Seasonal reproduction leads to population collapse and an Allee effect in a stage-structured consumer-resource biomass model when mortality rate increases

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Supplementary material to Sun and De Roos: Seasonal reproduction leads to population collapse and an Allee effect in a stage-structured consumer-resource biomass model when mortality rate increases

Here we present the derivation of $R_0$ shown in Eq. 2 in the main text.

For $\theta = 1$ juveniles and adults have the same net biomass productivity. To simplify the discussion we denote the net biomass productivity of the consumer individuals as $\nu (R)$, defined as:

$$\nu (R) = \sigma I_{\max} R - Q, \quad (1)$$

Because consumers never starve during the season (Figure 4a), the mortality rates of juveniles and adults are also the same and equal to $\mu$. The probability, denoted by $P_j(t)$, that the consumer individual is still in the juvenile stage at time $t$ follows a dynamics described by the following ordinary differential equation:

$$\frac{dP_j}{dt} = (-\gamma (\nu (R(t)), \mu) - \mu) P_j, \quad P_j(0) = 1. \quad (2)$$

This ODE can be formally solved to yield as solution:

$$P_j(t_m) = \exp \left( \int_0^{t_m} -\mu - \gamma (\nu (R(\tau)), \mu) \, d\tau \right)$$

$$= \exp (-\mu t_m) \exp \left( -\int_0^{t_m} \gamma (\nu (R(\tau)), \mu) \, d\tau \right), \quad (3)$$

The maturation rate at time $t_m$ out of the juvenile stage is then given by

$$h(t_m) = \gamma (\nu (R(t_m)), \mu) P_j(t_m). \quad (4)$$

If we would consider a cohort of $N$ newly produced offspring, $h(t_m) \cdot N$ would represent the expected rate of maturation of these individuals at time $t_m$.

The underlying assumption in the model about consumer life history [1] [2] is that juvenile individuals are born at the beginning of a season with body size $s(0) = s_b$, which we set without loss of generality equal to 1. They subsequently grow following:

$$\frac{ds}{d\tau} = \nu (R(\tau)) s, \quad s(0) = s_b.$$  

Individual body size at $t_m$ hence equals:

$$s(t_m) = \exp \left( \int_0^{t_m} \nu (R(\tau)) \, d\tau \right). \quad (5)$$

In the period from $t_m$ to the end of the season an individual maturing at $t_m$ accumulates and stores reproductive energy at a rate $s(t_m) \nu (R(\tau))$, which is proportional to its size at maturation. The contribution to the reproductive output at the end of the season by the individuals that mature at time $t_m$ in the growing season is thus given by
\[ R_0(t_m) = h(t_m) \int_{t_m}^{1} s(t_m) \nu(R(\tau)) \, d\tau \cdot \exp(-\mu(1-t_m)) \]
\[ = h(t_m) s(t_m) \int_{t_m}^{1} \nu(R(\tau)) \, d\tau \cdot \exp(-\mu(1-t_m)). \]

Using Equations (3)-(5) this expression can be rewritten as:

\[ R_0(t_m) = \gamma(\nu(R(t_m)), \mu) \exp \left(-\mu + \int_{0}^{t_m} \nu(R(\tau)) - \gamma(\nu(R(\tau)), \mu) \, d\tau \right) \int_{t_m}^{1} \nu(R(\tau)) \, d\tau. \] 

References
