.$$

Clearly, since $\mathcal{A}$ satisfies E, every $\mathcal{A}_i$ does so. Hence $\mathcal{A}$ is a subdirect product of a family of $(\mathcal{A}_i)_{i\in I}$ with $\mathcal{A}_i \in \mathfrak{Alg}_\Sigma(\mathsf{Md}+\mathsf{IL}+E)$ for every $i \in I$. □

As a consequence we obtain the following semantic version of the *Generic Basis Theorem* proved in [1].

Definition 4. *Let Σ be a set of new function symbols and let E be a set of identities. (Σ, E)* admits pointwise definitions *if every $\mathcal{A} \in \mathfrak{Alg}_\Sigma(\mathsf{Md}+E)$ has pointwise definitions for every $f \in \Sigma$.*

Corollary 4. *Let Σ be a set of new function symbols and let E be a set of identities. If (Σ, E) admits pointwise definitions, then*

$$\mathfrak{Alg}_\Sigma(\mathsf{Md}+E) \models s=t \quad \textit{iff} \quad \mathfrak{Alg}_\Sigma(\mathsf{Md}+\mathsf{IL}+E) \models s=t$$

for all Σ-meadow terms s,t.

Proof: $(\Rightarrow)$: Immediate.
$(\Leftarrow)$: Suppose $\mathfrak{Alg}_\Sigma(\mathsf{Md}+\mathsf{IL}+E) \models s=t$ and let $\mathcal{A} \in \mathfrak{Alg}_\Sigma(\mathsf{Md}+E)$. By the previous proposition, $\mathcal{A}$ is a subdirect product of a family of cancellation meadows $(\mathcal{A}_i)_{i\in I}$ with $\mathcal{A}_i \in$

$\mathfrak{Alg}_\Sigma(\mathsf{Md}+\mathsf{IL}+E)$. By the assumption, every $\mathcal{A}_i$ satisfies $s=t$. Thus $\prod_{i\in I}\mathcal{A}_i$ satisfies $s=t$. It follows that $\mathcal{A}$ satisfies $s=t$, since $\mathcal{A}$ is a subalgebra of $\prod_{i\in I}\mathcal{A}_i$. □

The next theorem gives an upper bound to the number of equations needed for the axiomatization of finitely based subvarieties of $\mathfrak{Alg}_\Sigma(\mathsf{Md})$.

Theorem 7. *Let Σ be a set of new function symbols and let E be a finite set of identities. If (Σ,E) admits pointwise definitions, then there exists an equation e such that $\mathfrak{Alg}_\Sigma(\mathsf{Md}+E)=\mathfrak{Alg}_\Sigma(\mathsf{Md}+e)$.*

Proof: We first prove that

$$\mathfrak{Alg}_\Sigma(\mathsf{Md}+\{r=0,t=0\})=\mathfrak{Alg}_\Sigma(\mathsf{Md}+(1-t\cdot t^{-1})(1-r\cdot r^{-1})=1). \qquad (\ddagger)$$

Clearly, $\mathfrak{Alg}_\Sigma(\mathsf{Md}+\{r=0,t=0\})\subseteq\mathfrak{Alg}_\Sigma(\mathsf{Md}+(1-t\cdot t^{-1})(1-r\cdot r^{-1})=1)$. In order to prove the converse inclusion, we have to show

$$\mathfrak{Alg}_\Sigma(\mathsf{Md}+(1-t\cdot t^{-1})(1-r\cdot r^{-1})=1)\models\{r=0,t=0\}.$$

By the previous corollary, it suffices to prove

$$\mathfrak{Alg}_\Sigma(\mathsf{Md}+\mathsf{IL}+(1-t\cdot t^{-1})(1-r\cdot r^{-1})=1)\models\{r=0,t=0\}.$$

This follows immediately, because if $\mathcal{A}\in\mathfrak{Alg}_\Sigma(\mathsf{Md}+\mathsf{IL}+(1-t\cdot t^{-1})(1-r\cdot r^{-1})=1)$ and e.g. $\mathcal{A}\not\models t=0$ then there are $t^*,r^*\in A$ with $t^*\neq 0$ and $(1-t^*\cdot t^{*-1})(1-r^*\cdot r^{*-1})=1$. It follows that $\mathcal{A}\models 0=1$, since $t^*\cdot t^{*-1}=1$.

Now let E be a finite set of equations. We may assume that every equation is of the form $s=0$. By $(\ddagger)$ we can code two equations of the form $r=0$ and $t=0$ in the equation $(1-t\cdot t^{-1})(1-r\cdot r^{-1})-1=0$. By the same argument we can proceed reducing smaller sets of equations until only a single equation e is left. □

In particular, we have

Corollary 5. *Let E be a set of meadow equations. Then*

1. *$\mathfrak{Alg}(\mathsf{Md}+E)\models e$ if and only if $\mathfrak{Alg}(\mathsf{Md}+\mathsf{IL}+E)\models e$ for all meadow equations e, and*
2. *if E is finite then $\mathfrak{Alg}(\mathsf{Md}+E)$ can be axiomatized by at most eleven equations.* □

A particular expansion with the sign function $\mathbf{s}$ was studied in [2] where a finite axiomatization of formally real meadows was given. We define the sign function in an equational manner by the set Signs of axioms given in Table 3. Here we write 1_x for $x\cdot x^{-1}$ and 0_x for $1-1_x$. Clearly, $\mathfrak{Alg}_{\{\mathbf{s}\}}(\mathsf{Md}+\mathsf{Signs})$ is a variety. But also the class of all reducts of signed meadows is a variety. This means that the sign function has an equational specification in the signature of the meadows.

Theorem 8. $\{\mathcal{A}\restriction_{\mathsf{Md}} \mid \mathcal{A}\in\mathfrak{Alg}_{\{\mathbf{s}\}}(\mathsf{Md}+\mathsf{Signs})\}$ *is a variety.*

Proof: Let $K=\{\mathcal{A}\restriction_{\mathsf{Md}} \mid \mathcal{A}\in\mathfrak{Alg}_{\{\mathbf{s}\}}(\mathsf{Md}+\mathsf{Signs})\}$. In [1] (Proposition 3.9), it is proved that $K\subseteq\mathfrak{Alg}(\mathsf{Md}+\mathsf{EFR})$ where $\mathsf{EFR}=\{0_{x_0^2+\cdots+x_n^2}\cdot x_0=0\mid n\in\mathbb{N}\}$ is an infinite axiomatization

$$\mathbf{s}(1_x) = 1_x \tag{S1}$$

$$\mathbf{s}(0_x) = 0_x \tag{S2}$$

$$\mathbf{s}(-1) = -1 \tag{S3}$$

$$\mathbf{s}(x^{-1}) = \mathbf{s}(x) \tag{S4}$$

$$\mathbf{s}(x \cdot y) = \mathbf{s}(x) \cdot \mathbf{s}(y) \tag{S5}$$

$$0_{\mathbf{s}(x)-\mathbf{s}(y)} \cdot (\mathbf{s}(x+y) - \mathbf{s}(x)) = 0 \tag{S6}$$

Table 3: The set Signs of axioms for the sign function

of formal realness. In order to prove that K is a variety, it therefore suffices to prove that $\mathfrak{Alg}(\mathsf{Md} + \mathsf{EFR}) \subseteq K$.

We have to show that every $\mathcal{A} \in \mathfrak{Alg}(\mathsf{Md} + \mathsf{EFR})$ can be expanded to a signed meadow. Thus pick $\mathcal{A} \in \mathfrak{Alg}(\mathsf{Md}+\mathsf{EFR})$. By Corollary 5, $\mathcal{A}$ is a subdirect product of a family $(\mathcal{A}_i)_{i\in I}$ of formally real cancellation meadows. Every $\mathcal{A}_i$ is therefore an expansion of a formally real field and can thus be ordered by an ordering $<$ satisfying the axioms (OF1)–(OF4) given below:

$$\begin{array}{rll}
x \neq 0 & \to (x < 0 \ \vee \ 0 < x) & \text{(OF1)} \\
x < y & \to \neg(y < x \ \vee \ x = y) & \text{(OF2)} \\
x < y & \to x + z < y + z & \text{(OF3)} \\
x < y \ \wedge \ 0 < z & \to x \cdot z < y \cdot z & \text{(OF4)}
\end{array}$$

W.l.o.g. we may assume that the ordering is such that $-1 < 0 < 1$. We define the sign function $\mathbf{s}_i$ on the carrier A_i of $\mathcal{A}_i$ by

$$\mathbf{s}_i(a) = \begin{cases} -1 & \text{if } a < 0, \\ 0 & \text{if } a = 0, \\ 1 & \text{if } a > 0. \end{cases}$$

Then $\mathbf{s}_i$ satisfies the axioms of Signs: (S1) and (S2) hold since $1_a, 0_a \in \{0,1\}$ for every $a \in A_i$. (S3) is immediate. To prove (S4), observe that $a = 0$ or $a < 0$ or $a > 0$ by (OF1). Moroever, $a = 0$ iff $a^{-1} = 0$ and, by (OF4), $a < 0$ iff $a^{-1} < 0$. Hence also (S4) holds. (S5) follows from a case distinction. Finally, (S6) is an equational representation of the conditional axiom $\mathbf{s}(x) = \mathbf{s}(y) \longrightarrow \mathbf{s}(x+y) = \mathbf{s}(x)$ which clearly holds too.

Since $\mathcal{A}$ is a subalgebra of $\prod_{i\in I} \mathcal{A}_i$, we can now define the sign function on $\mathcal{A}$ in the obvious way by $\mathbf{s}(\langle a_i\rangle_{i\in I}) = \langle \mathbf{s}_i(a_i)\rangle_{i\in I}$. Then $\mathcal{A}$ is a signed meadow. □

The situation is different if we expand signed meadows with a *signed square root* $\surd$ defined by the equations given in Table 4. Here we stipulate $\surd(x) = -\surd(-x)$ for $x < 0$. The class of reducts of signed meadows with square roots $K = \{\mathcal{A} \restriction_{\mathsf{Md}} | \ \mathcal{A} \in \mathfrak{Alg}_{\{\mathbf{s},\surd\}}(\mathsf{Md} + \mathsf{Signs} + \mathsf{Squareroots})\}$ is not a variety: clearly $\mathbb{R}_0 \in K$ but its subalgebra $\mathbb{Q}_0$ cannot be expanded with square roots. Hence K is not closed under subalgebras. It follows that unlike the sign function, the square roots have no equational specification in the signature of the meadows.

$$\sqrt{}(x^{-1}) = (\sqrt{}(x))^{-1} \tag{1}$$
$$\sqrt{}(x \cdot y) = \sqrt{}(x) \cdot \sqrt{}(y) \tag{2}$$
$$\sqrt{}(x \cdot x \cdot \mathbf{s}(x)) = x \tag{3}$$
$$\mathbf{s}(\sqrt{}(x) - \sqrt{}(y)) = \mathbf{s}(x - y) \tag{4}$$

Table 4: The set SR of axioms for the signed square root

5 Open questions

We suspect that the question whether a finite axiomatization of $\mathbb{Q}_0$ exists can be solved if the subvarieties of the meadows were well understood and fully classified. We end this paper with some open questions which we believe can approach the problem from different angles.

In Theorem 5, it is proved that $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$ has no finite basis. Clearly, there are finitely based subvarieties $V \subseteq \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$. E.g. in [3] it is shown that $\mathfrak{Alg}(\mathsf{Md} + (1 + x_1^2 + x_2^2) \cdot (1 + x_1^2 + x_2^2)^{-1}) \subset \mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$. The question then arises whether there exists a maximal finitely based subvariety of $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$. Another interesting question is whether there exists a minimal finitely based subvariety of $\mathfrak{Alg}(\mathsf{Md} + \mathsf{Inv}_P)$. If the answer to the latter question is affirmative, then a finite complete axiomatization of the equational theory of the rational numbers exists. In particular, by Corollary 5 a complete axiomatization of the form $\mathsf{Md} + e$ can be given. It is unclear whether e can be a one-variable equation.

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