On the spectral decomposition of affine Hecke algebras

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ON THE SPECTRAL DECOMPOSITION OF AFFINE HECKE ALGEBRAS

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Abstract. An affine Hecke algebra $\mathcal{H}$ contains a large abelian subalgebra $\mathcal{A}$ spanned by Lusztig's basis elements $\theta_\lambda$, where $\lambda$ runs over the root lattice. The center $\mathcal{Z}$ of $\mathcal{H}$ is the subalgebra of Weyl group invariant elements in $\mathcal{A}$. The trace of the affine Hecke algebra can be written as an integral of a rational $n$ form (with values in the linear dual of $\mathcal{H}$) over a certain cycle in the algebraic torus $T = \text{spec}(\mathcal{A})$. We shall derive the Plancherel formula of the affine Hecke algebra by localization of this integral on a certain subset of $\text{spec}(\mathcal{Z})$.

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1. Introduction

In this paper I will discuss the spectral decomposition of affine Hecke algebras. It is the natural sequel to the paper [24], in which I made a basic study of the Eisenstein functionals of an affine Hecke algebra $\mathcal{H}$. These Eisenstein functionals are holomorphic functions of a spectral parameter $t \in T$, where $T$ is a complex $n$-dimensional algebraic torus naturally associated to $\mathcal{H}$. In [24], I derived a representation of “the” trace functional $\tau$ of the Hecke algebra, as the integral of the normalized Eisenstein functional times the holomorphic extension of the Haar measure of the compact form of $T$, against a certain “global $n$-cycle” (a coset of the compact form of $T$) in $T$. The kernel of this integral is a meromorphic $(n,0)$-form on $T$.

The present paper takes off from that starting point, and refines step-by-step the above basic complex function theoretic representation
formula for the tracial state $\tau$ of the Hecke algebra until we reach the level of the spectral decomposition formula for $\mathcal{H}$, or rather for $\mathcal{C}$, the $C^*$-algebra hull of the regular representation of $\mathcal{H}$. On the simpler level of the spherical or the anti-spherical subalgebra, a similar approach can be found in [21] and [12]. In the case of the spherical algebra one should of course also mention the classical work [20], although the point of view is different there, and based on analysis on a reductive $p$-adic group.

A natural incarnation of the affine Hecke algebra is the centralizer algebra of a special kind of induced representations of a $p$-adic reductive group $\mathcal{G}$, notably representations induced from a cuspidal unipotent representation of a parahoric subgroup (see [23] and [19]). The constituents of these induced representations are called unipotent representations by Lusztig [19]. In this context, the explicit decomposition of the trace $\tau$ in terms of irreducible characters of the affine Hecke algebra should be naturally equivalent to the disintegration of the corresponding induced representation of $\mathcal{G}$ as a subrepresentation of the left regular representation of $\mathcal{G}$. In this way one can view the restriction of the Plancherel measure of $\mathcal{G}$ to the collection of irreducible tempered unipotent representations as a juxtaposition of Plancherel measures coming from various Iwahori-Hecke algebras (see for instance [12]). This is by itself already sufficient motivation for this work.

In doing these computations, I have tried to maintain a fresh and personal approach to the problems that arise at various stages. I did not study in detail the enormous amount of classical material which is relevant to the problems at hand, such as the works of Harish Chandra [10], Arthur [1], [2], Langlands [16], Moeglin and Waldspurger [22], Van den Ban and Schlichtkrull [4] and many others, dealing with general principles of a similar nature, but in situations of greater complexity, where explicit answers are not even to be expected. In all, I hope that this has lead to a valuable approach, which may perhaps be useful even beyond the relatively simple context of the affine Hecke algebra. As a drawback, I may have re-invented the wheel at points, exhibiting ignorance and disregard for existing wisdom. I apologize beforehand to every specialist who is offended by this style.

As a result, the paper is lengthy but completely elementary and almost self contained. To understand this paper in detail one needs a decent education in Fourier analysis and the theory of distributions, and some additional basic knowledge of commutative ring theory and representation theory of linear algebraic groups and $C^*$-algebras. No reference is made to more advanced machinery from algebraic geometry, non-commutative geometry or K-theory. This represents, more than anything else, my personal limitations. On the other hand, it is
It is however clear to me that this “elementary approach” is insufficient to understand the complete story of the spectral theory of the affine Hecke algebra—thereby acknowledging the fact that my understanding is more advanced than my technical skills. In any case, the paper hopefully brings out clearly the distinction between what is elementary and well understood, what is elementary but not understood, and what is not at all elementary. It may even provide some hints as to what one should look for if one is well prepared to apply high powered equipment to the remaining issues.

The explicit spectral decomposition of $\mathcal{H}$ is completely reduced in this paper to the analogous problem for certain finite dimensional symmetric algebras. I called these the “residue Frobenius algebras” of the affine Hecke algebra. They are associated to the so-called residual points of the spectrum of the Bernstein center $Z$ of $\mathcal{H}$. The understanding of these residue Frobenius algebras themselves seems to be a problem of a different level. A fundamental approach to this problem should perhaps involve techniques like cyclic cohomology or $K$-theory of $C^*$-algebras, but this is mere speculation. I simply ignored this problem altogether in this paper, except for one important fact, saying that these residue algebras are invariant for “base change transformations”. This invariance property is perhaps the most important point of this paper.

It may be helpful to give the reader a rough outline of the story in this paper, and an indication of the guiding principles in the various stages.

(1). The definition of the Eisenstein functional of the affine Hecke algebra, in [24]. This is closely related to the study of intertwining operators in minimal principal series modules. It naturally leads to a representation of the trace $\tau$ of $\mathcal{H}$ as an integral of a certain rational kernel over a “global” cycle.

(2). The study of the locus of the singularities of the above kernel. It leads to the notion “residual coset”, analogous to the notion of residual subspace which was introduced in [11]. This collection of cosets can be classified, and from this classification we verify some important geometric properties of this collection.

(3). The study of the residues of the rational kernel for $\tau$. This involves a general (but basic) scheme for the calculation of multivariable residues. After symmetrization over the Weyl group, the result is a decomposition of $\tau$ as an integral of local tracial states against an
explicit probability measure on the spectrum $W_0 \backslash T$ of the Bernstein center $Z$ of $\mathcal{H}$. The main tools in this process are the positivity of $\tau$, and the geometric properties of the collection of residual cosets (2). I called this step the “localization of the trace $\tau$”.

(4). The local trace (as was mentioned in (3)) defined at an orbit $W_0 t \subset T$, arises as an integral of the Eisenstein kernel over a “local cycle” which is defined in an arbitrarily small neighborhood of the orbit $W_0 t$. This gives a natural extension of the local trace to localizations of the Hecke algebra itself (localization as a module over the sheaf of analytic functions on $W_0 \backslash T$).

(5). The analytic localization of the Hecke algebra has a remarkable structure discovered by Lusztig in [17]. This part of the paper is not self-contained, but draws heavily on the paper [17] (which is by the way itself completely elementary). By Lusztig’s wonderful structure theorem we can now investigate the local traces. We discover in this way that everything is organized in accordance with Harish-Chandra parabolic induction. The problem of finding the Plancherel measure for $\mathcal{H}$ reduces completely to the study of the finite dimensional residue Frobenius algebras defined at the 0-dimensional residual cosets in this way.

(6). If we assume that the root labels defining the Hecke algebra are all of the form $q^{n_\alpha}$ for certain non-negative integers $n_\alpha$, we prove that the residue Frobenius algebras are independent of $q \in \mathbb{R}_{>1}$. It proves that the set of irreducible discrete series representations of $\mathcal{H}$ associated with a central character $r$, all have a formal dimension which is proportional to the mass of $W_0 r$ with respect to the explicit probability measure on $W_0 \backslash T$ mentioned in (3), with a ratio of proportionality which is independent of $q$.

2. Preliminaries and description of results

The algebraic background for our analysis was discussed in the paper [24]. The main result of that paper is an inversion formula (see equation 4.1) which will be the starting point in this paper. The purpose of this section is to define the affine Hecke algebra $\mathcal{H}$ and to review the relevant notations and concepts involved in the above result. Moreover we introduce a $C^*$-algebras hull $\mathcal{C}$ of $\mathcal{H}$, which will be the main object of study in this paper. Finally we will give a more precise outline of the results in the paper. We refer the reader to [17] and [24] for a more systematic introduction of the basic algebraic notions.
2.1. The affine Weyl group and its root datum

A reduced root datum is a 5-tuple \( \mathcal{R} = (X, Y, R_0, R_0^\vee, F_0) \), where \( X \) and \( Y \) are free Abelian groups with perfect pairing \( \langle \cdot , \cdot \rangle \) over \( \mathbb{Z} \), \( R_0 \subset X \) is a reduced integral root system, \( R_0^\vee \subset Y \) is the dual root system of coroots of \( R_0 \), and \( F_0 \subset R_0 \) is a basis of fundamental roots. Each element \( \alpha \in R_0 \) determines a reflection \( s_\alpha \in GL(X) \) by
\[
s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha.
\]
The group \( W_0 \) in \( GL(X) \) generated by the \( s_\alpha \) is called the Weyl group. As is well known, this group is in fact generated by the set \( S_0 \) consisting of the reflections \( s_\alpha \) with \( \alpha \in F_0 \). The set \( S_0 \) is called the set of simple reflections in \( W_0 \).

By definition the affine Weyl group \( W \) associated with a reduced root datum \( \mathcal{R} \) is the group \( W = W_0 \ltimes X \). This group \( W \) naturally acts on the set \( X \). We can identify the set of integral affine linear functions on \( X \) with \( Y \times \mathbb{Z} \) via \( (y,k)(x) := \langle y, x \rangle + k \). It is clear that \( w \cdot f(x) := f(w^{-1}x) \) defines an action of \( W \) on \( Y \times \mathbb{Z} \). The affine root system is by definition the subset \( R = R_0^\vee \times \mathbb{Z} \subset Y \times \mathbb{Z} \). Notice that \( R \) is a \( W \)-invariant set in \( Y \times \mathbb{Z} \) containing the set of coroots \( R_0^\vee \). Every element \( a = (\alpha^\vee, k) \in R \) defines an affine reflection \( s_a \in W \), acting on \( X \) by
\[
s_a(x) = x - a(x)\alpha.
\]
The reflections \( s_a \) with \( a \in R \) generate a normal subgroup \( W_{aff} = W_0 \ltimes Q \) of \( W \), where \( Q \subset X \) denotes the root lattice \( Q = \mathbb{Z}R_0 \). We can choose a basis of fundamental affine roots \( F \) by
\[
F := \{ (\alpha^\vee, 1) \mid \alpha \in S^m \} \cup \{ (\alpha^\vee, 0) \mid \alpha \in F_0 \},
\]
where \( S^m \) consists of the set of minimal coroots with respect to the dominance ordering on \( Y \). It is easy to see that every affine root is an integral linear combination of elements from \( F \) with either all nonnegative or all nonpositive coefficients. The set \( R \) of affine roots is thus a disjoint union of the set of positive affine roots \( R_+ \) and the set of negative affine roots \( R_- \). The set \( S \) of simple reflections in \( W \) is by definition the set of reflections in \( W \) associated with the fundamental affine roots. They constitute a set of Coxeter generators for the normal subgroup \( W_{aff} \subset W \).

There exists an Abelian complement to \( W_{aff} \) in \( W \). This is best understood by introducing the important length function \( l \) on \( W \). The splitting \( R = R_+ \cup R_- \) described above implies that \( R_+ \cap s_a(R_-) = \{ a \} \) when \( a \in F \). Define, as usual, the length of an element \( w \in W \) by
\[
l(w) := |R_+ \cap w^{-1}(R_-)|.
\]
It follows that, when \( a \in F \),

\[
    l(s_aw) = \begin{cases} 
        l(w) + 1 & \text{if } w^{-1}(a) \in R_+ \\
        l(w) - 1 & \text{if } w^{-1}(a) \in R_- 
    \end{cases}
\]

For any \( w \in W \) we may therefore write \( w = \omega \hat{w} \) with \( \hat{w} \in W_{\text{aff}} \) and with \( l(\omega) = 0 \) (or equivalently, \( \omega(F) = F \)). This shows that the set \( \Omega \) of elements of length 0 is a subgroup of \( W \) which is complementary to the normal subgroup \( W_{\text{aff}} \). Hence \( \Omega \cong W/W_{\text{aff}} \cong X/Q \) is a finitely generated Abelian group. There is a natural map \( m : X \to P \) which restricts to the inclusion \( Q \subset P \) on \( Q \). If we write \( Z_X \subset X \) for its kernel, we have \( \Omega/Z_X = \Omega_f \) where \( Z_X \) is free and \( \Omega_f = m(X)/Q \subset P/Q \) is finite. It is easy to see that \( Z_X \) is the subgroup of elements in \( X \) that are central in \( W \). The finite group \( \Omega_f \) acts faithfully on \( S \) by diagram automorphisms.

2.2. Root labels

The second ingredient in the definition of \( \mathcal{H} \) is a function \( q \) on \( S \) with values in the group of invertible elements of a commutative ring, such that

\[
    q(s) = q(s') \text{ if } s \text{ and } s' \text{ are conjugate in } W.
\]

A function \( q \) on \( S \), satisfying 2.5, can clearly be extended uniquely to a length-multiplicative function on \( W \), also denoted by \( q \). By this we mean that the extension satisfies

\[
    q(sw) = q(w)q(s')
\]

whenever

\[
    l(sw) = l(w) + l(s').
\]

Conversely, every length multiplicative function on \( W \) restricts to a function on \( S \) that satisfies 2.5. Another way to capture the same information is by assigning labels \( q_a \) to the affine roots \( a \in R \). These labels are uniquely determined by the rules

\[
    \begin{array}{l}
    (i) \quad q_{wa} = q_a \quad \forall w \in W, \text{ and} \\
    (ii) \quad q(s_a) = q_{a+1} \quad \forall a \in F.
    \end{array}
\]

Note that a translation \( t_x \) acts on an affine root \( a = (\alpha, k) \) by \( t_x a = a - \alpha(x) \). Hence by \( (i) \), \( q_a = q_{\alpha} \), except when \( \alpha \in 2Y \), in which case \( q_a = q_{(\alpha, k, (mod2))} \). This last case occurs iff \( W \) contains direct factors which are isomorphic to the affine Coxeter group whose diagram equals \( C_n^{\text{aff}} \). In such a direct factor we find 3 conjugacy classes of reflections.
Yet another manner of labeling will play an important role. It involves a possibly non-reduced root system $R_{\text{nr}}$, which is defined by:

(2.9) \[ R_{\text{nr}} := R_0 \cup \{2\alpha \mid \alpha^\vee \in R_0^\vee \cap 2Y\}. \]

Now define labels for the roots $\alpha^\vee/2$ in $R_{\text{nr}}^\vee \setminus R_0^\vee$ by:

\[ q_{\alpha^\vee/2} := \frac{q_{1+\alpha^\vee}}{q_{\alpha^\vee}}. \]

This choice is natural, because it implies the formula

(2.10) \[ q(w) = \prod_{\alpha \in R_{\text{nr},+} \cap w^{-1}R_{\text{nr},-}} q_{\alpha^\vee}, \]

for all $w \in W_0$. We denote by $R_1$ the root system of long roots in $R_{\text{nr}}$. In other words

(2.11) \[ R_1 := \{\alpha \in R_{\text{nr}} \mid 2\alpha \not\in R_{\text{nr}}\}. \]

2.3. The Iwahori-Hecke algebra as a Hilbert algebra

Let $\mathcal{R}$ be a root datum, and $q = (q_s)_{s \in S}$ a set of root labels as in the previous subsection. We assume throughout in this paper that the labels are real numbers satisfying

(2.12) \[ q(s) > 1 \ \forall s \in S, \]

The following theorem is well known.

**Theorem 2.1.** There exists a unique complex associative algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ with $\mathbb{C}$-basis $(T_w)_{w \in W}$ which satisfy the following relations:

(a) If $l(wu') = l(w) + l(w')$ then $T_wT_{w'} = T_{wu'}$.

(b) If $s \in S$ then $(T_s + 1)(T_s - q(s)) = 0$.

The algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ is called the affine Hecke algebra (or Iwahori-Hecke algebra) associated to $(\mathcal{R}, q)$.

We equip the Hecke algebra $\mathcal{H}$ with an anti-linear anti-involutive $*$ operator defined by

$T_w^* = T_{w^{-1}}$.

In addition, we define a trace functional $\tau$ on $\mathcal{H}$, by means of $\tau(T_w) = \delta_{w,e}$. It is a well known basic fact that

$\tau(T_w^*T_{w'}) = \delta_{w,w'}q(w)$,

implying that $\tau$ is positive and central. Hence the formula

$(h_1, h_2) := \tau(h_1^*h_2)$,
defines an Hermitian inner product satisfying the following rules:

\[(2.13) \quad (i) \quad (h_1, h_2) = (h_1^* h_2^*), \]
\[\quad (ii) \quad (h_1 h_2, h_3) = (h_2, h_1^* h_3). \]

The basis $T_w$ is orthogonal for $\langle \cdot, \cdot \rangle$. We put

\[(2.14) \quad N_w := q(w)^{-1/2} T_w \]

for the orthonormal basis of $\mathcal{H}$ that is obtained from the orthogonal basis $T_w$ by scaling. Let us denote by $\lambda(h)$ and $\rho(h)$ the left and right multiplication operators on $\mathcal{H}$ by an element $h \in \mathcal{H}$. Let $\mathcal{H}$ be the Hilbert space obtained from $\mathcal{H}$ by completion; in other words, $\mathcal{H}$ is the Hilbert space with Hilbert basis $N_w$. Then $\lambda(h)$ and $\rho(h)$ extend uniquely to elements of $\mathcal{B}(\mathcal{H})$, the space of bounded operators on the Hilbert space $\mathcal{H}$.

The operator $\ast$ extends to an isometric involution on $\mathcal{H}$.

**Lemma 2.2.** We have $\| \lambda(h) \| = \| \rho(h) \|$, and for a simple reflection $s \in S$, $\| \lambda(N_s) \| = q(s)^{1/2}$.

**Proof.** For every $w$ such that $l(sw) > l(w)$, $\lambda(N_s)$ acts on the two dimensional subspace $V_w$ of $\mathcal{H}$ spanned by $N_w$ and $N_{sw}$ as a self-adjoint operator with eigenvalues $q(s)^{1/2}$ and $-q(s)^{-1/2}$. Since $\mathcal{H}$ is the Hilbert sum of the subspaces $V_w$, we see that the operator norm of $\lambda(N_s)$ equals $q(s)^{1/2}$. The fact that $\| \lambda(h) \| = \| \rho(h) \|$ follows from

\[
\| \lambda(h) \| = \| \lambda(h)^* \|
\]
\[= \| \lambda(h^*) \|
\]
\[= \| \rho(h) \|.
\]

\[\square\]

The above lemma shows that $\mathcal{H}$ has the structure of a Hilbert algebra in the sense of Dixmier [8]. Moreover, this Hilbert algebra is unital, and the Hermitian product is defined with respect to the trace $\tau$.

We define the operator norm $\| \cdot \|_o$ on $\mathcal{H}$ by $\| h \|_o := \| \lambda(h) \| = \| \rho(h) \|$. The closure of $\mathcal{H}$ with respect to $\| \cdot \|_o$ is called $\mathcal{C}$. Clearly, $\| h \|_o \geq \| h \|$, and thus we can identify $\mathcal{C}$ with a subset of $\mathcal{H}$. We can equip $\mathcal{C}$ with the structure of a $C^*$-algebra by the product $c_1 c_2 := \lambda(c_1)(c_2) = \rho(c_2)(c_1)$ and the $*$-operator coming from $\mathcal{H}$. We extend $\lambda$ (and $\rho$) to a faithful left (right) representation of $\mathcal{C}$ in the Hilbert space $\mathcal{H}$.

An element $a \in \mathcal{H}$ is called *bounded* if there exists an element $\lambda(a) \in \mathcal{B}(\mathcal{H})$ such that for all $h \in \mathcal{H}$,

\[(2.15) \quad \lambda(a)(h) = \rho(h)(a).
\]

By continuity we see that $\lambda(a)$ is uniquely determined by $a = \lambda(a)(1)$.
When $a \in \mathcal{H}$ is bounded, there also exists a unique $\rho(a)$ such that for all $h \in \mathcal{H}$,
\begin{equation}
\rho(a)(h) = \lambda(h)(a).
\end{equation}
It is obvious that the elements of $\mathcal{C}$ are bounded. Let us denote by $\mathcal{M} \subset \mathcal{H}$ the subspace of bounded elements. We equip $\mathcal{M}$ with the involutive algebra structure defined by the product $n_1n_2 := \lambda(n_1)(n_2) = \rho(n_2)(n_1)$ and the $*$ operator as before.

**Proposition 2.3.** 1. The subspace $\lambda(\mathcal{M}) := \{\lambda(a) \mid a \in \mathcal{M}\} \subset \mathcal{B}(\mathcal{H})$ is the Neumann algebra completion of $\lambda(\mathcal{H})$. In other words, $\lambda(\mathcal{M})$ is the closure of $\lambda(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ with respect to the strong topology (defined by the semi-norms $T \to \|T(x)\|$ with $x \in \mathcal{H}$). The analogous statements hold when we replace $\lambda$ by $\rho$.

The Neumann algebras $\lambda(\mathcal{M})$ and $\rho(\mathcal{M})$ are mutually centralizing.

2. An element $a \in \mathcal{H}$ is bounded exactly when it is the limit in $\mathcal{H}$ of a sequence $h_i \in \mathcal{H}$ such that the sequence $\|h_i\|_a$ is bounded. In this case, $\lambda(h_i)$ converges to $\lambda(a)$ in the strong topology (similarly, $\rho(h_i)$ converges strongly to $\rho(a)$).

**Proof.** All this can be found in [8], Chapitre I, paragraphe 5. In general, $\lambda(\mathcal{M})$ is a two-sided ideal of the Neumann algebra hull of $\lambda(\mathcal{H})$, but in the presence of the unit $1 \in \mathcal{H}$ the two spaces coincide. In fact, when $A \in \mathcal{B}(\mathcal{H})$ and $A$ is in the strong closure of $\lambda(\mathcal{H})$, it is simple to see that $A(1) \in \mathcal{H}$ is bounded.

The pre-Hilbert structure coming from $\mathcal{H}$ gives $\mathcal{M}$ itself the structure of a unital Hilbert algebra. The algebra $\mathcal{M}$ can and will be identified with its associated standard Neumann algebra $\lambda(\mathcal{M})$. In this situation, $\mathcal{M}$ is said to be a saturated Hilbert algebra (with unit element).

Let $\mathcal{H}^*$ denote the algebraic dual of $\mathcal{H}$, equipped with its weak topology. Notice that $\tau$ extends to $\mathcal{H}^*$ by the formula $\tau(\phi) := \phi(1)$. The $*$-operator can be extended to $\mathcal{H}^*$ by $\phi^*(h) := \phi(h^*)$. We have the following chain of inclusions:
\begin{equation}
\mathcal{H} \subset \mathcal{C} \subset \mathcal{M} \subset \mathcal{H} \subset \mathcal{H}^*.
\end{equation}

**Proposition 2.4.** The restriction of $\tau$ to $\mathcal{M}$ is central, positive and finite. It is the natural trace of the Hilbert algebra $\mathcal{M}$, in the sense that
\begin{equation}
\tau(a) = (b, b)
\end{equation}
for every positive $a \in \mathcal{M}$, and $b \in \mathcal{M}$ such that $a = b^2$.

**Proof.** A square root $b$ is in $\mathcal{M}$ and is Hermitian. Then $(b, b) = (1, a) = \tau(a)$.
\qed
Corollary 2.5. The Hilbert algebra $\mathcal{H}$ is finite.

2.4. The Plancherel measure

By a well known (unpublished) result of J. Bernstein (see [17]), $\mathcal{H}$ can be viewed as the product $\mathcal{H}_0 \mathcal{A}$ or $\mathcal{A} \mathcal{H}_0$ of an abelian subalgebra $\mathcal{A}$ (isomorphic to the group algebra of the lattice $X$), and the Hecke algebra $\mathcal{H}_0$ of the finite Weyl group $W_0$. Both product decompositions $\mathcal{H}_0 \mathcal{A}$ and $\mathcal{A} \mathcal{H}_0$ give a linear isomorphism of $\mathcal{H}$ with the tensor product $\mathcal{H}_0 \otimes \mathcal{A}$.

The relations between products in $\mathcal{H}_0 \mathcal{A}$ and in $\mathcal{A} \mathcal{H}_0$ are described by Lusztig’s relations (see for example [24], Theorem 1.10), and with the above additional description of the structure of $\mathcal{A}$ and $\mathcal{H}_0$ these give a complete presentation of $\mathcal{H}$.

The algebra $\mathcal{A}$ has a $\mathbb{C}$-basis of invertible elements $\theta_x$ (with $x \in X$) such that $x \to \theta_x$ is a monomorphism of $X$ into the group of invertible elements of $\mathcal{A}$. This basis is uniquely determined by the additional property that $\theta_x = N_{x^-}$ (see 2.14) when $x \in X^+$. As an important corollary of this presentation of $\mathcal{H}$, Bernstein identified the center $Z$ of $\mathcal{H}$ as the space $Z = \mathcal{A}^{W_0}$ of $W_0$-invariant elements in $\mathcal{A}$ (see [24], Theorem 1.11).

Proposition 2.6. The Hecke algebra $\mathcal{H}$ is finitely generated over its center $Z$. At a maximal ideal $m = m_t$ (with $t \in T$) of $Z$, the local rank equals $|W_0|^2$ if and only if the stabilizer group $W_t \subset W_0$ is generated by reflections.

Proof. It is clear that $\mathcal{H} \simeq \mathcal{H}_0 \circ \mathcal{A}$ is finitely generated over $Z = \mathcal{A}^{W_0}$. When $W_t$ is generated by reflections, it is easy to see that the rank of $m$-adic completion $\mathcal{A}_m$ over $\mathcal{Z}_m$ is exactly $|W_0|$ (see Proposition 2.23(4) of [24]).

Lemma 2.7. The algebra $\mathcal{C}$ is of finite type. In fact every irreducible representation of $\mathcal{C}$ has dimension at most equal to the order $|W_0|$ of the finite Weyl group. Furthermore, restriction to $\mathcal{H}$ induces a bijection from the set $\mathcal{E}$ onto the set $\mathcal{H}$ of irreducible representations of $\mathcal{H}$.

Proof. Let $\pi \in \mathcal{C}$ be a nonzero irreducible representation in a Hilbert space $H$. Then for every nonzero $x \in H$, $\pi(\mathcal{H})(x)$ is a dense subspace of $H$. Also, the center $Z$ of $\mathcal{H}$ acts by scalars in $H$. By Proposition 2.6 we see that the dimension of $\pi(\mathcal{H})$ is finite, and thus that $\pi(\mathcal{H}) = \text{End}(H)$. We can find a simultaneous eigenvector $v$ with eigenvalue $t \in T$ say, for $\pi(\mathcal{A})$. This induces an epimorphism $\phi : I_t \to H$, where $I_t$ denotes the minimal principal series module $I_t = \mathcal{H} \circ \mathcal{A} t$. Hence the dimension
of \( H \) is at most \(|W_0|\). Therefore \( \mathcal{C} \) is liminal in the terminology of \([9]\). In particular, \( \mathcal{C} \) has type I. \( \square \)

Since \( \mathcal{C} \) is separable, liminal and unital, the spectrum \( \hat{\mathcal{H}} \) is a compact \( T_1 \) space with countable base. Moreover it contains an open dense Hausdorff subset. Because \( \mathcal{C} \) is of finite type I and unital, every irreducible representation of \( \mathcal{C} \) has a continuous character.

The general theory of the central disintegration of separable, liminal \( C^* \)-algebras (see \([9]\), paragraphe 8.8), equipped with a tracial state \( \tau \) (i.e. \( \tau \) is a trace such that \( \tau(1) = 1 \)), asserts that there exists a unique Borel measure \( \mu_\pi \) on \( \mathcal{C} = \hat{\mathcal{H}} \) such that

\[
\mathfrak{N} \cong \int_{\hat{\mathcal{H}}} \End(V_\pi) d\mu_\pi(\pi)
\]

and such that

\[
\tau(h) = \int_{\hat{\mathcal{H}}} \Tr(\pi(h)) d\mu_\pi(\pi).
\]

The measure \( \mu_\pi \) is called the Plancherel measure.

The center \( \mathfrak{Z} \) of \( \mathfrak{M} \) will be mapped onto the algebra of diagonalizable operators \( L^\infty(\hat{\mathcal{H}}, \mu_\pi) \). This is an isomorphism of algebras, continuous when we give \( \mathfrak{Z} \) the weak operator topology and \( L^\infty(\hat{\mathcal{H}}, \mu_\pi) \) the weak topology of the dual of \( L^1(\hat{\mathcal{H}}, \mu_\pi) \). It is an isometry.

The general theory implies in addition that there exists a compact metrizable space \( Z \), with a “basic measure” \( \mu \), and a \( \mu \)-negligible space \( N \subset Z \), such that \( Z - N \) is isomorphic as a Borel space to the complement of a \( \mu_\pi \)-negligible subset of the support \( \Omega \) of \( \mu_\pi \). The isomorphism transforms \( \mu \) to a measure which is equivalent to \( \mu_\pi \).

There may exist several (but finitely many, by 2.6) inequivalent irreducible \(*\)-representations of \( \mathcal{H} \) with the same central character \( t \in \Spec(Z) \cong W_0 \setminus T \), where \( T \) is the “algebraic torus” \( T = \Hom(X, \mathbb{C}^\times) = \Spec(\mathcal{A}) \). We consider \( T \) with its analytic topology. Let us put \( p_\pi : \Omega \to W_0 \setminus T \) for the finite map which associates to each irreducible \(*\)-representation its central character. In fact there may exist non negligible subsets \( V \) of \( \Omega \) such that \( 1 < |p_\pi^{-1}(y)|(\leq \infty) \) for all \( y \in p_\pi(V) \). In view of the above remarks this implies in particular that the closure \( \overline{Z} \) of \( Z \) in \( \mathfrak{M} \) is strictly smaller than \( \mathfrak{Z} \), in general.

2.5. **Description of the main results**

It is our goal to describe the Plancherel measure of \( \mathcal{H} \) explicitly. We distinguish between the following steps. Recall that \( \Omega \) denotes the
support of $\mu_p$, and $p_z$ denotes the projection of $\Omega$ onto the spectrum $W_0 \setminus T$ of $Z$.

2.5.1. Describe the set $S := p_z(\Omega) \subset W_0 \setminus T$. The set $S := p_z(\Omega)$ turns out to be a finite union of cosets of compact subtori of the algebraic torus $T$, modulo the action of $W_0$. The complexifications of these cosets are called “residual cosets” of $T$. Conversely, each residual coset $L$ has a unique compact form which is a subset of $S$. This is called the “tempered form” of $L$, and is denoted by $L^{\text{temp}}$. The residual cosets will be described and classified in Section 3, and the proof that the support of the Plancherel measure is restricted to central characters in this set is given in Section 4.

2.5.2. Describe the map $p_z : \Omega \rightarrow S$. The fibers of $p_z$ are finite everywhere, and $p_z$ is continuous and open. For each tempered coset $L^{\text{temp}}$, put $S_L := W_0 \setminus W_0 L^{\text{temp}}$ and $\Gamma_L := p_z^{-1}(S_L)$. Then there exists an open dense subset $\Gamma'_L \subset \Gamma_L$ such that the restriction of $p_z$ to $\Gamma'_L$ is a trivial finite covering of its image. The description of the fibers of $p_z$ seems to be out of the scope of the methods used in this paper. As we will see in Theorem 5.17, the description of the generic fibers $p_z^{-1}(p_z(\pi))$ with $\pi \in \Gamma'_L$ reduces, by the usual method of unitary parabolic induction, to the case of the residual points (0-dimensional residual cosets) of parabolic subsystems of roots. The fiber at residual points $r \in T$ consists of a collection $\Pi_r$ of equivalence classes of irreducible discrete series representations with central character $r$. The further description of the special fibers boils down to the study of the reducibility of representations that are induced from discrete series representations of “parabolic subalgebras”, at special unitary induction parameters. Since this phenomenon of reducibility occurs only at a measure 0 subset of $\Omega$, this problem is less urgent for our purpose.

For special cases these problems were solved, using geometric methods. Kazhdan and Lusztig [15] solved the problem using equivariant K-theory in the case when the labels $q_i$ are equal and $X = P$. In the appendix Section 7 one can find an account of these results, and the relation with residual cosets. Lusztig [18] solved more general situations using equivariant homology.

2.5.3. Describe the Plancherel measure $\mu_F$. We first describe a “basic measure” $\mu_0$ on $\Omega$, such that $\mu_F$ is absolutely continuous relative to $\mu_0$. As was mentioned, $S$ is a finite union of the tempered residual cosets of $T$, modulo the action of $W_0$. Take the normalized Haar measure on each of the tempered residual cosets, take the push forward of this measure to $S$ and add these measures. Since $p_z$ is almost a homeomorphism on
each of the connected components of $\Omega$, we can use $p_2$ to lift this to a measure on $\Omega$. The measure $\mu_0$ is the measure thus obtained.

We remark that the tempered residual cosets are never nested, and thus there is no “embedded spectrum”. This fact is a fundamental property of residual cosets, but unfortunately our only proof of rests on the classification of residual subspaces for graded Hecke algebras [11]. See Section 3.1.

We will find explicitly a $W_0$ invariant, continuous, positive function $m_S$ on $S$ such that the Plancherel measure $\mu_P$ on $\Omega$ can be expressed as follows. For each connected component $\Gamma_{L,\delta}$ of $\Gamma_L$ there exists a constant $d_{L,\delta} \in \mathbb{R}_+$ such that

$$\mu_P|_{\Gamma_{L,\delta}} = d_{L,\delta} p_2^*(m_S) \mu_0$$

(see Section 5.3). The constants $d_{L,\delta}$ depend on the choice of $q$. We will show however (in Section 6) that the $d_{L,\delta}$ are invariant for “scaling” in the following sense. Assume that

$$q(s) = q^h \ \forall s \in S$$

for positive integer constants $f_s \in \mathbb{N}$, and $q \in \mathbb{R}_{>1}$ a parameter. Using the results of Lusztig [17] we will show that the support $\Omega_q$ of the Plancherel measure of the $C^*$- algebra $\mathcal{C}_q$ associated with $\mathcal{H}_q := \mathcal{H}(R, q)$ (with $q = (q(s))_{s \in S}$ as in 2.22) is independent of $q \in \mathbb{R}_{>1}$ up to homeomorphisms. The scaling invariance means that in formula 2.21 the constants $d_{L,\delta}$ will be independent of the parameter $q$.

This invariance is one of the most important points of our considerations, in view of the following application. Let $\mathcal{G}$ denote the group of points of the split adjoint simple algebraic group over a $p$-adic field $F$. We now denote by $q$ the cardinality of the residue field of $F$. The centralizer algebra of the induced representation from a cuspidal unipotent representation of a parahoric subgroup is an affine Hecke algebra with its labels given by 2.22 for suitable constants $f_s$. By the Plancherel formula of the affine Hecke algebra we can compute the formal dimension of a discrete series unipotent representation which arises as a summand in the induced module. The invariance implies that we determine this formal dimension as a function of $q$, up to the computation of the relevant constant $d_{L,\delta}$.

I would like to mention here that the constants $d_{L,\delta}$ have been computed explicitly in the cases where the Hecke algebra is of exceptional split adjoint type (i.e. the constants $f_s$ in 2.22 are all equal to 1, and $X = P$) by Mark Reeder [26]. He conjectures an interpretation (in the general split adjoint case) of the constants $d_{L,\delta}$ (see also [25]) in terms of the Kazhdan-Lusztig parameters of the corresponding irreducible
discrete series representation. In the exceptional cases he verifies this conjecture using a formula of Schneider and Stuhler [27] for the formal degree of a discrete series representation of an almost simple $p$-adic group which contains fixed vectors for the pro-unipotent radical $U$ of a maximal compact subgroup $K$. This formula of Schneider and Stuhler however is an alternating sum of terms which does not explain the product structure of the formal degrees. In addition, a clever algorithmic approach to the Schneider-Stuhler formula is necessary to fit the above case-by-case verifications into a computer.

3. Definition and classification of residual cosets

We fix once and for all a rational, positive definite, $W_0$-invariant symmetric form on $X$. This defines an isomorphism between $X \otimes \mathbb{Q}$ and $Y \otimes \mathbb{Q}$, and thus also a rational, positive definite symmetric form on $Y$. We extend this form to a positive definite Hermitian from on $\mathfrak{t}_\mathbb{C} := \text{Lie}(T) = Y \otimes \mathbb{C}$, where $T$ is the complex torus $T = \text{Hom}(X, \mathbb{C}^\times)$. Via the exponential covering map $\exp : \mathfrak{t}_\mathbb{C} \to T$ this determines a distance function on $T$.

Let $q$ be a set of root labels. If $2\alpha \not\in R_{\text{nr}}$ we formally put $q_{\alpha^{\vee}/2} = 1$, and always $q_{\alpha^{\vee}/2}$ denotes the positive root square root of $q_{\alpha^{\vee}/2}$. Let $L$ be a coset for a subtorus $T^L \subset T$ of $T$. Put $R_L := \{\alpha \in R_0 \mid \alpha(T^L) = 1\}$. This is a parabolic subsystem of $R_0$. The corresponding parabolic subgroup of $W_0$ is denoted by $W_L$. Define

$$R^0_L := \{\alpha \in R_L \mid \alpha(L) = -q^{1/2}_\alpha \text{ or } \alpha(L) = q^{1/2}_\alpha q^{\vee}_\alpha\}$$

and

$$R^\pm_L := \{\alpha \in R_L \mid \alpha(L) = \pm 1\}.$$

We write $R^0_L \supset R^\pm_L$ and $R^0_L \cap R^\pm_L = R^0_L \cap R^\pm_L$. We define an index $i_L$ by

$$i_L := |R^0_L| - |R^\pm_L|.$$

We give the following recursive definition of the notion residual coset.

**Definition 3.1.** A coset $L$ of a subtorus of $T$ is called residual if either $L = T$, or else if there exists a residual coset $M \supset L$ such that $\dim(M) = \dim(L) + 1$ and

$$i_L \geq i_M + 1.$$

Notice that the collection of residual cosets is closed for the action of the group of automorphisms of the root system preserving $q$ (in particular the elements of $W_0$, but also for example $-\text{Id}$).
Proposition 3.2. If $L$ is residual, then

(i) $R_{L}^{\text{ess}}$ spans a subspace $V_{L}$ of dimension $\text{codim}(L)$ in the $\mathbb{Q}$ vectorspace $V = X \otimes \mathbb{Q}$.

(ii) We have $R_{L} = V_{L} \cap R_{0}$, and the rank of $R_{L}$ equals $\text{codim}(L)$.

(iii) Put $LX := V_{L} \cap X$ and $X^{L} := X/LX$. Then $T^{L} = \{t \in T \mid x(t) = 1 \forall x \in LX\} = \text{Hom}(X^{L}, \mathbb{C}^{	imes}) = (T_{W_{L}}^{L})^{0}$.

(iv) Put $Y_{L} := Y \cap Q R_{L}^{v}$ and $LX := Y_{L}^{\perp} \cap X$. Let $X_{L} := X/LX$. We identify $R_{L}$ with its image in $X_{L}$. Let $F_{L}$ be the basis of $R_{L}$ such that $F_{L} \subset R_{0,+}$. Then $R_{L} := (X_{L}, Y_{L}, R_{L}, R_{L}^{v}, F_{L})$ is a root datum.

(v) Put $T_{L} := \text{Hom}(X_{L}, \mathbb{C}^{	imes}) \subset T$ (we identify $T_{L}$ with its canonical image in $T$). Then $T_{L}$ is the subtorus in $T$ perpendicular to $L$. Define $K_{L} := T^{L} \cap T_{L} = \text{Hom}(X/(LX), \mathbb{C}^{	imes}) \subset T$, a finite subgroup of $T$. The intersection $L \cap T_{L}$ is a $K_{L}$-coset consisting of residual points in $T_{L}$ with respect to the root datum $R_{L}$ and the root label $q_{L}$ obtained from $q$ by restriction to $R_{L,\text{nr}}^{v} \subset R_{\text{nr}}^{v}$. When $r_{L} \in T_{L} \cap L$, we have $L = r_{L}T^{L}$. Such $r_{L}$ is determined up to multiplication by elements of $K_{L}$.

Proof. By induction on $\text{codim}(L)$ we may assume that the assertions of (i) and (ii) hold true for $M$ in 3.4. From the definition we see that $R_{L}^{\text{ess}} \setminus R_{M}^{\text{ess}}$ is not the empty set. An element $\alpha$ of $R_{L}^{\text{ess}} \setminus R_{M}^{\text{ess}}$ can not be constant on $M$, and hence $\alpha \not\in R_{M} = V_{M} \cap R_{0}$. Thus

$$\dim(V_{L}) \geq \dim(V_{M}) + 1 = \text{codim}(M) + 1 = \text{codim}(L).$$

Since also

$$V_{L} \subset \text{Lie}(T^{L})^{\perp},$$

equality has to hold. Hence $R_{L} \subset V_{L}$ and $R_{L}$ spans $V_{L}$. Since $R_{L}$ is parabolic, we conclude that $R_{L} = V_{L} \cap R_{0}$. This proves (i) and (ii). The subgroup $\{t \in T \mid x(t) = 1 \forall x \in LX\} \subset T$ is isomorphic to $\text{Hom}(X^{L}, \mathbb{C}^{	imes})$, which is a torus because $X^{L}$ is free. By (ii) then, its dimension equals $\dim(T^{L})$. It contains $T^{L}$, hence is equal to $T^{L}$. It follows that $T^{L}$ is the connected component of the group of fixed points for $W_{L}$, proving (iii). The statements (iv) and (v) are trivial. 

For later reference we introduce the following notation. A residual coset $L$ determines a parabolic subsystem $R_{L} \subset R_{0}$, and we associated with this a root datum $R_{L}$. When $\Sigma \subset R_{0}$ is any root subsystem, not necessarily parabolic, we associate to $\Sigma$ two new root data, namely $R_{\Sigma}^{\Sigma} := (X, Y, \Sigma, \Sigma^{\perp}, F_{\Sigma})$ with $F_{\Sigma}$ determined by the requirement $F_{\Sigma} \subset R_{0,+}$, and $R_{\Sigma} := (X_{\Sigma}, Y_{\Sigma}, \Sigma, \Sigma^{\perp}, F_{\Sigma})$ where the lattice $X \rightarrow X_{\Sigma}$ is the
quotient of $X$ by the sublattice perpendicular to $\Sigma^\vee$, and $Y_\Sigma \subset Y$ is the sublattice of elements of $Y$ which are in the $\mathbb{R}$-linear span of $\Sigma^\vee$.

There is an obvious converse to 3.2:

**Proposition 3.3.** Let $R' \subset R_0$ be a parabolic subsystem of roots, and let $T_L' \subset T$ be the subtorus such that $R' = R_L'$. Let $T_L \subset T$ be the subtorus whose Lie algebra $\text{Lie}(T_L')$ is spanned by $R_L'^\vee$. Let $r \in T_L$ be a residual point with respect to $(R_L, q_L)$ as in Proposition 3.2(v). Then $L := rT_L$ is a residual coset for $(\mathcal{R}, q)$.

The recursive nature of the definition of residual cosets makes it feasible to give a complete classification of them. By Lemma 3.2, this classification problem reduces to the classification of the residual points. In turn, Lusztig [17] indicates how the classification of residual points reduces to the classification of residual points in the sense of [11] for certain graded affine Hecke algebras. This classification is known by the results in [11]. Let us explain this in detail. Following [17] we call a root datum $\mathcal{R} = (X, Y, R_0, R_0^\vee, F_0)$ primitive if one of the following conditions is satisfied:

1. $\forall \alpha \in R_0 : \alpha^\vee \not\in 2Y$.
2. There is a unique $\alpha \in F_0$ with $\alpha^\vee \in 2Y$ and $\{w(\alpha) \mid w \in W_0\}$ generates $X$.

A primitive root datum $\mathcal{R}$ satisfying (2) is of the type $C_n^{\text{aff}}$, by which we mean that

$$\mathcal{R} = (Q(B_n) = \mathbb{Z}^n, P(C_n) = \mathbb{Z}^n, B_n, C_n, \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}).$$

By [17] we know that every root datum is a direct sum of primitive summands.

**Proposition 3.4.** Let $r \in T$ be a residual point, and write $r = se \in T_s T_{rs}$ for its polar decomposition (with $T_s = \text{Hom}(X, S^1)$ and $T_{rs} = \text{Hom}(X, \mathbb{R}_+)$. The root system

$$R_{s,1} := \{\alpha \in R_1 \mid \alpha(s) = 1\}$$

has rank $\dim(T)$. The system

$$R_{s,0} := \{\alpha \in R_0 \mid k\alpha \in R_{s,1} \text{ for some } k \in \mathbb{N}\}$$

contains both $R_{\text{r, ess}}^{\text{r}}$ and $R_{\text{r, ess}}^s$, and $r$ is residual with respect to the affine Hecke subalgebra $\mathcal{H}' \subset \mathcal{H}$ whose root datum is given by $\mathcal{R}' := (X, Y, R_{s,0}, R_{s,0}^\vee, F_{s,0})$ (with $F_{s,0}$ the basis of $R_{s,0}$ contained in $R_{0,1}$).

**Proof.** It is clear from the definitions that $R_{s,0}$ contains $R_{\text{r, ess}}^{\text{r}}$ and $R_{\text{r, ess}}^s$, and hence has maximal rank. Given a full flag of $\mathcal{R}$-residual subspaces $\{c\} = L_0 \subset L_1 \subset \cdots \subset L_n = T$, satisfying 3.4 at each level,
we see that the sets $R_{L_i}^p, R_{L_i}^z$ are contained in $R_{s,0}$. It follows by reverse induction on $i$ (starting with $L_n = T$) that each element of the flag is $R^a$-residual. \hfill \Box

**Lemma 3.5.** Given a residual point $r = sc$, let $s_0 \in T_u = \text{Hom}(X, S^1)$ be the element which coincides with $s$ on each primitive summand of type $C_n^{\text{aff}}$ and is trivial on the complement of these summands. Then $s_0$ has order 2.

**Proof.** To see this we may assume that $R$ is of type $C_n^{\text{aff}}$. Then $R_1$ is of type $C_n$, $s = s_0$, and $R_{s,1} = \{ \alpha \in R_1 \mid \alpha(s_0) = 1 \}$, being of maximal rank in $R_1$, is of type $C_k + C_{n-k}$ for some $k$. In particular, $\pm 2e_i \in R_{s,1}$ for all $i = 1, \ldots, n$. Moreover the index of $Z R_{s,1}$ in $Z R_1$ is at most 2. Thus $s_0$ takes values in $\{ \pm 1 \}$ on $R_1$, and is trivial on elements of the form $\pm 2e_i$. It follows that $s_0$ has order 2 on $X = Z^n$. \hfill \Box

Denote by $h \in \text{Hom}(Q, S^1)$ the image of $s_0$ in $\text{Hom}(Q, S^1)$. Choose root labels $k_\alpha = k_{s,\alpha} \in \mathbb{R}$ with $\alpha \in R_{s,0}$ by the requirement ($k_\alpha$ depends on the image of $s$ in $\text{Hom}(Q, S^1)$, but we suppress this in the notation if there is no danger of confusion)

$$e^{k_\alpha} = q_{\alpha}^{h(\alpha)/2} q_{\alpha^v + 1}^{1/2}$$

(3.5)

$$= \begin{cases} q_{\alpha}^{1/2} & \text{if } h(\alpha) = +1 \\ q_{\alpha^v}^{1/2} & \text{if } h(\alpha) = -1 \end{cases}$$

**Theorem 3.6.** Let $r = sc$ be a $(R, q)$-residual point. Then $\gamma := \log(c) \in t := \text{Lie}(T_s)$ is a residual point in the sense of [11] for the graded Hecke algebra $H^* = \mathbb{C}[W(R_{s,0})] \otimes \text{Sym}(t)$ with root system $R_{s,0}$ and root labels $k_\alpha := (k_{s,\alpha})_{\alpha \in R_{s,0}}$. This means explicitly that there exists a full flag of affine linear subspaces $\{ \gamma \} = l_n \subset l_{n-1} \subset \cdots \subset l_0 = t$ such that the sequence

$$i_{s,t} := |R_{s,0,i}^p| - |R_{s,0,i}^z|$$

(3.6)

is strictly increasing, where

$$R_{s,0,i}^p = \{ \alpha \in R_{s,0} \mid \alpha(l_i) = k_{s,\alpha} \},$$

and

$$R_{s,0,i}^z = \{ \alpha \in R_{s,0} \mid \alpha(l_i) = 0 \}.$$

Conversely, given a $s \in T_u$ such that $R_{s,1} \subset R_1$ has rank equal to rank$(X)$, and a residual point $\gamma \in t$ for the root system $R_{s,0}$ with labels $(k_{s,\alpha})$ defined by 3.5, the point $r := s \exp \gamma$ is $(R, q)$-residual. This sets up a 1–1 correspondence between $W_0$-orbits of $(R, q)$-residual points and the collection of pairs $(s, \gamma)$ where $s$ runs over the $W_0$-orbits of
elements of $T_u$ such that $R_{s,1}$ has rank equal to rank($X$), and $\gamma \in t$ runs over the $W(R_{s,0})$-orbits of residual points (in the sense of [11]) for $R_{s,0}$ with the labels $k_s$.

\textbf{Proof.} Straightforward from the definitions. \hfill $\Box$

For convenience we include the following lemma:

\textbf{Lemma 3.7.} If the rank of $R_0$ equals the rank of $X$ (a necessary condition for existence of residual points!), the $W_0$-orbits of points $s \in T_u$ such that $R_{s,1} \subset R_1$ has maximal rank correspond 1-1 to the $\text{Hom}(P(R_1)/X, S^1) \simeq Y/Q(R_i^{(1)})$-orbits on the affine Dynkin diagram $R_i^{(1)}$.

\textbf{Proof.} In the compact torus $\text{Hom}(P(R_1), S^1)$, the $W_0$-orbits of such points correspond to the vertices of the fundamental alcove for the action of the affine Weyl group $W_0 \ltimes 2\pi iQ(R_i^{(1)})$ on $Y \otimes 2\pi i\mathbb{R}$. Now we have to restrict to $X \subset P(R_1)$. \hfill $\Box$

With the results of this subsection at hand, the classification of residual cosets is now reduced to the classification of residual subspaces as was given in [11].

\section{Properties of residual and tempered cosets}

In the derivation of the Plancherel formula of the affine Hecke algebra, some properties of residual cosets will play an important role. Unfortunately, I have no direct proof of these properties. With the classification at hand they can be checked on a case-by-case basis. By the previous subsection this verification reduces to the case of residual subspaces for graded affine Hecke algebras. In [11] (cf. Theorem 3.9, 3.10 and Remark 3.14) these matters have indeed been verified.

\textbf{Theorem 3.8.} Define $*: T \to T$ by $x(t^*) = \overline{x(t)^{-1}}$. If $r = cs \in T$ is a residual point, then $r^* \in W(R_{s,0})r$.

\textbf{Theorem 3.9.} For each residual coset $L \subset T$ we have

\begin{equation}
(3.9) \quad \text{dim}(L) = \text{codim}(L).
\end{equation}

In other words, for every inclusion $L \subset M$ of residual cosets with $\text{dim}(L) = \text{dim}(M) - 1$, the inequality 3.4 is actually an equality. Note that Theorem 3.9 reduces to the case of residual points by Proposition 3.2 and Proposition 3.3. This reduces to the case of residual points in the sense of [11] by Theorem 3.6. This was proved in [11] by classification of the residual subspaces.
Theorem 3.9 has important consequences, as we will see later. At this point we show that the definition of residual cosets can be simplified as a consequence of Theorem 3.9. We begin with a simple lemma:

**Lemma 3.10.** Let $V$ be a complex vector space of dimension $n$, and suppose that $\mathcal{L}$ is the intersection lattice of a set $\mathcal{P}$ of linear hyperplanes in $V$. Assume that each hyperplane $H \in \mathcal{P}$ comes with a multiplicity $m_H \in \mathbb{Z}$, and define the multiplicity $m_L$ for $L \in \mathcal{L}$ by $m_L := \sum m_H$, where the sum is taken over the hyperplanes $H \in \mathcal{P}$ such that $L \subseteq H$. Assume that $\{0\} \in \mathcal{L}$ and that $m_{\{0\}} \geq n$. Then there exists a full flag of subspaces $V = V_0 \supset V_1 \cdots \supset V_n = \{0\}$ such that $m_k := m_{V_k} \geq k$.

**Proof.** We construct the sequence inductively, starting with $V_0$. Suppose we already constructed the flag up to $V_k$, with $k \leq n - 2$. Let $\mathcal{P}_k \subseteq \mathcal{L}$ denote the set of elements of $\mathcal{L}$ of dimension $n - k - 1$ contained in $V_k$, and let $N_k$ denote the cardinality of $\mathcal{P}_k$. By assumption, $N_k \geq n - k \geq 2$. Since every $H \in \mathcal{P}$ either contains $V_k$ or intersects $V_k$ in an element of $\mathcal{P}_k$, we have

$$\sum_{L \in \mathcal{P}_k} (m_L - m_k) = m_n - m_k.$$  

Assume that $\forall L \in \mathcal{P}_k : m_L \leq k$. Then, because $m_k \geq k$ and $N_k \geq 2$,

$$m_n \leq kN_k + (1 - N_k)m_k \leq k \leq n - 2,$$

contradicting the assumption $m_n \geq n$. Hence there exists a $L \in \mathcal{P}_k$ with $m_L \geq k + 1$, which we can define to be $V_{k+1}$. \qed

**Corollary 3.11.** For every coset $L \subseteq T$ one has $i_L \leq \text{codim}(L)$, and $L$ is residual if and only if $i_L = \text{codim}(L)$.

**Proof.** Define $\mathcal{P}$ to be the list of codimension 1 cosets of $T$ arising as connected components of the following codimension 1 sets:

$$\begin{align}
L_{\alpha,1}^+ := \{ t \in T \mid \alpha(t) = q_{\alpha^\vee}^{1/2} \} \\
L_{\alpha,2}^+ := \{ t \in T \mid \alpha(t) = -q_{\alpha^\vee}^{1/2} \} \\
L_{\alpha,1}^- := \{ t \in T \mid \alpha(t) = 1 \} \\
L_{\alpha,2}^- := \{ t \in T \mid \alpha(t) = -1 \}
\end{align}$$

Here $\alpha \in R_0$, and $q_{\alpha^\vee}^{1/2} = 1$ when $2\alpha \not\in R_1$. We give the components of $L_{\alpha,1}^+, L_{\alpha,2}^+$ the index $+1$, and we give the components of $L_{\alpha,1}^-, L_{\alpha,2}^-$ index $-1$.

Suppose that $L$ is any coset of a subtorus $T^k$ in $T$. Then $i_L$ is equal to the sum of the indices the elements of $\mathcal{P}$ containing $L$.

Assume that $i_L \geq \text{codim}(L) = k$. By Lemma 3.10 there exists a sequence $L \subseteq L_{k-\varepsilon} \subseteq L_{k-\varepsilon-1} \cdots \subseteq L_0 = T$ of components of intersections
of elements of $\mathcal{P}$ such that $i_{L_L} = i_L \geq k$ and $i_{L_j} \geq j = \text{codim}(L_j)$ (we did not assume that $L$ is a component of an intersection of elements in the list $\mathcal{P}$, hence $e > 0$ may occur). If $k(0)$ is the smallest index such that $i_{L_k(0)} > k(0)$, then $L_k(0)$ is by definition residual, and thus violates Theorem 3.9. Hence such $k(0)$ does not exist and we conclude that $i_{L_k} = k$ for all $k$. This proves that $e = 0$ and that $L$ is residual. □

Remark 3.12. This solves the question raised in Remark 3.11 of [11].

Theorem 3.13. Suppose that $L \subset M$ are two residual cosets. Write $L = r_L T_L = s_L \exp(\gamma_L) T_L$ and $M = r_M T_M = s_M \exp(\gamma_M) T_M$ as before, with $r_L \in T_L$ and $r_M \in T_M$. If $\gamma_L = \gamma_M$ then $L = M$.

Definition 3.14. Let $L$ be a residual coset, and write $L = r_L T_L = s_L \exp(\gamma_L) T_L$ with $r_L \in T_L$. This is determined up to multiplication of $r_L$ by elements of $K_L = T_L \cap T_L$. We call $c_L := \exp(\gamma_L)$ the “center” of $L$, and we call $L^{\text{temp}} := r_L T_L^{\text{u}}$ the tempered compact form of $L$ (both notions are independent of the choice of $r_L$ since $K_L \subset T_L^{\text{u}}$). The cosets of the form $L^{\text{temp}}$ in $T$ will be called “tempered cosets”.

Theorem 3.13 follows from Remark 3.14 in [11], and shows in particular that a tempered coset can not be a subset of a strictly larger tempered coset.

4. Localization of $\tau$ on $\text{Spec}(\mathcal{Z})$

Recall the decomposition of $\tau$ we derived in [24], Theorem 3.7:

\begin{equation}
\tau = \int_{t \in T_u} \left( \frac{E_i}{q(u_0) \Delta(t)} \right) \frac{dt}{q(u_0)c(t^{-1})c(t)}.
\end{equation}

Let us briefly review the various ingredients of this formula. First of all, $T_u = \text{Hom}(X, S^1)$, the compact form of the algebraic torus $T = \text{Hom}(X, \mathbb{C}^*)$, and $t_0 \in T_u$, the real split part of $T$, such that the following inequality is satisfied (cf. [24] Definition 1.4 and Corollary 3.2):

\begin{equation}
t_0 < \delta^{-1/2}.
\end{equation}

In other words, $t_0$ is in the shifted chamber defined by

\begin{equation}
\forall i \in \{1, \ldots, n\} : \alpha_i(t_0) < q(s_i)^{-1}.
\end{equation}

The form $dt$ denotes the holomorphic $n$-form on $T$ which restricts to the normalized Haar measure on $T_u$. 
The function $\Delta(t) := \prod_{\alpha \in R_{1,+}} \Delta_\alpha$ with
\begin{equation}
\Delta_\alpha := 1 - \theta_{-\alpha} \in \mathcal{A}
\end{equation}
is the Weyl denominator, and $c(t) = c(t, q)$ is Macdonald’s $c$-function.

This $c$-function is introduced as an element in $\mathcal{F} \mathcal{A}$, the field of fractions of $\mathcal{A}$, and will be interpreted as rational function on $T$ (cf. [24], Definition 1.13). Explicitly, we put
\begin{equation}
    c := \prod_{\alpha \in R_{1,+}} c_\alpha,
\end{equation}
where, for $\alpha \in R_{1}$, we define $c_\alpha$ by
\begin{equation}
c_\alpha := \frac{(1 + q_{\alpha}^{-1/2} \theta_{-\alpha/2})(1 - q_{\alpha}^{-1/2} q_{2\alpha}^{-1} \theta_{-\alpha/2})}{1 - \theta_{-\alpha}} \in \mathcal{F} \mathcal{A}.
\end{equation}

**Remark 4.1.** It is handy to write the formulas in the above form, but strictly speaking incorrect if $\alpha/2 \not\in R_{0}$. However, we formally put $q_{2\alpha} = 1$ if $\alpha/2 \not\in R_{0}$, and with this substitution the above formula reduces to an expression containing only $\theta_{\alpha}$. Here and below we use this convention.

The expression $E_t \in \mathcal{H}^*$ is the holomorphic *Eisenstein series* for $\mathcal{H}$, with the following defining properties (cf. [24], Propositions 2.23 and 2.24):
\begin{align}
    (i) \quad & \forall h \in \mathcal{H}, \text{ the map } T \ni t \to E_t(h) \text{ is regular.} \\
    (ii) \quad & \forall x, y \in X, h \in \mathcal{H}, \quad E_t(h \theta_y) = t(x + y) E_t(h). \\
    (iii) \quad & E_t(1) = q(w_0) \Delta(t).
\end{align}

We want to rewrite the integral 4.1 representing the trace functional as an integral over the collection of tempered residual cosets, by a contour shift. It turns out that such a representation exists and is unique. To find it, we need an intermediate step. We will first rewrite the integral as a sum of integrals over a larger set of tempered “quasi-residual cosets”, and then we will show that if we symmetrize the result over $W_0$, all the contributions of non-residual cosets cancel.

### 4.1. Quasi residual cosets

The basic scheme to compute residues has nothing to do with the properties of root systems. It is therefore convenient to formulate everything in a more general setting first. Later we will consider the consequences that are specific to our context.
Definition 4.2. Let $\omega = p dt/q$ be a rational $(n, 0)$-form on $T$. Assume that $p, q$ are relatively prime, and are of the form

$$q(t) = \prod_{m \in M} (x_m(t) - d_m); \quad p(t) = \prod_{m' \in M'} (x_{m'}(t) - d_{m'})$$

where the products are taken over finite sets $M, M'$. The index sets $M$ and $M'$ come equipped with maps $m \to (x_m, d_m) \in X \times \mathbb{C}^\times$. An $\omega$-residual coset $L$ is a connected component of an intersection of codimension 1 sets of the form $L_m = \{ t \mid x_m(t) = d_m \}$ with $m \in M$, such that the order $i_L$ of $\omega$ along $L$ satisfies

$$i_L := |\{ m \in M \mid L \subseteq L_m \}| - |\{ m' \in M' \mid L \subseteq L_{m'} \}| \geq \text{codim}(L).$$

The list of $\omega$-residual cosets is denoted by $\mathcal{L}^\omega$ (it includes by definition the empty intersection $T$).

We define the notions $T^L$, $T_L$ as we did in the case of the residual subspaces of section 3. We note that the “perpendicular torus” $T_L$ exists because the $W_0$-invariant inner product on $X \subseteq t^\ast$ has values in $\mathbb{Q}$. We denote by $\mathcal{M}_L \subseteq M$ the subset $\{ m \in M \mid x_m(L) = d_m \}$. We choose an element $r_L = s_L c_L \in T^L \cap L$ for each $L$ so that we can write $L = r_L T^L$. We call $c_L \in T$, the center of $L$, and note that this center is determined uniquely by $L$. We write $c_L = \exp \gamma_L$ with $\gamma_L \in t_L$. The set of centers of the $\omega$-residual cosets is denoted by $\mathcal{C}^\omega$. The tempered form of a $\omega$-residual $L = r_L T^L$ is by definition $L^\text{temp} := r_L T_u^L$ (which is independent of the choice of $r_L$), and such a coset will be called an $\omega$-tempered coset.

Basically, the only properties of the collection $\mathcal{L}^\omega$ we will need are

Proposition 4.3. (i) If $c \in \mathcal{C}^\omega$ then the union

$$S_c := \cup_{L \in \mathcal{L}^\omega} L^\text{temp} \subseteq c T_u$$

is a regular support in the sense of [28] in $c T_u$. This means that a distribution on $c T_u$ with support in $S_c$ can be written as a sum of derivatives of measures supported on $S_c$.

(ii) If $c = \exp \gamma \in T_{rs}$, and $L$ is $\omega$-residual with $|\gamma| \geq |\gamma|_{L}$ but $\gamma_L \neq \gamma$, then there exists a $m \in \mathcal{M}_L$ such that $f(t) = x_m(t) - d_m$ is non-vanishing on $c T_u$.

Proof. The set $S_c$ is a finite union of smooth varieties, obviously satisfying the condition of [28], Chapitre III, §9 for regularity. Hence (i) is trivial. As for (ii), first note that the assumption implies that $|\gamma_L| > 0$, hence that $L \neq T$. Thus the codimension of $L$ is positive, and $\mathcal{M}_L \neq \emptyset$. Clearly $\gamma \not\in \gamma_L + t^L = \log(T_{rs} \cap L T_u)$ since $\gamma_L$ is the unique smallest vector in this affine linear space. Because $\{ x_m \mid m \in \mathcal{M}_L \}$ spans
\[ t^*_L = (t^L)^{-1}, \] we can find a \( m \in \mathcal{M}_L \) such that \( x_m(\gamma) \neq x_m(\gamma_L) \). This implies the result. \qed

4.2. The contour shift and the local contributions

The following theorem is essentially the same as Lemma 3.1 of [11], but because of its basic importance I include the proof here. See also [3] for a more general method.

**Theorem 4.4.** Let \( \omega \) be as in Definition 4.2 and let \( t_0 \in T_{r_s} \cup (T_{r_s} \cap T_u L_m) \). There exists a unique collection of distributions \( \{ \mathcal{X}_c \in C^{-\infty}(c T_u) \}_{c \in \mathcal{C}} \) such that the following conditions hold:

(i) The support of \( \mathcal{X}_c \) satisfies \( \text{supp}(\mathcal{X}_c) \subset S_c \).

(ii) For every \( a \in \mathcal{A}^m(T) \) we have

\[
\int_{t \in \mathbb{I}_0 T_u} a(t) \omega(t) = \sum_{c \in \mathcal{C}} \mathcal{X}_c(a|_{c T_u}).
\]

**Proof.** The existence is proved by induction on the dimension \( n \) of \( T \), the case of \( n = 0 \) being trivial. Suppose that the result is true for tori of dimension \( n - 1 \). Choose a smooth path in \( T_{r_s} \) from \( t_0 \) to the identity \( e \) which intersects each coset \( T_{r_s} \cap T_u L_m \) transversally and in at most one point \( t_m \). We may assume the \( t_m \) to be distinct unless \( t_m = e \). When we move \( t_0 \) along this curve to \( e \) we pick up a residue when we cross at a point \( t_m \neq e \). For simplicity of notation we write \((x,d)\) instead of \((x_m,d_m)\), \( L \) instead of \( L_m \), \( t \) instead of \( t_m \) etc. Recall that we have the decomposition \( L = r_L T^L = s_L c_L T^L \) with \( r_L \in T_L \).

Let \( D \) be the unit constant vector field on \( T \) which is perpendicular to \( L \). Write \( d^L t = \langle D, dt \rangle \) for the invariant \((n - 1,0)\)-form which restricts to Haar measure on \( T^L_u \). The residue contribution we pick up when we cross \( L \) is then equal to

\[
\int_{t \in \mathbb{I}_0 T^L_u} D^{j-1} ((x-d)^{j_z} p/q) a(t) dt.
\]

This decomposes as a finite sum of the form

\[
\sum_{j=0}^{i_z} \sum_k \int_{T^L_u} (D^j(a)|_L) \omega_{j,k},
\]

where \( \omega_{j,k} \) is itself a rational \((n-1,0)\)-form on \( L \) in the product form as in Definition 4.2, with poles along the intersections \( L_n = L \cap L_n \) (with \( n \in M \) which are of codimension 1 in \( L \)). A simple computation shows that for every \( j, k \) and every connected component \( H \) of an intersection of cosets of the form \( L_n \subset L \), the index \( i_{\omega_{j,k}} \) of \( H \subset L \) satisfies
followers that the union over all \( j, k \) of the \( \omega_{j,k} \)-residual cosets of \( L \) is contained in the collection of \( \omega \)-residual cosets of \( H \).

By the induction hypotheses we can rewrite such residues in the desired form, where the role of the identity element in the coset \( L \) is now played by \( r_L \). At the identity \( e \) itself we have to take a boundary value of \( \omega \) towards \( T_u \), which defines a distribution on \( T_u \). This proves the existence.

The uniqueness is proved as follows. Suppose that we have a collection \( \{ \mathfrak{Z}_c \in C_c^\infty(\mathfrak{c}T_u) \}_{c \in \mathcal{C}} \) of distributions such that

(i) \( \text{supp}(\mathfrak{Z}_c) \subset S_c \).

(ii) \( \forall a \in \mathbb{C}[T] : \sum_{c \in \mathcal{C}} \mathfrak{Z}_c(a|_{\mathfrak{c}T_u}) = 0. \)

We show that \( \mathfrak{Z}_c \equiv 0 \) by induction on \( |\gamma| = \log(c)| \). Choose \( c \in \mathcal{C} \) such that \( \mathfrak{Z}_{c'} = 0 \) for all \( c' \) with \( |\gamma'| < |\gamma| \). For each \( L \in \mathcal{L} \) with \( |\gamma_L| \geq |\gamma| \) and \( \gamma_L \neq \gamma \) we choose a \( l \in \mathcal{M} \) such that \( x_l(t) - d_l \) does not vanish on \( cT_u \) (Proposition 4.3) and we set

\[
\nu(t) := \prod_{\{ L : |\gamma_L| \geq |\gamma| \text{ and } \gamma_L \neq \gamma \}} (x_l(t) - d_l).
\]

It is clear that for sufficiently large \( N \in \mathbb{N} \), \( \mathfrak{Z}_c(\nu^N a) = 0 \) for all \( a \in \mathbb{C}[T] \). On the other hand, by the theory of Fourier series of distributions on \( T_u \), \( \mathbb{C}[T] \mathfrak{c}T_u \) is a dense set of test functions on \( cT_u \). Since \( \nu^N \) is nonvanishing on \( cT_u \), this function is a unit in the space of test functions in \( cT_u \). Thus also \( \nu^N \mathbb{C}[T]_{cT_u} \) is dense in the space of test functions. It follows that \( \mathfrak{Z}_c \equiv 0. \)

4.2.1. Approximating sequences. There is an “analytically dual” formulation of the result on residue distributions that will be useful later on. The idea to deal with the residue distributions in this way was inspired by the approach in [13] to prove the positivity of certain residual spherical functions.

**Lemma 4.5.** For all \( N \in \mathbb{N} \) there exists a collection of sequences \( \{ a_n^{N,c} \}_{n \in \mathbb{N}} \) (\( c \in \mathcal{C} \)) in \( \mathcal{A} \) with the following properties:

(i) For all \( n \in \mathbb{N} \), \( \sum_{c \in \mathcal{C}} a_n^{N,c} = 1. \)

(ii) For every constant coefficient differential operator \( D \) of order at most \( N \) on \( T \), \( D(a_n^{N,c}) \to D(1) \) uniformly on \( S_c \) and \( D(a_n^{N,c}) \to 0 \) on \( S_c \) if \( c' \neq c \).

**Proof.** We construct the sequences with induction on the norm \( |\gamma| = \log(c)| \). We fix \( N \) and suppress it from the notation. Let \( c \in \mathcal{C} \) and assume that we have already constructed such sequences \( a_n^{N,c'} \) satisfying (ii) for all \( c' \) with \( |\gamma'| < |\gamma| \). Consider the function \( \nu \) constructed in
the second part of the proof of Theorem 4.4. By Fourier analysis on $cT_n$ it is clear that there exists a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ in $\mathbb{C}[T]$ such that for each constant coefficient differential operator $D$ of order at most $N$ there exists a constant $c_D$ such that

$$\| (D(\phi_n) - D(\nu^{-(N+1)})) \|_{L^\infty} < c_D/n.$$  

Applying Leibniz’ rule to $\nu^{-(N+1)}\phi_n - 1 = \nu^{(N+1)}(\phi_n - \nu^{-N+1})$ repeatedly we see that this implies that there exists a constant $c'_D$ for each constant coefficient differential operator $D$, such that

$$\| (D(\nu^{(N+1)}\phi_n) - D(1))\|_{L^\infty} < c'_D/n.$$  

Notice that $D(\nu^{(N+1)}\phi_n) = 0$ on all $S_{\gamma'}$ with $|\gamma'| \geq |\gamma|$ but $\gamma' \neq \gamma$. On the other hand, for each constant coefficient differential operator $E$ the function $E(1 - \sum_{|c'|| \gamma' < |\gamma|} a_{\gamma'}^c)$ converges uniformly to zero on each $S_{\gamma'}$ with $|\gamma'| < |\gamma|$. Again applying Leibniz’ rule repeatedly we see that there exist a $k \in \mathbb{N}$ (depending on $n$) such that the function

$$a_n^c := \nu^{(N+1)}\phi_n(1 - \sum_{|c'|| \gamma' < |\gamma|} a_{\gamma'}^c)$$

has the property that

$$\| D(a_n^c) \|_{L^\infty} < c'_D/n,$$

where the union is taken over all $c'$ with $|\gamma'| < |\gamma|$. It is clear that the sequence $a_n^c$ thus constructed satisfies (ii). We continue this process until we have only one center $c$ left. For this last center we can simply put

$$a_n^c := 1 - \sum_{c' \neq c} a_{c'}^c.$$  

This satisfies the property (ii), and forces (i) to be valid. \hfill \square

The use of such collections of sequences is the following:

**Proposition 4.6.** In the situation of Theorem 4.4 and given any collection of sequences $\{a_n^c\}$ as constructed in Lemma 4.5 we can express the residue distributions as (with $a \in A$):

$$\mathcal{X}_c(a) = \lim_{n \to \infty} \int_{10T_n} a_n^c a \omega,$$

provided $N$ (in Lemma 4.5) is chosen sufficiently large.

**Proof.** Because we are working with distributions on compact spaces, the orders of the distributions are finite. Take $N$ larger than the maximum of all orders of the $\mathcal{X}_c$. By Proposition 4.3 we can thus express
as a sum of derivatives of order at most $N$ of measures supported on $S_c$. The result now follows directly from the defining properties of the sequence $\alpha_n^c$. \hfill $\Box$

### 4.2.2. Cycles of integration

Yet another useful way to express the residue distribution is by means of integration of $a\omega$ over a suitable $n$-cycle. We will need this representation later on.

In the next proposition we will use the distance function on $T$ which measures the distance between $2\pi iY$-orbits in $t_c$. For $\delta > 0$ and each $L$ which is a connected component of an intersection of codimension 1 cosets $L_m \subset T$ with $m \in \mathcal{M}$, we denote by $\mathcal{B}_L(r_L, \delta)$ a ball in $T_L$ with radius $\delta$ and center $r_L$, and by $\mathcal{B}_L^L(\delta)$ a ball with radius $\delta$ and center $d$ in $T_L^L$. We put $\mathcal{M}_L \subset \mathcal{M}$ for the $m \in \mathcal{M}$ such that $L \subset L_m$, and $\mathcal{M}_L \subset \mathcal{M}$ for the $m \in \mathcal{M}$ such that $L_m \cap L$ has codimension 1 in $L$. We write $T^m = \{ t \mid x_m(t) = 1 \}$.

Let $U^L(\delta) \subset T^L$ be the open set $\{ t \in T_{L_m}^L \mid \forall m \in \mathcal{M}^L : t \mathcal{B}_L(r_L, \delta) \cap L_m = \emptyset \}$. Note that $U^L(\delta_1) \subset U^L(\delta_2)$ if $\delta_1 > \delta_2$, and that the union of these open sets is equal to the complement of union of the codimension 1 subsets $r^{-1}_L(L \cap L_m) \subset T^L$ with $m \in \mathcal{M}^L$.

**Proposition 4.7.** Let $\epsilon > 0$ be such that for all $m \in \mathcal{M}$ and $L \in \mathcal{L}^\omega$, $L_m \cap \mathcal{B}_L(r_L, \epsilon) \mathcal{B}_L^L(\epsilon) \cap T^m \neq \emptyset$ implies that $L_{\text{temp}} \cap L_m \neq \emptyset$. Denote by $\mathcal{M}^{L,\text{temp}}$ the set of $m \in \mathcal{M}^L$ such that $L_{\text{temp}} \cap L_m \neq \emptyset$. There exist

(i) $\forall L \in \mathcal{L}^\omega$, a point $e^L \in \mathcal{B}_L^L(\epsilon) \setminus \bigcup_{m \in \mathcal{M}^{L,\text{temp}}} T^m$,

(ii) $a 0 < \delta < \epsilon$ such that $\forall L \in \mathcal{L}^\omega, e^L T^L_u \subset U^L(\delta)$, and

(iii) $\forall L \in \mathcal{L}^\omega$, a cycle $\xi_L \subset \mathcal{B}_L(r_L, \delta) \setminus \bigcup_{m \in \mathcal{M}_L} L_m$,

such that

\begin{equation}
\forall \epsilon \in \mathcal{C}^\omega, \forall \phi \in C^\infty(c T_u) : \mathcal{X}_c(\phi) = \sum_{L : \epsilon^L = \epsilon} \mathcal{X}_L(\phi),
\end{equation}

where $\mathcal{X}_L$ is the distribution on $c T_u$ defined by

\begin{equation}
\forall a \in \mathcal{A} : \mathcal{X}_L(a) = \int_{c^L T_u^L \times \xi_L} a \omega,
\end{equation}

If $\mathcal{M}^{L,\text{temp}} = \emptyset$ we may take $\epsilon^L = \epsilon$.

**Proof.** We begin the proof by remarking that (i), (ii) and (iii) imply that the functional $\mathcal{X}_L$ on $\mathcal{A}$ indeed defines a distribution on $c T_u$, supported on $L_{\text{temp}}$. Consider for $t \in U^L(\delta)$ the inner integral

\begin{equation}
\int_{t^L} a \omega := i(a, t) d^L t.
\end{equation}
Then $i(a, t)$ is a linear combination of derivatives $D_n a$ (normal to $L$) of $a$ at $r, t$ with coefficients in the ring of meromorphic functions on $T^L$ which are regular outside the codimension 1 intersections $r^{-1}_L (L \cap L_m)$:

$$i(a, t) = \sum f_n D_n a.$$  

(4.13)  

Hence $\mathcal{X}_L(a)$ is equal to the sum of the boundary value distributions $B_{p, f_n}$ of the meromorphic coefficient functions, applied to the corresponding normal derivative of $a$, restricted to $L^{temp}$:

$$\mathcal{X}_L(a) = \sum f_n B_{p, f_n} (D_n a|_{L^{temp}}).$$  

(4.14)  

We see that $\mathcal{X}_L$ is a distribution supported in $L^{temp} \subset c_L T_u$, which only depends on $\xi_L$ and on the component of $\mathcal{B}^L (e) \setminus \cup_{m \in M, L^{temp}} T^m$ in which $e^L$ lies.

Hence, by the uniqueness assertion of Theorem 4.4, we conclude that it is sufficient to prove that we can choose $e^L$, $\delta$, $\xi_L$ in such a way that

$$\forall a \in A : \int_{t_0 T_u} a \omega = \sum_{L \in \mathcal{L}} \mathcal{X}_L(a).$$  

(4.15)  

In order to prove this it is enough to show that we can choose $e^L$, $\delta$, $\xi_L$ as in (i), (ii) and (iii) for the larger collection $\mathcal{L}$ of all the connected components of intersections of the $L_m$ (with $m \in M$), such that

$$t_0 T_u \sim \cup_{L \in \mathcal{L}} e^L T^L_u \times \xi_L.$$  

(4.16)  

Here $\sim$ means that the left hand side and the right hand side are homologous cycles in $T \setminus \cup_{m \in M} L_m$. The desired result follows from this, since the functional $\mathcal{X}_L$ is equal to 0 unless $L$ is $\omega$-residual (because the inner integral 4.12 is identically equal to 0 for non-residual intersections, by an elementary argument which is given in detail in the proof of Theorem 4.26).

Let $k \in \{0, 1, \ldots, n-1\}$. Denote by $\mathcal{L}^k$ the collection of connected components of intersections of the $L_m$ ($m \in M$) such that $\text{codim}(L) < k$. Assume that we already have constructed points $e^L$, $\delta$, $\xi_L$ satisfying (i), (ii) and (iii) for all $L \in \mathcal{L}^k$ and in addition, for each $L \in \mathcal{L}^k$ with $\text{codim}(L) = k$, a finite set of points $\Omega_L \subset T^L_u$ such that $\Omega_L T^L_u \subset U^L(\delta)$ and a cycle $\xi_{L,w} \subset \mathcal{B}(r_L, \delta) \setminus \cup_{m \in M, L_m}$ for each $w \in \Omega_L$, such that $t_0 T_u$ is homologous to

$$\cup_{L \in \mathcal{L}^k} (e^L T^L_u \times \xi_L) \cup \cup_{L \in \mathcal{L}^k} (e^L T^L_u \times \xi_{L,w}) \subset \Omega_L (w T^L_u \times \xi_{L,w}).$$  

(4.17)  

This equation holds for $k = 0$, with $\Omega_T = \{ t_0 \}$, which is the starting point of the inductive construction to be discussed below. We will construct $e^L$, $\delta_1$ and $\xi_L$ for $L \in \mathcal{L}^{k+1} \setminus \mathcal{L}^k$, and finite sets $\Omega_L$ for
$L \in \hat{\mathcal{L}}^\omega(k) \setminus \hat{\mathcal{L}}^\omega(k + 1)$, with a cycle $\xi_w$ for each $w \in \Omega_L$ such that equation 4.17 holds with $k$ replaced by $k + 1$, and $\delta$ by $\delta_1$.

First of all, notice that we may replace $\delta$ by any $0 < \delta' < \delta$ in equation 4.17, because we can shrink the $\xi_L$ and $\xi_{L,w}$ within their homology class to fit in the smaller sets $\mathcal{B}_L(r_L, \delta') \cup_{m \in \mathcal{M}_L} L_m$. Choose $\delta'$ small enough such that for each $L \in \hat{\mathcal{L}}^\omega(k + 1) \setminus \hat{\mathcal{L}}^\omega(k)$ there exists a point $e^L \in \mathcal{B}^L_{\delta}(e)$ with the property that $e^L T_u \subset U^L(\delta')$.

The singularities of the inner integral are located at codimension 1 cosets in $T^L$ of the form $r^{-1}_L N$, where $N$ is a connected component of $L \cap L_m$ for some $m \in \mathcal{M}^L$. We have $r^{-1}_L N = r^{-1}_L r_N T^N \subset T^L$, and thus $\gamma^{-1}_L c_N T_{rs} \subset T^L_{rs}$. Choose paths inside $T^L_{rs}$ from $w \in \Omega_L$ to the point $e^L$.

We choose each path such that it intersects the real cosets $\gamma^{-1}_L c_N T^N_{rs}$ transversally and in at most one point, and such that these intersection points are distinct. If $p$ is the intersection point with the path $\gamma$ from $w \in \Omega_L$ to $e^L$ then $p$ is of the form $p = \gamma^{-1}_L c_N w_L, w_{L,w} \in \gamma^{-1}_L c_N T^N_{rs}$ with $w_L, w_{L,w} \in T^N_{rs}$. Given $N \in \hat{\mathcal{L}}^\omega(k)$, we denote by $\Omega_N$ the set of all $w_{L,N,w}$ arising in this way, for all the $L \in \hat{\mathcal{L}}^\omega(k + 1) \setminus \hat{\mathcal{L}}^\omega(k)$ such that $L \supset N$, and $w \in \Omega_L$.

Notice that if $v = w_{L,N,w} \in \Omega_N$ and $u \in r^{-1}_N \subset (N \cap L_m)$ for some $m \in \mathcal{M}^N$ and $s \in T_u$, we have that $\gamma^{-1}_L c_N u \in \gamma^{-1}_L (c_N T^N \cap c_N T^N_{rs})$ where $N' = L \cap L_m$. Since $T^{N'} \not= T^N$, this contradicts the assertion that the intersection points of the paths in $T^L_{rs}$ and the cosets $\gamma^{-1}_L c_N T^N_{rs}$ are distinct. We conclude in particular that the compact set $\Omega_N T^N_u$ is contained in the union of the open sets $U^N(\delta')$. We can thus choose $\delta'$ small enough such that in fact $\Omega_N T^N_u \subset U^N(\delta')$, as required in equation 4.17.

Write $T^N_{N,L}$ for the identity component of the 1-dimensional intersection $T^N \cap T^L$, and decompose the torus $T^L$ as the product $T^N \times T^N_{N,L}$. Let $v = w_{L,N,w} \in \Omega_N$ and put $p = c^{-1}_L c_N v$ for the corresponding intersection point in $T^L_{rs}$. Notice that for a codimension 1 coset $r^{-1}_L N' \subset T^L$ with $N' \in \hat{\mathcal{L}}^\omega$ we have that

$$p T^N_{N,L,u} \cap r^{-1}_L N' = \begin{cases} \emptyset & \text{if } c^{-1}_L c_N T^{N'} \not= c^{-1}_L c_N T^N, \\ G_{L,N',w} & \text{otherwise} \end{cases}$$

where $G_{L,N',w}$ is a coset of the subgroup $T^L_{N,L} \cap T^N$ of the finite group $K_N = K_N = T_N \cap T^N \subset T^N_u$, of the form

$$G_{L,N',w} = (T^N_{N,L} \cap T^N) r^{-1}_L r_N v.$$  

The cosets $G_{L,N',w}$ are disjoint. Let $\delta(L, w)$ be the minimum distance of two points in the union of these cosets, and let $\delta(k + 1)$ denote the minimum of the positive real numbers $\delta(L, w)$ when we vary over all
the $L$ and $w \in \Omega_L$. Choose $\delta_1 > 0$ smaller than the minimum of $\delta'$ and $\delta(k + 1)$. Let $\eta$ be a circle of radius $\delta_1/2$ with center $e$ in $T_{NL \cup L}$. Next we make $\delta'$ sufficiently small so that $\bigcup N' G_{L, N', w} \eta \subset U^L(\delta')$. For $x_-, x_+ < x_0 < x_+$ we have in $U^L(\delta')$:

\begin{equation}
(4.20) \quad \gamma(x-) T_{NL \cup L, w} \sim \gamma(x+) T_{NL \cup L, w} \cup \bigcup N' G_{L, N', w} \eta,
\end{equation}

where the union is over all $N' \subset L$ such that $c_{L}^{-1} c_{N'} T_{N'} = c_{L}^{-1} c_{N} T_{N}$.

Define

\begin{equation}
(4.21) \quad \xi_{L, N', v} := r_{L}^{-1} r_{N} \eta \times \xi_{L, w}.
\end{equation}

We then have

\begin{equation}
(4.22) \quad \gamma(x-) T_{u}^{L} \times \xi_{L, w} \sim \gamma(x+) T_{u}^{L} \times \xi_{L, w} \cup \bigcup N' u T_{u}^{N} \times \xi_{L, N', v}.
\end{equation}

By possibly making $\delta'$ smaller we get that $\xi_{L, N', v} \subset B_{N}(r_{N}, \delta_1)$ for all possible choices $N, L$ and $w$. If $L_{m} \subset N$ but $L_{m} \not\subset L$, then, since $r_{L}^{-1} r_{N} \eta \subset U^L(\delta')$ and $\xi_{L, w} \subset B_{L}(r_{L}, \delta')$, we have $\xi_{L, N, w} \cap L_{m} = \emptyset$. If on the other hand $L_{m} \subset L$ then $\xi_{L, N, w} \cap L_{m} = r_{L}^{-1} r_{N} \eta \times (\xi_{L, w} \cap L_{m}) = \emptyset$. Finally we put

\begin{equation}
(4.23) \quad \xi_{N, v} := \cup (L, w) \xi_{L, N, v},
\end{equation}

where we take the union over all pairs $(L, w)$ with $L \in \hat{L}^{w}(k + 1) \setminus \hat{L}^{w}(k)$ such that $L \supset N$ and $w \in \Omega_{L}$ such that there is an intersection point $w_{L, N, w}$ with $w_{L, N, w} = v$. We have shown that

\begin{equation}
(4.24) \quad \xi_{N, v} \subset B_{N}(r_{N}, \delta_1) \setminus \cup_{m \in M} L_{m},
\end{equation}

as required in equation 4.17.

Applying equation 4.22 for all the intersections of all the paths we chose, we obtain equation 4.17 with $k$ replaced by $k + 1$ and $\delta$ by $\delta_1$. We thus take $\xi_{L} = \cup_{w \in \Omega_{L}} \xi_{L, w}$ for $L \in \hat{L}^{w}(k + 1) \setminus \hat{L}^{w}(k)$, and for $N \in \hat{L}^{w}(k + 2) \setminus \hat{L}^{w}(k + 1)$ we take $\Omega_{N}$ and $\xi_{N, v}$ as constructed above.

This process continues until we have $k = n - 1$ in equation 4.17. In the next step we proceed in the same way but with $\Omega_{N} = \{e\}$ for all $N \in \hat{L}^{w}(n + 1) \setminus \hat{L}^{w}(n)$. The process now stops, since also $\epsilon_{N} = e$. This proves the desired result, with $\delta$ equal to the $\delta_1$ obtained in the last step of the inductive construction.

\begin{remark}
4.8. The homology classes of the cycles $\xi_{L}$ are not uniquely determined by the above algorithm. In fact the splitting $\mathcal{X} = \sum_{\{L | c_{L} = c\}} X_{L}$ is not unique without further assumptions. Nonetheless, we will later study a situation where the decomposition $\mathcal{X} = \sum_{\{L | c_{L} = c\}} X_{L}$ is such that each $X_{L}$ is a regular measure supported on $L^{\text{temp}}$, and such a decomposition is of course unique.
\end{remark}
We list some useful properties of the cycles $\xi_L$. We fix $\omega$, and suppress it from the notation.

**Definition 4.9.** Let $L \in \mathcal{L}$. Denote by $\mathcal{L}^L$ the configuration of real cosets $M^L := c_LT^M$, where $M \in \mathcal{L}$ such that $M \ni L$, $M \neq T$. The “dual” configuration, consisting of the cosets $M_L := c_LT^M \subset T_L$ with $M \in \mathcal{L}$ such that $M \ni L$, is denoted by $\mathcal{L}_L$. Given an (open) chamber $C$ in the complement of $\mathcal{L}^L$, we call $C^d = \{c_L \exp(v) \mid (v, w) < 0 \forall w \in \log(c_L^{-1}C) \setminus \{0\}\}$ the anti-dual cone. This anti-dual cone is the interior of the closure of a union of chambers of the dual configuration $\mathcal{L}_L$ in $T_L$.

**Proposition 4.10.** If $t_0$ is moved inside a chamber of $\mathcal{L}^L$ we can leave $\xi_L$ unchanged.

*Proof.* If $t_0$ is moved within a chamber of $\mathcal{L}^L$, the path from $t_0$ to $e$ can be chosen equal to the original path up to a path which only crosses codimension one cosets of the form $L_mT_u \cap T_{rs}$ which do not contain $c = c_L$. Therefore this does not change $\xi_L$. \hfill $\Box$

**Proposition 4.11.** Write $L = r_LT^L = c_LT^M$ as usual, and let $M \in \mathcal{L}^L$. Then $M^{\text{temp}} \supset L^{\text{temp}}$ if and only if $e \in M_L$. In particular, $L^{\text{temp}}$ is maximal in the collection of $\omega$-tempered cosets if and only if $e$ is regular with respect to the configuration $\mathcal{L}_L$.

*Proof.* If $M \in \mathcal{L}^L$, then $L^{\text{temp}} \subset M^{\text{temp}} \Leftrightarrow c_L = c_M$ (since then $s_L \in (c^{-1}_LM) \cap T_u = s_MT^M_u$, implying that $r_L \in M^{\text{temp}}$). Now $c_L = c_M \Leftrightarrow c_L \in T_M \Leftrightarrow e \in M_L$. \hfill $\Box$

**Proposition 4.12.** (cf. [11], Lemma 3.3.) If $e$ is not in the closure of the anti-dual cone of the chamber of $\mathcal{L}^L$ in which $t_0$ lies, we can take $\xi_L = \emptyset$.

*Proof.* The possible contributions to the cycle $\xi_L$ come from contour shifts in the central configuration $\mathcal{L}^L$. We can therefore replace $\mathcal{L}$ by the configuration $\mathcal{L}(L)$ of connected components of intersections of cosets $L_m$ with $m \in M_L$, and apply the inductive process of Proposition 4.7 to $\mathcal{L}(L)$.

We identify $T_{rs}$ with the real vector space $t$ via the map $t \mapsto \log(c_L^{-1}t)$, and we denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product thus obtained on $T_{rs}$. Notice that the role of the origin is played by $c_L$. The sets $c_MT^M_{rs}$ with $M \in \mathcal{L}(L)$ satisfy $c_MT^M_{rs} \ni c_L$, and are equipped with the induced Euclidean inner product.

By the assumption and Proposition 4.10 we can choose $t_0$ within its chamber such that $\langle t_0, e \rangle > 0$. Assume by induction that in the $k$-th
step of the inductive process of Proposition 4.7 we have, \( \forall N \in \mathcal{L}(L) \) with \( \text{codim}(L) = k \) and \( \forall w \in \Omega_N \), that

\[
\langle c_N w, c_N \rangle > 0
\]  

(see equation 4.17 for the meaning of \( \Omega_N \)). Notice in particular that this implies that \( \Omega_N = \emptyset \) if \( c_N = c_L \). By choosing \( \epsilon \) sufficiently small, we also have \( \langle c_N \epsilon^N, c_N \rangle > 0 \). The path \( \gamma \) in \( T^N_{rs} \) from \( w \in \Omega_N \) to \( \epsilon^N \) is chosen equal to \( c_N^{-1} [c_N w, c_N \epsilon^N] \), where \( [c_N w, c_N \epsilon^N] \) denotes the (geodesic) segment from \( c_N w \) to \( c_N \epsilon^N \) in the Euclidean space \( c_N T^N_{rs} \).

Note that \( \langle x, c_N \rangle > 0 \) for all \( x \in [c_N w, c_N \epsilon^N] \). Let \( M \subseteq \mathcal{L}(L) \) with \( \text{codim}(M) = k + 1 \) and \( M \subseteq N \). If \( \gamma \) intersects \( c_N^{-1} c_M T^M_{rs} \) in \( c_N^{-1} c_M w_{N,M,w} \), then we have \( 0 < \langle c_M w_{N,M,w}, c_N \rangle = \langle c_M w_{N,M,w}, c_M \rangle \).

By induction on \( k \) this proves that we can perform the contour shifts in such a way that 4.25 holds for each \( k \in \{0, \ldots, \text{codim}(L)\} \). This implies that \( \Omega_L = \emptyset \), and thus that \( \xi_L = \emptyset \). \( \Box \)

In the next proposition we view the constants \( d_m \) as variables. We choose a continuous path \( [0,1] \ni \epsilon \rightarrow (d_m(\epsilon))_{m \in \mathcal{M}} \) from \( (d_m)_{m \in \mathcal{M}} \) to \( (d'_m)_{m \in \mathcal{M}} \), and consider the resulting deformation of \( \omega \) and \( \mathcal{L} \). The end point of the path corresponds to the form \( \omega' \) and its collection of \( \omega' \)-residual cosets, denoted by \( \mathcal{L}' \). Recall that \( \mathcal{M}_L \) denotes the set of \( m \in \mathcal{M} \) such that \( L_m \supset L \). Assume that \( \cap_{m \in \mathcal{M}_L} L_m(\epsilon) \neq \emptyset \) for all \( \epsilon \).

In this situation there exists a continuous path \( \epsilon \rightarrow r_L(\epsilon) \) such that \( L(\epsilon) := r_L(\epsilon) T^L \) is a connected component of \( \cap_{m \in \mathcal{M}_L} L_m(\epsilon) \). We may take \( r_L(\epsilon) \in T_L \cap L(\epsilon) \). We put \( L' = L(1) \). Assume that \( \{m \in \mathcal{M} \mid x_m(L') = d'_m\} = \{m \in \mathcal{M} \mid x_m(L) = d_m\} \).

**Proposition 4.13.** Assume that \( e(\epsilon) := c_L c_{L(\epsilon)}^{-1} \) stays within a facet of \( \mathcal{L}_L \) for all \( \epsilon \), and \( t_0(\epsilon) := t_0 c_L c_{L(\epsilon)}^{-1} \) stays within a chamber of \( \mathcal{L}_L \). With these assumptions we can take \( \xi_{L'} = r_L^{-1} r_L \xi_L \).

**Proof.** The only contributions to \( \xi_L \) come from contour shifts inside residual cosets of the configuration \( \mathcal{L}(L) \) as in the proof of Proposition 4.12. Likewise, for the construction of \( \xi_{L'} \) we only need to consider the translated configuration \( r_L^{-1} r_L \mathcal{L}(L) \). By the assumption on \( t_0 \) and Proposition 4.10 we can construct \( r_L^{-1} r_L \xi_L \), by working with \( \mathcal{L}(L) \) and \( t_0 \), but with the center \( e \) of \( T \) replaced by \( e' := e(1) \).

We now follow the deformations of the centers \( c_M(\epsilon) \) with \( \epsilon \in [0,1] \) and \( M \in \mathcal{L}(L) \). The assumption on \( e(\epsilon) \) implies that \( c_M \neq c_L \Leftrightarrow \forall \epsilon : c_M(\epsilon) \neq c_L \). This implies we can use \( \xi_L \) also as the cycle associated with \( L \) relative to the center \( e' \). \( \Box \)
Remark 4.14. Note that for some \( \epsilon \in (0, 1) \) there may be additional \( M \in L(\epsilon) \) such that \( L(\epsilon) \subset M \). It may also happen that for some values of \( \epsilon \in [0, 1] \), \( L(\epsilon)^{temp} \) contains smaller tempered cosets. We may need to adjust \( c_{l(\epsilon)} \) accordingly.

4.3. Application to the trace functional

We will now apply the above results to the integral 3.7. We thus use the rational \((n, 0)\)-form

\[
(4.26) \quad \eta(t) := \frac{dt}{q(u)2\Delta(t)c(t, q)c(t^{-1}, q)}
\]

and define the notion of quasi-residual coset as the \( \eta \)-residual cosets introduced above. We write \( L^{\nu} \) for the collection of these \( \eta \)-residual spaces, \( C^{\nu} \) for their centers etc. Note that the residual cosets of subsection 3 are included in this collection.

Apply theorem 4.4 to \( \eta \) of equation 4.26, with \( t_0 \) such that \( \alpha(\Re(t_0)) < q(s_{\alpha})^{-1} \) for all \( \alpha \in F_0 \). Denote the resulting local distributions by \( X_{\eta, c} \).

Proposition 4.15. The collection \( \{X^h \} \in C^{-\infty}(cT_u) \) of distributions \( X^h \in C^{-\infty}(cT_u) \) defined by \( X^h(a) := X_{\eta, c}(\{ t \to a(t)E_1(h) \}) \) satisfies

(i) \( \text{supp}(X^h) \subset S_{\nu}^n \).
(ii) \( \forall a \in A : \tau(ah) = \sum c_{\nu} X^h(a) \) (where \( X_c(a) \) means \( X_c(a_{|T_u}) \)).
(iii) The application \( h \to X^h \) is \( \mathbb{C} \)-linear.
(iv) \( \forall a, b \in A, h \in H : X^h(ab) = X^h(a)X^h(b) \).

Proof. These properties are simple consequences of 4.7. \( \square \)

4.3.1. Symmetrization. The main objects of this section are the \( W_0 \)-symmetric versions of the local distributions \( X^h \).

Definition 4.16. Let \( C_{\nu}^\infty \) denote the set of elements in \( C^n \) which lie in \( T_{r, \nu} = \{ t \in T_r \mid \forall a \in R_{0, \nu} : \alpha(t) \leq 1 \} \). For \( h \in H, a \in A, \) and \( c \in C_{\nu}^\infty \) put:

\[
(4.27) \quad \mathbb{P}^h_c(a) := \sum_{c' \in W_0} X^h(c') a,
\]

where \( a := |W_0|^{-1} \sum_{w \in W_0} a^w \). Then \( \mathbb{P}^h_c \) is a \( W_0 \)-invariant distribution on \( \cup_{c' \in W_0 c} T_u \), with support in \( W_0 S_{\nu}^\infty \), such that for all \( z \in Z' \):

\[
(4.28) \quad \tau(z h) = \sum_{c \in C_{\nu}^\infty} \mathbb{P}^h_c(z).
\]
It is elementary to compute the distribution $\mathfrak{Q}^h_c$ when $c = e$. Recall that ([24], Corollary 2.26) we have the following identity for the character of the minimal principal series $I_t$:

\[
\chi_{I_t} = q(w_0)^{-1} \sum_{\omega \in W_0} \Delta(wt)^{-1} E_{\omega t}.
\]

Hence we can write for all $z \in \mathbb{Z}$:

\[
\mathfrak{Q}^h_c(z) = \int_{T_u} z(t) E_i(h) \eta(t) \, dt = \int_{T_u} z(t) \chi_{I_t}(h) d\mu_T(t),
\]

where $\mu_T$ is the positive measure on $T_u$ given by

\[
d\mu_T(t) := \frac{dt}{q(w_0)c(t)c(t^{-1})}.
\]

Here we used the $W_0$-invariance of $c(t)c(t^{-1})$, and the fact that for $t \in T_u$ we have

\[
c(t)c(t^{-1}) = c(t)c(t) = c(t)^2.
\]

We see that $h \mapsto \mathfrak{Q}^h_c(1)$ is the integral of the function $T_u = S \ni t \to \chi_{I_t}(h)$ against a positive measure on $T_u$. Moreover, for every $t \in T_u$, the function $h \mapsto \chi_{I_t}(h)$ is positive and central, and is a $\mathbb{Z}$-eigenfunction with character $t$. Our first task will be to prove these properties for arbitrary centers $c \in C^{au}$. The main tools we will employ are the approximating sequences.

4.3.2. Positivity and centrality of the kernel. Let us choose, for a suitably large $N$, approximating sequences $\alpha^c_n$ for the distributions $\mathfrak{X}_{\eta,c}$. We remark that the group $\pm W_0$ acts on the collection of quasi-residual subspaces. In addition, complex conjugation also leaves this collection stable. We define an action $\cdot$ on $A$ of the group $G$ of automorphisms of $T$ generated by $W_0$, inv : $t \to t^{-1}$ and conj : $t \to t$. For elements $g \in \pm W_0$ this action is given by $g \cdot a := a^g$, and $(\text{conj} \cdot a)(t) := a(\bar{t})$.

**Lemma 4.17.** We can choose the $\alpha^c_n$ in a $G$-equivariant way, i.e. such that $\forall g \in G : a^g_n = g \cdot (\alpha^c_n)$.

**Proof.** Just notice that for any given collection of approximating sequences $A := \{a^c_n\}$ and any $g \in G, g \cdot A = \{g \cdot a_n^g\}$ is also a collection of approximating sequences for the distributions $\mathfrak{X}_{\eta,c}$, and this defines an action of $G$ on the set of collections of approximating sequences for the $\mathfrak{X}_{\eta,c}$. Hence we can take the average over $G$. \(\square\)
For $c \in C^w_n$ we now define
\begin{equation}
  z_n^c := \sum_{c' \in W_{c\otimes}} a_{n}^{c'}.
\end{equation}

Then these sequences in the center $Z$ of $H$ have the property that for all $c \in C^w_n$, $z \in Z$ and $h \in H$:
\begin{equation}
  \mathcal{Q}_c^h(z) = \lim_{n \to \infty} \tau(z_n^c z h).
\end{equation}

It is easy to see that the map $h \to \mathcal{Q}_c^h$ is central.

**Proposition 4.18.** For all $c \in C^w_n$, we have $\mathcal{Q}_c^h = 0$ if $h$ is a commutator.

**Proof.** We compute $\mathcal{Q}_c^h(z) = \lim_{n \to \infty} \tau(z_n^c z h) = 0$, because $z_n^c z h$ is also a commutator and $\tau$ is central. \qed

We define an anti-holomorphic involutive map $t \to t^*$ on $T$ by $t^* := \overline{t^{-1}}$. In view of the action of conjugation on $A$, we see that for all $z \in Z$, $z^*(t) = z(t^*)$. By Lemma 4.17 we have, for each $c \in C^w_n$,
\begin{equation}
  z_n^c(t) = z_n^c(t^{-1}) = (z_n^c)^*(t).
\end{equation}

Now we embark on the proof that the distributions $\mathcal{Q}_c^h$ are in fact (complex) measures.

**Definition 4.19.** Let us denote by $H_+$ the set of elements $h \in H$ such that $\forall x \in H : (hx, x) \geq 0$. We call this the set of positive elements of $H$.

By spectral theory in the Hilbert completion $\mathcal{H} \supset H$, this is equivalent to saying that $\lambda(h) \in B(\mathcal{H})$ is hermitian and has its spectrum in $\mathbb{R}_{\geq 0}$. Thus $H_+$ is the intersection of $H$ with the usual positive cone $C_+$ of the completion $\mathcal{C}$. It is clear that for all $x \in H$, $x^*x \in H_+$ but not every positive element is of this form. In fact, $x \in H_+$ does not imply that $\pi(x) \geq 0$ for an arbitrary representation $\pi$ of the involutive algebra $H$. This is only true for representations $\pi$ that extend to $\mathcal{C}$, and the trivial representation is a counter example for this.

We write $H^{re}$ for the subspace of real or hermitian elements, i.e. $h \in H$ such that $h^* = h$.

**Lemma 4.20.**
\begin{enumerate}
  \item If $z \in Z_+$, $h \in H_+$ then $zh \in H_+$.
  \item If $h \in H^{re}$ and $A \in \mathbb{R}_+$ such that $A \geq \|h\|_o$, then $A + h \in H_+$.
\end{enumerate}

**Proof.** A square root $\sqrt{z} \in \mathcal{B}$ of $z$ has obviously the property that $\lambda(\sqrt{z}) = \rho(\sqrt{z})$. Hence for every $x \in H$, $h \in H_+$:
\[(zhx, x) = (h\lambda(\sqrt{z})x, \lambda(\sqrt{z})x) \geq 0.\]
The second assertion follows since $\text{Spec}(\lambda(x)) \subset [-\|x\|_0, \|x\|_0]$.

**Lemma 4.21.** (i) If $c^* \not\in W_0c$ then $\mathfrak{Y}^h_c = 0$.
(ii) Let $c^* \in W_0c$, and $cs \in S^\omega_0$ such that $(cs)^* = c^{-1}s \not\in W_0(cs)$. Then $cs \not\in \text{Supp}(\mathfrak{Y}^h_c)$.

**Proof.** (i). Any $h \in \mathcal{H}$ can be decomposed as $h = h_1 + ih_2$ with $h^+_2 = h_2$ and $h^-_2 = h_1$, so it suffices to prove the assertion for $h \in \mathcal{H}^+$. Thus by Lemma 4.20 it is sufficient to prove the assertion for a positive element $h \in \mathcal{H}_+$. Similarly $z \in Z$ is a linear combination of positive central elements, so that it is sufficient to show that $\mathfrak{Y}^h_c(z) = \mathfrak{Y}^{h^*}_c(1) = 0$ for each positive central element $z$. By Lemma 4.20 this reduces our task to proving that $\mathfrak{Y}^h_c(1) = 0$ for an arbitrary element $h \in \mathcal{H}_+$. Then

\begin{equation}
0 \leq \lim_{n \to \infty} \tau(h(z_n^c + uz_n^c)^* (z_n^c + uz_n^c)) = u \mathfrak{Y}^h_c(1) + \overline{\mathfrak{Y}}^h_c(1).
\end{equation}

It follows easily that $\mathfrak{Y}^h_c(1) = \overline{\mathfrak{Y}}^h_c(1) = 0$.

(ii). This is essentially the same argument that we used to prove (i). Since $(cs)^* \not\in W_0(cs)$, we can find an open neighborhood $U \ni cs$ in $cT_u$ such that $W_0U \cap U^* = \emptyset$. Let $\phi \in C^\infty(W_0U)\overline{W_0}$. Then $\phi^* \phi = 0$, where $\phi^*(x) := \overline{\phi(x^*)}$. We want to prove that $\mathfrak{Y}^h_c(\phi) = 0$ for $h \in \mathcal{H}_+$. By Fourier analysis on $cT_u$ we can find a sequence $f_n \in \mathcal{A}^{W_c}$ such that $D(f_n)$ converges uniformly to $D(\phi)$ on $cT_u$ for every constant coefficient differential operator $D$ on $T$ of order at most $N$ on $T$. We can then find a sequence $g_n$ of the form $g_n = f_n a_n^\omega_{k(n)}$ such that $D(g_n)$ converges uniformly to $D(\phi)$ on $S^\omega_c$, and to 0 on $S^\omega_{c'}$ for every $c' \neq c$. Hence if we put

\[ \phi_n = \sum_{w \in W_c} g_n^w \in Z, \]

then for each constant coefficient differential operator $D$ on $T$ of order at most $N$, $D(\phi_n) \to D(\phi)$ uniformly on $W_0S^\omega_{c'}$, and $D(\phi_n) \to 0$ uniformly on $S^\omega_{c'}$ for $c' \neq W_0c$. Hence $\forall h \in \mathcal{H}_+, u \in \mathbb{C}$,

\begin{equation}
0 \leq \lim_{n \to \infty} \tau(h(uz_n^c + \phi_n)^* (uz_n^c + \phi_n)) = u \mathfrak{Y}^h_c(1) + u \mathfrak{Y}^{h^*}_c(\phi^*) + \overline{\mathfrak{Y}}^h_c(\phi)
\end{equation}

If we divide this inequality by $|u|$ and send $|u|$ to 0, we get that $\forall \epsilon \in \mathbb{C}$ with $|\epsilon| = 1$,

\begin{equation}
0 \leq \epsilon \mathfrak{Y}^h_c(\phi^*) + \overline{\mathfrak{Y}}^h_c(\phi)
\end{equation}

It follows that $\forall h \in \mathcal{H}_+$, $\mathfrak{Y}^h_c(\phi) = \mathfrak{Y}^{h^*}_c(\phi^*) = 0$. Hence the same is true for arbitrary $h \in \mathcal{H}$. \hfill \Box
Corollary 4.22. If $h \in \mathcal{H}_+$, the distribution $\mathfrak{g}_c^h$ is a $W_0$-invariant positive measure on $W_0cT_u$, supported on $W_0S^{\alpha}_c$.

Proof. It suffices to show that $\mathfrak{g}_c^h$ is a positive distribution. Assume that $\phi \in C^\infty(W_0cT_u)^{W_0}$ and that $\phi > 0$. Then the positive square root $\sqrt{\phi}$ is also in $C^\infty(W_0cT_u)^{W_0}$. Using the approximating sequences as we did before, we can find a sequence $f_n \in \mathcal{Z}$ such that $D(f_n) \to D(\sqrt{\phi})$, uniformly on $W_0S^{\alpha}_c$, and to 0 on $S^{\alpha}_c$ for $c' \neq c$. By Lemma 4.21, the support of $\mathfrak{g}_c^h$ is contained in $W_0S^{\alpha}_c := W_0S^{\alpha}_c \cap T^{\text{herm}}$, where $T^{\text{herm}} := \{ t \in T \mid t^* \in W_0 t \}$. This is itself a regular support for distributions. On $W_0S^{\alpha}_c$, the sequence $\phi_n := f_n^* f_n \in \mathcal{Z}_+$ converges uniformly to $\phi$ up to derivatives of order $N$. Hence

$$0 \leq \lim_{n \to \infty} \tau(h f_n^* f_n) = \mathfrak{g}_c^h(\phi).$$

This proves the desired inequality. \hfill \Box

Corollary 4.23. Put $\nu_c := \mathfrak{g}_c^1$. Then for all $h \in \mathcal{H}$, $\mathfrak{g}_c^h$ is absolutely continuous with respect to $\nu_c$.

Proof. It is enough to prove this for $h \in \mathcal{H}$ which are hermitian, i.e. such that $h^* = h$. By Lemma 4.20 and Corollary 4.22 we see that for positive functions $\phi \in C^\infty(W_0cT_u)^{W_0}$,

$$-\|h\|_{\phi_c}(\phi) \leq \mathfrak{g}_c^h(\phi) \leq \|h\|_{\phi_c}(\phi).$$

\hfill \Box

Definition 4.24. Let $\nu := \sum_{c \in \mathcal{C}_c^{\alpha}} \nu_c$. We define a bounded, $W_0$-invariant function $t \to \chi_t \in \mathcal{H}^*$ on $T$ by

$$\sum_{c \in \mathcal{C}_c^{\alpha}} \mathfrak{g}_c^h(\phi|_{W_0S^{\alpha}_c}) = \int_T \phi(t) \chi_t(h) d\nu(t).$$

for each $\phi \in C_c(T)^{W_0}$. For $t$ outside the support of $\nu$ we set $\chi_t(h) = 0$.

Corollary 4.25. The function $t \to \chi_t \in \mathcal{H}^*$ satisfies

(i) The support of $t \to \chi_t$ is the support of $\nu$.

(ii) $\chi_t \in \mathcal{H}^*$ is a positive, central functional such that $\chi_t(1) = 1$, $\nu$ almost everywhere on $T$.

(iii) For $h \in \mathcal{H}$, $z \in \mathcal{Z}$ : $\chi_t(z h) = z(t) \chi_t(h)$, $\nu$ almost everywhere on $T$.

(iv) $\chi_t$ extends, for $\nu$-almost all $t$, to a tracial state of the $C^*$-algebra $\mathcal{C}$.

Proof. Everything is clear except perhaps the continuity assertion of (iv). By equation 4.40 and Definition 4.24 it follows that $|\chi_t(h)| \leq \|h\|_\nu$. 

for Hermitian $h \in \mathcal{H}$. By (ii) we see that $\langle x, y \rangle := \chi_l(x^* y)$ is a positive semi-definite Hermitian from on $\mathcal{H}$, and thus it satisfies the Schwartz inequality. For arbitrary $x \in \mathcal{H}$ we thus get $|\chi_l(x)|^2 \leq \chi_l(1)\chi_l(x^* x) \leq \|x^* x\|_o = \|x\|_o^2$. 

4.4. The probability measure $\nu$ and the $A$-weights of $\chi_l$

The measure $\nu$ can be computed almost explicitly, due to the properties of residual cosets which were discussed in subsection 3.1. As we remarked in that subsection, the proofs of these facts are based on classification. However, we will see the underlying reason for these properties in the present paragraph.

The results in this subsection are based on the fact that the Eisenstein kernel of 4.1 simplifies considerably when restricted to the subalgebra $A$ of $\mathcal{H}$. We will exploit this fact here to study the behaviour of the states $\chi_l$ on $A$.

**Theorem 4.26.** The measure $\nu$ is a $W_0$-invariant probability measure, whose support is contained in the union of the compact cosets $L^{temp}$, where $L$ runs over the collection of residual cosets as defined in Section 3. Let $d^L$ denote the normalized Haar measure on $T^L_w$, transported to the coset $L^{temp}$ by translation. On $L^{temp}$, the measure $\nu_L$ is given by a density function $\delta_L m_L := \frac{d\nu_L(t)}{d^L(t)}$, where $\delta_L \in \mathbb{Q}$ is a constant and where $m_L$ is of the form

$$m_L(t) = q(w_0) \frac{\prod_{\alpha \in R_1} (1 - \alpha(t))}{\prod_{\alpha \in R_0} (1 + q_0^{1/2} \alpha(t)^{1/2})(1 - q_0^{1/2} q_2 \alpha(t)^{1/2})}.$$

Here we used the convention of Remark 4.1. The constant $\delta_L$ is independent of $q$ if we assume $q$ to be of the form 2.22. The notation $\prod$ means that we omit the factors which are identically equal to 0 on $L$. The density $m_L$ is a smooth function on $L^{temp}$.

**Proof.** We apply Proposition 4.7 to the integral

$$\tau(a) = \int_{T^L_w} a(t) \frac{dt}{q(w_0)c(t)c(t^{-1})} = \sum_{c \in C_{qu}} \mathcal{X}^L_c(a).$$

Choose $\epsilon > 0$. For a suitably small $\delta > 0$ we can find, for each quasi residual subspace $L$, an $\ell^L \in T^{L'}_{\epsilon}$ in an $\epsilon$ neighborhood of $\ell$, and a cycle $\xi_L \subset B_L(r_L, \delta) \setminus \cup_{L' \supset L} L'_m$, where $B_L(r_L, \delta) \subset T_L$ denotes a ball of radius $\delta > 0$ centered around $r_L$, such that

$$\mathcal{X}^L_c(a) := \sum_{L \subset L' = \epsilon} \int_{\xi_L} \left\{ \int_{L' \in T_L} a'(t') \frac{d_L(t')}{q(w_0)c(t')c(t'^{-1})} \right\} d^L(t).$$

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Here $d^L(t)$ is the holomorphic extension to $L$ of $d^L$, and $d_L t'$ denotes the Haar measure on $T_{L,s'}$, also extended as a holomorphic form on $T_L$. We assume that $\delta$ is small enough to assure that log is well defined on $B_L(r_L, \delta)$. For the inner integral we use orthonormal linear coordinates $(x_i)$ centered at $\log(r_L)$. We note that we can write the integrand as:

\begin{equation}
(4.44)\quad a(t) m_L(t)(1 + f_1) \omega_L
\end{equation}

where $\omega_L$ is a rational homogeneous $(l := \dim(T_L), 0)$-form (independent of $t$), and $f_1$ is a power series in $x_i$ such that $f_1(0) = 0$. In fact, the form $\omega_L$ is easily seen to be

\begin{equation}
(4.45)\quad \omega_L(x) = \frac{\prod_{\alpha \in R^+_L} \alpha(x)}{i^l \text{vol}(t_L/(2\pi Y_L)) \prod_{\beta \in \mathfrak{r}_L^+} \beta(x)} dx_1 \wedge dx_2 \cdots \wedge dx_l.
\end{equation}

By Corollary 3.11 it follows that the form $\omega_L$ has homogeneous degree $\geq 0$ if $L$ is residual in the sense of Definition 3.1. A homogeneous closed rational form of positive homogeneous degree is exact. Hence the inner integral will be nonzero only if $L$ is in fact a residual coset. In that case the inner integral will have value

\begin{equation}
(4.46)\quad \kappa_L a(r_L t) m_L(r_L t)
\end{equation}

for some $\kappa_L \in \mathbb{Q}$ (since $\omega_L$ defines a rational cohomology class). Let us therefore assume that $L$ is residual from now on. Write $r_L = s c$. By Theorem 3.8 we know that $r_L^s = sc^{-1} = w_s(r_L)$ with $w_s \in W(R_{L,s,1})$. When $t \in L^{temp}$, the expression $m_L(t)$ can be rewritten as

\begin{equation}
(4.47)\quad q(w^L) m_{R_{L,1}}(r_L) \prod_{\alpha \in R_{L,1}^+} \frac{[1 - \alpha(t)]^2}{1 + q_{\alpha^1/2} \alpha(t)^{1/2} \alpha(t)^{1/2} [1 - q_{\alpha^1/2} q_{\alpha^2} \alpha(t)^{1/2}]^2}.
\end{equation}

Here we used that if $t = cu \in L^{temp}$ with $u \in s T_u^L$, we have $w_s c = c^{-1}$, $w_s u = u$, and $w_s(R_{L,1} \setminus R_{L,1}^+ = R_{L,1} \setminus R_{L,1}^+$. By the same argument as was used in Theorem 3.13 of [11] we see that this expression is real analytic on $L^{temp}$. This implies that we can in fact take $e^L = e$ for all residual $L$ in equation 4.43 after we evaluate the inner integrals. This leads to

\begin{equation}
(4.48)\quad \mathcal{F}_L(a) = \sum_{L \leq L = c} \kappa_L \int_{L^{temp}} a(t) m_L(t) d^L(t).
\end{equation}

where the sum is taken over residual cosets only. When we combine terms over $W_0$ orbits of residual cosets we find the desired result. Let $W_0 L$ denote the set of residual cosets in the orbit of $L$. We have to
take

\begin{equation}
\delta_{L} = \frac{1}{|W_{0L}|} \sum_{L' \in W_{0L}} \kappa_{L'}.
\end{equation}

We note in addition that \( \kappa_{L} = \kappa_{R_{L}, \{r_{L}\}} \), because the cycle \( \xi_{L} \) is constructed inside \( T_{L} \), entirely in terms of the root system \( R_{L} \). Also, it is clear that \( m_{R_{L}, \{r_{L}\}}(r_{L}) \) is independent of the choice of \( r_{L} \), because the finite group \( K_{L} = T_{L} \cap T^{L} \) is contained in the simultaneous kernel of the roots of \( R_{L} \). Finally, the independence of \( q \) is clear from Proposition 4.13. When we apply a scaling transformation \( q \to q' \), the point \( c_{L} \) moves such that the facet of the dual configuration containing \( e \) does not change. Hence \( r_{L}^{-1} \xi_{L} \) and \( \omega_{L} \) will be independent of \( \epsilon \). \( \square \)

**Proposition 4.27.** For \( L \) residual, denote by \( R_{L} = (X_{L}, Y_{L}, R_{L}, R_{L}, F_{L}) \) the root datum as was introduced in Proposition 3.2 (iv), and by \( q_{L} \) the restriction of the label function \( q \) to \( R_{L} \). Then \( \{r_{L}\} \subset T_{L} \) is a \((R_{L}, q_{L})\) residual point. Assume that \( R_{L} \) is a standard parabolic subsystem of roots, and thus that \( F_{L} \subset F_{0} \). Denote by \( W_{L} \) the standard parabolic subgroup \( W_{L} = W(R_{L}) \) of \( W_{0} \), and let \( W^{L} \) denote the set of minimal length representatives of the right \( W_{L} \) cosets in \( W_{0} \).

(i) When \( w \in W^{L} \), we may take \( \xi_{wL} = w(\xi_{L}) \). Consequently, \( \kappa_{L} = \kappa_{wL} \) if \( w \in W^{L} \).

(ii) Put

\begin{equation}
m^{L}(t) := q(w_{L}) \prod_{\alpha \in R_{L, +} / R_{L, -}} \frac{|1 - \alpha(t)|^{2}}{1 + q_{\alpha}^{1/2} \alpha(t)^{1/2} |1 - q_{\alpha}^{1/2} q_{2\alpha} \alpha(t)^{1/2}|^{2}}.
\end{equation}

Then \( m^{L} \) and \( m_{L} \) are \( \text{Aut}(W_{0}) \)-equivariant, i.e. \( m^{L}(t) = m^{\gamma L}(\gamma t) \) and \( m_{L}(t) = m_{\gamma L}(\gamma t) \) for every \( \gamma \in \text{Aut}(W_{0}) \). In particular, \( m^{L} \) and \( m_{L} \) are invariant for the stabilizer \( N_{L} \) of \( L \) in \( W_{0} \).

(iii) For \( z \in \mathcal{Z} \), we have

\begin{equation}
\frac{1}{|W_{0L}|} \int_{T} zdv_{L} = \nu_{R_{L}, \{r_{L}\}}(\{r_{L}\}) \int_{L_{\text{emp}}} z(t)m^{L}(t)d^{L}(t).
\end{equation}

**Proof.** (i). We note that for \( w \in W^{L} \), \( t_{0} \) and \( w^{-1}t_{0} \) are in the same chamber of \( \mathcal{L}^{L} \). Hence, by application of 4.10, we may replace \( \xi_{wL} \) by \( w(\xi_{L}) \).

(ii). This is trivial.

(iii). Let \( W_{L} = W(R_{L}) \). Let \( N_{T^{L}} \) be the stabilizer of \( T^{L} \) in \( W_{0} \). Observe that \( N_{L} \subset N_{T^{L}} \) and \( W_{L} \subset N_{T^{L}} \). If we define \( \Gamma_{L} = N_{T^{L}} \cap W_{L} \) then \( \Gamma_{L} \) is a complementary subgroup of \( W_{L} \) in \( N_{T^{L}} \). Using (i), (ii)
and the remark $\kappa_L = \kappa_{R_L,(r_L)}$ as in the proof of Theorem 4.26, we see that

$$
\delta_L = \frac{1}{|W_0L|} \sum_{L' \in W_0L} \kappa_{L'}
= \frac{1}{|W_0T_L|} \sum_{L' \in N_{T_L}L} \kappa_{L'}
= \frac{1}{|W_0T_L| |N_{T_L}L|} \delta_{R_L,(r_L)} = \delta_{R_L,(r_L)}
$$

(4.52)

Using Theorem 4.26 and equation 4.47 the result follows. 

The next proposition is a direct consequence of (the proof of) Theorem 4.26 and the definition of $\chi_1$.

**Proposition 4.28.** Let $r = sc \in T$ be a residual point, and let $a \in A$. Then

$$
\nu(W_0r)\chi_1(a) = m_{[r]}(r) \sum_{r' \in W_0r} \kappa_{[r']}a(r').
$$

(4.53)

**Theorem 4.29.** The support of $\nu$ is exactly equal to the union of the tempered residual cosets.

Proof. By Proposition 4.27 this reduces to the case of a residual point $r = sc$. By Proposition 4.28 it is enough to show that there exists at least one $r' = wr \in W_0r$ such that $\kappa_{[r']} \neq 0$. In other words, using Proposition 4.28 we single out the point residue at $r'$. In particular, we ignore all residues at residual cosets which do not contain $r'$ and thus do not contribute to $\kappa_{[r']}$ in the argument below.

By the $W_0$-invariance of $\omega$, we can formulate the problem as follows. Recall from the proof of Theorem 4.26 that

$$
\kappa_{[r]}m_{[r]}(r) = \int_{\xi} \omega,
$$

(4.54)

where $\xi$ is the residue cycle at $r$, which is obtained from Proposition 4.7 if we use the $n$-form

$$
\omega(t) = \frac{dt}{e(t)c(t^{-1})}
$$

(4.55)

and a base point $t_0 \in T_{rs}$ such that $\forall \alpha_i \in F_0 : \alpha_i(t_0) < q(s_i)$. By definition, $m_{[r]}(r) \neq 0$. For $r' = wr$ we have

$$
\kappa_{[r']}m_{[r']}(r) = \int_{\xi(w)} \omega,
$$

(4.56)

where $\xi(w)$ is the cycle near $r$ which we obtain in Proposition 4.7 when we replace $t_0$ by $w^{-1}t_0$. Hence we have to show that there exists a proper choice for $t_0$ such that when we start the contour shift algorithm from this point, the corresponding point residue at $r$ will be nonzero.
The problem we have to surmount is possible cancellation of nonzero contributions to $\kappa_r(\nu)$. We will do this by showing that there exists at least one chamber such that the residue at $r$ consists only of one nonzero contribution.

We consider the real arrangement $\mathcal{L}^{(r)}$ in $T_{rs}$, and transport the Euclidean structure of $t$ to $T_r$, by means of $t \to \log(c^{-1}t)$. Then $\mathcal{L}^{(r)}$ is the lattice of intersections of a central arrangement of hyperplanes with center $c$. We assign indices $i_L$ to the elements of $\mathcal{L}^{(r)}$ by considering the index of the corresponding complex coset containing $r$, and we note that by Corollary 3.11, $i_L = n : = \text{codim}(\{r\})$. From Corollary 3.11 we further obtain the result that there exist full flags of subspaces $c_L T_{rs} \in \mathcal{L}^{(r)}$ such that $i_L = \text{codim}(L)$. In particular, there exists at least one line $l$ through $r$ with $i_l = n - 1$.

By Theorem 3.13 we see that the centers $c_L, c_{L'}$ of two “residual subspaces” $c_L T^L \subset c_{L'} T^{L'}$ (i.e. $\text{codim}(T^L) = i_L$ and $\text{codim}(T^{L'}) = i_{L'}$) in $\mathcal{L}^{(r)}$ satisfy $c_L \neq c_{L'}$. Hence every residual line $l \in \mathcal{L}^{(r)}$ is divided in two half lines by $c$, and $c_l$ lies in one of the two halves. Let us color the half line containing $c_l$ blue, and the other half line red.

We want to find a chamber for $t_0$ such that the corresponding point residue $\kappa_r m_\nu(r)$ at $r$ is nonzero. We argue by induction on the rank. If the rank of $R_0$ is 1, obviously we get $\kappa_r(\nu) \neq 0$ if we choose $t_0$ in the red half line, because we then have to pass a simple pole of $\omega$ at $r$ when moving the contour $t_0 T_u$ to $T_u$ (since $t_0$ and $e = c_T$ are separated by $c$). Assume by induction that for any residual point $p$ of a rank $n - 1$ root system, we can choose a chamber for $t_0$ such that $\kappa_r(p) \neq 0$. Let $S \subset T_{rs}$ be a sphere centered at $r$ through $e$, and consider the configuration of hyperspheres in $\mathcal{L}^{(r)} \cap S$. If $L_S = c_L T_{rs} \cap S$ with $\text{dim}(T^L) > 1$, we denote by $c_{L_S}$ the intersection of the half line through $c_L$ beginning in $c$ (recall that $c \neq c_L$) and $L_S$. We call this point the center of $L_S$. In the case when $\text{dim}(T^L) = 1$, $L_S$ consists of two antipodal points $p$ and $\overline{p}$, one blue and the other red, as was described above. Both of these are considered as centers of $\mathcal{L}^{(r)} \cap S$.

Consider a closed (spherical) ball $D \subset S$ centered at $e$ such that $D$ contains a point $p$ of a red half line $l$ in its boundary sphere, but no red points in its interior. We take $t_0$ in $S$, and we apply the algorithm as described in the proof of Proposition 4.7, but now on the sphere $S$, and with respect to the the sets $L_S$ and their centers.

By the induction hypothesis, we can take $t_0 \in S$ close to $p$ in a chamber of the configuration $\mathcal{L}^{(r)} \cap S$ which contains $p$ in its closure, such that a nonzero residue at $l$ is picked up in $p$. As was explained in 4.13, the residue at $p$ does not change when we apply the contour shifts as in the proof of Proposition 4.7 to $\mathcal{L}^{(r)} \cap S$ with respect to a “fake
identity element” \( \check{e} \) if \( e \) and \( \check{e} \) belong to the same chamber of the dual configuration \((\mathfrak{S})\) of the central sub-arrangement of \( \mathcal{L}^{(r)} \cap \mathfrak{S} \) with center \( p \cap \pi \). We choose \( \check{e} \) close to \( p \), in the interior of \( D \). By Proposition 4.7 we can replace the integral over \( t_0 T_u \) by a sum of integrals over cosets of the form \( \check{c}_L \cap \mathfrak{S} \check{s}_L T_u^L \) (for some \( \check{s}_L \in T_u \)) of the residue kernel \( \check{c}_L m_L \) (cf. equation 4.46) on \( L \). As was mentioned above, we are only interested in such contributions when \( r \in L \), which means that we may take \( \check{s}_L = s \). The “fake centers” \( \check{c}_L \cap \mathfrak{S} \) are in the interior of \( D \).

Next we apply the algorithm of contour shifts as in 4.7 to move the cycles \( \check{c}_L \cap \mathfrak{S} \check{s}_L T_u^L \) to \( c_L \cap \mathfrak{S} \check{s}_L T_u^L \). Since both the fake centers \( \check{c}_L \cap \mathfrak{S} \) and the real centers \( c_L \cap \mathfrak{S} \) belong to the interior of \( D \), and since the intersection of \( D \) with \( L \) is connected if \( \dim(L) > 0 \), we can choose every path in the contour shifting algorithm inside the interior of \( D \). Thus, the centers \( c_L \cap \mathfrak{S} \) of the residual cosets \( L \) that arise in addition the one red center \( c_0 = p \) in the above process are in the interior of \( D \). In particular, with the exception of \( p s T_u^L \), the one dimensional cosets of integration which show up in this way, all have a blue center.

Finally, in order to compute the residue \( \kappa_{L \cap \mathfrak{S}}(r) \) at \( s \), we now have to move the center \( c_L \cap \mathfrak{S} \in L \) to the corresponding center \( c_L \in T_r \) of \( L \), for each residual coset \( L \) which contains \( r \) and which contributes to \( \int_{T_u} \omega \). The only such center of \( \mathcal{L}^{(r)} \cap \mathfrak{S} \) which will cross \( e \) is the red center \( p \). Since \( m_L \) has a simple pole at \( r = sc \), we conclude that this gives a nonzero residue at \( r \). Hence with the above choice of \( t_0 \) we get \( \kappa_{L \cap \mathfrak{S}} \neq 0 \), which is what we wanted to show.

\[ \kappa_{L \cap \mathfrak{S}}(r) \neq 0 \] for all \( r \in L \) and for all \( a \in \mathcal{A} \):

\[ \chi_r(a) = \frac{1}{\delta_{L \cap \mathfrak{S}} |W_0 r|} \sum_{r' \in W_0 r} \kappa_{L \cap \mathfrak{S}}(r') \] (4.57)

Moreover, \( \kappa_{L \cap \mathfrak{S}}(r) = 0 \) unless \( \forall x \in X^+ \setminus \{0\} : |x(r')| < 1 \) (where \( X^+ \) denotes the set of dominant elements in \( X \)).

**Proof.** This is now immediate, except for the last assertion. This fact follows from Proposition 4.12. We know that \( e \) is regular in \( L_{(r)} \) by Theorem 3.13. On the other hand, \( t_0 \) lies in \( c T_{r_s} \), which is clearly a subset of a chamber of \( \mathcal{L}^{(r')} \). The anti-dual of the chamber of \( \mathcal{L}^{(r')} \) containing \( t_0 \) is thus a subset of \( c T_{r_s}^+ \), with \( T_{r_s}^+ := \{ t \in T_{r_s} \mid \forall x \in X^+ \setminus \{0\} : x(t) > 1 \} \). Thus when \( e \) is contained in the anti-dual chamber we have \( e' \in T_{r_s}^- \) as desired.

For \( t \in \text{Supp}(\nu) \) we have defined a tracial state \( \chi_t \). We define the associated symmetric semi-definite Hermitian form \( (x, y)_t := \chi_t(x^* y) \)
Remark 4.34. We note in addition that if the restrictions \( \nu_{r_i} \) all have the same sign (equal to the sign of \( m(a) \)),

\[
\sum_{i} \dim (V'_r)^{\nu_{r_i}} = \sum_{r_i} \nu_{r_i} \mid a \mid.
\]  

(4.63)

In particular we conclude that the nonzero \( \nu_{r_i} \) all have the same sign to \( A \) of the characters \( \chi_i \) are linearly independent, it follows from the formal degrees \( d_i > 0 \) of the characters \( \chi_i \) of the residue Provenas algebra \( \mathcal{H}_f \) (with \( r \), a residual point, satisfies the following system of linear equations.

\[
\sum_{i} \dim (V'_r)^{\nu_{r_i}} = \sum_{r_i} \nu_{r_i} \mid a \mid.
\]  

(4.64)

Hence from \( d_i > 0 \),

\[
\sum_{r_i} \nu_{r_i} \mid a \mid = \sum_{r_i} \dim (V'_r)^{\nu_{r_i}}.
\]  

(4.65)

and Corollary 4.30 we conclude that the generalized weight spaces of \( \chi_i \) indeed satisfy the Casselman criterion for discrete series.

\[ x_i(a) = \sum_{r_i} \dim (V'_r)^{\nu_{r_i}}. \]  

(4.66)

Corollary 4.32. If \( H_f \) is a (finite dimensional) Provenas algebra with trace \( x_i \), we refer to this algebra as the residual Provenas algebra at \( x_i \).

Let \( x_i \) denote the set of minimal central elements of \( H_f \) and the associated irreducible characters 

\[ \chi_i \in H_f. \]  

We define \( d_i := \dim (H_f) \), \( x_i \in H_f \) the associated irreducible characters of \( H_f \) in fact elements of the Hilbert completion \( 5 \) of \( H_f \).

Proof. By Casselman's criterion (4.11) and Lemma 4.11 it is enough to show that the representation \( (\chi_i, \chi_i) \) of \( H_f \) afforded by \( \chi_i \) has nonzero generalized weight spaces \( V' \), only for \( r \) in \( W_f \) such that \( V' \neq 0 \).

4.12

Definition 4.31. The algebra \( H_f \) is a (finite dimensional) Provenas algebra with trace \( x_i \) if it is \( H_f \) such that \( V' \neq 0 \).

Let \( x_i \) denote the set of minimal central elements of \( H_f \) and the associated irreducible characters of \( H_f \) in fact elements of the Hilbert completion \( 5 \) of \( H_f \).

Proof. By Casselman's criterion (4.11) and Lemma 4.11 it is enough to show that the representation \( (\chi_i, \chi_i) \) of \( H_f \) afforded by \( \chi_i \) has nonzero generalized weight spaces \( V' \), only for \( r \) in \( W_f \) such that \( V' \neq 0 \).

4.12

Definition 4.31. The algebra \( H_f \) is a (finite dimensional) Provenas algebra with trace \( x_i \) if it is \( H_f \) such that \( V' \neq 0 \).

Let \( x_i \) denote the set of minimal central elements of \( H_f \) and the associated irreducible characters of \( H_f \) in fact elements of the Hilbert completion \( 5 \) of \( H_f \).
4.4.1. Temperedness of $\chi_t$. In this subsection we discuss the tempered
growth behaviour of the $\chi_t$ on the orthonormal basis $N_w$ of $H$, as a
corollary of the analysis of the $A$-weights of $\chi_t$. The results of this
section are supplementary, they will not be used in the sequel. We can
avoid the use of temperedness because $C$ is unitary and of finite type
I, implying that every irreducible representation of $C$ has a continuous
character. Nevertheless it is natural to briefly discuss this issue here.

**Definition 4.35.** A functional $f \in H^*$ is called tempered if there exists
an $N \in \mathbb{N}$ and constant $C > 0$ such that for all $w \in W$,
\[
|f(N_w)| \leq C(1 + l(w))^N.
\]

Here $l(w)$ denotes the length function, and the $N_w = q(w)^{-1/2}T_w$ are
the orthonormal basis elements of $H$ introduced in 2.14.

**Lemma 4.36.** (Casselman’s criterion) The character $\chi$ of a finite
dimensional representation $(V, \pi)$ of $H$ is tempered if and only if the
weights $t$ of the generalized $A$-weight spaces of $V$ satisfy $\forall x \in X^+$ :
$|x(t)| \leq 1$.

**Proof.** If there exists a weight $t$ of $V$ violating the condition, then there
exists a $x \in X^+$ such that $x(t) > 1$. Recall that for $x \in X^+$, $N_x = \theta_x$.
Hence $|\chi(N_x)| = |\chi(\theta_x)|$ has exponential growth in $n$. Conversely,
assume that all the weights of $V$ satisfy the condition. Recall that the
elements $T_w\theta_xT_w$ with $x \in X^+$, $u, v \in W_0$ span the subspace of $H$ with
basis $N_w$, where $w$ runs over the double coset $W_0xW_0 \subset W$ (see the
proof of Lemma 3.1 of [24]). It is not difficult to see that in fact we
can write, for $w = uxx \in W_0xW_0$ with $x \in X^+$,
\[
N_w = \sum_{u', u''} c_{w, (u', u'')} T_{u'} \theta_x T_{u''},
\]
such that the coefficients $c_{wxu, (u', u'')} = c_{wxu, (u', u'')}$ are equal if $x$ and $y$
belong to the same facet of the cone $X^+$. Moreover, by the length
formula [24], equation 1.1, we have
\[
l(x) - |R_{0,+}| \leq l(uxv) \leq l(x) + |R_{0,+}|
\]
with $l(x) = x(2\rho^\vee)$. Therefore it is enough to show that there exists a
constant $C > 0$ and $N \in \mathbb{N}$ such that for all $x \in X^+$ and $u, v \in W_0$,
\[
|\chi(T_u \theta_x T_v)| \leq C(1 + x(2\rho^\vee))^N.
\]

It suffices to show that the matrix entries of $\pi(\theta_x)$ in a generalized
$A$-weight space $V_t$ with weight $t$ are polynomially bounded in $x(2\rho^\vee)$. 
By Lie’s Theorem we can put the $\pi(\theta_x)$ simultaneously in upper triangular form. Choose $\mathbb{Z}_+$-generators $x_1, \cdots, x_m$ for the cone $X^+$, and commuting strictly upper triangular matrices $N_1, \cdots, N_m$ such that

$$\pi(\theta_x)|_{V_i} = x_i(t) \exp(N_i).$$

(4.66)

Hence for $x = \sum_{i=1}^m l_i x_i$ with $l_i \in \mathbb{Z}_+$ we have

$$\pi(\theta_x)|_{V_i} = x(t) \exp\left( \sum_{i=1}^m l_i N_i \right).$$

(4.67)

Since $|x(t)| \leq 1$ by assumption, and the exponential map is polynomial on the space of strictly upper triangular matrices, we see that the matrix entries are bounded by a polynomial in the $l_i$. Observe that $x_i(2\rho^\vee) \in \mathbb{Z}_{\geq 0}$. Since the coefficients $l_i$ are nonnegative this implies that $l_i \leq x(2\rho^\vee)$. This gives us the desired estimate of the matrix entries by a polynomial in $x(2\rho^\vee)$. \qed

**Corollary 4.37.** Let $L$ be residual such that $W_L$ is a standard parabolic subgroup of $W_0$. For $t \in L^{\text{temp}}$ we write $t = r_L t^L$, with $t^L \in T^L_n$. We consider $\chi_t|_A$ as a formal linear combination of elements of $T$. Likewise, let $A_L = \mathbb{C}[X_L]$ be the ring of regular functions on $T_L \subset T$. We consider $\chi_{r_L(t^L)}|_{A_L}$ as a formal linear combination of elements of $T_L$. In this sense we have, $\nu_L$-almost everywhere on $L^{\text{temp}}$,

$$\chi_t|_A = \frac{1}{|W_L|} \sum_{w \in W_L} w(t^L \chi_{r_L(t^L)}|_{A_L}).$$

(4.68)

Hence $\nu$-almost everywhere, $\chi_t$ is a nonzero tempered function on $\mathcal{H}$.

**Proof.** Equation 4.68 follows by a straightforward computation similar to Proposition 4.28, using Proposition 4.27 and the definition of $\chi_t$. Since $\chi_t$ is a positive combination of the irreducible characters of the residue Frobenius algebra $\overline{H}$, it follows that the weights $t \in W_{\text{red}}$ of the generalized $A$-eigenspaces of $\overline{H}$ satisfy $\forall x \in X^+: |x(t)| \leq 1$. This shows, by Casselman’s criterion, that $\chi_t$ is a tempered functional on $\mathcal{H}$. \qed

**Remark 4.38.** In a similar way we can show that the coefficients of $\chi_r$ (with $r$ a residual point of $T$) with respect to the basis $N_w$ have exponential decay. More precisely, there exist constants $C, \epsilon > 0$ such that

$$|\chi_r(N_w)| \leq C \exp(-\epsilon l(w)).$$

(4.69)
ON THE SPECTRAL DECOMPOSITION OF AFFINE HECKE ALGEBRAS

5. Localization of the Hecke algebra

In this section we review some important ideas of Lusztig [17] about localizations of the affine Hecke algebra. In fact Lusztig works with the less gentle procedure of $T_v$-adic completion of the center $\mathcal{Z}$ with respect to a maximal ideal $\mathcal{I}_v$. It is not difficult to see that Lusztig’s arguments can be adapted to (analytic) localization with respect to a suitably small open neighborhoods $U \supset W_0 t$ of orbits of points in $T$, and this will be discussed in this section.

When $s = s_\alpha \in S_0$ (with $\alpha \in F_1$), we define an intertwining element $R_s$ as follows:

$$R_s = (1 - \theta_{-\alpha})T_s + ((1 - q_\alpha q_{2\alpha}) + q_\alpha^{1/2}(1 - q_{2\alpha})\theta_{-\alpha/2})$$

$$= T_s(1 - \theta_\alpha) + ((q_\alpha q_{2\alpha} - 1)\theta_\alpha + q_\alpha^{1/2}(q_{2\alpha} - 1)\theta_{\alpha/2})$$

We remind the reader of the convention of Remark 4.1. These elements are important tools to study the Hecke algebra. We recall from [24], Theorem 2.8 that these elements satisfy the braid relations, and they satisfy (for all $x \in X$)

$$R_s \theta_x = \theta_{s(x)} R_s,$$

and finally they satisfy

$$R_s^2 = (q_\alpha^{1/2} + \theta_{-\alpha/2})(q_\alpha^{1/2} + \theta_{\alpha/2})(q_\alpha^{1/2} q_{2\alpha} - \theta_{-\alpha/2})(q_\alpha^{1/2} q_{2\alpha} - \theta_{\alpha/2}).$$

(where we have again used the convention of Remark 4.1!). Suitably normalized versions of the $R_s$ generate a group isomorphic to the Weyl group $W_0$. In order to normalize the intertwiners, we need to tensor $\mathcal{H}$ by the field of fractions $\mathcal{F}$ of the center $\mathcal{Z}$. So let us introduce the algebra

\begin{equation}
\mathcal{F}\mathcal{H} := \mathcal{F} \otimes_{\mathcal{Z}} \mathcal{H}
\end{equation}

with the multiplication defined by $(f \otimes h)(f' \otimes h') := ff' \otimes hh'$. Notice that this an algebra over $\mathcal{F}$ of dimension $|W_0|^2$. The subalgebra $\mathcal{F}A = \mathcal{F} \otimes_{\mathcal{Z}} A$ is isomorphic to the field of fractions of $A$. The field extension $\mathcal{F} \subset \mathcal{F}A$ has Galois group $W_0$, and we denote by $f \rightarrow f^w$ the natural action of $W_0$ on the field of rational functions on $T$. The elements $T_w$ with $w \in W_0$ form a basis for $\mathcal{F}H$ for multiplication on the left or multiplication on the right by $\mathcal{F}A$, in the sense that

\begin{equation}
\mathcal{F}\mathcal{H} = \bigoplus_{w \in W_0} \mathcal{F}A T_w = \bigoplus_{w \in W_0} T_w \mathcal{F}A.
\end{equation}
The algebra structure of $\mathcal{H}$ is determined by the Lusztig relations as before: when $f \in \mathcal{H}$ and $s = s_\alpha$ with $\alpha \in F_1$, we have

$$fT_s - T_sf^s = \left((q_{2\alpha}q_{\alpha}^\lambda - 1) + q_{\alpha}^{1/2}(q_{2\alpha} - 1)\theta_{-\alpha/2}\right)\frac{f - f^s}{1 - \theta_{-\alpha}}$$

We have identified $\mathcal{A}$ with the algebra of regular functions on $T$ in the above formula.

Let us introduce

$$n_\alpha := q(s_\alpha)\Delta_\alpha c_\alpha$$

$$= (q_{\alpha}^{1/2} + \theta_{-\alpha/2})(q_{\alpha}^{1/2}q_{2\alpha} - \theta_{-\alpha/2}) \in \mathcal{A},$$

where we used the Macdonald $\alpha$-function introduced in equation 4.4 and 4.5.

The normalized intertwiners are now defined by (with $s = s_\alpha$, $\alpha \in R_1$):

$$R_\alpha^0 := n_\alpha^{-1} R_s \in \mathcal{H}.$$  

By the properties of the intertwiners listed above it is clear that $(R_\alpha^0)^2 = 1$. In particular, $R^0_\alpha \in \mathcal{H}^\times$, the group of invertible elements of $\mathcal{H}$.

From the above we have the following result:

**Lemma 5.1.** The map $S_0 \ni s \mapsto R^0_\alpha \in \mathcal{H}^\times$ extends (uniquely) to a homomorphism $W_0 \ni w \mapsto R^0_w \in \mathcal{H}^\times$. Moreover, for all $f \in \mathcal{H}$ we have that $R^0_w f R^0_{w^{-1}} = f^w$.

Lusztig ([17], Proposition 5.5) proved that in fact

**Theorem 5.2.**

$$\mathcal{H} = \bigoplus_{w \in W_0} R^0_w \mathcal{A} = \bigoplus_{w \in W_0} \mathcal{A} R^0_w$$

Let $Z^m(T)$ denote the ring of $W_0$-invariant holomorphic functions of $T$, and consider the algebras $A^m(T) := Z^m(T) \otimes \mathcal{A}$ and $H^m(T) := Z^m(T) \otimes \mathcal{H}$. The algebra structure on $H^m(T)$ is defined by $(f \otimes h)(f' \otimes h') := ff' \otimes hh'$ (similar to the definition of $\mathcal{H}$).

Let us first remark that the finite dimensional representation theory of the “analytic” affine Hecke algebra $H^m(T)$ is the same as the finite dimensional representation theory of $H$. Every finite dimensional representation $\pi$ of $H$ determines a co-finite ideal $J_\pi \subset Z$, the ideal of central elements of $H$ which are annihilated by $\pi$. Denote by $J^m_\pi$ the ideal of $Z^m(T)$ generated by $J_\pi$. Because of the co-finiteness we have an isomorphism

$$Z/J_\pi \cong Z^m(T)/J^m_\pi(T).$$

This shows that $\pi$ can be uniquely lifted to a representation $\pi^m$ of $H^m(T)$ whose restriction to $H$ is $\pi$. The functor $\pi \mapsto \pi^m$ defines an
equivalence between the categories of finite dimensional representations of \( \mathcal{H} \) and \( \mathcal{H}^m(T) \) (with the inverse given by restriction).

For any \( W_0 \)-invariant nonempty open set \( U \subset T \) we define the localized affine Hecke algebra

\[
\mathcal{H}^m(U) := \mathcal{Z}^m(U) \otimes_{\mathbb{Z}} \mathcal{H}.
\]

This defines a presheaf of \( \mathcal{Z}^m \)-algebras on \( W_0 \setminus T \), which is finitely generated over the analytic structure sheaf \( \mathcal{Z}^m \) of the geometric quotient \( W_0 \setminus T \).

A similar argument as above shows that

**Proposition 5.3.** The category \( \text{Rep}(\mathcal{H}^m(U)) \) of finite dimensional modules \( \pi^m \) over \( \mathcal{H}^m(U) \) is equivalent to the category \( \text{Rep}_U(\mathcal{H}) \) of finite dimensional modules \( \pi \) over \( \mathcal{H} \) whose \( \mathbb{Z} \)-spectrum is contained in \( U \).

**Lemma 5.4.** For every \( W_0 \)-invariant nonempty open set \( U \) in \( T \), we have the isomorphism \( \mathcal{A}^m(U) \cong \mathcal{Z}^m(U) \otimes_{\mathbb{Z}} \mathcal{A} \), where \( \mathcal{A}^m(U) \) denotes the ring of analytic functions on \( U \).

**Proof.** Both the left and the right hand side are finitely generated modules over \( \mathcal{Z}^m(U) \), and the right hand side is naturally contained in the left hand side. Hence to prove that these modules are equal it suffices to show that the stalks of the corresponding sheaves are equal in each point of \( W_0 \setminus U \). Let \( \mathcal{I}_t \) denote the maximal ideal in \( \mathcal{Z} \) corresponding to \( W_0 t \), and let \( \hat{\mathcal{Z}}_t \) denotes the \( \mathcal{I}_t \)-adic completion. Because \( \hat{\mathcal{Z}}_t \) is faithfully flat over \( \mathcal{Z}^{m_0} \), it suffices to check that for each \( t \in U \), we have

\[
\hat{\mathcal{Z}}_t \otimes_{\mathcal{Z}^{m_0}} \mathcal{A}^{m_0} = \hat{\mathcal{Z}}_t \otimes_{\mathbb{Z}} \mathcal{A}.
\]

The left hand side is easily seen to be equal to \( \oplus_{t' \in W_0 t} \mathcal{A}_{t'} \), where \( \mathcal{A}_{t'} \) is the completion of \( \mathcal{A} \) with respect to the maximal ideal \( m_{t'} \) of \( \mathcal{A} \) corresponding to \( t' \). The right hand side is equal to the the completion \( \mathcal{A}_{\mathcal{I}_t} \). Let \( M_t = \prod_{t' \in W_0 t} m_{t'} \supset \mathcal{I}_t \mathcal{A} \). Since \( \mathcal{A}/\mathcal{I}_t \mathcal{A} \) is finite dimensional over \( \mathbb{C} \), we have \( M_t^k \subset \mathcal{I}_t \mathcal{A} \) for sufficiently large \( k \). Hence \( \mathcal{A}_{\mathcal{I}_t} = \mathcal{A}_{M_t} \). By the chinese remainder theorem, \( \mathcal{A}_{M_t} \simeq \oplus_{t' \in W_0 t} \mathcal{A}_{t'} \). This finishes the proof. \( \square \)

**Proposition 5.5.** The algebra \( \mathcal{H}^m(U) \) is a free \( \mathcal{A}^m(U) \) module of rank \( |W_0| \), with basis \( T_w \otimes 1 \) (\( w \in W_0 \)). When \( f \in \mathcal{A}^{m_0}(U) \) and \( s = s_\alpha \) with \( \alpha \in F_1 \) we have again Lusztig's relation

\[
f T_s - T_s f^s = \left((q_{2\alpha^\vee} q_{\alpha^\vee} - 1) + q_{\alpha^\vee}^{1/2} (q_{2\alpha^\vee} - 1) \theta_{-\alpha/2}\right) \frac{f - f^s}{1 - \theta_{-\alpha}}.
\]

This describes the multiplication in the algebra \( \mathcal{H}^m(U) \). The center of \( \mathcal{H}^m(U) \) is equal to \( \mathcal{Z}^m(U) \).
Similarly we have the localized meromorphic affine Hecke algebra $\mathcal{H}^{me}(U)$, which is defined by

\begin{equation}
\mathcal{H}^{me}(U) := \mathcal{F}^{me}(U) \otimes_{\mathbb{Z}^{an}(U)} \mathcal{H}^{an}(U),
\end{equation}

where $\mathcal{F}^{me}(U)$ it the quotient field of $\mathbb{Z}^{an}(U)$. We write $\mathcal{A}^{me}(U) := \mathcal{F}^{me}(U) \otimes_{\mathbb{Z}^{an}(U)} \mathcal{A}^{an}(U)$. It is the ring of meromorphic functions on $U$.

**Theorem 5.6.**

\begin{equation}
\mathcal{H}^{me}(U) = \bigoplus_{w \in W_0} \mathcal{A}^{me}(U) R^0_w = \bigoplus_{w \in W_0} R^0_w \mathcal{A}^{me}(U)
\end{equation}

**Proof.** This is clear from Theorem 5.2 by the remark that $\mathcal{H}^{me}$ arises from the $\mathcal{F}$-algebra $\mathcal{F}$ by extension of scalars according to

\begin{equation}
\mathcal{H}^{me}(U) = \mathcal{F}^{me}(U) \otimes_{\mathbb{Z}^{an}(U)} \mathbb{Z}^{an}(U) \otimes_{\mathcal{F}} \mathcal{H}
\end{equation}

\begin{equation}
\mathcal{F}^{me}(U) \otimes_{\mathbb{Z}^{an}(U)} \mathbb{Z}^{an}(U) \otimes_{\mathcal{F}} \mathcal{F} \otimes_{\mathcal{F}} \mathcal{H} = \mathcal{F}^{me}(U) \otimes_{\mathcal{F}} \mathcal{F} \mathcal{H}.
\end{equation}

\[ \square \]

5.1. **Lusztig’s structure theorem and parabolic induction**

We shall investigate the structure of the tracial states $\chi_t$, using Lusztig’s technique of localisation of $\mathcal{H}$ as discussed above. The results in the present subsection are substitutes for the usual techniques of parabolic induction for reductive groups. The results in this subsection are closely related to the results on parabolic induction in the paper [5].

We assume from now on that the labels $q$ are of the form $q(s) = q^{f_s}$ for a suitable $q > 1$ and positive integers $f_s$ as in 2.22.

We say that $t_1, t_2 \in T$ are equivalent modulo $q$ if for all $\alpha \in R_1$, $\alpha(t_1^{-1} t_2) \in \langle q \rangle$. We denote this by $t_1 \sim_q t_2$.

Let $R \subset R_0$ denote the roots $\alpha \in R_0$ such that $t \sim_q s_\alpha(t)$, and let $W_t$ be its Weyl group. Explicitly, $\alpha \in R_t$ if and only if $\alpha(t) \in \langle q \rangle$ when $2\alpha \not\in R_1$, and $\alpha(t) \in \pm \langle q \rangle$ if $2\alpha \in R_1$. Note that $R_t = R_t'$ if $t \sim_q t'$. Following Lusztig [17], we say that two elements $t_1, t_2$ in the orbit $W q t$ are equivalent if $t_2 \in W t_1$. We denote this equivalence relation by $t_1 \sim t_2$. Let $\varpi = \varpi t = W t$ be the equivalence class of $t$.

By the previous remarks we may use the notation $R_\varpi$ and $W_\varpi$ instead of $R_t$ and $W_t$. Note that $W_0$ acts transitively on the set of equivalence classes.

Denote by $W(\varpi) \subset W_\varpi$ the stabilizer in $W_0$ of the equivalence class $\varpi$. Put $R_{\varpi,+} = R_{\varpi} \cap R_{0,+}$ and let $\Gamma(\varpi) = \{ w \in W(\varpi) \mid w(R_{\varpi,+}) = R_{\varpi,+} \}$. Then $W_\varpi \triangleleft W(\varpi)$, and $W(\varpi) = W_\varpi \rtimes \Gamma(\varpi)$.
Consider the algebra $\mathcal{H}^t := \mathcal{H}(X, Y; R_t, R_t^\vee, F_t)$ where $F_t$ is the basis of roots in $R_t$ such that $R_{t, +} \subset R_{0, +}$. Note that $\Gamma(\varpi)$ acts by means of automorphisms on $\mathcal{R}^t = (X, Y; R_t, R_t^\vee, F_t)$, compatible with the root labels $q$. Thus we may define an action of $\Gamma(\varpi)$ on $\mathcal{H}^t$ by $\gamma(T_w \theta_x) = T_{\gamma w \theta_{\gamma x}}$. In this way we form the algebra $\mathcal{H}^\varpi := \mathcal{H}^t[\Gamma(\varpi)]$, with its product being defined by $(h_1 \gamma_1)(h_2 \gamma_2) = h_1 \gamma_1 (h_2 \gamma_2) \gamma_2$.

Let $B$ be an open ball in $\mathfrak{t}_\mathbb{C}$ around the origin, with radius small enough such that (with $t \in T$ given):

**Conditions 5.7.** $B$ satisfies:

(i) $\forall \alpha \in R_\alpha, b \in B : |\text{Im}(\alpha(b))| < \pi$. In particular, the map $\exp : \mathfrak{t}_\mathbb{C} \to T$ restricted to $B$ is an analytic diffeomorphism onto its image $\exp(B)$ in $T$.

(ii) If $w \in W_0$ and $t \exp(B) \cap w(t \exp(B)) \neq \emptyset$ then $wt = t$.

(iii) For all $t' \in t \exp(B)$, we have $R_{t'} \subset R_t$.

Lusztig [17] proves the following important structure theorem:

**Theorem 5.8.** For $\varpi \subset W_0 t$ an equivalence class, we put $U_{\varpi} := \varpi \exp(B)$. We define $1_{\varpi} \in \mathbb{A}_{\varpi}^\text{an}(U)$ by $1_{\varpi}(u) = 1$ if $u \in U_{\varpi}$ and $1_{\varpi}(u) = 0$ if $u \notin U_{\varpi}$. The elements $1_{\varpi}$ are mutually orthogonal idempotents. Let $t \in \varpi$.

(i) We have $\mathcal{H}^\varpi,\text{an}(U_{\varpi}) := \mathcal{H}^{t,\text{an}}(U_{\varpi})[\Gamma(\varpi)] \simeq 1_{\varpi} \mathcal{H}^\text{an}(U)1_{\varpi}$.

(ii) We can define linear isomorphisms

$$\Delta_{\varpi_1, \varpi_2} : \mathcal{H}^\varpi,\text{an}(U_{\varpi}) \to 1_{\varpi_1} \mathcal{H}^\text{an}(U)1_{\varpi_2}. $$

such that $\Delta_{\varpi_1, \varpi_3}(h) \Delta_{\varpi_3, \varpi_4}(h') = \Delta_{\varpi_1, \varpi_4}(hh')$ if $\varpi_2 = \varpi_3$, and $\Delta_{\varpi_1, \varpi_2}(h) \Delta_{\varpi_3, \varpi_4}(h') = 0$ else.

(iii) The center of $\mathcal{H}^\varpi,\text{an}(U_{\varpi})$ is $\mathbb{Z}^\varpi,\text{an}(U_{\varpi}) := (\mathbb{A}_{\varpi}^\text{an}(U_{\varpi}))^{W(\varpi)}$. This algebra is isomorphic to $\mathbb{Z}^\text{an}(U)$ via the map $z \to 1_{\varpi}z$, and this gives $\mathcal{H}^\varpi,\text{an}(U_{\varpi})$ the structure of a $\mathbb{Z}^\text{an}(U)$-algebra.

(iv) Let $N$ denote the number of equivalence classes in $W_0 t$. There exists an isomorphism $\mathcal{H}^\text{an}(U) \simeq (\mathcal{H}^\varpi,\text{an}(U_{\varpi}))^N$, the algebra of $N \times N$ matrices with entries in $1_{\varpi} \mathcal{H}^\text{an}(U)1_{\varpi} \simeq \mathcal{H}^\varpi,\text{an}(U_{\varpi})$. It is an isomorphism of $\mathbb{Z}^\text{an}(U)$-algebras.

**Proof.** The difference with Lusztig's approach is that he works with the $\mathcal{L}$-adic completions of the algebras instead of the localizations to $U$. However, we can copy his arguments without change, because $U$ is a “good” neighborhood of $W_0 t$, by our careful choice of $B$. The crucial point is that because of (iii), the function $c_\alpha$ is analytic and invertible on $U_{\varpi} \cup U_{\alpha, \varpi}$ for all $\alpha \notin R_{\varpi}$ (compare [17], Lemma 8.9).
Corollary 5.9. The functor $V \to V_{\varpi} := 1_{\varpi}V$ defines an equivalence between the representation category of $\mathcal{H}^{an}(U)$ and the representation category of $\mathcal{H}^{\varpi,an}(U_{\varpi}) = \mathcal{H}^{l,an}(U_{\varpi})[\Gamma(\varpi)]$. We have $\dim(V) = N_{\varpi} \dim(V_{\varpi})$ where $N_{\varpi}$ denotes the number of equivalence classes in $W_0t$.

Let $R_P \subset R_0$ be a parabolic root subsystem. We denote the corresponding parabolic subgroup of $W_0$ by $W_P := W(R_P)$. We call $t \in T$ a $R_P$-generic point if $t \sim wt$ for $w \in W_0$ implies that $w \in W_P$. If $t$ is $R_P$-generic we have $R_0 \subset R_P$. Let $F_P$ denote the basis of $R_P$ such that $F_P \subset R_{0,+}$. Define $\mathcal{H}^P = \mathcal{H}(X, Y, R_P, R_{P}, F_P)$.

Assume that $B$ satisfies (i), (ii) and (iii). Note that all $t' \in t\exp(B)$ are $R_P$-generic. Indeed, if $t' \sim wt'$ then there exists a $w' \in W_t \subset W$ (by (iii)) such that $w't' = wt'$. By (ii), also $t \sim w't = wt$. Hence $w' \in W_P$ as required.

We now put $U = W_0t\exp(B)$, $U_P = W_Pt\exp(B)$ and consider the localization $\mathcal{H}^{P,an}(U_P)$.

Corollary 5.10. Assume that $t \in T$ is $R_P$-generic. We have $\mathcal{H}^{an}(U) \simeq (\mathcal{H}^{P,an}(U_P))_{N_P}$, where $N_P = \frac{W_0}{W_P}$. Moreover, when we define $1_P := \sum_{\varpi \in W_P} 1_{\varpi}$ then $\mathcal{H}^{P,an}(U_P) \simeq 1_P\mathcal{H}^{an}(U)1_P$. These are isomorphisms of $\mathcal{Z}^{an}(U)$-algebras.

Proof. This is a consequence of the genericity of $t$. The genericity implies that the $W_0$-equivalence classes of the elements of $W_Pt$ are equal to the $W_P$-equivalence classes of these elements. Therefore we have, by the above theorem, $\mathcal{H}^{P,an}(U_P) \simeq (\mathcal{H}^{\varpi,an}(U_{\varpi}))_{n_P}$, where $n_P$ is the number of equivalence classes $\varpi' \in \Gamma(\varpi)$ in the orbit $W_Pt$. We denote by $\varpi$ the equivalence class of $t$. Then $w'\varpi \subset wW_Pt$ if and only if $w'^{-1}w' \in W_P$, since the stabilizer of $t$ in $W_0$ is contained in $W_P$ (because $t$ is generic). This gives a natural partitioning of the equivalence classes in $W_0t$ by right $W_P$-cosets. Hence $N_{\varpi} = n_PN_P$, and the result follows. \qed

Recall that, by Proposition 5.3, a finite dimensional representation $(V, \pi)$ of $\mathcal{H}$ with its $\mathcal{Z}$-spectrum contained in $U$ extends uniquely to a representation $(V^{an}, \pi^{an})$ of $\mathcal{H}^{an}(U)$.

Corollary 5.11. In the situation of Corollary 5.10, there exists an equivalence $(V, \pi) \to (V_P, \pi_P)$ between $\text{Rep}_P(\mathcal{H})$ and $\text{Rep}_{U_P}(\mathcal{H}^P)$, characterized by $V^{an}_P = 1_PV^{an}$. We have $\dim(V) = |N_P| \dim(V_P)$, and the inverse functor is given by $V_P \mapsto \text{Ind}_{h_P}^{H_P}(V_P) = \mathcal{H} \otimes_{h_P} V_P$. The character $\chi^P$ of the module $(V_P, \pi_P)$ of $\mathcal{H}^P$ is given in terms of the character $\chi_{\pi}$ of $(V, \pi)$ by the formula $\chi^P(h) = \chi_{\pi}(1_P h)$. 
Proof. We localize both the algebras $\mathcal{H}$ and $\mathcal{H}^P$ and use Proposition 5.3 and Corollary 5.10. Using Corollary 5.10 we see that the functor $V \to 1_PV^{\text{an}}|_{\mathcal{H}^P}$ is the required equivalence. The relation between the dimensions of $V$ and $V_P$ is obvious from this definition. Conversely, again using Corollary 5.10, we have

\begin{equation}
1_P(\text{Ind}^\mathcal{H}_{\mathcal{H}^P} V_P)^{\text{an}} = 1_P(\mathcal{H} \otimes_{\mathcal{H}^P} V_P)^{\text{an}}
= 1_P\left(\sum_{P^*} 1_P \mathcal{H}^{P^*}\left(U_P\right)1_{P^*}\right) \otimes_{\mathcal{H}^{P^*}\left(U_P\right)1_{P^*}} 1_P V^{\text{an}}
= 1_P V^{\text{an}} = V_P^{\text{an}},
\end{equation}

finishing the proof. \hfill \Box

**Proposition 5.12.** Let $F_P \subset F_0$ be a subset, and let $R_P \subset R_0$ be the corresponding standard parabolic subsystem. We define the subtori $T_P$, $T^P$ and the lattices $X_P$, $Y_P$ as in Proposition 3.2. Put $R_P = (X_P, Y_P, R_P, R_P^\vee, F_P)$, and let $t \in T^P$. There exists a surjective homomorphism $\phi_t : \mathcal{H}^P \rightarrow \mathcal{H}_P$ which is characterized by (1) $\phi_t$ is the identity on the finite dimensional Hecke algebra $\mathcal{H}(W_P)$, and (2) $\phi_t(\theta_x) = x(t)\theta_{tx}$, where $x \in X_P$ is the natural image of $x$ in $X_P = X/F_P X = X/(X \cap Y_P)$. 

Proof. We have to check that $\phi_t$ is compatible with the Lusztig relations. Let $s = s_\alpha$ with $\alpha \in F_P \subset F_0$. Then

\begin{equation}
\theta_s T_s - T_s \theta_s(x) = \left\{ \begin{array}{ll}
(q_\alpha - 1)\frac{\theta_{s^{-\alpha}} - \theta_{s(x)}}{\theta_{-\alpha}} & \text{if } 2\alpha \not\in R_{nr}.

((q_\alpha - 1)q_\alpha^{1/2} - 1)q_\alpha^{1/2} \frac{\theta_{s^{-\alpha}} - \theta_{s(x)}}{\theta_{-\alpha}} & \text{if } 2\alpha \in R_{nr}.
\end{array} \right.
\end{equation}

Since $s$ acts trivially on $T^P$, we have $x(t) = sx(t)$. This implies the result. \hfill \Box

**Definition 5.13.** Let $F_P \subset F_0$ be a subset. In this case we identify the algebra $\mathcal{H}^P = \mathcal{H}(X, Y, R_P, R_P^\vee, F_P)$ with the subalgebra in $\mathcal{H}$ generated by $\mathcal{H}(W_P)$ and $\mathbb{C}[X]$. Let $(V, \delta)$ be a representation of $\mathcal{H}$ with central character $r \in T_P$, and let $t \in T^P$. Denote by $\delta_t$ the representation $\delta_t = \delta \circ \phi_t$ of $\mathcal{H}^P$. We define a representation $\pi(R_P, r, \delta, t)$ of $\mathcal{H}$ by $\pi(R_P, r, \delta, t) = \text{Ind}^\mathcal{H}_{\mathcal{H}^P}(\delta_t)$. We refer to such representations as parabolically induced representations.

**Corollary 5.14.** In the situation of Definition 5.13, assume that $\delta$ is irreducible with central character $r \in T_P$. Let $t^P \in T^P$ be such that $t = rt^P \in T$ is $R_P$-generic. Then $\pi(R_P, r, \delta, t^P)$ is irreducible with
central character \( t \). All irreducible representations of \( \mathcal{H} \) with central character \( t \) are of this form.

Proof. This is now a simple consequence of the above results. \( \square \)

5.2. The tracial states \( \chi_t \)

From now on, we assume that the coordinates of the centers \( c_L \) of residual subspaces \( L \) satisfy the property that for all \( x \in X \), \( x(c_L) \in \langle q \rangle \), the multiplicative group generated by \( q \). This is harmless, since we can always achieve this if we replace \( q \) by a suitable radical, and the integers \( f_i \) accordingly by a suitable multiple.

Let \( L \) be a residual coset such that \( R_L \subset R_0 \) is a standard parabolic subsystem. In other words, \( F_L \subset F_0 \). Let us denote by \( \mathcal{H}_L \) the affine Hecke algebra with root datum \( \mathcal{R}_L := (X_L, Y_L, R_L, R^\vee_L, F_L) \) (see Proposition 3.2). Let \( r_L = c_L s_L \in T_L \) be the corresponding residual point of \( \mathcal{R}_L \).

Lemma 5.15. Let \( U \subset T \) be a nonempty \( W_0 \)-invariant open subset. Let \( t \in U \). There exists a unique extension of the Eisenstein functional (cf. 4.7) \( E_t \) (which we will also denote by \( E_t \)) to the localization \( \mathcal{H}^m(U) \), such that \( E_t(fh) = E_t(hf) = f(t)E_t(h) \) for all \( f \in \mathcal{A}^m(U) \).

Proof. The functional \( E_t \) factors to a functional of the finite dimensional \( \mathbb{C} \)-algebra \( \mathcal{H}/I_i \mathcal{H} \), where \( I_i \) is the maximal ideal in \( \mathcal{Z} \) corresponding to \( W_0 t \). We have \( \mathcal{H}/I_i \mathcal{H} = \mathcal{H}^m(U)/I_i^m(U) \mathcal{H}^m(U) \) for \( t \in U \), and this defines the extension with the required property uniquely. \( \square \)

Lemma 5.16. Let \( L \) be such that \( R_L \subset R_0 \) is a standard parabolic subset of roots, and let \( t \in T \). Set \( U = W_0 t \exp(B) \) with \( B \) satisfying the conditions (i), (ii), and (iii). As before, we put \( U_L = W_L t \exp(B) \).

We denote by \( E^L_t \) the Eisenstein functional of the subalgebra \( \mathcal{H}^L \subset \mathcal{H} \). For \( t_L \in T_L \), we write \( E_{L,t_L} \) to denote the Eisenstein functional at \( t_L \in T_L = \text{Hom}(X_L, \mathbb{C}^\times) \) of the algebra \( \mathcal{H}_L \). Let \( t = t^L t_L \in U \) with \( t^L \in T^L \) and \( t_L \in T_L \). Put \( \delta_U \) for the characteristic function of \( U_L \).

We have, for all \( h \in \mathcal{H}^L \):  
(i) \( E_t(1_L h 1_L) = q(w^L) \delta_U(t) \Delta^L(t) E^L_t(h) \).
(ii) \( E^L_t(h) = E_{L,t_L}(\phi_L(h)) \).

Proof. Because these are both equalities of holomorphic functions of \( t \) it suffices to check them for \( t \) regular in \( T \), and outside the union of the residual cosets (in other words, \( c(t)c(t^{-1}) \neq 0 \)).

(i). By the defining properties 4.7 and [24], 2.23(4) we need only to show that the left hand side satisfies the properties \( E_t(1_L x h 1_L) = \)
\[ E_t(1_L h x 1_L) = t(x)E_t(h) \] and \[ E_t(1_L) = q(u_0)\delta_{u_1}(t)\Delta(t) \]. These facts follow from Lemma 5.15.

(ii). We see that
\[
E_{\ell, t} (\phi_{t \ell}(\theta_x h)) = E_{\ell, t} (x(t^L)\theta_x \phi_{t \ell}(h)) \\
= x(t^L)\pi(t_L)E_{\ell, t} (\phi_{t \ell}(h)) \\
= x(t)E_{\ell, t} (\phi_{t \ell}(h)),
\]
and similarly for \( E_{\ell, t} (\phi_{t \ell}(h \theta_x)) \). The value at \( h = 1 \) is equal to \( q(w_0)\Delta_L(t) \) on both the left and the right hand side. As in the proof of (i), this is enough to prove the desired equality.

**Theorem 5.17.** Let \( L \) be a residual coset such that \( R_L \subset R_0 \) is a standard parabolic subset of roots. Let \( t^L \in T_u^L \) be such that \( t := rLt^L \in L^{\text{emp}} \) is \( R_L \)-generic. Notice that this condition is satisfied outside a finite union of real codimension one subsets in \( T_u^L \). Let \( \{e_i\}_{i=1}^l \) be the set of central idempotents of the residue Frobenius algebra \( \overline{H}_L^{\ell} \), and let \( \chi_{r_L} = \sum_{i=1}^l \chi_{r_L,i} d_{r_L,i} \) be the corresponding decomposition in irreducible characters of the tracial state \( \chi_{r_L} \) of \( H_L \). Let \((V_{r_L,i}, \delta_{r_L,i})\) be the irreducible discrete series module of \( H_L \) corresponding to \( e_i \).

(i) For all \( i \), \( \pi_{i,i} := \pi(R_L, r_L, \delta_{r_L,i}, t^L) \) is irreducible, with central character \( t \). These representations are mutually inequivalent.

(ii) Let \( L' \) be \( W \)-conjugate to \( L \) such that \( R_{L'} \) is also a standard parabolic subsystem of roots. Choose \( r_{L'} \in T_{u'} , L' \) and let \( t^{L'} \in T_u^{L'} \) be such that \( t = r_L t^{L'} \) and \( t = r_{L'} t^{L'} \) are \( W \)-conjugate in \( T \). There exists a unique ordering \( \{\delta_{r_{L'},i} , \ldots , \delta_{r_{L'},i} \} \) of the central idempotents of \( \overline{H}_L^{L'} \) such that \( \pi(R_L, r_L, \delta_{r_L,i}, t^L) \simeq \pi(R_{L'}, r_{L'}, \delta_{r_{L'},i}, t^{L'}) \).

(iii) The character \( \chi_{r_L,i} \) of \( \pi_{i,i} \) is a positive trace on \( H \).

(iv) We have \( \chi_i = \sum \chi_{i,i} d_{r_L,i} \), where \( \chi_{i,i} \) denotes the character of \( \pi_{i,i} \).

**Proof.** (i) This is Corollary 5.14.

(ii) and (iv). Recall the definition of the states \( \chi_L \). Recall that the support of the measure \( \nu \) is the union of the tempered residual cosets. We combine Definition 4.16, Proposition 4.15, and Proposition 4.7 to see that (with \( N_L \) the stabilizer of \( L \) in \( W_0 \))

\[
(5.17) \quad \frac{W_0}{N_L} \int_{1 \in L^{\text{emp}}} z(t) \chi_0(h) d\nu_L(t) = \sum_{M \in W_0} \int_{t^M \in T_u^M} \int_{t' \in M^e \mu} z(t') E_{\nu}(h) \eta(t')
\]
for all $h \in \mathcal{H}$ and $z \in \mathcal{Z}$. Rewrite the right hand side as

$$
(5.18) \quad \frac{1}{|N_L|} \int_{t^L \in T^L} \sum_{w \in W_0} J_{w,L}(e^{u_Lr_w L}w(t^L))d^L(t^L)
$$

where the inner integral equals, with $s \in wT^L$,

$$
(5.19) \quad J_{w,L}(r_w L s) = \int_{t' \in \xi_{wL}} z(t') \frac{E_{t'}(h)}{q(u_0)\Delta_{w, L}(t')} m_{w, L}(t') \frac{1}{\Delta_{w, L}(t')} \omega_{w, L}(t'),
$$

which is a linear combination of derivatives (normal to $wT^L$) of the kernel

$$
(5.20) \quad z(t') \frac{E_{t'}(h)}{q(u_0)\Delta_{w, L}(t')} m_{w, L}(t'),
$$

evaluated at $r_w L s$. In other words, the $N_L$-invariant measure on $L^{emp}$ on the left hand side is obtained by taking the boundary values $e^{u_L} \to 1$ of the $J_{w,L}(e^{u_Lr_w L}w(t))$, and then sum over the Weyl group as in equation 5.18. Notice that the collection of $R_L$-generic points in $L$ is the complement of a union of algebraic subsets of $L$ of codimension $\geq 1$.
The kernel 5.20 is regular at such points of $L$. The boundary values at $R_L$-generic points are therefore computed simply by specialization at $e^{u_L} = 1$. We thus have

$$
(5.21) \quad z(t)\chi_t(h)\delta_{L,ML}(t) = \frac{1}{|W_0|} \sum_{w \in W_0} J_{w,L}(r_w L w(t^L)).
$$

The expression on the left hand side can be extended uniquely to $z \in \mathcal{Z}^{an}(U)$ and $h \in \mathcal{H}^{an}(U)$. By equation 5.19, each summand in the expression on the right hand can also be extended uniquely to such locally defined analytic $z$ and $h$.

Take $U = W_0 t \exp(B)$. We restrict both sides to $1_L \mathcal{H}L1_L \subset \mathcal{H}^{an}(U_L)$. Substitute $h$ by $1_L h_1 L$ with $h \in \mathcal{H}L$. On the left hand side we get, by Corollary 5.11,

$$
(5.22) \quad \frac{1}{N_L} z(t)\chi_t^L(h)\delta_{L,ML}(t)
$$

where $\chi_t^L$ is a central functional on $\mathcal{H}L$, normalized by $\chi_t^L(e) = 1$.

On the other hand, by Lemma 5.16, if $h = 1_L h_1 L$ with $h \in \mathcal{H}L$ then

$$
(5.23) \quad J_{w,L}(w(r_L t^L)) = \int_{t' \in w(t^L) \xi_{wL}} z(t') \frac{E_{t'}(\phi_{w,t}(h))}{q(w_L)\Delta_{L}(t') m_{w, L}(t') \omega_{w, L}(t')}
$$

if $w(r_L t^L) \in U_L$, and $J_{w,L}(w(r_L t^L)) = 0$ otherwise.
Observe that, because of condition (ii) for $B$, $wt \in U_L$ implies that there exists a $w' \in W_L$ such that $wt = w't$. Since $t = r_L t^L$ is $R_L$-generic, we see in particular that the stabilizer of $t$ is contained in $W_L$. Thus $wt \in U_L$ implies that $w \in W_L$, and hence that $wt = w(r_L)t^L$.

Therefore the sum at the right hand side of equation 5.21 reduces, if $h$ is of the form $1_L h 1_L$ with $h \in \mathcal{H}^L$, to

$$
(5.24) \quad \frac{1}{|W_0|} \sum_{w \in W_L} \int_{t' \in \xi_w} \mathcal{L} (t^L t') m^{wL}(t^L t') \frac{E_{L,t'}(\phi_{t^L}(h))}{q(w_L) \Delta_L(t')} \omega_{wL}(t')
$$

The function $\alpha w_L \exp(B \cap t_L) \ni t' \rightarrow m^{wL}(t^L t')$ is $W_L$-invariant on $W_L r_L \exp(B \cap t_L) = (t_L)^{-1} U_L \cap T_L$, because $m^L(t)$ is clearly $W_L$-equivariant (i.e. $m^w(t) = m^L(t)$ when $w \in W_L$). In other words, this function is in the center of $H_{\mathcal{L}}^m((t_L)^{-1} U_L \cap T_L)$. By Definition 4.16, Corollary 4.22, Definition 4.24, and Theorem 4.26 applied to $\mathcal{H}_{\mathcal{L}}$ therefore, this sum reduces to

$$
(5.25) \quad \frac{|W_L|}{|W_0|} \mathcal{L} (t) \chi_{t^L}(\phi_{t^L}(h)) m^L(t) \delta_{L,m} \delta_{R_L r_L}(r_L),
$$

which we can rewrite as

$$
(5.26) \quad \frac{|W_L|}{|W_0|} z(t) \chi_{t^L}(\phi_{t^L}(h)) \delta_{L,m} m_L(t).
$$

Comparing this with the left hand side 5.22 we see that, in view of equation 4.60, this implies that for $h \in \mathcal{H}^L$,

$$
(5.27) \quad N^L \chi_L(1_L h) = \chi_L^L(h) = \sum_{i=1}^{\ell} \chi_{r,L,i}(\phi_{t^L}(h)) d_{r,L,i}.
$$

Applying Corollary 5.11, Definition 5.13, and Corollary 5.14 obtain the desired results (iii) and (iv).

(ii). Apply the above computation to $R_L$, $r_L$, and $t^{L'}$. Comparing the results we conclude the required correspondence.

\textbf{Corollary 5.18.} Let $t = r_L t^L \in L^\text{temp}$. Let $V_{i,i} = \mathcal{H} \otimes H^L$ $V_{r,L,i}$ be the complex vector space underlying the parabolically induced module $\pi_{i,i}$. Then $V_{i,i}$ carries a positive definite hermitian form $(v, w)$ such that for all $h \in \mathcal{H}$, $\pi_{i,i}(h^*) = \pi_{i,i}(h^*)$. Moreover, $\pi_{i,i}$ defines a representation of the $C^*$-algebra $\mathfrak{C}$, and is tempered.

\textbf{Proof.} The temperedness of $\pi_{i,i}$ follows from Corollary 4.37 in combination with the previous Theorem (or can be shown directly, cf. [5]). In addition, the positivity of $\chi_{i,i}$ ($\nu_{L,i}$-almost everywhere) induces a positive definite Hermitian form on $V_{i,i}$ which is compatible with $\ast$. By
continuity, it is true for all \( t \in L^\text{temp} \). The extension of \( \pi_{i,i} \) to \( \mathcal{C} \) follows from Corollary 4.25(iv). \( \square \)

**Remark 5.19.** In the context of graded affine Hecke algebras, the Hermitian form on \( V_{i,i} \) which makes \( \pi_{i,i} \) a representation of the involutive algebra \( (\mathcal{H}, \ast) \) is constructed directly from the Hermitian form on the representation space \( V_{\delta_i} \) of the discrete series representation \( \delta_i \) of \( \mathcal{H}_L \) in the paper [5]. We did not pursue this direct approach here.

**Corollary 5.20.** We have the following isomorphism of Hilbert spaces:

\[
\mathcal{F}_{\mathfrak{H}} = \int_{T} \mathcal{H}_{l} \text{d} \nu(t).
\]

The support of the probability measure \( \nu \) is the union of the tempered residual cosets. If \( t = r_{L} \delta_{L} \in L^\text{temp} \) is \( R_{L} \) generic, then the residue Frobenius algebra \( \mathcal{H}_{\mathfrak{F}} \) has the structure

\[
\mathcal{H}_{\mathfrak{F}} \simeq (\mathcal{H}_{L}^\mathfrak{F})_{N_{L}}.
\]

Finally, the residue Frobenius algebra \( \mathcal{H}_{\mathfrak{F}} \) at a residual point \( r \in T \) is of the form

\[
\mathcal{H}_{\mathfrak{F}} \simeq \bigoplus_{i=1}^{l_r} \text{End}(V_{r,i})
\]

with the hermitian form on the summand \( \text{End}(V_{r,i}) \) given by

\[
(A, B) = d_{r,i} \text{trace}(A^\ast B),
\]

where the positive real numbers \( d_{r,i} \) are defined in Definition 4.31.

**Corollary 5.21.** The formal dimension of the irreducible square integrable representation \( (V_{r,i}, \delta_{r,i}) \) of \( \mathcal{H} \) associated with the minimal central idempotent \( e_{r} \) of the residue Frobenius algebra \( \mathcal{H}_{\mathfrak{F}} \) of a residual point \( r \in T \) equals

\[
\text{fdim}(\delta_{r,i}) = \mu_{R_{L}, P}(\delta_{r,i}) = d_{r,i} \nu(\{ W_{0r} \}).
\]

### 5.3. The Plancherel decomposition of the trace \( \tau \)

When \( k \in K_L = T_L \cap T_L \), we have an isomorphism \( \phi_k : \mathcal{H}_L \to \mathcal{H}_L \) defined by \( \phi_k(\theta_s T_w) = k(x) \theta_s T_w \). This induces an isomorphism \( \phi_k : \mathcal{H}_{l_i}^k \to \mathcal{H}_{l_i}^k \), and we can take \( \delta_{r,L,i} := \delta_{r,L,i} \circ \phi_k \) as the collection of irreducible representations of \( \mathcal{H}_{l_i}^k \). With this indexing we have \( \pi(R_L, kr_L, \delta_{kr,L,i}, t^L) \simeq \pi(R_L, r_L, \delta_{r,L,i}, k t^L) \). This is a particular case of Theorem 5.17(ii).
If \( n \in N_L \), the stabilizer of \( L \) in \( W_0 \), then \( n \in N_{T_L} = N_{T_L} \), and thus \( n(r_L) = k_n r_L \) for some \( k_n \) in \( K_L = T_L \cap T^L \). Note that \( K_L \) has a natural action of \( N_L \) by conjugation. Thus we obtain a 1 cocycle \( n \rightarrow k_n \) of \( N_L \) with values in \( K_L \). We define an action \( \ast \) of \( N_L \) on \( T^L \) by \( n \ast t^L := k_n n(t^L) \). By Theorem 5.17(ii) we can define an action of \( N_L \) on the index set \{1, \ldots, l\} of the set of irreducible discrete series representations \( \delta_{r_L,i} \) with central character \( r_L \) of \( \mathcal{H}_L \) by requiring that

\[
\pi(\mathcal{R}_L, r_L, \delta_{r_L,n(i)}, n \ast t^L) \simeq \pi(\mathcal{R}_L, r_L, \delta_{r_L,i}, t^L).
\]

Assume that \( t^L \in T^L \) is such that \( W_0 t \cap L = N_L t \). This condition is again satisfied outside a subset of codimension at least 1 in \( L_{\text{temp}} \). It implies that if \( \pi_{t,i} \simeq \pi_{t',i'} \) if and only if there exists an \( n \in N_L \) such that \( t' = n(t) \) and \( i' = n(i) \).

**Proposition 5.22.** We have, for all \( n \in N_L \), \( d_{r_L,i} = d_{r_L,n(i)} \).

**Proof.** Since \( \chi_{n(t)} = \chi_t \), we have, by the Theorem 5.17(iv),

\[
\sum_{i=1}^l \chi_{t,i} d_{r_L,n(i)} = \sum_{i=1}^l \chi_{n(t),n(i)} d_{r_L,n(i)} = \sum_{i=1}^l \chi_{t,i} d_{r_L,i}.
\]

Recall the projection \( p_\circ : \Omega \to S \), with \( S = \bigcup L_{\text{temp}} \) (union over the residual subspaces) modulo the action of \( W_0 \), as was discussed in 2.5.2. For \( L \) a residual subspace with \( R_L \) standard parabolic, define \( L_{\text{temp}, \text{reg}} \) be the open dense subset of \( t \in L_{\text{temp}} \) such that \( t \) is \( R_L \)-generic and \( W_0 t \cap L = N_L t \). Notice that the inertia group \( W_t \) is equal to \( |W_L \cap W_{r_L}| \) with these assumptions. The components of \( p_\circ^{-1}(S_L) \) with \( S_L = W_0 \cap L_{\text{temp}} \) are the closures (in \( \hat{\Omega} \)) \( \Gamma_{L,i} = \Gamma_{L,i}^\circ \) with \( \Gamma_{L,i} = \{ \pi_{t,i} \mid t \in N_L \} \).

The Plancherel measure \( \mu_P |_{r_L} \) is given by the following measure, using the parametrization of \( \Gamma_{L,i} \) by \( t \in N_L \setminus L_{\text{temp}, \text{reg}} \).

\[
d\mu_P(\pi_{t,i}) = \frac{|W_0|}{|W_L \cap W_{r_L}|} d_{r_L,i} d\nu_L(t)
\]

\[
= \frac{|W_0|}{|W_L \cap W_{r_L}|} \nu_{R_L}(\{r_L\}) d_{r_L,i} m_L(t) d_L(t^L)
\]

\[
= N_L \mu_{R_L,P}(\delta_{r_L,i}) m_L(t) d_L(t^L),
\]

where \( d_{r_L,i} > 0 \) is the formal dimension of \( \delta_{r_L,i} \) in \( \mathcal{H}^L \), and \( \mu_{R_L,P} \) is given in Corollary 5.21.
Let us fix a set $\Lambda$ of representatives of residual cosets $L$ modulo the action of $W_0$, such that each chosen representative $L$ is such that $R_L$ is a standard parabolic subset of $R_0$. For each $L \in \Lambda$, choose a $r_L \in L \cap T_L$, and denote by $\Pi_L$ the set of discrete series representations $\delta$ of $H_L$ with central character $r_L$. Write $\Gamma_{L,\delta}$ instead of $\Gamma_{L,i}$ if $\delta = \delta_{r_L,i}$. We have shown:

**Theorem 5.23.** The disintegration of the tracial state $\tau$ of $\mathcal{C}$ in irreducible, mutually distinct characters of $\mathcal{C}$ is given by

$$\tau = \sum_{L \in \Lambda} \sum_{\delta \in \Pi_L} \int_{\pi \in \Gamma_{L,\delta}} \chi_{\pi} d\mu_{\pi}(\pi),$$

where the measure $\mu_{\pi}$ is described in equation 5.35.

6. Base change invariance of the residue Frobenius algebra

Thus far we have found the spectral decomposition for $H$ in terms of the formal degrees $d_{r_L,i}$ of the residue Frobenius algebras $H_L^{r_L}$. We assume throughout in this section that the labels $q(s)$ are of the form $q(s) = q^{f_s}$ for $q > 1$ and nonnegative integers $f_s$. We shall prove in this section that the residue Frobenius algebras are independent of $q$.

6.1. Scaling of the root labels

Let $r = sc \in T$ be fixed, with $s \in T_u$ and $c = \exp(\gamma)$ with $\gamma \in t$. Assume that $B \subseteq t_C$ is an open ball centered around the origin such that the conditions 5.7 (with respect to $r \in T$) are satisfied.

The second condition implies that each connected component of the union $U := W_0(r \exp(B))$ contains a unique element of the orbit $W_0r$. Given $u \in U$ there is a unique $r' = s'c' \in W_0r$ such that $u \in r' \exp(B)$. By (i) there is a unique $b \in B$ such that $u = s'c' \exp(b) = s' \exp(b + \gamma')$.

Now let $\epsilon \in (0, 1]$ be given. We define an analytic map $\sigma_\epsilon$ on $U$ by

$$\sigma_\epsilon(u) := s' \exp(\epsilon \log((-1)^{-1}u)) = s' \exp(\epsilon(b + \gamma')).$$

**Lemma 6.1.** The map $\sigma_\epsilon$ is an analytic, $W_0$-equivariant diffeomorphism from $U$ onto $U_\epsilon := W_0(s \exp(\epsilon B))$. The inverse of $\sigma_\epsilon$ will be denoted by $\sigma_{-\epsilon}$.

**Proof.** On the connected component $r' \exp(B)$ the map $\sigma_\epsilon$ is equal to $\sigma_\epsilon = \mu_{r'} \circ \exp \circ M_\epsilon \circ \log \circ \mu_{r'}^{-1}$ where $\mu_{r'}$ is the multiplication in $T$ by $s'$, and $M_\epsilon$ is the multiplication in $t_C$ by $\epsilon$. These are all analytic diffeomorphisms, because of condition (i). The $W_0$ equivariance follows
from the fact that log is well defined (and thus equivariant, since exp is equivariant) from \( W_0 \exp(B + \gamma) \) to \( W_0(B + \gamma) \), and that \( M_\gamma \) is \( W_0 \)-equivariant. This implies that for \( w \in W_0 \), \( w \exp(\varepsilon \log((s')^{-1}u)) = \exp(\varepsilon \log((ws')^{-1}wu)) \). It follows that

\[
\sigma_\varepsilon(wu) = ws' \exp(\varepsilon \log((ws')^{-1}wu)) = w(\sigma_\varepsilon(u)).
\]

\[\Box\]

**Lemma 6.2.** Denote by \( q^* \) the label function \( q^*(s) = q(s)^* = q^{f \alpha} \), and denote by \( \mathcal{H}_{q^*} \) the affine Hecke algebra with root datum \( \mathcal{R} \) (same as the root datum of the affine Hecke algebra \( \mathcal{H} = \mathcal{H}_q \)), but with the labels \( q \) replaced by \( q^* \). Let \( c_{\alpha, \varepsilon} \in \mathcal{A}_{q^*} \subset \mathcal{A}_{q^*} \) be the corresponding Macdonald c-functions. For every root \( \alpha \in R_1 \) we have:

\[
U \ni u \mapsto (c_{\alpha, \varepsilon}((\sigma_\varepsilon(u))c_{\alpha}(u)^{-1})^{\pm 1} \in \mathcal{A}^{m}(U).
\]

**Proof.** For \( u \) in the connected component \( r' \exp(B) \) of \( U \) we write \( u = s'v \) with \( v \in c' \exp(B) \). We have

\[
c_{\alpha, \varepsilon}((\sigma_\varepsilon(u))c_{\alpha}(u)^{-1}) = \frac{(1 + q_{\alpha}^{-1/2} q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2} (1 - q_{\alpha}^{-1/2} q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2})) (1 - q_{\alpha}^{-1/2} q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2} (1 - q_{\alpha}^{-1/2} q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2}))}{(1 + q_{\alpha}^{-1/2} q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2} (1 - q_{\alpha}^{-1/2} q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2}) (1 - q_{\alpha}^{-1/2} q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2})) (1 - q_{\alpha}^{-1/2} q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2} (1 - q_{\alpha}^{-1/2} q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2}))}
\]

We remind the reader of the convention Remark 4.1: in particular, the expression \( \alpha(s')^{1/2} \) occurs only if \( \alpha/2 \in R_0 \), in which case this expression stands for \( (\alpha/2)(s') \). If \( \alpha/2 \not\in R_0 \), we should reduce formula 6.4 to

\[
c_{\alpha, \varepsilon}((\sigma_\varepsilon(u))c_{\alpha}(u)^{-1}) = \frac{(1 - q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2} (1 - \alpha(v)^{-1} \alpha(s')^{-1})) (1 - q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2} (1 - \alpha(v)^{-1} \alpha(s')^{-1}))}{(1 - q_{\alpha}^{-1/2} \alpha(v)^{-1/2} \alpha(s')^{-1/2} (1 - \alpha(v)^{-1} \alpha(s')^{-1}) (1 - \alpha(v)^{-1} \alpha(s')^{-1}))}
\]

By conditions (i) and (iii) it is clear that poles and zeroes of these functions will only meet \( U \) if \( \alpha(s') = 1 \) when \( \alpha \in R_0 \cap R_1 \) or \( \alpha(s') = \pm 1 \) if \( \alpha \in 2R_0 \). In these cases the statement we want to prove reduces to the statement that the function

\[
f(x) := \frac{1 - \exp(-\varepsilon x)}{1 - \exp(-x)}
\]

is holomorphic and invertible on the domain \( x \in p + \alpha(\gamma' + B) \), where \( p \) is a real number and \( \alpha \in R_0 \). By condition (i) both the denominator and the numerator of \( f \) have a zero in this domain only at \( x = 0 \) (if this belong to the domain), and this zero is of order 1 both for the numerator and the denominator. The desired result follows. \( \Box \)
Recall Theorem 5.6. This result tells us that the structure of algebra with coefficients in the locally defined meromorphic functions on $U$ is independent of the root labels. We will now show that the subalgebra with analytic coefficients (defined locally on $U$) is invariant for scaling transformations.

**Theorem 6.3.** The map

\[
j_e : \mathcal{H}^{m_{	ext{e}}} (U) \to \mathcal{H}^{m_{	ext{e}}} (U) \quad \sum_{w \in W_0} f_w R_w^0 \to \sum_{w \in W_0} (f_w \circ \sigma_e) R_w^0,
\]

defines an isomorphism of $\mathbb{C}$-algebras, with the property that $j_e (\mathcal{F}^{m_{	ext{e}}} (U)) = \mathcal{F}^{m_{	ext{e}}} (U)$ and $j_e (\mathcal{A}^{m_{	ext{e}}} (U)) = \mathcal{A}^{m_{	ext{e}}} (U)$. Moreover (and most significantly), $j_e (\mathcal{H}^m (U)) = \mathcal{H}^m (U)$.

**Proof.** The map $j_e$ as defined above is clearly a $\mathbb{C}$-linear isomorphism by Theorem 5.6. It is an algebra homomorphism because we have

\[
j_e (\sum_{u \in W_0} f_u R_u^0 \sum_{v \in W_0} g_v R_v^0) = j_e (\sum_{u, v \in W_0} f_u g_v R_u^0) = \sum_{u, v \in W_0} (f_u \circ \sigma_e) (g_v \circ \sigma_e) R_{u, v}^0 \quad j_e (\sum_{u \in W_0} f_u R_u^0) = \sum_{u \in W_0} (f_u \circ \sigma_e) R_{u, e}^0 \quad j_e (\sum_{v \in W_0} g_v R_v^0) = \sum_{v \in W_0} (g_v \circ \sigma_e) R_{v, e}^0.
\]

What remains is the proof that $j_e (\mathcal{H}^m (U)) = \mathcal{H}^m (U)$. Notice that $\mathcal{H}^m (U)$ is the subalgebra generated by $\mathcal{A}^m (U)$ and the elements $T_s$ where $s = s_\alpha$ with $\alpha \in R_1$. The $j_e$-image of $\mathcal{A}^m (U)$ equals $\mathcal{A}^m (U)$ since $\sigma_e$ is an analytic diffeomorphism. To determine the image of $T_s$, we use formula Lemma 2.27(2) of [24], applied to the situation $W_0 = \{e, s\}$. This tells us that

\[
(1 + T_s) = q_{s \alpha} q_{2s \alpha} c_\alpha (1 + R_s^0).
\]

Hence we see that

\[
j_e (T_s) = q_{s \alpha} q_{2s \alpha} (c_\alpha \circ \sigma_e) (1 + R_{s, e}) - 1 = q_{s \alpha} q_{2s \alpha} (c_\alpha \circ \sigma_e) c_\alpha (1 + T_{s, e}) - 1.
\]

By Lemma 6.2 it is clear that this is indeed in $\mathcal{H}^m (U)$, and that these elements together with $\mathcal{A}^m (m_{	ext{e}} (U))$ generate $\mathcal{H}^m (U)$. \hfill \square

### 6.2. Application to the residue Frobenius algebra

In order to prove that the residue algebras $\mathcal{R}$ are invariant for the scaling transformation $q \to q^e$ it suffices to consider the case $\mathcal{R}$ for a residual point $r \in T$. This follows from Theorem 5.17, expressing $\chi_t$...
in terms of characters induced from discrete series characters of proper parabolic subalgebras.

When \( r = sc \in T \) is a residual point, the state \( \chi_r \) has a natural extension to the localized algebras \( \mathcal{H}^{an}(U) \) where \( U = W_0 r \exp(B) \), with \( B \) an open ball in \( \mathfrak{t}_C \) satisfying the conditions \( 5.7 \) with respect to the point \( r \in T \). Because the radical \( \text{Rad}_r^{an}(U) \) of the biframe \( (x, y)_r := \chi_r(x^* y) \) on \( \mathcal{H}^{an}(U) \) is contained in the maximal ideal \( I_r^{an}(U) \) of functions in the center \( Z^{an}(U) \) which vanish in the orbit \( W_0 r \), we clearly have

\[
\mathcal{H}^{an}(U) = \mathcal{H}^{an}(U)/\text{Rad}_r^{an}(U).
\]

The structure of this algebra as a Frobenius algebra is given by the biframe defined by \( \chi_r \). Therefore, we need to prove independence of \( \chi_r \) for the scaling transformation. We start with a simple lemma:

**Lemma 6.4.** Let \( h \in \mathcal{H}^{an}(U) \) We have

\[
(6.12) \quad \frac{E_{\theta x, \sigma_t}(j_e(h))}{q^{\theta}(w_0)\Delta(\sigma_t(t))} = \frac{E_t(h)}{q(w_0)\Delta(t)}
\]

**Proof.** For all \( x \in X \) we have

\[
(6.13) \quad E_{\theta x, \sigma_t}(j_e(\theta_x h)) = E_{\theta x, \sigma_t}(j_e(\theta_x)j_e(h))
\]

\[
= (x \circ \sigma_x)(\sigma_e(t))E_{\theta x, \sigma_t}(j_e(h))
\]

\[
= x(t)E_{\theta x, \sigma_t}(j_e(h)),
\]

showing that the left hand side has the correct eigenvalue for multiplication of \( h \) by \( \theta_x \) on the left. For the multiplication of \( h \) by \( \theta_x \) on the right a similar computation holds. This shows, in view of Lemma 5.15 and [24], Proposition 2.23(3) that, for regular \( t \) and outside the union of all residual cosets, the left and the right hand side are equal up to normalization. But both the left and the right hand side are equal to 1 if \( h = T_1 = 1 \). Hence generically in \( t \), we have the desired equality. Since both expressions are holomorphic in \( t \), the result extends to all \( t \in T \).

**Theorem 6.5.** Let \( \epsilon \in (0, 1] \) be given. We have, for all \( h \in \mathcal{H}^{an}(U) \),

\[
(6.14) \quad \chi_{\theta x, \sigma_t}(j_e(h)) = \chi_r(h).
\]

**Proof.** Take a neighborhood \( U = W_0 r \exp(B) \) with \( B \) satisfying \( 5.7 \) relative to \( r \). Let \( \Xi \in \mathcal{H}_n(U) \) be the \( n \)-cycle defined by \( \Xi = \cup_{r \in W_0 r} \xi_r \).

In view of Proposition 4.7, Definition 4.16 and Definition 4.24 we see that, for all \( h \in \mathcal{H} \),

\[
(6.15) \quad \nu(\{W_0 r \})\chi_r(h) = \int_{\Xi} \left( \frac{E_t(h)}{q(w_0)\Delta(t)} \right) \frac{dt}{q(w_0)c(c)(t^{-1})}
\]
Let $r' \in W_0 r$. The scaling operation sends the root labels $q$ to $q'$, and follows the corresponding path of $\epsilon \to \sigma_r (r')$ of the residual point $r'$. Obviously the position of $t_0$ (in equation 4.1) relative to $\mathcal{L}^{[\sigma_r (r')]}$ is independent of $\epsilon$. And also, the position of $e$ relative to the facets of the dual configuration $\mathcal{L}_{[\sigma_r (r')]}$ is independent of $\epsilon$, since the effect of the scaling operation on $\mathcal{L}_{[r']} \subset T_r$ simply amounts to the application of the map $c \to c^\epsilon$. In view of Proposition 4.10 and Proposition 4.13, we can take the cycle $\sigma_\epsilon (\Xi) \in H_n (\sigma_r (U))$ in order to define the state $\chi_{\sigma_\epsilon (r)}$ of $\mathcal{H}_q^n (\sigma_r (U))$. In other words, we have, for $h \in \mathcal{H}_q^n (U)$,

$$
(6.16) \quad \nu_q' (\{W_0 \sigma_\epsilon (r)\}) \chi_{q', \sigma_\epsilon (r)} (\dot{j}_e (h)) = \int_{\sigma_\epsilon (\Xi)} \left( \frac{E_i (j_e (h))}{q^\epsilon (w_0) \Delta (t)} \right) q^\epsilon (w_0) c_\epsilon (t)c_\epsilon (t^{-1}) dt
$$

$$
= \int_{\Xi} \left( \frac{E_i (j_e (h))}{q^\epsilon (w_0) \Delta (\sigma_\epsilon (t))} \right) q^\epsilon (w_0) c_\epsilon (\sigma_\epsilon (t))c_\epsilon (\sigma_\epsilon (t^{-1})) dt
$$

$$
= \int_{\Xi} \left( \frac{E_i (h)}{q (w_0) \Delta (t)} \right) \phi_\epsilon (t) \frac{dt}{q (w_0) c(t) c(t^{-1})},
$$

where

$$
(6.17) \quad \phi_\epsilon (t) := \frac{e^n q (w_0) c(t) t^{-n}}{q^\epsilon (w_0) c_\epsilon (\sigma_\epsilon (t)) c_\epsilon (\sigma_\epsilon (t^{-1}))}.
$$

By Lemma 6.2, the function $t \to \phi_\epsilon$ extends, for all $\epsilon \in (0, 1]$, to a regular holomorphic function on $U$. Clearly, $\phi_\epsilon$ is $W_0$-invariant. In other words, $\phi_\epsilon$ is an element of $\mathcal{Z}^m (U)$. Its value in $W_0 r$ can be computed easily, if we keep in mind that the index $i_{r'} = n$ (by Theorem 3.9 applied to the residual coset $r'$). We obtain, by a straightforward computation:

$$
(6.18) \quad \phi_\epsilon (W_0 r) = \frac{m_{q', \sigma_\epsilon (r)} (r)}{m_r (r)} = \frac{\nu_q' (\{W_0 \sigma_\epsilon (r)\})}{\nu (\{W_0 r\})}.
$$

We now continue the computation which we began in equation 6.16, using the fact that $\phi_\epsilon \in \mathcal{Z}^m (U)$ and the fact that $\chi_r$ extends uniquely to $\mathcal{H}_q^n (U)$ in such a way that for all $\phi \in \mathcal{Z}^m (U)$ and $h \in \mathcal{H}_q^n (U)$, $\chi_r (\phi h) = \phi (r) \chi (h)$. We get

$$
(6.19) \quad \nu_q' (\{W_0 \sigma_\epsilon (r)\}) \chi_{q', \sigma_\epsilon (r)} (\dot{j}_e (h)) = \nu (\{W_0 r\}) \chi_r (\phi_\epsilon h)
$$

$$
= \nu_q' (\{W_0 \sigma_\epsilon (r)\}) \chi_r (h).
$$

This gives the desired result. $\square$
Corollary 6.6. The “base change” isomorphism $j_\epsilon$ induces an isomorphism

$$j_\epsilon : \mathcal{H}_e \longrightarrow \mathcal{H}_{q^r}$$

of Frobenius algebras.

7. Appendix: the Kazhdan-Lusztig parameters

Let $G$ be a split semisimple algebraic group over a $p$-adic field $F$ of adjoint type, and let $I$ be an Iwahori subgroup of $G$. The centralizer algebra of the representation of $G$ induced from the trivial representation of $I$ is isomorphic to an affine Hecke algebra $\mathcal{H}$ with “equal labels”, that is, the labels are given by 2.22 with $q$ equal to the cardinality of the residue field of $F$, and the integers $f_i$ all equal to 1. Moreover, the lattice $X$ is equal to the weight lattice of $R$ in this case. The Langlands dual group $G$ is the simply connected semisimple group with root system $R$, and the torus $T$ can be viewed as a maximal torus in $G$.

In this situation Kazhdan and Lusztig [15] have given a complete classification of the irreducible representations of $\mathcal{H}$, and also of the tempered and square integrable irreducible representations. Their results solve completely the problem mentioned in 2.5.2. Let us explain the connection with residual cosets explicitly.

We assume that we are in the “equal label case” in this subsection, unless stated otherwise. We put $k = \log(q)$. Let $G$ be a connected semisimple group over $\mathbb{C}$, with fixed maximal torus $T = \text{Hom}(X, \mathbb{C}^\times)$. We make no assumption on the isogeny class of $G$ yet.

Proposition 7.1. (i) If $r$ is a residual point with polar decomposition $r = sc = s \exp(\gamma) \in T_n T_r$, and $\gamma$ dominant, then the centralizer $C_G(s)$ of $s$ in $g := \text{Lie}(G)$ is a semisimple subalgebra of $g$ of rank equal to $\text{rk}(g)$, and $2\gamma/k$ is the Dynkin diagram of a distinguished nilpotent class of $C_G(s)$.

(ii) Conversely, let $s \in T_n$ be such that the centralizer algebra $C_G(s)$ is semisimple and let $e \in C_G(s)$ be a distinguished nilpotent element. If $h$ denotes the Dynkin diagram (cf. [6]) of $e$ then $r = sc$ with $c := \exp(\frac{k h}{2})$ is a residual point.

(iii) The above maps define a $1-1$ correspondence between $W_0$-orbits of residual points on the one hand, and conjugacy classes of pairs $(s, e)$ with $s \in G$ semisimple such that $C_G(s)$ is semisimple, and $e$ a distinguished nilpotent element in $C_G(s)$.
(iv) Likewise there is a $1 - 1$ correspondence between $W_0$-orbits of residual points and conjugacy classes of pairs $(s, u)$ with $C_G(s)$ semisimple and $u$ a distinguished unipotent element of $C_G(s)^0$.

Proof. (i). We already saw in subsection 3 that the rank of $C_G(s)$ is indeed maximal. So we are reduced to the case $s = 1$. Let $\langle q \rangle$ denote the group of integer powers of $q$, and denote by $R_q \subset R$ the root subsystem of roots $\alpha \in R$ such that $\alpha(c) \in \langle q \rangle$. Now $R_q$ is a root subsystem of rank equal to $\text{rk}(R)$, with the property that for all $\alpha, \beta \in R_q$ such that $\alpha + \beta \in R$ we have $\alpha + \beta \in R_q$. Of course, $c$ is a residual point of $R_q$. By an elementary result of Borel and De Siebenthal there exists a finite subgroup $Z \subset T_u$ such that $C_G(Z)$ is semisimple with root system $R_q$.

We claim that for every simple root $\alpha$ of $R_q$ we have $\alpha(c) = 1$ or $\alpha(c) = q$. To see this, observe that all the roots $\alpha \in R_q$ with $\alpha(c) = q$ are in the parabolic system obtained from $R_q$ by omitting the simple roots $\alpha$ such that $\alpha(c) = q^l$ with $l > 1$. If this would be a proper parabolic subsystem, $c$ would violate Theorem 3.9 in this parabolic. This proves the claim.

Define the element $h := 2\gamma/k$. Note that $h$ belongs to $2P(R_q^\vee)$ by the previous remarks. Consider the grading of $R_q$ given by this element, and define a standard parabolic subalgebra $p$ of $C_G(Z)$ by

$$p := t \oplus \sum_{\{\alpha \in R_q: \alpha(h) \geq 0\}} g_\alpha = \sum_{i \geq 0} C_G(Z)(i).$$

Its nilpotent radical $n$ is

$$n := \sum_{i \geq 2} C_G(Z)(i),$$

and by the definition of residual points we see that $P \subset C_G(Z)$ is a distinguished parabolic subalgebra (see [6], Corollary 5.8.3.). According to ([6], Proposition 5.8.8.) we can choose $e \in n(2)$ in the Richardson class associated with $p$, and $f \in C_G(Z)(-2)$, such that $(f, h, e)$ from a $sL$-triple in $C_G(Z)$. By $sL$-representation theory it is now clear that $h \in P(R^\vee)$. Consider the grading of $g$ and $R$ induced by $h$. By definition of $Z$ we see that $g(0) = C_G(Z)(0)$ and $g(2) = C_G(Z)(2)$. Hence $e$ is distinguished in $g$ by ([6], Proposition 5.7.5.), proving the desired result. Note also that, by ([6], Proposition 5.7.6.), in fact $g(1) = 0$, and hence that $R_q = R$.

(ii). Is immediate from the defining property

$$\dim(C_g(s)(0)) = \dim(C_g(s)(2))$$
of the grading with respect to the Dynkin diagram of a distinguished class.

(iii). Is clear by the well known 1 – 1 correspondence between distinguished classes and their Dynkin diagrams.

(iv). The result follows from the well known 1–1 correspondence between unipotent classes and nilpotent classes for connected semisimple groups over $\mathbb{C}$. \hfill \square

**Corollary 7.2.** From the proof of Proposition 7.1(i) we see that if $r = sc$ is a residual point, then $\alpha(c) \in \langle q^{1/2} \rangle$ for all $\alpha \in R$. If $s = 1$ we have $\alpha(c) \in \langle q \rangle$ for all $\alpha \in R$.

Let $M \subset T$ be a residual coset. Write $M = rT^M \subset T \subset G$ with $r \in T_M$ as in Proposition 3.2. Let $r = sc = s \exp(kh/2)$ be the polar decomposition of $r$ in $T_M$. Let $L_M \subset G$ be the Levi subgroup $L_M := C_G(T^M)$ and let $L'_M$ denote its semisimple part. By Proposition 3.2 we see that the root system of $L'_M$ is $R_M$, $T_M$ is a maximal torus of $L'_M$, and the connected center of $L_M$ is $T^M$. Moreover, $r \in T_M$ is a residual point with respect to $R_M$. Thus by Proposition 7.1, $C_{L'_M}(s)$ is semisimple, and there exists a distinguished unipotent element $u = \exp(e)$ in $C_{L'_M}(s)^{0}$ such that $[h,e] = 2e$. This implies that the set $N = N_u$ of all elements $t \in G$ such that

\[ t u t^{-1} = u^q. \]

is of the form $N = rC_G(u)$. The centralizer $C_G(r,u) = C_G(s,c,u)$ is known to be maximal reductive in $C_G(s,u)$, and it contains $T^M$. Its intersection with $L'_M$ is also reductive but, since $u$ is distinguished in $C_{L'_M}(s)^{0}$, the rank of this intersection is 0. Hence $L'_M \cap C_G(r,u)$ is finite. We conclude that $T^M$ is a maximal torus in $C_G(s,u)$. Let $u'$ be another unipotent element in $G$ such that $M \subset N' = N_u'$ and such that $T^M$ is a maximal torus of $C_G(s,u')$. We see that $u' \in C_{L'_M}(s)^{0}$ is distinguished and associated to the Dynkin diagram $h$. Hence $u'$ is conjugate to $u$ in $C_{L'_M}(s)^{0}$ by an element of $C_{L'_M}(r)$. We have shown:

**Proposition 7.3.** For each residual coset $M = rT^M = scT^M \subset T$ there exists a unipotent element $u$ such that $t u t^{-1} = u^q$ for all $t \in M$, and such that $T^M$ is a maximal torus of $C_G(s,u)$. This $u$ is an element of $C_{L'_M}(s)$ with $L_M := C_G(T^M)$, and is distinguished in this semisimple group. It is unique up to conjugation by elements of the reductive group $C_{L'_M}(r)$.

Let us consider the converse construction. From now in this subsection we assume that $G$ is simply connected. We will be interested in conjugacy classes of pairs $(t,u)$ with $t$ semisimple and $u$ unipotent,
satisfying 7.1. We choose an element \((t, u)\) in the conjugacy class. By Jacobson-Morozov’s theorem there exists a homomorphism

\[
\phi: SL_2(\mathbb{C}) \to G
\]

such that

\[
u = \phi\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)
\]

Define

\[
c := \phi\left(\begin{array}{cc} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{array}\right),
\]

and put

\[
h := d\phi\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right),\ e := d\phi\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right).
\]

Denote by \(C_G(\phi)\) the centralizer of the image of \(\phi\). We have \(C_G(\phi) = C_G(d\phi)\), and by \(sl_2\) representation theory we see that \(C_G(d\phi) = C_G(h, e)\). Hence \(C_G(\phi) = C_G(c, u)\). By [15], Section 2, this is a maximal reductive subgroup of \(C_G(u)\), and we can choose \(\phi\) in such a way that \(t \in cC_G(\phi)\).

In this case \(t\) commutes with \(c\), and thus \(t_1 := tc^{-1} \in C_G(\phi)\) commutes with \(c, t\), and is semisimple. It follows that \(C_G(t_1, \phi) = C_G(\phi) \cap C_G(t_1)\) is reductive in \(C_G(t_1)\), and contains \(t_1\) in its center. According to [15], the choice of \(\phi\) such that \(t_1 \in C_G(\phi)\) is unique up to conjugation by elements in \(C_G(t, u)\).

By conjugating \((t, u)\) and \(\phi\) suitably we can arrange that \(\mathcal{T} := (T \cap C_G(t_1, \phi))^0\) is a maximal torus of \(C_G(t_1, \phi)\). Put \(L = C_G(\mathcal{T})\), a Levi group of \(G\). We claim that \(L\) is minimal among the Levi groups of \(G\) containing \(\phi\) and \(t_1\). Indeed, if \(N\) would be a strictly smaller Levi group of \(G\) also containing \(\phi\) and \(t_1\), then its connected center \(T^N\) would be a torus contained in \(C_G(t_1, \phi)\) on the one hand, but strictly larger than \(\mathcal{T}\) on the other hand. This contradicts the choice of \(\mathcal{T}\), proving the claim. In particular, since the connected center \(T^L\) of \(L\) satisfies \(\mathcal{T} \subset T^L \subset C_G(t_1, \phi)\), we have the equality \(\mathcal{T} = T^L\).

Note that maximal tori of \(L\) are the maximal tori of \(G\) containing \(T^L\), and these are conjugate under the action of \(L\). The derived group \(L'\) is simply connected, because the cocharacter lattice \(Y_L\) of its torus \(T_L\) equals \(Y_L = Q(R^+) \cap Q'R^+_L = Q(R^+_L)\). Hence, by a well known result of Steinberg, \(C := C_L(t_1) \subset L\) is connected, and reductive. This implies that there exist maximal tori of \(C\) containing \(t\). Thus there exist maximal tori of \(L\) containing both the commuting semisimple elements \(t_1\) and \(t\). Therefore we may and will assume (after conjugation of \((t, u)\)
and $\phi$ by a suitable element of $L$) that $T^L$ and the elements $t_1, t$ are inside $T$.

Both the image of $\phi$ and $t_1$ are contained in $C$. Let $C' \subset L'$ denote its derived group. If the semisimple rank of $C$ would be strictly smaller than that of $L$, there would exist a Levi group $N$ such that $C \subset N \nsubseteq L$, a contradiction. Hence $C'$ has maximal rank in $L'$.

Choose $s_L$ in the intersection $t_1 T^L \cap L'$. By the above, $s_L$ is in $T_{L,u}$, the compact form of the maximal torus $T_L := (L' \cap T)^0$ of $L'$. We put $r_L = s_L c \in L'$, and we claim that this is a $R_L$-residual point of $T_L$. By Proposition 7.1 this is equivalent to showing that $u$ is a distinguished unipotent element of $C' = C_{L'}(s_L)$. This means that we have to show that $C_{L'}(s_L, \phi)$ does not contain a nontrivial torus. But $L = C_G(T^L)$ with $T^L$ a maximal torus in $C_G(t_1, \phi)$. Hence $C_G(t_1, \phi)^0 \cap L = T^L$, and thus

$$C_G(s_L, \phi)^0 \cap L' = C_G(t_1, \phi)^0 \cap L' = T_L \cap T^L,$$

proving the claim.

This proves that $M := t T^L = r_L T^L \subset T$ is a residual coset, by application of Proposition 3.3.

Notice that 7.3 shows that $T^L$ is also a maximal torus of $C_G(s_L, \phi)$, and thus of $C_G(s, u)$.

Finally notice that the $W_0$-orbit of the pair $(t, M)$ is uniquely determined by the conjugacy class of $(t, u)$ by the above procedure. We have shown:

**Proposition 7.4.** For every pair $(t, u)$ with $t$ semisimple and $u$ unipotent satisfying 7, we can find a homomorphism $\phi$ as in 7.2 such that $t$ commutes with $c$. Let $T^L$ be a maximal torus of $C_G(t_1 = tc^{-1}, u)$ and put $M = t T^L$. By suitable conjugation we can arrange that $t, c$ and $M$ are in $T$. Then $M \subset T$ is a residual coset. If we write $t = rt^L$ with $r = sc \in T_{L,r} T_{L,s}$ and $t^L \in T^L$, then $T^L$ is also a maximal torus of $C_G(s, c, u)$. The $W_0$-orbit of the pair $(t, M)$ is uniquely determined by $(t, u)$.

**Corollary 7.5.** There is a one-to-one correspondence between conjugacy classes of pairs $(t, u)$ satisfying 7.1 and $W_0$-orbits of pairs $(t, M)$ with $M \subset T$ a residual coset, and $t \in M$.

**Proof.** The maps between these two sets as defined in 7.3 and 7.4 are clearly inverse to each other. \qed

**Remark 7.6.** Let $(c, u)$ (with $c \in T_{L,r}$) be a pair satisfying 7, with $u$ a distinguished unipotent element of $G$. Then $u$ will be distinguished in $C_G(s)$ for each $s$ in the finite group $C_G(c, u)$. In particular, $C_G(s)$ is
semisimple. Hence s gives rise to a residual point s' in T where s' ∈ T is conjugate with s in G. This defines a one-to-one correspondence between the orbits in \(C_G(c, u)\) with respect to the normalizer \(N_G(C_G(c, u))\) and the residual points in T with split part c.

The Kazhdan-Lusztig parameters for irreducible representations of \(H\) consist of triples \((t, u, \rho)\) where \((t, u)\) is as above, and \(\rho\) is a geometric irreducible representation of the finite group

\[ A(t, u) = C_G(t, u)/(Z_G C_G(t, u)^0), \]

where \(Z_G\) is the center of \(G\). Moreover, Kazhdan and Lusztig show that the irreducible representation \(\pi(t, u, \rho)\) is tempered if and only if \(t \in M^\text{temp}\), where \(M\) is the residual subspace associated to the pair \((t, u)\). In this way we obtain a precise geometric description of the set of minimal central idempotents \(\{e_i\}_{i=1}^n\) of the residue Frobenius algebra \(H'\) for \(R_M\)-generic \(t \in M^\text{temp}\).

References


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