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RESEARCH ARTICLE

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The master relation for polynomiality and equivalences of integrable systems

Xavier Blot[1] | **Adrien Sauvaget**[2] | **Sergey Shadrin**[1]

[1]Korteweg-de Vriesinstituut voor Wiskunde, Universiteit van Amsterdam, Postbus, Amsterdam, Netherlands

[2]CNRS, Université de Cergy-Pontoise, Laboratoire AGM, Chauvin, Cergy-Pontoise Cedex, France

Correspondence
Sergey Shadrin, Korteweg-de Vriesinstituut voor Wiskunde, Universiteit van Amsterdam, Postbus 94248, 1090GE Amsterdam, Netherlands.
Email: s.shadrin@uva.nl

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Abstract

We prove the so-called master relation in the tautological ring of the moduli space of curves that implies polynomial properties of the Dubrovin–Zhang hierarchies associated to different versions of cohomological field theories as well as their equivalences to the corresponding double ramification hierarchies.

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1 | INTRODUCTION

The main goal of the paper is to prove the so-called *master relation* in $R^*(\overline{\mathcal{M}}_{g,n+m})$, $g \geqslant 0, n, m \geqslant 1$, $2g-2+n+m>0$, conjectured in [3, Conjecture 3.4]. This master relation is proven in [3] in the Gorenstein quotient and used to imply the strong form of the *DR/DZ equivalence* conjecture due to Buryak–Dubrovin–Guéré–Rossi [5, 6] (see also the foundational work of Buryak [4] for an earlier weaker version of the DR/DZ equivalence conjecture).

The master relation in the Gorenstein quotient is sufficient for the most important applications to integrable systems associated with ordinary cohomological field theories (CohFTs), partial CohFTs, and F-CohFTs. Indeed, the examples considered in the literature in the context of integrable systems are constructed in terms of the tautological classes. Yet, it was believed for a long time that polynomiality and equivalence properties of integrable systems of topological type are governed by actual relations in the tautological ring of the moduli space of curves. To this end, a number of conjectures were made, see [7, Conjecture 2.5] and [8, Conjectures 1, 2, 3]. All these

conjectures were shown through a sequence of works [1–3, 8] to follow from the master relation. Therefore, the present paper settles these conjectures as well.

Despite the fact that the master relation that we prove in this paper is of crucial importance for integrable systems, we would not survey these applications as it is already done in a very detailed way in [8, Section 4]. In order to keep this paper short, we focus on the statement and proof of the master relation itself. It is worth noticing that our proof follows from a direct application of the virtual localization formula for moduli spaces of (relative) stable maps to $\mathbb{P}^1$ [11], while the previous proof in the Gorenstein quotient relied on a series of reduction steps to control the intersection of the master relation with generators of the tautological rings.

2 | THE MASTER RELATION

2.1 | Definition

We follow the exposition in [3].

2.1.1 | Prestable star rooted trees

We fix integers $m \geqslant 1$, $n \geqslant 1$, and $g \geqslant 0$, such that $2g-2+n+m>0$. We denote by $\mathrm{PSSRT}_{g,n,m}$ the set of *$n+m$-prestable star rooted trees of genus g*, which is the set of prestable graphs

$$T = (V, H, \iota : H \to H, g : V \to \mathbb{Z}_{\geqslant 0}, H^\iota \simeq \{\sigma_1, \dots, \sigma_{n+m}\})$$

of genus g with $n+m$ legs, satisfying the following constraints:

- If $E(T)$ stands for the set of *edges* (the cycles of length two of ι), then the graph (V,E) is a rooted tree, and all edges are between the root and another vertex (hence the term "star").
- The legs $\sigma_{n+1}, \dots, \sigma_{n+m}$ are attached to the root vertex, while $\sigma_1, \dots, \sigma_n$ are attached to nonroot vertices. In particular, the root vertex v_r is uniquely determined by the rest of the data.
- For each $v \in V(T)$, let $H(v)$ be the set of half-edges incident to v (including the legs). The prestability condition means that $2g(v)-2+|H(v)| \geqslant 0$ for every $v \in V(T)$.
- There is at least one leg attached to each vertex.

We will consider the ring $Q = \mathbb{Q}(a_1, \dots, a_n)$ (where the a_i's are formal variable). If T is a graph in $\mathrm{PSSRT}_{g,n,m}$, then each edge e connects the root vertex v_r to a nonroot vertex that we denote v_e. We set $a(e)$ to be the sum of a_i's associated with the legs attached to v_e. If a half-edge h is the leg σ_i, then $a(h)$ stands for a_i.

2.1.2 | Classes assigned to a tree

We want to assign to T a class $\Xi(T)$ in $R^*(\overline{\mathcal{M}}_{g,n+m}) \otimes_{\mathbb{Q}} Q[u,u^{-1}]$ (where u is an extra formal variable used to control the degree). In order to construct this class, we first assign a class to each vertex.

Let v be a vertex of T. We will consider the moduli space of curves $\overline{\mathcal{M}}_{g(v),|H(v)|}$. If v is the root vertex v_r, we have a natural isomorphism $H(v_r) \cong E(T) \sqcup \{\sigma_{n+1}, \dots, \sigma_{n+m}\}$ and we associate the first $|E(T)|$ marked points in the corresponding space $\overline{\mathcal{M}}_{g(v_r),|H(v_r)|}$ with the edges, and the remaining

m points with the legs $\{\sigma_{n+1}, \dots, \sigma_{n+m}\}$. We set

$$\Psi(v_r) := \prod_{e \in E(T)} \frac{1}{1 - a(e)\psi_e} \in R^*(\overline{\mathcal{M}}_{g(v_r),|E(T)|+m}) \otimes_{\mathbb{Q}} Q, \tag{2.1}$$

where ψ_e stands for the ψ-class attach the half-edge part of the edge e. In the exceptional unstable case $g(v_r) = 0, m = 1, |E(T)| = 1$, we formally assign to the root vertex the following class:

$$\Psi(v_r) := a(e)^{-1} \in R^{-1}(\overline{\mathcal{M}}_{0,2}) \otimes_{\mathbb{Q}} Q, \tag{2.2}$$

where $R^{-1}(\overline{\mathcal{M}}_{0,2})$ is identified with $\mathbb{Q}$ and the negative cohomological degree is just formally assigned to allow to treat this case nonexceptionally in what follows.

If v is a nonroot vertex $v_e \neq v_r$ for some edge e, the we associate the first $|H(v_e)| - 1$ marked points in the space $\overline{\mathcal{M}}_{g(v_e),|H(v_e)|}$ with the legs attached to v_e, and the last marked point with the edge e that connects v_e to v_r. We set

$$\mathbb{D}(v_e) := \frac{\lambda_{g(v_e)} \mathrm{DR}_{g(v_e)}\Big(a(h_1), \dots, a(h_{|H(v_e)|-1}), -a(e)\Big)}{1 - a(e)\psi_{|H(v_e)|}} \in R^*(\overline{\mathcal{M}}_{g(v_e),|H(v_e)|}) \otimes_{\mathbb{Q}} Q. \tag{2.3}$$

As $a(e) = a(h_1) + \cdots + a(h_{|H(v_e)|-1})$, the class $\mathrm{DR}_{g(v_e)}\big(a(h_1), \dots, a(h_{|H(v_e)|-1}), -a(e)\big)$ is the corresponding double ramification cycle. In the exceptional unstable case $g(v_e) = 0, H(v_e) = 2$, we formally assign to this vertex the following class:

$$\mathbb{D}(v_e) := a(e)^{-1} \in R^{-1}(\overline{\mathcal{M}}_{0,2}) \otimes_{\mathbb{Q}} Q, \tag{2.4}$$

where, as it was for the root vertex, the negative cohomological degree is just formally assigned to allow the treatment of this case nonexceptionally. With this system of notation, we set

$$\Xi(T) := u^{2g-2+m} \left(\prod_{e \in E(T)} \frac{a(e)}{u} \right) (b_T)_* \left(\left(\sum_{d=-1}^{\infty} \frac{\Psi(v_r)_d}{(-u)^d} \right) \otimes \bigotimes_{e \in E(T)} \left(\sum_{d=-1}^{\infty} \frac{\mathbb{D}(v_e)_d}{u^d} \right) \right), \tag{2.5}$$

where $(b_T)_* : \bigotimes_{v \in V(T)} R^*(\overline{\mathcal{M}}_{g(v),|H(v)|}) \to R^*(\overline{\mathcal{M}}_{g,n+m})$ is the boundary pushforward (composed with the contraction of unstable components) map extended by linearity with the coefficients in $Q[u, u^{-1}]$, and α_d stands for the projection to $R^*(\overline{\mathcal{M}}_{g(v),|H(v)|})$ of a tautological class. Note that the resulting formula is a polynomial in a_i's of degree bounded by $3g - 3 + n + m$.

2.2 | Statement and proof

For any $g \geqslant 0, m, n \geqslant 1, 2g - 2 + n + m > 0$, we set

$$\Xi^m_{g,n} := \sum_{T \in \mathrm{PSSRT}_{g,n,m}} \Xi(T). \tag{2.6}$$

Theorem 2.1. *We have*

$$\Xi^m_{g,n} \in R^*(\overline{\mathcal{M}}_{g,n+m}) \otimes_{\mathbb{Q}} \mathbb{Q}[a_1, \dots, a_n, u] \tag{2.7}$$

(in other words, the coefficients of all negative degrees of u vanish).

Proof. Since $\Xi^m_{g,n}$ is a polynomial in $a_1, \dots, a_n$ of degree at most $3g-3+m+n$, it is sufficient to prove that $\Xi^m_{g,n}(a_1, \dots, a_n)$ is a polynomial in u for each $a_1, \dots, a_n \in \mathbb{Z}_{>0}$. Note also that it is sufficient to consider cohomology with complex coefficients. Then, the proof directly generalizes the argument in [8, Proof of Theorem 5.3] along the lines of a computation done in [10, Section 3.2].

We consider the moduli space $\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}^1, a_1, \dots, a_n)$ of relative stable maps to $(\mathbb{P}^1, \infty)$: the first n marked points are mapped to ∞ with contact orders prescribed by the a_i's. We consider the standard $\mathbb{C}^*$-action on $\mathbb{P}^1$ and induced action on $\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, a_1, \dots, a_n)$. Let $\pi : U \to \overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, a_1, \dots, a_n)$ be the universal curve and $f : U \to \mathbb{P}^1$ the universal map. The curve U is endowed with the $\mathbb{C}^*$-action making f equivariant. For $i \in \{n+1, \dots, n+m\}$, we denote $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n+m}(\mathbb{P}^1, a_1, \dots, a_n) \to \mathbb{P}^1$ the evaluation morphism at the ith marking.

We will consider the line bundle $\mathcal{O}(-1) \to \mathbb{P}^1$. We lift the $\mathbb{C}^*$-action to $\mathcal{O}(-1) \to \mathbb{P}^1$ with the fiber weights at 0 and ∞ equal to -1 and 0, respectively. We set

$$I^m_{g,n} := (-1)^{g+m} \mathrm{ev}^*_{n+1}([0]) \cup \dots \cup \mathrm{ev}^*_{n+m}([0]) \cup e_{\mathbb{C}^*}(R^1\pi_* f^* \mathcal{O}(-1)) \tag{2.8}$$

in $H^{\mathbb{C}^*}_*(\overline{\mathcal{M}}_{g,n+m}) \cong H_*(\overline{\mathcal{M}}_{g,n+m}) \otimes_{\mathbb{C}} \mathbb{C}[u]$. In this expression, the class $[0] \in H^2_{\mathbb{C}^*}(\mathbb{P}^1)$ stands for the $\mathbb{C}^*$-equivariant class dual to the point 0, while $e_{\mathbb{C}^*}$ is the $\mathbb{C}^*$-equivariant Euler class (here, the derived pushforward $R^1\pi_* f^* \mathcal{O}(-1)$ is a vector bundle of rank $g + a_1 + \dots + a_n - 1$).

Let $\epsilon : \overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, a_1, \dots, a_n) \to \overline{\mathcal{M}}_{g,n+m}$ and $[\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, a_1, \dots, a_n)]^{\mathrm{vir}}$ denote the $\mathbb{C}^*$-equivariant virtual fundamental class of $\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, a_1, \dots, a_n)$. We will prove that

$$\Xi^m_{g,n}(a_1, \dots, a_n, -u) = \epsilon_* \Big(I^m_{g,n} \cap [\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, a_1, \dots, a_n)]^{\mathrm{vir}} \Big). \tag{2.9}$$

To this end, we need to apply the localization formula to (2.9) along the lines of [11] (see also a survey in [8, Appendix]), and the computation of $e_{\mathbb{C}^*}(R^1\pi_* f^* \mathcal{O}(-1))$ performed in [10, Equation (3.13)]. As the right-hand side of (2.9) belongs to $H^{\mathbb{C}^*}_*(\overline{\mathcal{M}}_{g,n+m}) \cong H_*(\overline{\mathcal{M}}_{g,n+m}) \otimes_{\mathbb{C}} \mathbb{C}[u]$, the theorem follows.

The localization formula expresses $[\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, a_1, \dots, a_n)]^{\mathrm{vir}}$ as a sum over bipartite graphs with some vertices corresponding to the components mapped to 0 and ∞. We allow these vertices to correspond to the unstable moduli spaces as well — formally speaking, in such cases, a separate computation is required, but the final result can be written uniformly with the contributions of the unstable vertices introduced formally as we did above in the definition of $\Xi^m_{g,n}$. Only part of these bipartite graphs contributes when we intersect the virtual fundamental class with $I^m_{g,n}$.

- The first n legs are incident to vertices over ∞ because of the relative conditions, while the last m legs are over 0 because of the factors $\mathrm{ev}^*_i([0])$.
- Our specific choice of lift of the $\mathbb{C}^*$-action to $\mathcal{O}(-1)$ reduces the computation to an expression on star trees. Indeed, the expression of $e_{\mathbb{C}^*}(R^1\pi_* f^* \mathcal{O}(-1))$ has a factor $0^{|E(v)|-1}$ for each vertex v over infinity, where $E(v)$ is the set of edges attached to v (see [10, Equation (3.13)]).[†] Therefore, each vertex over ∞ should have exactly one edge (otherwise, the class vanishes).

[†] This observation was used in [12] and [9] to obtain the multiple cover formula for rigid rational curves in CY 3-folds.

Hence, the bipartite graphs that we consider are, in fact, the graphs in $\mathrm{PSSRT}_{g,n,m}$: the root is the unique vertex over 0, while the other vertices are over ∞. Note that the degree assignment at edges, which is part of the bipartite graph data, is uniquely determined by the relative conditions at infinity and the fact that the graphs are of compact type. Altogether, the virtual localization formula provides an expression of the form

$$\epsilon_*\left(I^m_{g,n}\cap[\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1,a_1,\dots,a_n)]^{\mathrm{vir}}\right)=\sum_{T\in\mathrm{PSSRT}_{g,n,m}}(b_T)_*(\alpha_1(T)\alpha_2(T)),\tag{2.10}$$

where $\alpha_1(T)$ corresponds to the contribution of $[\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1,a_1,\dots,a_n)]^{\mathrm{vir}}$ from [11] (see also a convenient reminder in [8, Appendix]), while $\alpha_2(T)$ corresponds to the contribution of $I_{m,n}$ deduced from [10, Equation (3.13)]. The explicit expression of these classes is

$$\alpha_1(T)=-u^{-|E(T)|-1}\left(\sum_{i=0}^{g(v_r)}(-1)^i\lambda_i u^{g(v_r)-i}\prod_{e\in E(T)}\frac{a(e)^{a(e)+1}}{a(e)!}\frac{u^{-a(e)}}{1-u^{-1}a(e)\psi_e}\right)\tag{2.11}$$

$$\bigotimes_{e\in E(T)}\left(\frac{\mathrm{DR}_{g(v_e)}\left(a(h_1),\dots,a(h_{|H(v_e)|-1}),-a(e)\right)}{1+u^{-1}a(e)\psi_{|H(v_e)|}}\right),$$

and

$$\alpha_2(T)=(-u)^{|E(T)|-1}(-1)^{g+m}u^m\left((-1)^{g(v_r)}\sum_{i=0}^{g(v_r)}\lambda_i u^{g(v_r)-i}\prod_{e\in E(T)}\frac{a(e)!}{a(e)^{a(e)}}u^{a(e)-1}\right)\tag{2.12}$$

$$\bigotimes_{e\in E(T)}(-1)^{g(v_e)}\lambda_{g(v_e)}.$$

Now we use Mumford's relation [13]

$$\left(\sum_{i=0}^{g(v_r)}(-1)^i\lambda_i u^{g(v_r)-i}\right)\left(\sum_{i=0}^{g(v_r)}\lambda_i u^{g(v_r)-i}\right)=u^{2g(v_r)}\tag{2.13}$$

to deduce

$$\alpha_1(T)\alpha_2(T)=(-u)^{2g-2+m}\left(\prod_{e\in E(T)}\frac{a(e)}{-u}\frac{1}{1-u^{-1}a(e)\psi_e}\right)\tag{2.14}$$

$$\bigotimes_{e\in E(T)}\left(\frac{(-u)^{2g(v_r)}\lambda_{g(v_r)}\mathrm{DR}_{g(v_e)}\left(a(h_1),\dots,a(h_{|H(v_e)|-1}),-a(e)\right)}{1+u^{-1}a(e)\psi_{|H(v_e)|}}\right).$$

Together with (2.10), we indeed obtain the identity (2.9). □

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