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## SLOW TIME-PERIODIC SOLUTIONS OF THE GINZBURG–LANDAU EQUATION

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In this paper we study the behaviour of solutions of the form  $\psi(z, t) = \phi(z) e^{-i\omega t}$  ( $\epsilon \ll 1$ ) of the rescaled Ginzburg–Landau equation,  $\psi_t = [1 - (1 + iB)|\psi|^2]\psi + (1 + iA)\psi_{zz}$ , for  $A = \epsilon a$ ,  $B = \epsilon b$ ,  $\omega$  plays the role of free parameter. This leads to a perturbation analysis on a complex Duffing equation (similar to the analysis of Holmes [Physica D 23 (1986) 84]). We show that the spatial quasiperiodic solutions (of the unperturbed,  $\epsilon = 0$ , case) disappear due to the perturbation and prove the existence of degenerated periodic solutions which oscillate through the origin. We also establish the existence of several types of heteroclinic orbits connecting counterrotating periodic patterns.

### 1. Introduction

The Ginzburg–Landau equation is a modulation equation describing the nonlinear development of unstable waves in many physical systems (such as hydrodynamics) or chemical systems in which some kind of turbulence appears. The Ginzburg–Landau equation governs (at least) the first steps in the transition process leading to turbulence. In its most general form it reads:

$$\frac{\partial \psi}{\partial t} = (\alpha + \beta |\psi|^2) \psi + \gamma \frac{\partial^2 \psi}{\partial z^2} \quad (1.1)$$

with  $\psi(z, t): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ ,  $\alpha, \beta, \gamma \in \mathbb{C}$  constants which can be computed explicitly for a given (practical) problem.

The Ginzburg–Landau equation has been studied by many authors, from many different viewpoints. This can be illustrated by the following (incomplete) enumeration of papers on the subject. Eq. (1.1) with periodic boundary conditions is studied by Doering et al. [1] and Ghidaglia and Héron [2]: in these papers (sharp) estimates on the dimension of the (chaotic) attractor are derived; numerical studies of this situation appeared in Moon et al. [3] and Keefe [4] (and other papers). The stability of periodic wave solutions ( $\psi = R e^{i(kx + \omega t)}$ ) is investigated in Stuart and DiPrima [5]; slowly varying waves are studied by Bernhoff [6]. Other types of solutions (bursting solutions, quasiperiodic solutions, homoclinic solutions) are discussed by Hocking and Stewartson [7], Holmes [8], Kramer and Zimmerman [9], Landman [10] and other authors.

The equation has been derived by many authors, in various ways. The earliest papers in which the Ginzburg–Landau equation is derived are: Newell and Whitehead [11], Stewartson and Stuart [12], DiPrima, Eckhaus and Segel [13]. In these papers the authors start the analysis by investigating the linear stability of a basic (laminar) solution as a control parameter  $R$  is varied. The stability against spatial periodic disturbances (in the  $z$  direction) is determined by the sign of the real part of the (largest) eigenvalue  $\omega(k, R)$  ( $k$  is wavenumber). The curve  $\text{Re } \omega(k, R) = 0$  is (locally) a parabola in the  $k, R$  plane with a minimum at  $(k_c, R_c)$ : the basic flow is unstable against disturbances with wavenumbers in a (small) interval centered around  $k_c$  for  $R > R_c$  (supercritical bifurcation). The Ginzburg–Landau modulation equation governs the nonlinear evolution of the linear unstable disturbances (for  $R$  close to  $R_c$ ). In the derivation of the modulation equation one finds that the linear coefficients  $\alpha, \gamma$  (in eq. (1.1)) are determined by the linear analysis: in the case of supercritical bifurcation one finds  $\text{Re } \alpha > 0$  and  $\text{Re } \gamma > 0$ . As in many physical examples and mathematical studies (see refs. [1–4, 6, 9, 11]) we consider in this paper  $\text{Re } \beta < 0$  (the nonlinear term “balances” the linear instability). Thus eq. (1.1) can be transformed into

$$\frac{\partial \phi}{\partial t} = [1 - (1 + iB)|\phi|^2] \phi + (1 + iA) \frac{\partial^2 \phi}{\partial z^2} \tag{1.2}$$

with  $A, B \in \mathbb{R}$ . If one considers the case of real coefficients, i.e.  $A = B = 0$  one obtains, for stationary  $\phi$ , a complex Duffing equation

$$\phi_{zz} + \phi - |\phi|^2 \phi = 0, \tag{1.3}$$

which is integrable. This was already observed by Newell and Whitehead [11]; Kramer and Zimmerman [9] also analysed the case of real coefficients. There are four types of solutions of eq. (1.3):

- (a) Periodic solutions  $\phi(z) = R e^{ikz}$  with  $R, k \in \mathbb{R}$ ,  $R^2 + k^2 = 1$ .
- (b) Quasiperiodic solutions  $\phi(z) = \rho(z) e^{i\theta(z)}$  with  $\rho(z)$  and  $d\theta(z)/dz$  periodic (with the same period).
- (c) Periodic solutions  $a(z) \in \mathbb{R}$  with  $a'' + a - a^3 = 0$ . These solutions can be considered as degenerated quasiperiodic solutions of type (b) (see section 2.2).
- (d) Homoclinic solutions  $\phi(z) = \rho(z) e^{i\theta(z)}$  with  $\rho \rightarrow \rho_0$  as  $z \rightarrow \pm \infty$ .

For a more detailed discussion of these solutions see section 2.2. Remark that every solution belongs to a one-parameter family of solutions, due to the invariance of eqs. (1.3) and (1.1) under multiplication of solutions  $\phi$  by a factor  $e^{i\varphi}$ ,  $\varphi \in \mathbb{R}$ .

In this paper we examine the Ginzburg–Landau equation with coefficients with small imaginary parts

$$\frac{\partial \phi}{\partial t} = [1 - (1 + i\epsilon b)|\phi|^2] \phi + (1 + i\epsilon a) \frac{\partial^2 \phi}{\partial z^2} \tag{1.4}$$

with  $a, b \in \mathbb{R}$ ,  $0 < \epsilon \ll 1$ . In the sequel we name (1.4) the Ginzburg–Landau equation with small complex coefficients. The main questions are:

- (1) What happens to the special solutions of the stationary Ginzburg–Landau equation with real coefficients (i.e. the unperturbed situation)?
- (2) Are there other types of special solutions of (1.4)?

If one searches for periodic solutions  $\phi(z, t) = R e^{i(kz - wt)}$  of (1.4) one obtains

$$0 = 1 - R^2 - k^2, \quad w = \varepsilon(bR^2 + ak^2).$$

Thus for  $\varepsilon = 0$  this type of periodic solutions is stationary, for  $\varepsilon \neq 0$  there is a slow oscillation in time:  $w = \mathcal{O}(\varepsilon)$ . In this paper we study slow time periodic solutions of the Ginzburg–Landau equation with small complex coefficients, i.e.  $\phi(z, t)$  is a solution of (1.4) of the form

$$\phi(z, t) = \rho(z) e^{i[\theta(z) - \varepsilon wt]}, \quad (1.5)$$

$w$  plays the role of a free parameter.

*Remark.* Holmes studied in ref. [8] eq. (1.1) for small values of the real parts of constants  $\alpha, \beta, \gamma$ , i.e. (1.1) is close to the nonlinear Schrödinger equation. This yields equations very similar to the ones we study in this paper (see also remark 4, section 3).

We tackle the problem (stated in the questions above) by performing a perturbation analysis on the integrable (Hamiltonian) system (1.3). We apply similar approximation techniques as were used by Holmes in ref. [8]: we define a Poincaré map  $P$  and approximate  $P$  using the unperturbed system. However, one has to be very careful in using this approximation: it degenerates in important situations. The study of these degenerations leads to solutions  $\phi$  of (1.4) which cross through the origin of the complex plane (i.e.  $\rho(z)$  in (1.5) switches from positive to negative). The nature of the perturbation is of significant importance: the perturbation preserves symmetries of the unperturbed system, which are essential in the study of (the degenerations of) Poincaré map  $P$ .

The main results of the perturbation analysis are

(1) None of the quasiperiodic solutions (of type (b)) “survives” the perturbation, i.e. (1.4) has no solutions of the form  $\rho(z) e^{i[\theta(z) - \varepsilon wt]}$  with periodic  $\rho(z)$  and  $d\theta(z)/dz$ .

(2) The Ginzburg–Landau equation with small complex coefficients has periodic solutions which are perturbations of the (degenerated) periodic solutions of type (c).

It should be remarked that these periodic solutions are very different from the other type of periodic solutions,  $R e^{i(kz - \varepsilon wt)}$ : solutions of type (c) oscillate through the origin of the complex plane (see fig. 4).

We also show that there are values of the free parameter  $w$  (see (1.5)) such that there exist heteroclinic solutions which “start” (i.e. for  $z \rightarrow -\infty$ ) at a periodic solution  $R(w) e^{i[k(w)z - \varepsilon wt]}$  and spiral towards the counterrotating periodic solution  $R(w) e^{i[-k(w)z - \varepsilon wt]}$  for  $z \rightarrow +\infty$ . There are two types of these heteroclinic solutions, one which never crosses the origin of the complex plane, the other crosses the origin once. The set of values of  $w$  for which heteroclinic orbits exist consists of a continuous part (an interval) and a collection of discrete points. The continuous part corresponds to heteroclinic orbits connecting unstable periodic solutions of (1.4), the discrete points correspond to heteroclinic connections between stable periodic patterns.

In section 2 we formulate the basic results and construct the basic tools necessary for the perturbation analysis. Section 3 is dedicated to the study of the (quasi) periodic solutions. In section 4 we deal with the heteroclinic solutions, in section 5 we show pictures of numerical simulations.

## 2. Perturbation analysis: the setting of the problem

### 2.1. Derivation of the equations

We substitute  $\phi(z, t) = \rho(z) e^{i\theta(z) - \varepsilon w t}$  into the Ginzburg–Landau equation with small complex coefficients (1.4). This leads to

$$\begin{aligned} \rho_{zz} - \rho \theta_z^2 + \rho \frac{1 + \varepsilon^2 a w}{1 + \varepsilon^2 a^2} - \rho^3 \frac{1 + \varepsilon^2 a b}{1 + \varepsilon^2 a^2} &= 0, \\ 2\rho_z \theta_z + \rho \theta_{zz} - \varepsilon \rho \frac{a - w}{1 + \varepsilon^2 a^2} + \varepsilon \rho^3 \frac{a - b}{1 + \varepsilon^2 a^2} &= 0. \end{aligned} \quad (2.1)$$

Setting  $\varepsilon = 0$  in (2.1) yields an integrable system

$$\begin{aligned} \rho_{zz} - \rho \theta_z^2 + \rho - \rho^3 &= 0, \\ 2\rho_z \theta_z + \rho \theta_{zz} &= 0 \end{aligned} \quad (2.2)$$

with integrals

$$\Omega = \rho^2 \theta_z, \quad (2.3)$$

$$K = \rho_z^2 + \rho^2 - \frac{1}{2} \rho^4 + \frac{\Omega^2}{\rho^2}. \quad (2.4)$$

It is natural to consider  $\Omega = \Omega(z)$  as a slow variable of the perturbed system. With  $V = \rho_z$  we derive from (2.1) the perturbed equations

$$\begin{aligned} \rho_z &= V, \\ V_z &= -\rho + \rho^3 + \frac{\Omega^2}{\rho^3} + \frac{a\varepsilon^2}{1 + \varepsilon^2 a^2} \rho [(a - w) - \rho^2(a - b)], \\ \Omega_z &= \frac{\varepsilon}{1 + \varepsilon^2 a^2} \rho^2 [(a - w) - \rho^2(a - b)]. \end{aligned} \quad (2.5)$$

### 2.2. The unperturbed system

It is useful to investigate more thoroughly the unperturbed, integrable system before one begins with the study of perturbed system (2.5),

$$\begin{aligned} \rho_z &= V, \\ V_z &= -\rho + \rho^3 + \frac{\Omega^2}{\rho^3}, \\ \Omega_z &= 0. \end{aligned} \quad (2.6)$$

System (2.6) has periodic solutions in a  $\Omega = \Omega_0$  plane for  $|\Omega_0| < \sqrt{4/27}$ . All solutions are unbounded for

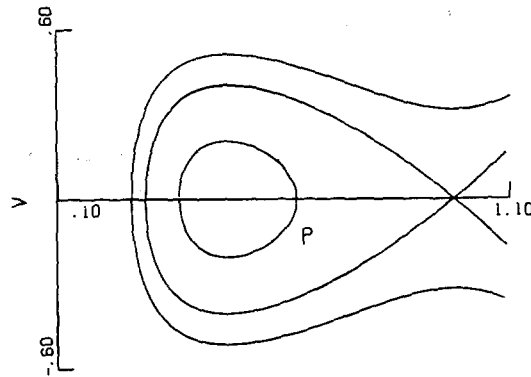


Fig. 1. The phase portrait of (2.6) in a  $\Omega = \Omega_0$  plane,  $0 < |\Omega_0| < \sqrt{4/27}$

$\Omega_0 > \sqrt{4/27}$ . For  $\Omega \neq 0$  the phase portrait in the  $\rho, V$  plane is sketched in fig 1: for  $\rho > 0$  there are two critical points:  $\rho_1(\Omega_0)$ , a center, inside the homoclinic saddle connection of saddle point  $\rho_2(\Omega_0)$ .

We define  $K_i(\Omega_0)$  as the value of the second integral  $K$  in the critical point  $\rho_i(\Omega_0)$ ,  $i = 1, 2$ . The periodic orbits in the  $\Omega_0$  plane correspond to values of  $K$  between  $K_1(\Omega_0)$  and  $K_2(\Omega_0)$  ( $K_1 < K_2$  for all  $\Omega_0$ ), these periodic solutions correspond to (families of) solutions  $\phi(z, t)$  of (1.4), quasiperiodic in  $z$ , slowly periodic in  $t$ . Solutions with values of  $K \notin [K_1, K_2]$  are unbounded. One easily obtains

$$0 \leq \rho_1(\Omega_0) \leq \sqrt{2/3} \leq \rho_2(\Omega_0) \leq 1$$

and

$$\begin{aligned} \lim_{|\Omega| \uparrow \sqrt{4/27}} \rho_i(\Omega) &= \sqrt{2/3}, & i = 1, 2, \\ \lim_{\Omega \rightarrow 0} \rho_i(\Omega) &= 0, & i = 1, \\ &= 1 & i = 2. \end{aligned}$$

Fig. 1 degenerates as  $\Omega_0$  becomes equal to zero (see fig. 2):  $\rho(z)$  changes sign in the  $\Omega_0 = 0$  plane. To compare between the degenerated and nondegenerated case we have sketched the phase portraits of (2.6) in the  $\Omega = 0$  plane and in a  $\Omega \neq 0, |\Omega| \ll 1$  plane in figs. 2a, 2b.

Solutions in the  $\Omega = 0$  plane correspond to periodic solutions of (1.4) (with  $\varepsilon = 0$ ) of the form  $a(z)e^{i\alpha}$  with  $a: \mathbb{R} \rightarrow \mathbb{R}$ , solution of  $a'' + a - a^3 = 0$  and  $\alpha$  a constant ( $\in \mathbb{R}$ ). These solutions oscillate through the origin on a line segment in the complex plane (see fig. 4 below).

The integral values  $(K, \Omega)$  of periodic orbits of (2.6) form a bounded region  $E$  in  $(K, \Omega)$  space (symmetrical with respect to the  $K$  axis). The boundaries of  $E, \partial E_i$ , consist of points corresponding to the critical points  $\rho_i(\Omega), i = 1, 2$ :

$$\partial E_i = \left\{ (K, \Omega) : \Omega^2 = Y^2 - Y^3, K = 2Y - \frac{3}{2}Y^2, Y = \rho_i^2 \right\}, \quad i = 1, 2.$$

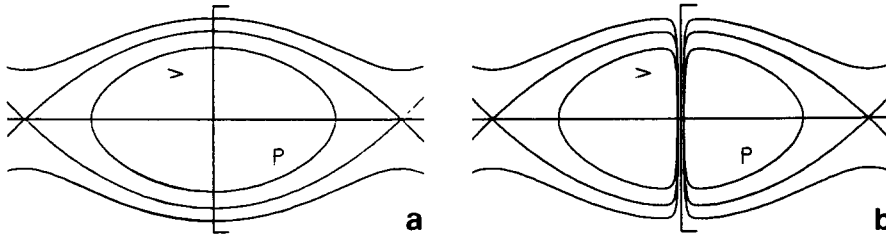


Fig. 2. The phase portraits of (2.6) in (a) the  $\Omega = 0$  and in (b) a  $\Omega \neq 0$ ,  $|\Omega| \ll 1$  plane.

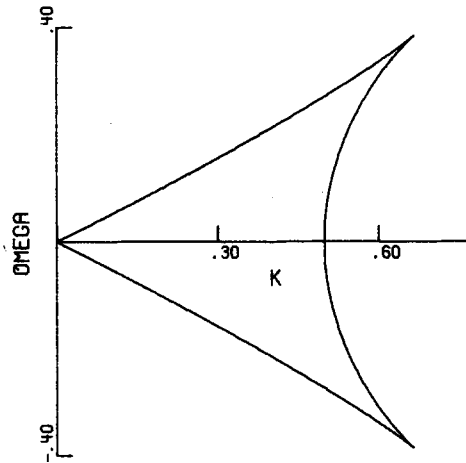


Fig. 3. Region  $E$ .

The left boundary of  $E$ ,  $\partial E_1$ , consists of center points, the right boundary,  $\partial E_2$ , consists of the homoclinic loops, see fig. 3.

Region  $E$ , in  $(K, \Omega)$  space, corresponds to a volume  $S$  in  $(\rho, V, \Omega)$  space, which boundary  $\partial S$  is given by the family of homoclinic loops. Inside  $S$  all solutions are periodic.

### 2.3. Basic properties of the perturbed system

Due to the special character of the perturbation, system (2.5) has some important, basic features in common with the unperturbed system:

- (I) The flow induced by (2.5) is volume preserving.
- (II) The flow is invariant under symmetry transformations:

$$z \rightarrow -z, \quad V \rightarrow -V, \quad \Omega \rightarrow -\Omega$$

and

$$\rho \rightarrow -\rho, \quad V \rightarrow -V.$$

(III) Critical points of (2.5), if they exist, are also critical points of the unperturbed equations. The position of a critical point does not depend on  $\epsilon$ .

The existence of critical points depends on the values of the parameters in the problem: let  $(\pm \rho_*, 0, \pm \Omega_*)$ ,  $\rho_*, \Omega_* > 0$  be a critical point, then

$$\rho_*^2 = \frac{a-w}{a-b}, \quad \Omega_*^2 = \rho_*^4 - \rho_*^6. \tag{2.7}$$

Thus, for  $a, b$  and  $w$  such that  $a < w < b$  or  $b < w < a$  system (2.5) has critical points.

$\Omega_z$  does not change sign inside the region  $S$  of unperturbed periodic solutions for  $w$  not between  $a$  and  $b$ , hence all solutions are unbounded. The situation  $b < a$  can be transformed to the situation  $a < b$  such that only the  $\mathcal{O}(\varepsilon^2)$  part of  $V_z$  changes sign, this does not influence the analysis. Thus we focus our attention to the case  $a < w < b$ . (Remark that the perturbation disappears for  $a = w = b$ .)

Critical point  $(\rho_*, 0, \Omega_*)$  is also a critical point of the unperturbed system, hence it is either a perturbed center of a perturbed saddle:

- $\frac{1}{3}(a + 2b) < w < b$ : perturbed saddle: a one-dimensional unstable manifold  $\Gamma_u^+$ ,  
a two-dimensional stable manifold  $\Gamma_s^+$ ,
- $a < w < \frac{1}{3}(a + 2b)$ : perturbed center: a (slow) one-dimensional unstable manifold  $\Gamma_u^+$ ,  
a two-dimensional stable manifold  $\Gamma_s^+$ .

Our main tool to handle the flow of (2.5) will be the Poincaré map  $P$ , a return map which is only defined close to periodic solutions of the unperturbed system. Due to the fact that system (2.5) is a perturbation of an integrable system we are able to compute this map  $P$  accurate up to  $\mathcal{O}(\varepsilon)$ .

Integrals  $K$  and  $\Omega$  of the unperturbed system (see (2.3), (2.4)) are slow variables of the perturbed system:  $K_z, \Omega_z = \mathcal{O}(\varepsilon)$ . Solutions of (2.5) with initial data  $K_0, \Omega_0$  not close ( $= \mathcal{O}(\varepsilon)$ ) to  $E$  will become unbounded: all (possible) bounded solutions of (2.5) have to remain near the periodic solutions of the unperturbed system. Hence we can describe the dynamics of possible bounded solutions of (2.5) by a Poincaré map  $P$  defined on a  $\mathcal{O}(\varepsilon)$  neighbourhood of  $E, E_\varepsilon$ . Let  $(K_0, \Omega_0) \in E_\varepsilon$ , consider  $\Gamma_\varepsilon(z)$  the solution of (2.5) with initial values  $V(0) = 0, \Omega(0) = \Omega_0$  and  $\rho(0)$  such that  $K(0) = K_0$ . Let  $(\bar{\rho}, 0, \bar{\Omega})$  be the next intersection point of  $\Gamma_\varepsilon$  with the  $V = 0$  plane with  $dV/dz < 0$  (if such a point exists). Then

$$P(K_0, \Omega_0) = (\bar{K}(\bar{\rho}, 0, \bar{\Omega}), \bar{\Omega}) = (\bar{\rho}^2 - \frac{1}{2}\bar{\rho}^4 + \bar{\Omega}^2/\bar{\rho}^2, \bar{\Omega}).$$

Define  $\Delta K(K_0, \Omega_0)$  and  $\Delta \Omega(K_0, \Omega_0)$  by

$$P(K_0, \Omega_0) = (\bar{K}, \bar{\Omega}) = (K_0 + \Delta K(K_0, \Omega_0), \Omega_0 + \Delta \Omega(K_0, \Omega_0)).$$

The expressions  $\Delta K$  and  $\Delta \Omega$  can be calculated up to  $\mathcal{O}(\varepsilon)$ ,

$$\Delta K(K_0, \Omega_0) = \int_0^{Z_\varepsilon(K_0, \Omega_0)} K_z(\Gamma_\varepsilon(z)) dz \tag{2.8}$$

with  $Z_\varepsilon(K_0, \Omega_0)$  the return time of  $\Gamma_\varepsilon$  (“time =  $z$ ”). Using (2.4) we derive

$$K_z = -2\varepsilon(b-a)(\rho_*^2 - \rho^2)\Omega + \mathcal{O}(\varepsilon^2) \tag{2.9}$$

with  $\rho_*^2$  as in (2.7). Let  $\Gamma_0(z) = (\rho_0(z), V_0(z), \Omega_0)$  be the periodic solution of the unperturbed system (2.6) with the same initial data as  $\Gamma_\varepsilon(z)$  and period  $Z_0(K_0, \Omega_0)$  (the initial data have to be adapted a little



(=  $\mathcal{O}(\varepsilon)$ ) for  $(K_0, \Omega_0) \in E_\varepsilon - E$ ). Since  $|\Gamma_0 - \Gamma_\varepsilon| = \mathcal{O}(\varepsilon)$  on  $\mathcal{O}(1)$  timescale, we obtain from (2.8) and (2.9)

$$\Delta K(K_0, \Omega_0) = -2\varepsilon(b-a)\Omega_0 \int_0^{Z_0(K_0, \Omega_0)} [\rho_\star^2 - \rho_0^2(z)] dz + \mathcal{O}(\varepsilon^2). \tag{2.10}$$

The orbit  $(\rho_0(z), V_0(z))$  in the  $\Omega_0$  plane is described by the  $K$  integral (2.4), it intersects the  $\rho$  axis ( $V = \rho_z = 0$ ) in two points  $0 < \hat{\rho}(K_0, \Omega_0) < \check{\rho}(K_0, \Omega_0)$ , see fig. 1. Hence we find, setting  $\rho^2 = R$  (and  $\hat{\rho}^2 = \hat{R}$ ,  $\check{\rho}^2 = \check{R}$ )

$$\Delta K(K_0, \Omega_0) = -4\varepsilon(b-a)\Omega_0 \int_{\hat{R}(K_0, \Omega_0)}^{\check{R}(K_0, \Omega_0)} \frac{\rho_\star^2 - R}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}} dR + \mathcal{O}(\varepsilon^2). \tag{2.11}$$

(Remark that  $\Delta K = \mathcal{O}(\varepsilon^2)$  for  $\Omega_0 = \mathcal{O}(\varepsilon)$ .) Analogously we derive

$$\Delta \Omega(K_0, \Omega_0) = -2\varepsilon(b-a) \int_{\hat{R}(K_0, \Omega_0)}^{\check{R}(K_0, \Omega_0)} \frac{R(\rho_\star^2 - R)}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}} dR + \mathcal{O}(\varepsilon^2). \tag{2.12}$$

### 3. Periodic orbits

First we state an important observation. Due to symmetry we only consider  $\Omega \geq 0$ .

*Property 3.1.* ( $\Omega > 0$ ) if  $\Delta K(K_0, \Omega_0) = 0$  then  $\Delta \Omega(K_0, \Omega_0) > 0$ ,  
if  $\Delta \Omega(K_0, \Omega_0) = 0$  then  $\Delta K(K_0, \Omega_0) < 0$ .

This result can be obtained by combining the expressions of  $\Omega_z$  and  $K_z$  (see (2.9)):

$$\frac{d}{dz}(\Omega^2 - \rho_\star^2 K) = 2\varepsilon(b-a)(\rho_\star^2 - \rho^2)^2 \Omega + \mathcal{O}(\varepsilon^2). \tag{3.1}$$

Hence:  $2\Omega \Delta \Omega - \rho_\star^2 \Delta K > 0$  for  $\Omega > 0$ ,  $\Omega \neq \mathcal{O}(\varepsilon)$ . Close to  $\Omega = 0$ ,  $\Omega = \mathcal{O}(\varepsilon)$ , one needs to be more careful: rescaling  $\Omega$  leads to an expression which also satisfies property 3.1 for  $\Omega \neq 0$ .

Property 3.1 expresses that for  $\Omega_0 \neq 0$   $\Delta K = \Delta \Omega = 0$  is impossible. Hence Poincaré map  $P$  has no fixed points for  $\Omega \neq 0$ , and thus, since fixed points would have corresponded to quasiperiodic solutions  $\phi$  of (1.4), for  $\Omega \neq 0$ , all quasiperiodic solutions break open due to the perturbation.

It should be remarked that it is easy to show that for every value of  $\Omega_0$ ,  $|\Omega_0| < \Omega_\star$  (see (2.7)) there exist values  $K_\Omega$  and  $K_K$  such that

$$\Delta \Omega(K_\Omega, \Omega_0) = 0, \quad \Delta K(K_K, \Omega_0) = 0.$$

(Consider  $K$  close to  $K_1(\Omega_0)$ , i.e.  $\hat{\rho}, \check{\rho}$  close to the centerpoint  $\rho_1(\Omega_0)$ , then  $\Delta K, \Delta \Omega < 0$  ( $\rho_1(\Omega_0) < \rho_\star$ ); for  $K$  in the neighbourhood of  $K_2(\Omega_0)$  one deduces that  $\Delta K, \Delta \Omega > 0$ .)

Thus, also for  $\Omega = 0$ , there is a  $K_\Omega$  such that expressions (2.11) and (2.12) are equal to  $\mathcal{O}(\varepsilon^2)$  in  $(K_\Omega, 0)$ . One has to be careful in concluding that this is a fixed point of  $P$ . The point  $(K_\Omega, 0)$  cannot correspond to a perturbation of a nondegenerated periodic solution of the unperturbed system (as sketched in fig. 2b):

this type of periodic orbits has to intersect the positive  $\rho$  axis (in the  $\Omega = 0$  plane) twice, due to the symmetry  $z \rightarrow -z$ ,  $V \rightarrow -V$ ,  $\Omega \rightarrow -\Omega$ . One can see from fig. 2a that a solution of (2.5) can only cross the positive  $\rho$  axis (in the  $\Omega = 0$  plane) with  $V_z < 0$ . Hence a periodic solution which intersects the positive  $\rho$  axis twice is impossible. Remark that this type of periodic solutions corresponds to the quasiperiodic solutions of the Ginzburg–Landau equation. The point  $(K_\rho, 0)$  corresponds to a solution which crosses the  $V$  axis (i.e.  $\rho = 0 \cap \Omega = 0$ ): expressions (2.11) and (2.12) describe the evolution of a solution of (2.5) close to a degenerated periodic solution of the unperturbed system for *half a period* (see again fig. 2a). A solution which starts at the  $\rho$  axis and intersects the  $V$  axis is, due to the symmetries of the system, periodic.

**Theorem 3.2.** The perturbation causes the disappearance of periodic solutions of (2.6), corresponding to quasiperiodic solutions of the Ginzburg–Landau equation.

For every value of  $w \in (a, b)$ , one of the degenerated periodic solutions of the unperturbed system survives the perturbation. Thus there exist solutions of the Ginzburg–Landau equation with small complex coefficients, which are slowly periodic in time and space periodic, not of the type  $\text{Re}^{i(kx - wt)}$  (see fig. 4).

**Remark 1.** It is clear that the flow induced by (2.5) can only skip from the  $\rho > 0$  halfspace to the  $\rho < 0$  halfspace through the  $V$  axis, the intersection of the  $\rho = 0$  and  $\Omega = 0$  planes. One can write down the beginning of a power series development of the solution  $\Gamma_A(z)$  with initial data  $(0, A, 0)$ , parametrized by  $\Omega(z) = -\varepsilon s(z)$  (for  $z > 0$ :  $\Omega(z) < 0$ ):

$$\begin{aligned}\Omega(z) &= -\varepsilon s, \\ \rho(z) &= \left(\frac{3A}{w-a}\right)^{1/3} s^{1/3} + \mathcal{O}(s^{2/3}, \varepsilon^2), \\ V(z) &= A - \frac{1}{2A} \left(\frac{3A}{w-a}\right)^{2/3} s^{2/3} + \mathcal{O}(s^{2/3}, \varepsilon^2).\end{aligned}\tag{3.2}$$

Remark that  $\theta_z = \Omega/\rho^2$  is well defined as  $s$  passes through 0. Using shooting arguments one can (also) show that there has to be a degenerated solution of the unperturbed system which does not break open under the perturbation: for  $A$  small  $\Gamma_A$  intersects the  $V = 0$  plane in a point with negative  $\Omega$  coordinate (since  $\Omega_z < 0$  for  $\rho < \rho_*$ ), for  $A$  close to  $\frac{1}{2}\sqrt{2}$ , i.e. close to the homoclinic loop (fig. 2a)  $\Gamma_A$  crosses the  $V = 0$  plane in a point with positive  $\Omega$  coordinate. There has to be a value  $A_*$  in between such that  $\Gamma_{A_*}$  crosses the  $\rho$  axis in the  $V = 0$  plane.

**Remark 2.** The perturbed periodic orbit corresponds to a (family of) solution(s)  $\phi = \rho(z) e^{i[\theta(z) - \varepsilon wt] + i\alpha}$  with  $\rho(z)$  periodic and  $\theta_z = \Omega(z)/\rho^2(z)$ . The average of  $\theta_z$  over one period is equal to zero, hence  $\theta(z)$  is also periodic with the same period as  $\rho$ . So, the solution  $\phi$  of the Ginzburg–Landau equation with small complex coefficients is also space periodic. As the degenerated periodic solution of the Ginzburg–Landau equation with real coefficients:  $\phi$  oscillates through the origin of the complex plane (see figs. 4a, 4b).

**Remark 3.**  $w$  may not be chosen  $\mathcal{O}(\varepsilon)$  close to  $a$  or  $b$  (then approximations (2.11) and (2.12) may become invalid, see section 4).

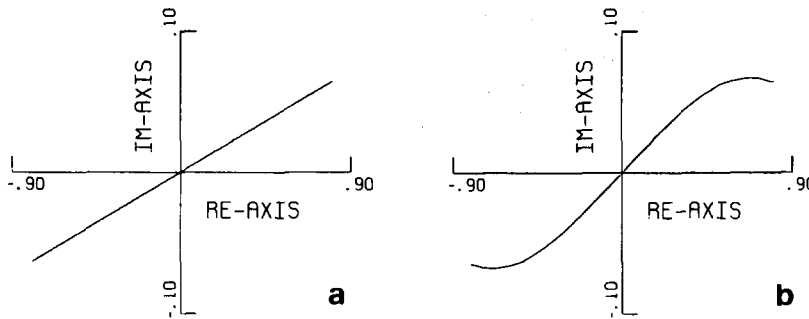


Fig. 4. (a) The line segment on which a degenerated periodic solution oscillates (unperturbed). (b) The space-dependent part (i.e.  $\rho(z)e^{i\theta(z)}$ ) of a perturbed periodic solution.

**Remark 4.** Holmes studied in ref. [8] a system similar to (2.5): only the signs in front of the  $\rho$  and  $\rho^3$  terms in the  $V_z$  expression of the unperturbed problem (2.6) are reversed (then for all  $\Omega \neq 0$  all solutions of the unperturbed problem are periodic in the  $\Omega$  is fixed plane). Holmes studied this system by means of the variables  $R = \rho^2$ ,  $v = V/\rho$  and  $m = \Omega$ . Due to the denominator  $\rho$  in  $v$  the  $v$  coordinate of a solution will tend to  $\infty$  as  $\rho$  decreases towards zero. Thus, due to the choice of variables (and the subtleties of the flow close to the  $m = \Omega = 0$  plane), Holmes did not find solutions which cross the  $\rho = 0$  plane. One can show, using variables  $\rho$ ,  $V$  and  $\Omega$ , that solutions which cross the  $\rho = 0$  plane exist in the (perturbed) system studied in ref. [8]. Holmes did find quasiperiodic solutions of the perturbed nonlinear Schrödinger equation, these solutions correspond to degenerated points of “his” Poincaré map: the map is not defined close to these points (using variables  $R, v, m$ ). Hence the quasiperiodic solutions do not exist (for all values of the parameters), the “fixed points” correspond to degenerated periodic solutions which oscillate through the origin. Professor Holmes agrees with this amendment of his analysis.

**Remark 5.** In the analysis we focused on periodic orbits which are fixed points of map  $P$ . Of course it is still possible that (2.6) has periodic orbits which are periodic points of  $P$ . Numerical simulations show these periodic points, see section 5, fig. 6.

## 4. Heteroclinic orbits

### 4.1. Preliminaries

Eq. (2.5) has many features in common with the perturbed central-force problem studied by Kopell and Howard in ref. [14]. This system exhibits heteroclinic orbits for special (discrete) values of a free parameter. In this section we search for heteroclinic orbits connecting two critical points of system (2.5). Since the critical points correspond to the periodic orbits,

$$\phi(z, t) = \rho_* \exp \left[ i \left( \pm \frac{\Omega_*}{\rho_*^2} z - \varepsilon \omega t \right) \right],$$

the heteroclinic orbits of (2.5) connecting two critical points correspond to solutions of the Ginzburg–Landau equation which connect wave solutions with reversed (spatial) wave number.

Again the analysis relies heavily on the Poincaré map  $P$  (defined inside  $S$ , the region of unperturbed periodic solutions). The critical point  $(\rho_*, 0, \pm \Omega_*)$  on  $\partial S$  corresponds with a degenerated point  $(K_*, \pm \Omega_*)$  of  $P$  on  $\partial E$ . One easily checks that

*Property 4.1.* For  $\frac{1}{3}(a + 2b) < w < b$ : the two-dimensional stable manifold  $\Gamma_s^+$  of  $(\rho_*, 0, \Omega_*)$  does not enter  $S$ .

Hence for  $w$  in this range we fix our attention to the one-dimensional unstable manifold  $\Omega_u^+$  which can be controlled by the following properties (these properties also hold for  $w \in (a, \frac{1}{3}(a + 2b))$ ).

Define in  $(\rho, V, \Omega)$  space ( $\rho \geq 0$ ) the volume  $S_*$ :  $S_* \subset S$ ,  $S_*$  consists of all periodic orbits of the unperturbed equation,  $\Gamma_0(z) = (\rho_0(z), V_0(z), \Omega_0)$ , which satisfy  $\rho_0(z) \leq \rho_*$  for all  $z$ . Region  $E_* \subset E$  corresponds with  $S_* \subset S$ . One easily deduces

*Property 4.2.* The linear approximation of  $\Gamma_u^+$  around  $(\rho_*, 0, \Omega_*)$  points into  $S_*$ .

And, using expressions (2.11) and (2.12):

*Property 4.3.* For  $(K_0, \Omega_0) \in E_*$ :  $\Delta\Omega(K_0, \Omega) < 0$ ,  $\Delta K(K_0, \Omega_0) < 0$ .

Analysing the flow of the perturbed system near the boundary of  $S_*$  one finds that the part of  $E_*$  with positive  $\Omega$  coordinates can only be left through the  $\Omega = 0$  plane:

*Property 4.4.* If  $(K_0, \Omega_0) \in \{E_* \cap \Omega > 0\}$  then either  $P(K_0, \Omega_0) \in \{E_* \cap \Omega > 0\}$  or  $P(K_0, \Omega_0) \in \{\Omega < 0\}$ .

Thus,  $\Gamma_u^+$  remains inside  $S_*$  as long as the  $\Omega$  coordinate remains positive; inside  $S_*$   $\Omega_z < 0$ :  $\Gamma_u^+$  can be handled by Poincaré map  $P$ . However, one needs to be very careful in using approximations (2.11) and (2.12) of  $\Delta K$  and  $\Delta\Omega$ :  $\rho_z$  and  $V_z$  are  $\mathcal{O}(\varepsilon)$  in a  $\mathcal{O}(\varepsilon)$  neighbourhood of a critical point of the unperturbed system. Hence  $\Omega_z$  is not small compared to  $\rho_z$  and  $V_z$ : one cannot approximate the solutions of (2.5) by solutions of the unperturbed system. This degeneration of the approximation can be illustrated by a special solution which was first noted by Hocking and Stewartson [7]. Hocking and Stewartson searched for solutions of the Ginzburg–Landau equation of the form

$$\phi(z, t) = \lambda L e^{i\nu t} [\operatorname{sech}(\lambda z)]^{1+iM} \quad (4.1)$$

for some values of  $\lambda$ ,  $L$ ,  $\nu$  and  $M$ . In our variables (4.1) transforms to

$$\rho(z) = P \operatorname{sech}(\lambda z) \quad (V(z) = \rho_z), \quad \Omega(z) = Q \sinh(\lambda z) \operatorname{sech}^3(\lambda z) \quad (4.2)$$

for some values of  $P$ ,  $Q$ ,  $\lambda$  and  $w$  (corresponding to  $\nu$ ). Substituting (4.2) into eq. (2.5) one observes that a solution of this type exists for

$$P = \pm 1 + \mathcal{O}(\varepsilon^2), \quad Q = \pm 1 + \mathcal{O}(\varepsilon^2), \quad \lambda = \pm \frac{1}{3}(b - a)\varepsilon + \mathcal{O}(\varepsilon^3), \quad w = \frac{1}{3}(a + 2b) + \mathcal{O}(\varepsilon^2).$$

Thus, for the critical value of  $w$  (i.e.  $\Omega_*(w) = \sqrt{4/27} + \mathcal{O}(\varepsilon^2)$ ) there exist homoclinic solutions connecting

one-dimensional stable and unstable manifolds of the degenerated critical point  $(0, 0, 0)$  of (2.5). In  $(K, \Omega)$  space this solution is precisely the boundary of  $E$  (accurate up to  $\mathcal{O}(\varepsilon^2)$ ): thus for all  $z$  the homoclinic loop remains  $\mathcal{O}(\varepsilon)$  close ( $V(z) = \mathcal{O}(\varepsilon)$ ) to the curve of critical points of the unperturbed system. This homoclinic orbit does not oscillate in  $(\rho, V, \Omega)$  space:  $V$  changes sign only ones. Thus, Poincaré map  $P$  is not (always) defined  $\mathcal{O}(\varepsilon)$  close to  $\partial E$ .

The following property enables us to control map  $P$  inside  $E_*$ .

*Property 4.5.* For  $(K_0, \Omega_0) \in E_*$ ,  $\Omega_0 > 0$

$$\frac{\rho_1^2(\Omega_0)}{2\Omega_0} \leq \left| \frac{\Delta\Omega(K_0, \Omega_0)}{\Delta K(K_0, \Omega_0)} \right| \leq \frac{\rho_*^2}{2\Omega_0}.$$

*Proof.* The inequality on the right is a direct consequence of (3.1). The inequality on the left follows from a similar observation:

$$2\Omega\Omega_z - \rho_1^2(\Omega) K_z = 2\varepsilon(b - a)(\rho_*^2 - \rho^2)[\rho_1^2(\Omega) - \rho^2] \tag{4.3}$$

$(\rho_1(\Omega))$  is the position of the centerpoint of the unperturbed equation in the  $\Omega$  plane). Hence property 4.5 is proven if, for all  $(K_0, \Omega_0) \in E_*$ :

$$\int_{\hat{R}(K_0, \Omega_0)}^{\check{R}(K_0, \Omega_0)} \frac{R - \rho_1^2(\Omega_0)}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}} dR \geq 0. \tag{4.4}$$

This inequality can be deduced from (4.3) by the approximation technique of section 2.3, using the fact that  $\Omega_z < 0$ ,  $K_z < 0$  and  $\rho_*^2 - \rho^2 \geq 0$  in  $S_*$ . Remark that  $R_1 = \rho_1^2(\Omega_0)$  is the minimum of  $F(R) = (\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2})^{-1}$ . Inequality (4.4) follows from

$$\int_{\hat{R}}^{\check{R}} \frac{R}{R_1} F(R) dR \geq \int_{\hat{R}}^{\check{R}} F(R) dR, \tag{4.5}$$

which, again, is a consequence of

$$\int_{\hat{R}}^{R_1} F(R) dR \leq \int_{R_1}^{\check{R}} F(R) dR. \tag{4.6}$$

One easily deduces:

$$F(\check{R} - S) > F(\hat{R} + S) \quad \text{for } S \in [0, \frac{1}{2}(\check{R} - \hat{R})],$$

$$R_1 < \frac{1}{2}(\check{R} - \hat{R}),$$

which yields (4.6), and thus (4.4) (via (4.5)).

The boundaries  $\partial E_i$  of  $E$  are described by  $d\Omega/dK = \rho_i^2(\Omega)/2\Omega$ ,  $i = 1, 2$ . Hence, due to property 4.5:

*Corollary 4.6.* Let  $(K_0, \Omega_0) \in E_*$ ,  $\Omega_0 > 0$ ,  $(K_0, \Omega_0)$  not  $\mathcal{O}(\varepsilon)$  close to  $\partial E_1$ , then: the image of  $(K_0, \Omega_0)$  under map  $P$ ,  $P(K_0, \Omega_0)$ , can be approximated using (2.11) and (2.12);  $P(K_0, \Omega_0)$  is not closer to  $\partial E_1$  than  $(K_0, \Omega_0)$ .

For  $\frac{1}{3}(a + 2b) < w < b$ , unstable manifold  $\Gamma_u^+$  is a perturbation of the unstable manifold of unperturbed saddle in the  $\Omega_*$  plane ( $(K_*, \Omega_*) \in \partial E_2$ ). Hence  $\Gamma_u^+$  can be studied using the approximations of  $P$  for all  $\Omega_*$  not  $\mathcal{O}(\varepsilon)$  close to  $\sqrt{4/27}$  (i.e.  $w$   $\mathcal{O}(\varepsilon)$  close to  $\frac{1}{3}(a + 2b)$ ). For  $a < w \leq \frac{1}{3}(a + 2b)$  ( $(K_*, \Omega_*) \in \partial E_1$ : averaging of system (2.5) in the neighbourhood of  $(\rho_*, 0, \Omega_*)$  shows that the solutions of (2.5) tend towards a nonoscillating orbit ( $\mathcal{O}(\varepsilon)$  close to the orbit of unperturbed centerpoints): map  $P$  cannot be used,  $\Gamma_u^+$  has a behaviour similar to the special orbit (4.2) found by Hocking and Stewartson (or remains  $\mathcal{O}(\varepsilon)$  close to such an orbit). Remark also that  $\Delta\Omega/\Delta K$  tends to  $\rho_1^2/2\Omega$  as  $(K_0, \Omega_0)$  tends towards  $\partial E_1$ .

Thus:

$$\frac{1}{3}(a + 2b) < w < b:$$

The one-dimensional unstable manifold  $\Gamma_u^+$  can be studied using map  $P$ . In section 4.2 we prove that for  $\varepsilon$  small enough there exist discrete values of  $w$  such that there is a heteroclinic orbit in system (2.5).

$$a < w < \frac{1}{3}(a + 2b):$$

We study the two-dimensional stable manifold  $\Gamma_s^+$ . In section 4.3 we show that there is an interval such that for all  $w$  inside the interval system (2.5) has a heteroclinic orbit.

#### 4.2. $\frac{1}{3}(a + 2b) < w < b$ : two types of heteroclinic orbits

Define

$$\gamma_u^+ = \{(K_i^+, \Omega_i^+)\}_{i \in I}, \quad I = 1, 2, \dots \quad (4.7)$$

with  $(K_1^+, \Omega_1^+) \in E_*$  the first intersection point of  $\Gamma_u^+$  with the  $V = 0$  plane (and  $V_z < 0$ ),  $(K_{i+1}^+, \Omega_{i+1}^+) = P(K_i^+, \Omega_i^+)$ . Remark that  $K_i^+, \Omega_i^+$  depend on  $\varepsilon$  and  $w$ . Likewise one defines  $\gamma_s^-$ , the set of points in  $E$  symmetric (in the  $K$  axis) to  $\gamma_u^+$ ;  $\gamma_s^-$  corresponds to the one-dimensional stable manifold  $\Gamma_s^-$  of  $(\rho_*, 0, -\Omega_*)$ . We search for values of  $w$  such that  $\Gamma_u^+ = \Gamma_s^-$ ; the symmetry  $z \rightarrow -z$ ,  $V \rightarrow -V$ ,  $\Omega \rightarrow -\Omega$  is essential in the analysis:

$$\Gamma_u^+ = \Gamma_s^- \Leftrightarrow \Gamma_u^+(z) \text{ intersects the } \Omega = 0 \text{ plane in a point on the } \rho \text{ axis.} \quad (4.8)$$

And, since  $\Gamma_u^+$  can only cross the  $\rho$  axis with  $V_z < 0$ , (4.8) yields

$$\Gamma_u^+ = \Gamma_s^- \Leftrightarrow \gamma_u^+ \cap \{\Omega = 0\} \neq \emptyset. \quad (4.9)$$

Define  $N^+ = N^+(\varepsilon, w)$  by: for  $i \leq N^+$   $\Omega_i^+ \geq 0$ ,  $\Omega_{N^++1}^+ < 0$ . This  $N^+$  exists due to the properties in section 4.1. Now (4.9) can be reformulated more precisely:

$$\Gamma_u^+ = \Gamma_s^- \Leftrightarrow \Omega_{N^+}^+ = 0. \quad (4.10)$$

First we consider  $w$  fixed,  $w$  not close to the boundaries of the interval  $(\frac{1}{3}(a + 2b), b)$ . We observe, using the properties of section 4.1 and approximation (2.12) of  $\Delta\Omega$ :

**Property 4.7.**  $N^+(\varepsilon, w) = \mathcal{O}(1/\varepsilon)$

Now fix an  $\varepsilon_0$  small enough, define  $N_0^+ = N^+(\varepsilon_0)$ . Due to property 4.7 here exists an  $\varepsilon_1 > \varepsilon_0$  such that  $\Omega^+(\varepsilon_1)_{N_0^+} < 0$  (“ $\gamma_u^+$  needs less steps to arrive at  $\Omega = 0$ ”). Hence there is an  $\varepsilon^*, \varepsilon_1 > \varepsilon^* > \varepsilon_0$  such that  $\Omega^+(\varepsilon^*)_{N_0^+} = 0$  (and thus  $N^+(\varepsilon^*) = N_0^+$ ). This  $\varepsilon^*$  is isolated since  $d\Delta\Omega/d\varepsilon \neq 0$ .

**Proposition 4.8.** There is a discrete set  $\{\varepsilon_j\}_{j=1}^\infty$ ,  $\varepsilon_j > \varepsilon_{j+1}$ ,  $\varepsilon_0$  small enough,  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow \infty$ , such that  $\Omega^+(\varepsilon_j)_{N^+(\varepsilon_j)} = 0$ . Thus for all  $\varepsilon_j$  there exists a heteroclinic orbit in (2.5), connecting  $(\rho_*, 0, \Omega_*)$  and  $(\rho_*, 0, -\Omega_*)$ .

Remark that these heteroclinic orbits remain in the  $\rho > 0$  halfspace for all  $z$ . A priori one would expect (see section 3) that there are also values of  $\varepsilon$  such that  $\Gamma_u^+(\varepsilon)$  crosses the  $\rho = 0$  plane and enters the  $\rho < 0$  halfspace.

Consider the heteroclinic orbit  $\Gamma_u^+(\varepsilon_j)$  for some  $j$ :  $\Omega_{N^+(\varepsilon_j)}^+ = 0$ ,  $\Omega_{N^+(\varepsilon_j)+1}^+ < 0$ . There is an  $\varepsilon_{j'} < \varepsilon_j$  such that  $N^+(\varepsilon_{j'}) \geq N^+(\varepsilon_j) + 1$ . Let  $\varepsilon$  decrease from  $\varepsilon_j$ . There has to be an  $\bar{\varepsilon} \in (\varepsilon_{j'}, \varepsilon_j)$  such that

$$\lim_{\varepsilon \downarrow \bar{\varepsilon}} \Omega^+(\varepsilon)_{N^+(\varepsilon)} = - \lim_{\varepsilon \downarrow \bar{\varepsilon}} \Omega^+(\varepsilon)_{N^+(\varepsilon)+1} < 0.$$

Define for  $\varepsilon \in (\bar{\varepsilon}, \varepsilon_j)$   $\Omega_{\text{CR}}(\varepsilon)$ : the  $\Omega$  coordinate of the first intersection of  $\Gamma_u^+(\varepsilon)$  with the  $V = 0$  plane (with  $V_z > 0$ ) after the intersection corresponding to  $\Omega_{N^+(\varepsilon)}^+$ . The symmetries of system (2.5) yield

$$\lim_{\varepsilon \downarrow \bar{\varepsilon}} \Omega_{\text{CR}}(\varepsilon) = 0.$$

Since  $V_z > 0$  in the point corresponding to  $\Omega_{\text{CR}}(\varepsilon)$  we conclude (see figs. 2a, 2b and section 3):  $\Gamma_u^+(\bar{\varepsilon})$  crosses the  $\rho = 0$  plane. Due to symmetry  $\rho \rightarrow -\rho$ ,  $\Omega \rightarrow -\Omega$ ,  $z \rightarrow -z$  we immediately observe:  $\Gamma_u^+(\bar{\varepsilon})$  connects  $(\rho_*, 0, \Omega_*)$  and  $(-\rho_*, 0, -\Omega_*)$ .

**Proposition 4.9.** There is a discrete set  $\{\bar{\varepsilon}_j\}_{j=1}^\infty$ ,  $\bar{\varepsilon}_{j+1} < \bar{\varepsilon}_j$ ,  $\bar{\varepsilon}_j \downarrow 0$  as  $j \rightarrow \infty$  such that  $\Gamma_u^+(\bar{\varepsilon}_j)$  is a heteroclinic orbit (of system (2.5)) connecting  $(\rho_*, 0, \Omega_*)$  and  $(-\rho_*, 0, -\Omega_*)$ . A solution of Ginzburg–Landau equation (1.4) corresponding to  $\Gamma_u^+(\bar{\varepsilon}_j)$  crosses once through the origin of the complex plane.

Using the following basic observation we are able to apply the techniques above to the more important case  $\varepsilon$  fixed,  $w$  (free) parameter.

**Property 4.10.** For  $w = b - \mathcal{O}(\varepsilon^\gamma)$ ,  $0 < \gamma < 1$ , and  $\varepsilon$  small enough:  $N^+(\varepsilon, w) = \mathcal{O}(\varepsilon^{\gamma-1})$ .

Thus we can handle the variation of  $N^+$  as a function of  $w$ , for  $\varepsilon$  fixed, and thus we can apply the reasoning above.

**Theorem 4.11.** For every  $\varepsilon$  small enough there are (at least) two values of the free parameter,  $w_0$  and  $w_1$ , such that for  $w = w_0$  system (2.5) has a heteroclinic orbit connecting  $(\rho_*, 0, \Omega_*)$  and  $(\rho_*, 0, -\Omega_*)$  and for  $w = w_1$  there is a heteroclinic orbit connecting  $(\rho_*, 0, \Omega_*)$  and  $(-\rho_*, 0, -\Omega_*)$ .

*Remark 1.* Define  $N(\varepsilon)$  as the number of  $w$ 's for which, for given  $\varepsilon$ , heteroclinic orbits exist. Property 4.10 yields:  $N(\varepsilon) \rightarrow \infty$  as  $\varepsilon \downarrow 0$ .

*Remark 2.* As was already remarked in section 1: the heteroclinic orbits correspond to one-parameter families of solutions of Ginzburg–Landau equation (1.4). For every  $\Phi_u^+(z, t)$  of such a family holds:

$$\lim_{z \rightarrow -\infty} \Phi_u^+ = \rho_* \exp[i(k_* z - \varepsilon w t + \varphi_0)],$$

$$\lim_{z \rightarrow +\infty} \Phi_u^+ = \rho_* \exp[i(-k_* z - \varepsilon w t + \varphi_0)],$$

for some  $\varphi_0$ , with  $k_* = \Omega_*/\rho_*^2$ . These periodic solutions are stable solutions of (1.4), stable against arbitrary spatial periodic perturbations, for  $w$  not  $\mathcal{O}(\varepsilon)$  close to  $\frac{1}{3}(a + 2b)$  or  $b$  (see refs. [5, 9]).

Thus solutions  $\Phi_u^+$  are solutions connecting two stable periodic, spatially counterrotating, patterns. One type of solutions  $\Phi_u^+$  crosses through the origin of the complex plane.

#### 4.3. $a < w < \frac{1}{3}(2b + a)$ : heteroclinic orbits for a range of $w$

We now fix our attention on the two-dimensional stable manifold  $\Gamma_s^+$  of  $(\rho_*, 0, \Omega_*)$ . Since  $(\rho_*, 0, \Omega_*)$  is a centerpoint we have  $\Gamma_s^+ \cap S \neq \emptyset$  ( $S \subset (\rho, V, \Omega)$  space defined in section 2.1). It is clear that if  $\Gamma_s^+$  intersects the  $\Omega = 0$  plane then, due to symmetry, there exists a heteroclinic orbit: the intersection of  $\Gamma_s^+$  and  $\Gamma_u^-, \Gamma_u^-$  is the two-dimensional unstable manifold of  $(\rho_*, 0, -\Omega_*)$ . Reasoning as in section 4.2 one concludes that, in this case, there exists also a heteroclinic orbit which crosses the  $\rho = 0$  plane.

In order to obtain a more transparent view of the behaviour of the flow induced by (2.5) we examine the special case  $w - a = \mathcal{O}(\varepsilon^\gamma)$ ,  $0 < \gamma < 1$ , i.e. we set  $\rho_*^2 = P_*^2 \varepsilon^\gamma$ ,  $P_*$  plays the role of free parameter. We scale  $\rho = \varepsilon^{\gamma/2} P$ ,  $V = \varepsilon^{\gamma/2} v$ ,  $\Omega = \varepsilon^\gamma W$ ,  $K = \varepsilon^\gamma k$ . Expressions (2.11) and (2.12) become, with  $s = P^2$

$$\Delta k(k, W) = 4\varepsilon^{1+\gamma}(b-a)W \int_s^{\bar{s}} \frac{s - P_*^2}{\sqrt{sk - s^2 - W^2 + \frac{1}{2}\varepsilon^\gamma s^3}} ds + \mathcal{O}(\varepsilon^{2+\gamma}) \quad (4.11)$$

$$\Delta W(k, W) = 2\varepsilon^{1+\gamma}(b-a) \int_s^{\bar{s}} \frac{s(s - P_*^2)}{\sqrt{sk - s^2 - W^2 + \frac{1}{2}\varepsilon^\gamma s^3}} ds + \mathcal{O}(\varepsilon^{2+\gamma}). \quad (4.12)$$

Approximations (4.11) and (4.12) can be computed, up to a certain accuracy, explicitly:

$$\Delta k(k, W) = \pi(b-a)\varepsilon^{1+\gamma}W(k - 2P_*^2) + o(\varepsilon^{1+\gamma}),$$

$$\Delta W(k, W) = \pi(b-a)\varepsilon^{1+\gamma}\left(\frac{3}{8}k^2 - \frac{1}{2}kP_*^2 - \frac{1}{2}W^2\right) + o(\varepsilon^{1+\gamma}).$$

*Remark that, as a consequence of the scaling, we have*

$$\partial E_* = \{(k, W) : W^2 = P_*^2 k - P_*^4 + \mathcal{O}(\varepsilon^\gamma)\},$$

$$\partial E_1 = \{(k, W) : W = \frac{1}{2}k + \mathcal{O}(\varepsilon^\gamma)\},$$

$$(k_*, W_*) = (2P_*^2 + \mathcal{O}(\varepsilon^\gamma), P_*^2 + \mathcal{O}(\varepsilon^\gamma)).$$

Now we can sketch a direction field for Poincaré map  $P$ , see fig. 5.



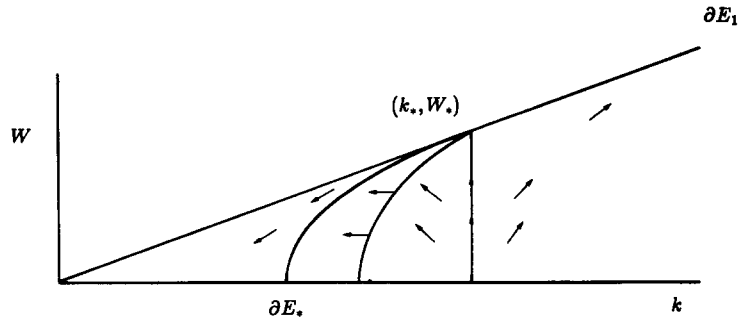


Fig. 5. The direction field induced by  $P$  (after scaling).

Reversing  $z$ ,  $z \rightarrow -z$ , shows (see fig. 5): the image of  $\Gamma_s^+$  in  $k, W$  space,  $\gamma_s^+$ , intersects the  $W = 0$  axis (it coincides, up to a certain accuracy with the line  $k = 2P_*^2$ ). For larger  $w$  a picture similar to fig. 5 can be produced. However, one needs to be careful:  $\gamma_s^+$  may leave  $E$  through  $\partial E_2$  for  $w$  close enough to  $\frac{1}{3}(a + 2b)$ .

**Theorem 4.12.** There is an interval  $(w_0, w_1) \subset (a, \frac{1}{3}(a + 2b))$  such that for every  $w \in (w_0, w_1)$  two heteroclinic orbits exist: one connecting  $(\rho_*, 0, \Omega_*)$  with  $(\rho_*, 0, -\Omega_*)$ , the other connecting  $(\rho_*, 0, \Omega_*)$  with  $(-\rho_*, 0, -\Omega_*)$ .

**Remark.** These heteroclinic orbits correspond to (families of) solutions of (1.4) which connect spatially counterrotating periodic solutions, which are unstable (see again refs. [5, 9]).

### 5. Some further remarks

If one wants to study the flow induced by (2.5) more thoroughly one will have to investigate expressions as  $d\Delta\Omega/dk$ ,  $d\Delta k/dK$  etc. These expressions are very hard to handle. Numerical simulations of (2.5) exhibit an interesting behaviour of map  $P$ , see figs. 6a, 6b.

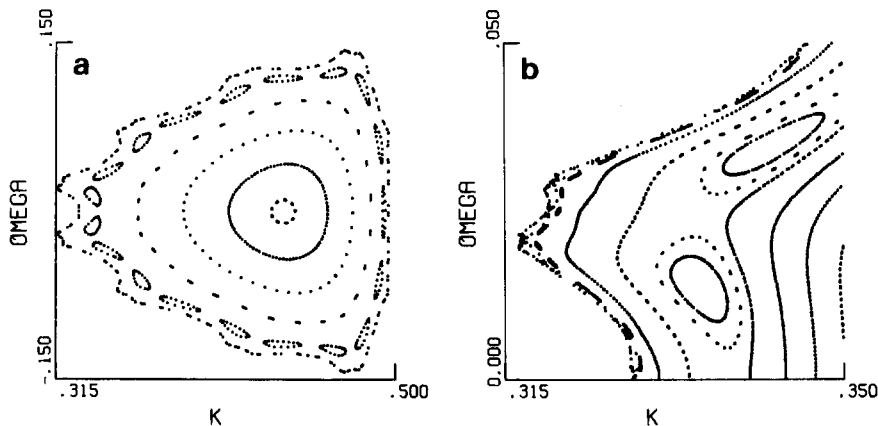


Fig. 6. Poincaré map  $P$  inside  $E$  for some choices of the initial data and  $a = -5$ ,  $b = -1$ ,  $w = -3$ ,  $\epsilon = 0.1$ . (b) is a magnification of a certain part of (a).

These figures show periodic points of  $P$  (with period  $\mathcal{O}(1/\varepsilon)$ ), invariant tori, and a thin strip of irregular behaviour: the Poincaré section of a perturbed integrable Hamiltonian system. However, the fact that solutions may cross through the  $\rho = 0$  plane and the special kind of the (non-Hamiltonian) perturbation make it hard to analyse the flow induced by (2.5) more deeply.

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