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ON THE EQUIVALENCE OF CONVERGENT KINETIC EQUATIONS FOR HOT DILUTE PLASMAS

II. GENERATING FUNCTIONS FOR COLLISION BRACKETS

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The generating functions for the collision brackets associated with two alternative convergent kinetic equations are derived for small values of the plasma parameter. It is shown that the first few terms in the asymptotic expansions of these generating functions are identical. Consequently, both kinetic equations give rise to the same transport coefficients in arbitrarily high order of the Chapman–Cowling truncation scheme.

1. Introduction

In the first paper[1]) of this series we have studied the heat conductivity and the viscosity of a hot dilute plasma by applying the Chapman–Enskog scheme to two alternative convergent kinetic equations. In lowest order of the Chapman–Cowling truncation procedure both equations turned out to lead to identical asymptotic forms of the transport coefficients for small values of the plasma parameter ϵ. The purpose of the present paper is to extend these results by including higher-order terms in the Chapman–Cowling approximation procedure.

In order to derive the higher-order contributions to the transport coefficients collision brackets of increasing complexity must be determined. A convenient tool to achieve this is furnished by the generating function for these brackets. Once the generating function is known general conclusions about the brackets of arbitrarily high order can be drawn. In particular, a comparison of the generating functions associated with the two convergent equations will enable us to establish the equivalence of these equations in so far as the ensuing transport properties are concerned.

In section 2 the definition and some general properties of the generating functions for collision brackets will be presented. These functions will be evaluated asymptotically in sections 3–5 for the various partial collision terms that occur in the convergent kinetic equations. In the final section it will be proved that the generating functions for the convergent kinetic equations are identical in the limit of small plasma parameter.

2. Definition of generating functions for collision brackets

For the calculation of the heat conductivity and the viscosity in arbitrary order of the Chapman–Cowling approximation one needs collision brackets[1,2]) of the general form

$$[S_p^{(l+1/2)}\{\tfrac{1}{2}\beta m(\boldsymbol{v}-\boldsymbol{V})^2\}\boldsymbol{T}^{(l)}(\boldsymbol{v}-\boldsymbol{V})\cdot, S_q^{(l+1/2)}\{\tfrac{1}{2}\beta m(\boldsymbol{v}-\boldsymbol{V})^2\}\boldsymbol{T}^{(l)}(\boldsymbol{v}-\boldsymbol{V})]. \tag{2.1}$$

Here l denotes the rank of the tensor $\boldsymbol{T}^{(l)}$ associated with the transport phenomenon. In particular, $l=1$ corresponds to heat conduction, with $\boldsymbol{T}^{(1)}(\boldsymbol{a})=\boldsymbol{a}$, and $l=2$ to viscosity, with $\boldsymbol{T}^{(2)}(\boldsymbol{a})=\boldsymbol{aa}-\frac{1}{3}a^2\boldsymbol{U}$. The dot $\cdot$ denotes the complete contraction of the tensors $\boldsymbol{T}^{(l)}$. The Sonine polynomials appearing in (2.1) are defined as:

$$S_p^{(l+1/2)}(z)=\sum_{r=0}^{p}\frac{\Gamma(l+p+\frac{3}{2})}{(p-r)!\,r!\,\Gamma(l+r+\frac{3}{2})}(-z)^r. \tag{2.2}$$

Both $S_p^{(l+1/2)}$ and $\boldsymbol{T}^{(l)}$ depend on the velocity $\boldsymbol{v}$ of an individual particle and the hydrodynamical velocity $\boldsymbol{V}$. Dimensionless velocities will be denoted by a bar, i.e. $\bar{\boldsymbol{v}}=(\frac{1}{2}\beta m)^{1/2}\boldsymbol{v}$, with β the inverse temperature and m the mass of the particles.

A more convenient form of the collision brackets is obtained by introducing the spherical-tensor formalism[3]). Then (2.1) becomes (apart from a trivial factor)

$$B_{pq}^{l}=\sum_{m=-l}^{l}[S_p^{(l+1/2)}\{(\bar{\boldsymbol{v}}-\bar{\boldsymbol{V}})^2\}\mathcal{Y}_{lm}(\bar{\boldsymbol{v}}-\bar{\boldsymbol{V}}),\, S_q^{(l+1/2)}\{(\bar{\boldsymbol{v}}-\bar{\boldsymbol{V}})^2\}\mathcal{Y}_{lm}^{*}(\bar{\boldsymbol{v}}-\bar{\boldsymbol{V}})], \tag{2.3}$$

with $\mathcal{Y}_{lm}$ the regular solid spherical harmonics. In (I.4.15) and (I.4.17) the quantities Λ^{pq} and H^{pq}, which determine the heat conductivity and the viscosity, have been defined. They are related to B_{pq}^{l} in the following way:

$$\begin{aligned}\Lambda^{pq}&=\frac{32\pi}{225}\frac{m\beta}{k_B}B_{pq}^{1},\\ H^{pq}&=\frac{16\pi}{75}\beta B_{pq}^{2}.\end{aligned} \tag{2.4}$$

Instead of discussing (2.3) for various values of l, p and q we will introduce a generating function which itself can be written as a collision bracket. We define:

$$\mathscr{B}^l(x, y) = \sum_{m=-l}^{l} [i_l(x|\bar{v} - \bar{V}|)Y_{lm}(\bar{v} - \bar{V}), i_l(y|\bar{v} - \bar{V}|)Y^*_{lm}(\bar{v} - \bar{V})]. \tag{2.5}$$

Here i_l is the modified spherical Bessel function of order l. In contrast to the solid harmonics appearing in (2.3) the spherical harmonics Y_{lm} occurring here depend only on the direction of their vectorial argument.

We will now demonstrate the generating character of $\mathscr{B}^l(x, y)$ by expanding it into a power series in x and y with coefficients that contain B^l_{pq}. Substitution of the expansion

$$i_l(z) = \frac{1}{2}\sqrt{\pi} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{l+2r}}{r!\Gamma(l + r + \frac{3}{2})} \tag{2.6}$$

in (2.5) yields

$$\mathscr{B}^l(x, y) = \frac{\pi}{4} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\frac{1}{2}x)^{l+2r}(\frac{1}{2}y)^{l+2s}}{r!s!\Gamma(l + r + \frac{3}{2})\Gamma(l + s + \frac{3}{2})} \hat{B}^l_{rs}, \tag{2.7}$$

where $\hat{B}^l_{rs}$ is the collision bracket

$$\hat{B}^l_{rs} = \sum_{m=-l}^{l} [(\bar{v} - \bar{V})^{2r}Y_{lm}(\bar{v} - \bar{V}), (\bar{v} - \bar{V})^{2s}Y^*_{lm}(\bar{v} - \bar{V})]. \tag{2.8}$$

By employing the relation[4])

$$z^n = n! \sum_{p=0}^{n} (-1)^p \frac{\Gamma(l + n + \frac{3}{2})}{(n - p)!\Gamma(l + p + \frac{3}{2})} S_p^{(l+1/2)}(z), \tag{2.9}$$

which is the inverse of (2.2), one may express $\hat{B}^l_{rs}$ in terms of B^l_{pq}. Subsequently (2.7) becomes

$$\mathscr{B}^l(x, y) = \frac{\pi}{4} e^{(x^2+y^2)/4} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} \frac{(\frac{1}{2}x)^{l+2p}(\frac{1}{2}y)^{l+2q}}{\Gamma(l + p + \frac{3}{2})\Gamma(l + q + \frac{3}{2})} B^l_{pq}. \tag{2.10}$$

Hence the relevant collision brackets B^l_{pq} can indeed by derived from $\mathscr{B}^l$.

The family of l-dependent generating functions (2.5) can be obtained from a "master" generating function as we will show by the following reasoning. From the Rayleigh-like expansion

$$e^{\boldsymbol{a}\cdot\boldsymbol{x}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i_l(ax)Y_{lm}(\boldsymbol{a})Y^*_{lm}(\boldsymbol{x}) \tag{2.11}$$

and the orthogonality of the Y_{lm} we have

$$i_l(ax)Y_{lm}(\boldsymbol{a}) = \int \frac{d\hat{\boldsymbol{x}}}{4\pi} Y_{lm}(\boldsymbol{x}) e^{\boldsymbol{a}\cdot\boldsymbol{x}}. \tag{2.12}$$

Here $\hat{\boldsymbol{x}}$ is the unit vector $\boldsymbol{x}/|\boldsymbol{x}|$ and $\int d\hat{\boldsymbol{x}}$ denotes the integral over all directions of $\hat{x}$. Upon insertion of (2.12) into (2.5) we get

$$\mathscr{B}^l(x, y) = \sum_{m=-l}^{l} \int \frac{\mathrm{d}\hat{x}}{4\pi} \frac{\mathrm{d}\hat{y}}{4\pi} Y_{lm}(\boldsymbol{x}) Y^*_{lm}(\boldsymbol{y}) [\mathrm{e}^{\boldsymbol{x}\cdot(\bar{v}-\bar{V})}, \mathrm{e}^{\boldsymbol{y}\cdot(\bar{v}-\bar{V})}]. \tag{2.13}$$

If we define

$$\mathscr{B}(\boldsymbol{x}, \boldsymbol{y}) = [\mathrm{e}^{\boldsymbol{x}\cdot(\bar{v}-\bar{V})}, \mathrm{e}^{\boldsymbol{y}\cdot(\bar{v}-\bar{V})}], \tag{2.14}$$

then (2.13) can be written as

$$\mathscr{B}^l(x, y) = \mathscr{P}_l[\mathscr{B}(\boldsymbol{x}, \boldsymbol{y})], \tag{2.15}$$

with

$$\mathscr{P}_l(\cdot) = \sum_{m=-l}^{l} \int \frac{\mathrm{d}\hat{x}}{4\pi} \frac{\mathrm{d}\hat{y}}{4\pi} Y_{lm}(\boldsymbol{x}) Y^*_{lm}(\boldsymbol{y})(\cdot). \tag{2.16}$$

Since [3])

$$\sum_{m=-l}^{l} Y_{lm}(\boldsymbol{a}) Y^*_{lm}(\boldsymbol{b}) = \frac{2l+1}{4\pi} P_l(\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}}), \tag{2.17}$$

whereP_l is the Legendre polynomial of order l, we have

$$\mathscr{P}_l(\cdot) = \frac{2l+1}{(4\pi)^3} \int \mathrm{d}\hat{x}\, \mathrm{d}\hat{y} P_l(\hat{x} \cdot \hat{y})(\cdot). \tag{2.18}$$

One may regard $\mathscr{P}_l$ as an operator, which selects the contribution of a specific value of l. By a rotational-symmetry argument it is easily seen that $[i_l Y_{lm}, i_{l'} Y_{l'm'}]$ vanishes, unless $l = l'$ and $m = -m'$; hence, from (2.5) and (2.14) the formal solution of (2.15) is found as

$$\mathscr{B}(\boldsymbol{x}, \boldsymbol{y}) = 4\pi \sum_{l=0}^{\infty} P_l(\hat{x} \cdot \hat{y}) \mathscr{B}^l(x, y). \tag{2.19}$$

The brackets for the convergent collision operators discussed in paper I are linear combinations of the Boltzmann-, Balescu–Guernsey–Lenard- and Landau-type brackets. From (2.14) it follows that the generating functions associated with the convergent collision terms are likewise combinations of those generating the partial brackets. The latter will be discussed in the following sections.

3. The generating function for the Boltzmann collision brackets

The collision bracket associated with the Boltzmann collision operator is given by (I.4.3). Hence the explicit expression for the generating function

(2.14) reads in the Boltzmann (B′) case:

$$\mathscr{B}_{B'}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4} \int \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{v}'\, \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} |\boldsymbol{v} - \boldsymbol{v}'| \bar{f}_M(\boldsymbol{v}) \bar{f}_M(\boldsymbol{v}')$$
$$\times [\mathrm{e}^{\boldsymbol{x}\cdot(\bar{\boldsymbol{v}}-\bar{\boldsymbol{V}})} + \mathrm{e}^{\boldsymbol{x}\cdot(\bar{\boldsymbol{v}}'-\bar{\boldsymbol{V}})} - \mathrm{e}^{\boldsymbol{x}\cdot(\bar{\boldsymbol{v}}_1-\bar{\boldsymbol{V}})} - \mathrm{e}^{\boldsymbol{x}\cdot(\bar{\boldsymbol{v}}_1'-\bar{\boldsymbol{V}})}]$$
$$\times [\mathrm{e}^{\boldsymbol{y}\cdot(\bar{\boldsymbol{v}}-\bar{\boldsymbol{V}})} + \mathrm{e}^{\boldsymbol{y}\cdot(\bar{\boldsymbol{v}}'-\bar{\boldsymbol{V}})} - \mathrm{e}^{\boldsymbol{y}\cdot(\bar{\boldsymbol{v}}_1-\bar{\boldsymbol{V}})} - \mathrm{e}^{\boldsymbol{y}\cdot(\bar{\boldsymbol{v}}_1'-\bar{\boldsymbol{V}})}], \tag{3.1}$$

where $\bar{f}_M$ is the local Maxwell–Boltzmann distribution, given by (I.3.3). Let us introduce the centre-of-mass and the relative velocities $\bar{\boldsymbol{U}} = \frac{1}{2}(\bar{\boldsymbol{v}} + \bar{\boldsymbol{v}}') = \frac{1}{2}(\bar{\boldsymbol{v}}_1 + \bar{\boldsymbol{v}}_1')$, $\bar{\boldsymbol{u}} = \bar{\boldsymbol{v}} - \bar{\boldsymbol{v}}'$ and $\bar{\boldsymbol{u}}_1 = \bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_1'$. Since $\mathrm{d}\sigma/\mathrm{d}\Omega$ depends only on $\bar{\boldsymbol{u}}_1$ and $\bar{\boldsymbol{u}}$, but not on $\bar{\boldsymbol{U}}$, the $\bar{\boldsymbol{U}}$-integration can be performed. The result reads:

$$\mathscr{B}_{B'}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2} \pi^{-3/2} (\beta m)^{-1/2}\, \mathrm{e}^{(\boldsymbol{x}+\boldsymbol{y})^2/8} \int \mathrm{d}\bar{\boldsymbol{u}}\, \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} |\bar{\boldsymbol{u}}|\, \mathrm{e}^{-\bar{u}^2/2}$$
$$\times \mathscr{S}(\boldsymbol{x}) \mathscr{S}(\boldsymbol{y}) [\mathrm{e}^{(\boldsymbol{x}\cdot\bar{\boldsymbol{u}}+\boldsymbol{y}\cdot\bar{\boldsymbol{u}})/2} - \mathrm{e}^{(\boldsymbol{x}\cdot\bar{\boldsymbol{u}}+\boldsymbol{y}\cdot\bar{\boldsymbol{u}}_1)/2}$$
$$- \mathrm{e}^{(\boldsymbol{x}\cdot\bar{\boldsymbol{u}}_1+\boldsymbol{y}\cdot\bar{\boldsymbol{u}})/2} + \mathrm{e}^{(\boldsymbol{x}\cdot\bar{\boldsymbol{u}}_1+\boldsymbol{y}\cdot\bar{\boldsymbol{u}}_1)/2}]. \tag{3.2}$$

Here we introduced the symmetry operator $\mathscr{S}(\boldsymbol{a})$ which is defined as $\mathscr{S}(\boldsymbol{a})F(\boldsymbol{a}) = \frac{1}{2}F(\boldsymbol{a}) + \frac{1}{2}F(-\boldsymbol{a})$ for arbitrary functions $F(\boldsymbol{a})$. We now apply the expansion (2.11) to the expression in brackets in (3.2). From the first term we obtain

$$(4\pi)^2 \sum_{l_1}{}' \sum_{l_2}{}' \sum_{m_1, m_2} i_{l_1}(\tfrac{1}{2}x\bar{u}) i_{l_2}(\tfrac{1}{2}y\bar{u}) Y_{l_1 m_1}(\boldsymbol{x}) Y_{l_2 m_2}(\boldsymbol{y}) Y^*_{l_1 m_1}(\bar{\boldsymbol{u}}) Y^*_{l_2 m_2}(\bar{\boldsymbol{u}}). \tag{3.3}$$

Here and in the following the primes at the summation signs indicate the restriction to even values of the summation variables. In the integrand of (3.2) the expression (3.3) is multiplied by a factor, which depends only on $|\bar{\boldsymbol{u}}|$ and $\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}}_1$. Hence only the part of (3.3) that is independent of the direction of $\bar{\boldsymbol{u}}$ contributes to the integral in (3.2). This rotational-invariant part reads

$$\sum_l{}' (2l+1) i_l(\tfrac{1}{2}x\bar{u}) i_l(\tfrac{1}{2}y\bar{u}) P_l(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}}), \tag{3.4}$$

as follows from (2.17). Since one has $|\bar{\boldsymbol{u}}| = |\bar{\boldsymbol{u}}_1|$ the fourth term in (3.2) yields an identical contribution. The second and the third term in (3.2) can be treated similarly. Since the argument of $Y^*_{l_2 m_2}$ in (3.3) is now $\bar{\boldsymbol{u}}_1$ instead of $\bar{\boldsymbol{u}}$, the part that is invariant with respect to simultaneous rotation of $\bar{\boldsymbol{u}}_1$ and $\bar{\boldsymbol{u}}$ (keeping $\hat{\boldsymbol{u}}_1 \cdot \hat{\boldsymbol{u}}$ fixed) is found to be of the form of (3.4), with $P_l(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}}_1)$ appearing as an extra factor in the summand. In this way (3.2) is seen to evolve into

$$\mathscr{B}_i(\boldsymbol{x}, \boldsymbol{y}) = 2\, \mathrm{e}^{(\boldsymbol{x}+\boldsymbol{y})^2/8} \sum_l{}' (2l+1) P_l(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}}) I_i^l(x, y), \tag{3.5}$$

where the index i equals B′ and where

$$I^{l}_{\mathrm{B'}}(x, y) = \tfrac{1}{2}\pi^{-3/2}(\beta m)^{-1/2} \int \mathrm{d}\bar{\boldsymbol{u}}\, \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\, |\bar{\boldsymbol{u}}|\, \mathrm{e}^{-\bar{u}^2/2}\, i_l(\tfrac{1}{2}x\bar{u})\, i_l(\tfrac{1}{2}y\bar{u})[1 - P_l(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}}_1)], \tag{3.6}$$

with even $l \geqq 2$. For $l = 0$ we trivially have $I^{0}_{\mathrm{B'}} = 0$. It should be noted that $I^{l}_{\mathrm{B'}}$, and hence $\mathscr{B}_{\mathrm{B'}}$, is symmetric under an interchange of its arguments.

The integrals $I^{l}_{\mathrm{B'}}$ can be linked to the Ω-integrals (I.5.3). To this end we express the Legendre polynomials for even l as a hypergeometric function[4]):

$$P_l(z) = (-1)^{l/2} \binom{l}{\frac{1}{2}l} 2^{-l}\, {}_2F_1(-\tfrac{1}{2}l, \tfrac{1}{2}l + \tfrac{1}{2}, \tfrac{1}{2}; z^2). \tag{3.7}$$

Using the power series expansion (2.6) for the i_l we obtain

$$I^{l}_{\mathrm{B'}}(x, y) = \pi(-1)^{l/2} \binom{l}{\frac{1}{2}l} 2^{-l+1} \sum_{m,n=0}^{\infty} \sum_{p=0}^{l/2} \times \frac{(-\frac{1}{2}l)_p(\frac{1}{2}l + \frac{1}{2})_p}{m!\,n!\,p!\,\Gamma(l + m + \frac{3}{2})\Gamma(l + n + \frac{3}{2})(\frac{1}{2})_p} (\tfrac{1}{8}x^2)^{l/2+m} (\tfrac{1}{8}y^2)^{l/2+n}\, \Omega^{(2p, l+m+n)} \tag{3.8}$$

for even $l \geqq 2$. The Ω-integrals satisfy an asymptotic expansion for small plasma parameter ϵ, which has been derived by Kihara[5]). In appendix A we show how his result can be written in a more elegant form, which reads

$$\Omega^{(l,r)} \simeq \frac{e^4\beta^{3/2}}{32\pi^{3/2}m^{1/2}}\, l(r-1)!(-\log\epsilon - 2\gamma + 2\log 2 + S_{r-1} - T_{l/2}), \tag{3.9}$$

for even $l \geqq 0$ and $r \geqq 2$. Here we have introduced the notations

$$S_l = \sum_{j=1}^{l} \frac{1}{j} = \psi(l+1) - \psi(1),$$

$$T_l = \sum_{j=1}^{l} \frac{1}{2j-1} = \tfrac{1}{2}\psi(l + \tfrac{1}{2}) - \tfrac{1}{2}\psi(\tfrac{1}{2}), \tag{3.10}$$

with $\psi(z) = \mathrm{d}\log\Gamma(z)/\mathrm{d}z$.

The asymptotic expression for $I^{l}_{\mathrm{B'}}$ follows by insertion of (3.9) into (3.8). The summation over p in (3.8) can then be performed. To that end we note that the first four terms in (3.9) give rise to

$$\sum_{p=1}^{n} \frac{(-n)_p(n + \frac{1}{2})_p}{p!(\frac{1}{2})_p}\, p, \tag{3.11}$$

which can be evaluated as a hypergeometric function:

$$-n(2n+1)_2F_1(-n+1, n+\tfrac{3}{2}, \tfrac{3}{2}; 1) = (-1)^n 2^{n-1}\frac{n(2n+1)n!}{(2n-1)!!}. \tag{3.12}$$

The last term in (3.9) leads to the sum

$$\tfrac{1}{2}\sum_{p=1}^{n}\frac{(-n)_p(n+\frac{1}{2})_p}{p!(\frac{1}{2})_p}p[\psi(p+\tfrac{1}{2})-\psi(\tfrac{1}{2})]. \tag{3.13}$$

It may be written in terms of a hypergeometric function and its derivative

$$\begin{aligned}
&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left[\sum_{p=1}^{n}\frac{(-n)_p(n+\frac{1}{2})_p}{p!(z+\frac{1}{2})_p}p\right]_{z=0} \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{n(2n+1)}{2z+1}{}_2F_1(-n+1, n+\tfrac{3}{2}, z+\tfrac{3}{2}; 1)\right]_{z=0} \\
&= (-1)^n 2^{n-2}\frac{n(2n+1)n!}{(2n-1)!!}(S_n+2T_n-1).
\end{aligned} \tag{3.14}$$

To obtain the last line we expressed the hypergeometric function in Γ-functions, which lead upon differentiation to the sums defined in (3.10).

With the use of (3.11)–(3.12) and (3.13)–(3.14) we find from (3.8) with (3.9) the asymptotic expression for $I^l_{\mathrm{B}'}$:

$$I^l_{\mathrm{B}'}(x, y) \simeq \frac{e^4\beta^{3/2}}{32\pi^{1/2}m^{1/2}}l(l+1)\sum_{m,n=0}^{\infty}c^l_{mn}(-\log\epsilon + \Gamma^{l,m+n}_{\mathrm{B}'})(\tfrac{1}{8}x^2)^{l/2+m}(\tfrac{1}{8}y^2)^{l/2+n}, \tag{3.15}$$

with the abbreviations

$$c^l_{mn} = \frac{(l+m+n-1)!}{m!n!\Gamma(l+m+\frac{3}{2})\Gamma(l+n+\frac{3}{2})}, \tag{3.16}$$

$$\Gamma^{lk}_{\mathrm{B}'} = -2\gamma + 2\log 2 + \tfrac{1}{2} + S_{l+k-1} - \tfrac{1}{2}S_{l/2} - T_{l/2}. \tag{3.17}$$

The asymptotic expression for the generating function is obtained by substituting (3.15) in (3.5).

4. The generating function for the modified Boltzmann collision brackets

The collision bracket (I.4.10) associated with the modified Boltzmann operator becomes upon insertion of exponential functions according to (2.14):

$$\begin{aligned}
\mathscr{B}_{\bar{\mathrm{B}}}(\mathbf{x}, \mathbf{y}) = &-\tfrac{1}{2}\int \mathrm{d}\boldsymbol{U}\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{r}\bar{f}_{\mathrm{M}}(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{u})\bar{f}_{\mathrm{M}}(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{u})\exp\left(-k_{\mathrm{D}}r-\frac{r_{\mathrm{L}}}{r}\right)k_{\mathrm{D}}\boldsymbol{u}\cdot\hat{\boldsymbol{r}} \\
&\times[\mathrm{e}^{\mathbf{x}\cdot(\bar{\boldsymbol{U}}+\bar{\boldsymbol{u}}/2-\bar{\boldsymbol{V}})}+\mathrm{e}^{\mathbf{x}\cdot(\bar{\boldsymbol{U}}-\bar{\boldsymbol{u}}/2-\bar{\boldsymbol{V}})}][\mathrm{e}^{\mathbf{y}\cdot(\bar{\boldsymbol{U}}+\bar{\boldsymbol{w}}/2-\bar{\boldsymbol{V}})}+\mathrm{e}^{\mathbf{y}\cdot(\bar{\boldsymbol{U}}-\bar{\boldsymbol{w}}/2-\bar{\boldsymbol{V}})}],
\end{aligned} \tag{4.1}$$

where $k_D = e(\beta n)^{1/2}$ is the reciprocal Debye length and $r_L = \beta e^2/4\pi$ the Landau length. The velocity $\boldsymbol{w}$ has been defined in (I.3.16). The $\boldsymbol{U}$-integration in (4.1) is readily performed. We obtain

$$\mathscr{B}_{\tilde{B}}(\boldsymbol{x}, \boldsymbol{y}) = -\pi^{-3/2}(\beta m)^{-1/2}\, e^{(x+y)^2/8} \int d\bar{\boldsymbol{u}}\, d\boldsymbol{r}\, \exp\left(-\tfrac{1}{2}\bar{u}^2 - k_D r - \frac{r_L}{r}\right) \times k_D \bar{\boldsymbol{u}} \cdot \hat{\boldsymbol{r}} \mathscr{S}(\boldsymbol{x}) \mathscr{S}(\boldsymbol{y}) \exp[\tfrac{1}{2}(\boldsymbol{x} \cdot \bar{\boldsymbol{u}} + \boldsymbol{y} \cdot \bar{\boldsymbol{w}})]. \tag{4.2}$$

By a reasoning, which is quite similar to that of the previous section, we find that $\mathscr{B}_{\tilde{B}}$ can again be written as in (3.5), with $\mathrm{i} = \tilde{\mathrm{B}}$. Instead of (3.6) we now have

$$I^l_{\tilde{B}}(x, y) = -\tfrac{1}{2}\pi^{-3/2}(\beta m)^{-1/2} \int d\bar{\boldsymbol{u}}\, d\boldsymbol{r}\, \exp\left(-\tfrac{1}{2}\bar{u}^2 - k_D r - \frac{r_L}{r}\right) \times k_D \bar{\boldsymbol{u}} \cdot \hat{\boldsymbol{r}}\, i_l(\tfrac{1}{2} x \bar{u}) i_l(\tfrac{1}{2} y \bar{w}) P_l(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{w}}), \tag{4.3}$$

for even $l \geqq 2$. In the case $l = 0$ one has $I^0_{\tilde{B}} = 0$ owing to the antisymmetry in the $\boldsymbol{r}$-integration. In contrast to (3.6) $I^l_{\tilde{B}}(x, y)$ is not symmetric under permutation of its arguments, since the norms of $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{w}}$ differ. Hence the generating function $\mathscr{B}_{\tilde{B}}(\boldsymbol{x}, \boldsymbol{y})$ itself is not symmetric with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$.

As in section 5 of paper I we introduce in (4.3) the variables $\xi = \beta E$, $\eta = \varphi/E$ (where $E = \frac{1}{4}mu^2 + \varphi$ and $\varphi = e^2/4\pi r$) and $\zeta' = \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{w}}$. Then we obtain the following expression for $I^l_{\tilde{B}}$:

$$I^l_{\tilde{B}}(x, y) = -\frac{e^4 \beta^{3/2}}{2\pi^{3/2} m^{1/2}} \epsilon \int_0^\infty d\xi \int_0^1 d\eta\, \frac{1}{\xi^2 \eta^2 (1-\eta)} \exp\left(-\frac{\epsilon}{\xi\eta} - \xi\right) \times i_l[(\tfrac{1}{2}\xi)^{1/2}(1-\eta)^{1/2} x]\, i_l[(\tfrac{1}{2}\xi)^{1/2} y] F_l(\eta), \tag{4.4}$$

with

$$F_l(\eta) = (1-\eta)^2 \int_{-1}^{1} d\zeta'\, \frac{2(1-\eta)^{1/2} - \zeta'(2-\eta)}{[2 - \eta - 2\zeta'(1-\eta)^{1/2}]^3} P_l(\zeta'). \tag{4.5}$$

The function $F_l(\eta)$ can be looked upon as a generalization of $F(\eta)$, which we introduced in (I.5.15); in particular, we have $F_2(\eta) = \frac{3}{2} F(\eta)$. On the interval $(0, 1]$ the function $F_l(\eta)$ is regular, while for $\eta \downarrow 0$ its asymptotic expression is:

$$F_l(\eta) \simeq \tfrac{1}{4} l(l+1)(\log \eta - 2 \log 2 + \tfrac{1}{2} S_{l/2} + T_{l/2}), \tag{4.6}$$

as will be derived in appendix B; the numbers S_l and T_l have been defined in (3.10). We will use this expression in appendix C to obtain the asymptotic form of the integral in (4.4) for small plasma parameter ϵ. From (C.9) we have

$$I^{l}_{\bar{B}}(x, y) \simeq \frac{e^4 \beta^{3/2}}{32\pi^{1/2} m^{1/2}} l(l+1) \sum_{m,n=0}^{\infty} c^{l}_{mn} (-\log \epsilon + \Gamma^{l,m+n}_{\bar{B}})(\tfrac{1}{8} x^2)^{l/2+m} (\tfrac{1}{8} y^2)^{l/2+n}, \tag{4.7}$$

where c^{l}_{mn} has been defined in (3.16) and where

$$\Gamma^{lk}_{\bar{B}} = -2\gamma + 2 \log 2 + S_{l+k-1} - \tfrac{1}{2} S_{l/2} - T_{l/2}. \tag{4.8}$$

We note that the difference between (4.7) and (3.15) is contained in the Γ^{lk}_{i}. In fact, we have from (3.17) and (4.8)

$$\Gamma^{lk}_{\bar{B}} = \Gamma^{lk}_{B'} - \tfrac{1}{2}. \tag{4.9}$$

The asymptotic expression (4.7) for $I^{l}_{\bar{B}}$ is invariant under the interchange of x and y. From (3.5) it follows that the generating function $\mathscr{B}^{l}_{\bar{B}}$ becomes likewise symmetric for $\epsilon \to 0$. Consequently, although the collision brackets B^{l}_{pq} associated with the modified Boltzmann collision operator are in general asymmetric in p and q, the symmetry is restored in leading order with respect to ϵ.

5. Generating functions for the Balescu–Guernsey–Lenard and the Landau collision brackets

In this section we shall derive expressions for the generating functions of the brackets associated with the BGL and L collision terms, both in their original and in their modified form.

The collision bracket that follows from the ordinary BGL equation has been given in (I.4.4). With the use of this expression the generating function (2.14) becomes

$$\begin{aligned}\mathscr{B}_{\mathrm{BGL}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{e^4}{16\pi^2 m^2} \int & \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{v}'\, \mathrm{d}\boldsymbol{q} \delta[\boldsymbol{q} \cdot (\boldsymbol{v} - \boldsymbol{v}')] \\ & \times \frac{1}{q^4 |\epsilon^{(0)}(\boldsymbol{q}, \boldsymbol{q} \cdot \boldsymbol{v})|^2} \bar{f}_{\mathrm{M}}(\boldsymbol{v}) \bar{f}_{\mathrm{M}}(\boldsymbol{v}') \\ & \times \boldsymbol{q} \cdot \left(\frac{\partial}{\partial \boldsymbol{v}} - \frac{\partial}{\partial \boldsymbol{v}'}\right) [\mathrm{e}^{\boldsymbol{x} \cdot (\bar{\boldsymbol{v}} - \bar{\boldsymbol{V}})} + \mathrm{e}^{\boldsymbol{x} \cdot (\bar{\boldsymbol{v}}' - \bar{\boldsymbol{V}})}] \\ & \times \boldsymbol{q} \cdot \left(\frac{\partial}{\partial \boldsymbol{v}} - \frac{\partial}{\partial \boldsymbol{v}'}\right) [\mathrm{e}^{\boldsymbol{y} \cdot (\bar{\boldsymbol{v}} - \bar{\boldsymbol{V}})} + \mathrm{e}^{\boldsymbol{y} \cdot (\bar{\boldsymbol{v}}' - \bar{\boldsymbol{V}})}].\end{aligned} \tag{5.1}$$

Here $\epsilon^{(0)}$ is the dielectric permeability of the plasma in local equilibrium. The integral over $\boldsymbol{q}$ is logarithmically divergent for large $|\boldsymbol{q}|$; for that reason a cut-off at $|\boldsymbol{q}| = k_{\mathrm{D}}\Lambda$, with $\Lambda \gg 1$, is implied in (5.1). For the evaluation of $\mathscr{B}_{\mathrm{BGL}}$

it is convenient to introduce instead of $\boldsymbol{v}$ and $\boldsymbol{v}'$ centre-of-mass and relative velocities $\bar{\boldsymbol{U}}$ and $\bar{\boldsymbol{u}}$, of which the components orthogonal to and parallel with $\boldsymbol{q}$ are denoted by labels $\perp$ and $\|$, respectively. The integral over $\bar{u}_\|$ then becomes trivial on account of the δ-function, while the $\bar{\boldsymbol{u}}_\perp$- and $\bar{\boldsymbol{U}}_\perp$-integrations are Gaussian. After a transformation to the new variables $\zeta = (\bar{U} - \bar{V})_\|$ and $\eta = q/k_{\mathrm{D}}$ one gets with the help of (I.5.19)

$$\mathscr{B}_{\mathrm{i}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sqrt{2} e^4 \beta^{3/2}}{8\pi^2 m^{1/2}} \int \frac{\mathrm{d}\hat{\boldsymbol{q}}}{4\pi}\, x_\| y_\| \, \mathrm{e}^{(x_\perp^2 + y_\perp^2)/4} (\mathrm{e}^{\boldsymbol{x}_\perp \cdot \boldsymbol{y}_\perp/2} - 1) I_{\mathrm{i}}(x_\| + y_\|), \tag{5.2}$$

with i = BGL. The η–ζ-integrations are contained in the integral

$$I_{\mathrm{BGL}}(z) = \int_0^{\Lambda} \mathrm{d}\eta \int_{-\infty}^{\infty} \mathrm{d}\zeta \, \frac{\eta^3 \, \mathrm{e}^{-2\zeta^2 + \zeta z}}{[F_1(\zeta) + \eta^2]^2 + F_2(\zeta)^2}. \tag{5.3}$$

where F_1 and F_2 have been defined in (I.5.20). The η-integral is elementary; for large Λ one finds

$$I_{\mathrm{BGL}}(z) \simeq \left(\frac{\pi}{2}\right)^{1/2} \mathrm{e}^{z^2/8} \log \Lambda - \int_{-\infty}^{\infty} \mathrm{d}\zeta \, \mathrm{e}^{-2\zeta^2 + \zeta z} G(\zeta), \tag{5.4}$$

with

$$G(\zeta) = \tfrac{1}{4} \log[F_1(\zeta)^2 + F_2(\zeta)^2] + \frac{F_1(\zeta)}{2F_2(\zeta)} \left\{ \frac{\pi}{2} - \operatorname{arctg}\left[\frac{F_1(\zeta)}{F_2(\zeta)}\right] \right\}. \tag{5.5}$$

The generating function for the brackets of the modified B$\tilde{\mathrm{G}}$L collision term follows directly from (5.1) by replacing a factor $e^2 q^{-2}$ by the Fourier transformed effective potential $\hat{\varphi}_{\mathrm{B\tilde{G}L}}(q)$, as defined in section 3 of paper I. Introducing in (I.3.23) the dimensionless variable $\xi = r/r_{\mathrm{L}}$ and taking the same steps as above we arrive again at (5.2), with $I_{\mathrm{B\tilde{G}L}}(z)$ given by

$$I_{\mathrm{B\tilde{G}L}}(z) = \epsilon \int_0^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta \int_{-\infty}^{\infty} \mathrm{d}\zeta \, \xi (1 - \mathrm{e}^{-1/\xi}) \frac{\eta^4 j_1(\epsilon\eta\xi) \, \mathrm{e}^{-2\zeta^2 + \zeta z}}{[F_1(\zeta) + \eta^2]^2 + F_2(\zeta)^2}. \tag{5.6}$$

In appendix B of paper I the asymptotic form of the ξ–η-integral for small values of ϵ has been derived. Inserting the result (I.B.13) into (5.6) we obtain:

$$I_{\mathrm{B\tilde{G}L}}(z) \simeq \left(\frac{\pi}{2}\right)^{1/2} \mathrm{e}^{z^2/8} (-\log \epsilon - 2\gamma + 2) - \int_{-\infty}^{\infty} \mathrm{d}\zeta \, \mathrm{e}^{-2\zeta^2 + \zeta z} G(\zeta). \tag{5.7}$$

Let us consider now the generating functions for the brackets associated with the Landau-type collision terms. The brackets for the ordinary Landau collision term L′ follow, according to the remark below (I.4.4), from those for the BGL case by putting $\epsilon^{(0)}$ equal to 1 and inserting a factor $q^4/(q^2+k_D^2)^2$. Carrying out these manipulations in (5.1) we are led to (5.2) with an $I_{L'}(z)$ that is found from (5.3) by deletion of both F_i in the denominator and insertion of a factor $\eta^4/(1+\eta^2)^2$:

$$I_{L'}(z) = \int_0^\Lambda d\eta \int_{-\infty}^{\infty} d\zeta \frac{\eta^3}{(1+\eta^2)^2} e^{-2\zeta^2+\zeta z} \simeq \left(\frac{\pi}{2}\right)^{1/2} e^{z^2/8}(\log \Lambda - \tfrac{1}{2}). \tag{5.8}$$

Likewise, the generating function for the modified Landau case $\tilde{L}$ is obtained from that of the $B\tilde{G}L$ case by the replacements $\epsilon^{(0)} \to 1$ and $\hat{\varphi}_{B\tilde{G}L}(q) \to \hat{\varphi}_{\tilde{L}}(q)$ (as given by (I.3.27)). Correspondingly, we put $F_1 = 0 = F_2$ and insert a factor $\exp(-\epsilon\xi)$ in (5.6):

$$I_{\tilde{L}}(z) = \epsilon \int_0^\infty d\xi \int_0^\infty d\eta \int_{-\infty}^{\infty} d\zeta\, \xi(1 - e^{-1/\xi})\, e^{-\epsilon\xi} j_1(\epsilon\eta\xi)\, e^{-2\zeta^2+\zeta z}. \tag{5.9}$$

The first terms in the asymptotic expansion of the ξ–η-integral for small ϵ have also been derived in appendix B of paper I. From (I.B.19) we get

$$I_{\tilde{L}}(z) \simeq \left(\frac{\pi}{2}\right)^{1/2} e^{z^2/8}\,(-\log \epsilon - 2\gamma + 1). \tag{5.10}$$

The linearized collision terms and the collision brackets associated with the convergent kinetic equations labelled P and $\tilde{P}$ both contain the difference of a BGL-like and a Landau-like contribution. For that reason we are interested in the generating functions $\mathcal{B}_{BGL-L'}$ and $\mathcal{B}_{B\tilde{G}L-\tilde{L}}$, which follow by substituting in (5.2) either the difference of (5.4) and (5.8), viz.

$$I_{BGL-L'}(z) \equiv I_{BGL}(z) - I_{L'}(z) = \tfrac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} e^{z^2/8} - \int_{-\infty}^{\infty} d\zeta\, e^{-2\zeta^2+\zeta z} G(\zeta), \tag{5.11}$$

or the difference of (5.7) and (5.10)

$$I_{B\tilde{G}L-\tilde{L}}(z) \equiv I_{B\tilde{G}L}(z) - I_{\tilde{L}}(z) \simeq \left(\frac{\pi}{2}\right)^{1/2} e^{z^2/8} - \int_{-\infty}^{\infty} d\zeta\, e^{-2\zeta^2+\zeta z} G(\zeta). \tag{5.12}$$

It should be noted that the cut-off parameter Λ has dropped out of (5.11).

6. Comparison of the generating functions associated with the convergent kinetic equations for plasmas with small ϵ

The generating functions discussed in the previous sections have the form of a collision bracket according to (2.14). Hence, we observe from (I.4.2) that the generating functions associated with the convergent kinetic equations labelled P and $\tilde{\mathrm{P}}$ in (I.3.1) follow directly from those generating the partial brackets. In particular, we have for the convergent equation with label P:

$$\mathscr{B}_{\mathrm{P}} = \mathscr{B}_{\mathrm{B}'} + \mathscr{B}_{\mathrm{BGL-L}'} . \tag{6.1}$$

Here $\mathscr{B}_{\mathrm{B}'}$ and $\mathscr{B}_{\mathrm{BGL-L}'}$ are given by (3.5) with (3.8), and (5.2) with (5.11), respectively. For small ϵ (3.8) may be replaced by its asymptotic version (3.15). The generating function for the convergent equation with subscript $\tilde{\mathrm{P}}$ reads likewise:

$$\mathscr{B}_{\tilde{\mathrm{P}}} = \mathscr{B}_{\tilde{\mathrm{B}}} + \mathscr{B}_{\mathrm{B\hat{G}L-\tilde{L}}} , \tag{6.2}$$

with $\mathscr{B}_{\tilde{\mathrm{B}}}$ following from (3.5) with (4.4) and $\mathscr{B}_{\mathrm{B\hat{G}L-\tilde{L}}}$ from (5.2) with (5.6) and (5.9). For small ϵ (4.4) has the asymptotic form (4.7), while (5.6) and (5.9) lead to (5.12).

In paper I it has been proved that the asymptotic expressions for the lowest-order collision brackets are the same for both convergent kinetic equations. In this section we shall prove that this is a property shared by all collision brackets of the general form of B^{l}_{pq} (2.3). In fact, it will be shown that the generating functions (6.1) and (6.2) for these brackets have identical asymptotic expressions for small ϵ. To establish this result we shall study the functions

$$\begin{aligned} \Delta\mathscr{B}_{\mathrm{B}} &= \mathscr{B}_{\mathrm{B}'} - \mathscr{B}_{\tilde{\mathrm{B}}} , \\ \Delta\mathscr{B}_{\mathrm{BGL-L}} &= \mathscr{B}_{\mathrm{BGL-L}'} - \mathscr{B}_{\mathrm{B\hat{G}L-\tilde{L}}} , \end{aligned} \tag{6.3}$$

which together determine the difference of (6.1) and (6.2):

$$\mathscr{B}_{\mathrm{P}} - \mathscr{B}_{\tilde{\mathrm{P}}} = \Delta\mathscr{B}_{\mathrm{B}} + \Delta\mathscr{B}_{\mathrm{BGL-L}} . \tag{6.4}$$

The asymptotic expression of $\Delta\mathscr{B}_{\mathrm{B}}$ is obtained from (3.5) with (3.15), (4.7) and (4.9) as

$$\Delta\mathscr{B}_{\mathrm{B}} \simeq \frac{e^4\beta^{3/2}}{16\pi^{3/2}m^{1/2}}\, \mathrm{e}^{(x+y)^2/8}\,\overline{\Delta\mathscr{B}}_{\mathrm{B}} , \tag{6.5}$$

where $\overline{\Delta\mathscr{B}}_{\mathrm{B}}$ is defined as

$$\overline{\Delta\mathscr{B}}_{\mathrm{B}} = \tfrac{1}{2}\pi \sum_{l}{}' (2l+1)l(l+1)P_l(\hat{x}\cdot\hat{y}) \sum_{m,n=0}^{\infty} c^{l}_{mn}(\tfrac{1}{8}x^2)^{l/2+m}(\tfrac{1}{8}y^2)^{l/2+n} . \tag{6.6}$$

Likewise, from (5.2) with (5.11) and (5.12) an asymptotic expression for $\Delta\mathscr{B}_{\mathrm{BGL-L}}$ is found that is analogous to (6.5). Instead of (6.6) one has in this case

$$\overline{\Delta\mathscr{B}}_{\mathrm{BGL-L}} = -\int \frac{\mathrm{d}\hat{q}}{4\pi}\, x_{\parallel}y_{\parallel}[\mathrm{e}^{(x_{\perp}+y_{\perp})^2/8} - \mathrm{e}^{(x_{\perp}-y_{\perp})^2/8}] . \tag{6.7}$$

The expressions (6.6) and (6.7) will be studied now consecutively.

The coefficient c^l_{mn} occurring in (6.6) has been defined in (3.16). It contains a product of Γ-functions, which have originated from the series expansion (2.6) for the modified spherical Bessel functions i_l. An expression that is closely related to the right-hand side of (6.6) follows by starting from the Rayleigh expansion (2.11). Using (2.17) and the orthogonality of the spherical harmonics one may prove the identity

$$\sum_l{}' (2l+1)P_l(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}})i_l(kx)i_l(ky) = \tfrac{1}{4}\int \frac{\mathrm{d}\hat{\boldsymbol{k}}}{4\pi}(\mathrm{e}^{\boldsymbol{k}\cdot\boldsymbol{x}}+\mathrm{e}^{-\boldsymbol{k}\cdot\boldsymbol{x}})(\mathrm{e}^{\boldsymbol{k}\cdot\boldsymbol{y}}+\mathrm{e}^{-\boldsymbol{k}\cdot\boldsymbol{y}}). \tag{6.8}$$

If the power series (2.6) for the i_l functions are inserted in the left-hand side we arrive at a triple sum of a similar type as in (6.6). However, one may observe the following differences. In the first place an extra factor $l(l+1)$ occurs in (6.6); furthermore the coefficient c^l_{mn} (3.16) contains the factorial $(l+m+n-1)!$, which is absent in (6.8). The factor $l(l+1)$ can be generated in (6.8) by making use of the Legendre differential equation in the form

$$\mathscr{L}(t)P_l(t) = l(l+1)P_l(t), \tag{6.9}$$

with the operator

$$\mathscr{L}(t) = (t^2-1)\,\mathrm{d}^2/\mathrm{d}t^2 + 2t\,\mathrm{d}/\mathrm{d}t. \tag{6.10}$$

If this operator acts on both sides of (6.8) one gets

$$\sum_l{}' (2l+1)l(l+1)P_l(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}})i_l(kx)i_l(ky)$$
$$= \mathscr{L}(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}})\tfrac{1}{2}\int \frac{\mathrm{d}\hat{\boldsymbol{k}}}{4\pi}[\mathrm{e}^{\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}+\mathrm{e}^{\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}]. \tag{6.11}$$

Since $\mathscr{L}(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}})$ is a differential operator one may add a constant to the integrand at the right-hand side. For reasons that will become clear presently we use this freedom to insert a term -2 between the square brackets.

The factorial $(l+m+n-1)!$, which is still missing in (6.11) as compared to (6.6), can be produced by making use of the identity

$$\Gamma(n) = \int_0^\infty \mathrm{d}t\, t^{n-1}\,\mathrm{e}^{-t} = 2^{n+1}\int_0^\infty \mathrm{d}k\, k^{2n-1}\,\mathrm{e}^{-2k^2}. \tag{6.12}$$

In fact, if we multiply both sides of (6.11) by $4k^{-1}\exp(-2k^2)$ and integrate the result over k, the left-hand side becomes identical to (6.6). At the right-hand side the integrals over $\hat{\boldsymbol{k}}$ and k combine to an integral over the vector $\boldsymbol{k}$, so that we find:

$$\overline{\Delta\mathscr{B}}_{\mathrm{B}} = \mathscr{L}(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}})\int \frac{\mathrm{d}\boldsymbol{k}}{2\pi k^3}\mathrm{e}^{-2k^2}[\mathrm{e}^{\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{y})}+\mathrm{e}^{\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}-2]. \tag{6.13}$$

Owing to the presence of the term -2 the integrand is finite in the origin. Substituting the power series expansions of the exponentials and using (6.12) we get

$$\overline{\Delta\mathscr{B}}_{\mathrm{B}} = \mathscr{L}(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}}) \sum_{n=1}^{\infty} 2^n \frac{(n-1)!}{(2n+1)!}(z_1^{2n}+z_2^{2n}), \tag{6.14}$$

where we introduced the vectors

$$\boldsymbol{z}_1 = \tfrac{1}{2}(\boldsymbol{x}+\boldsymbol{y}), \quad \boldsymbol{z}_2 = \tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{y}). \tag{6.15}$$

An expression for $\mathscr{L}(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}})z_i^{2n}$ may be found by writing z_i^2 as a function of x^2, y^2, $\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}}$ and performing the differentiations contained in $\mathscr{L}$. If z_1 and z_2 are reintroduced subsequently we get

$$\mathscr{L}(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}})z_i^{2n} = n[z_i^4 - nz_1^2z_2^2 + (n-1)(\boldsymbol{z}_1\cdot\boldsymbol{z}_2)^2]z_i^{2n-4}, \tag{6.16}$$

for $i=1,2$. The final form for $\overline{\Delta\mathscr{B}}_{\mathrm{B}}$ is obtained by substituting (6.16) in (6.14):

$$\overline{\Delta\mathscr{B}}_{\mathrm{B}} = \sum_{n=1}^{\infty} \frac{2^n n!}{(2n+1)!}[z_1^4 - nz_1^2z_2^2 + (n-1)(\boldsymbol{z}_1\cdot\boldsymbol{z}_2)^2]z_1^{2n-4} + (1\leftrightarrow 2), \tag{6.17}$$

where $(1\leftrightarrow 2)$ stands for the preceding terms with the indices 1 and 2 interchanged.

Let us consider now the BGL expression (6.7). Again the variables z_i (6.15) will be introduced; in particular, one has $x_\|y_\| = z_{1\|}^2 - z_{2\|}^2$. If the exponential functions are replaced by their power series, we obtain

$$\overline{\Delta\mathscr{B}}_{\mathrm{BGL,\,L}} = -\sum_{n=0}^{\infty} \frac{1}{2^n n!}(A_{11}^n z_1^2 - A_{12}^n z_2^2)z_1^{2n} + (1\leftrightarrow 2), \tag{6.18}$$

with the angular integrals

$$A_{ij}^n = \int \frac{\mathrm{d}\hat{\boldsymbol{q}}}{4\pi}\sin^{2n}\theta_i\cos^2\theta_j, \tag{6.19}$$

containing the angles θ_i made by $\boldsymbol{z}_i$ with $\hat{\boldsymbol{q}}$. For $i=j$ the integral is easily evaluated[6]):

$$A_{ii}^n = \frac{(2n)!!}{(2n+3)!!}. \tag{6.20}$$

For $i\neq j$ it is convenient to choose the polar axis of the spherical-coordinate system in the direction of $\boldsymbol{z}_i$. The polar angles of $\hat{\boldsymbol{q}}$ then are (θ_i, φ_i), while those of $\boldsymbol{z}_j$ will be denoted by (θ, φ). The integral (6.19) then becomes

$$\begin{aligned} A_{ij}^n &= \frac{1}{4\pi}\int \mathrm{d}\theta_i\,\mathrm{d}\varphi_i\,\sin^{2n+1}\theta_i[\cos\theta\cos\theta_i + \sin\theta\sin\theta_i\cos(\varphi-\varphi_i)]^2 \\ &= \frac{(2n)!!}{(2n+3)!!}(n+1-n\cos^2\theta). \end{aligned} \tag{6.21}$$

Upon substitution of (6.20) and (6.21) in (6.18) we arrive at an expression for $\overline{\overline{\Delta\mathscr{B}}}_{\mathrm{BGL-L}}$ which is the inverse of (6.17), so that one has

$$\overline{\overline{\Delta\mathscr{B}}}_{\mathrm{B}}+\overline{\overline{\Delta\mathscr{B}}}_{\mathrm{BGL-L}}=0. \tag{6.22}$$

In view of (6.4) and (6.5) this is equivalent to the statement

$$\mathscr{B}_{\mathrm{P}}\simeq\mathscr{B}_{\tilde{\mathrm{P}}} \tag{6.23}$$

for small values of ϵ. Hence we have established the equivalence of the convergent kinetic equations labelled P and $\tilde{\mathrm{P}}$ in so far as the asymptotic expressions for the ensuing collision brackets are concerned.

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Appendix A

Asymptotic expansion for the Ω-integrals

The asymptotic expression for $\Omega^{(l,r)}$ that has been derived by Kihara[5]) differs from (3.9) by the replacement of $T_{l/2}$ by $B_l+\frac{1}{2}$, where B_l is to be determined from

$$J_l^{\Lambda}=\int_0^{\Lambda}\left[1-\left(\frac{x-1}{x+1}\right)^l\right]\mathrm{d}x=2l(\log\Lambda-2B_l)+\mathcal{O}(\Lambda^{-1}), \tag{A.1}$$

for $\Lambda\gg1$. Following Kihara an explicit expression for B_l can be found by applying the binomial expansion of $(x-1)^l$ about $x=-1$ and integrating term by term. A more elegant form will be obtained by making use of a recursion argument.

After a partial integration in (A.1) and substitution of the variable $t=(x-1)/(x+1)$ we have:

$$J_l^{\Lambda}=\Lambda\left[1-\left(\frac{\Lambda-1}{\Lambda+1}\right)^l\right]+l\int_{-1}^{(\Lambda-1)/(\Lambda+1)}\mathrm{d}t\,t^{l-1}\frac{1+t}{1-t}. \tag{A.2}$$

For $l = 1, 2$ we obtain by separation into partial fractions:

$$\begin{aligned} J_1^\Lambda &\simeq 2 \log \Lambda, \\ J_2^\Lambda &\simeq 4 \log \Lambda - 4. \end{aligned} \tag{A.3}$$

To determine J_l^Λ for $l \geqq 3$ we now derive a recursion relation. From (A.2) it is easily seen that in the limit $\Lambda \to \infty$ the following equality holds:

$$\frac{J_{l+1}^\infty}{l+1} - \frac{J_l^\infty}{l} = -\int_{-1}^{1} \mathrm{d}t\, t^{l-1}(1+t) = \frac{-(2l+1)+(-1)^l}{l(l+1)}. \tag{A.4}$$

By repetition we arrive at

$$\frac{J_{l+2}^\infty}{l+2} - \frac{J_l^\infty}{l} = \begin{cases} -\dfrac{4}{l+1}, & \text{even } l \geqq 2, \\[2ex] -\dfrac{4(l+1)}{l(l+2)}, & \text{odd } l \geqq 1. \end{cases} \tag{A.5}$$

These recursion relations can be iterated to obtain (A.1) with

$$B_l = \begin{cases} T_{l/2} - \frac{1}{2}, & \text{even } l \geqq 2, \\[2ex] T_{(l-1)/2} - \dfrac{l-1}{2l}, & \text{odd } l \geqq 1, \end{cases} \tag{A.6}$$

where T_l has been defined in (3.10).

Appendix B

Asymptotic behaviour of the functions F_l

The functions F_l, which we have defined in (4.5), can be slightly generalized. We will consider

$$F(\eta) = \tfrac{1}{8}(1-\eta)^{1/2}(2-\eta) \int_{-1}^{1} \mathrm{d}\zeta\, \frac{\zeta - \zeta_0^{-1}}{(\zeta-\zeta_0)^3} P(\zeta), \tag{B.1}$$

where $P(\zeta)$ is a polynomial function and $\zeta_0 = (1 - \frac{1}{2}\eta)(1-\eta)^{-1/2}$. In order to obtain an asymptotic expansion for $\eta \to 0$ (and hence $\zeta_0 \to 1$) we first note

$$\int_{-1}^{1} \mathrm{d}\zeta\, \frac{\zeta - \zeta_0^{-1}}{(\zeta-\zeta_0)^3} = 0. \tag{B.2}$$

Consequently, $P(\zeta)$ in (B.1) can be replaced by $P(\zeta)-P(\zeta_0)$. From the Taylor expansion of $P(\zeta)$ about ζ_0 we have

$$\frac{P(\zeta)-P(\zeta_0)}{(\zeta-\zeta_0)^3}=\frac{P'(\zeta_0)}{(\zeta-\zeta_0)^2}+\frac{1}{2}\frac{P''(\zeta_0)}{\zeta-\zeta_0}+R(\zeta,\zeta_0), \tag{B.3}$$

where R is the polynomial function

$$R(\zeta,\zeta_0)=(\zeta-\zeta_0)^{-3}[P(\zeta)-P(\zeta_0)-(\zeta-\zeta_0)P'(\zeta_0)-\tfrac{1}{2}(\zeta-\zeta_0)^2P''(\zeta_0)]. \tag{B.4}$$

Substitution of (B.3) in (B.1) and separation into partial fractions yields for $\eta\to 0$

$$F(\eta)=\tfrac{1}{2}P'(1)\left[\log\left(\frac{\eta}{4}\right)+1\right]+\tfrac{1}{4}P''(1)+\tfrac{1}{4}\int_{-1}^{1}\mathrm{d}\zeta(\zeta-1)R(\zeta,1)+\mathcal{O}(\eta). \tag{B.5}$$

Let us consider the special case that P is a simple even power of ζ, i.e. $P(\zeta)=\zeta^{2n}$. Then the integrand in (B.5) reads:

$$(\zeta-1)R(\zeta,1)=\sum_{k=1}^{2n-1}k\zeta^{2n-k-1}-n(2n-1). \tag{B.6}$$

Only the terms in (B.6) with odd $k=2l-1$ contribute to the integral; in fact, we have

$$\tfrac{1}{4}\int_{-1}^{1}\mathrm{d}\zeta(\zeta-1)R(\zeta,1)=\tfrac{1}{2}\sum_{l=1}^{n}\frac{2l-1}{2(n-l)+1}-\tfrac{1}{2}n(2n-1)=nT_n-n^2. \tag{B.7}$$

Hence in this special case (B.5) becomes

$$F(\eta)\simeq n(\log\eta-2\log 2+T_n+\tfrac{1}{2}). \tag{B.8}$$

From (B.8) the corresponding asymptotic expression for $F_l(\eta)$ follows, since F_l is obtained by inserting a Legendre polynomial of even order l into (B.1). Using (3.7) we arrive at

$$F_l(\eta)\simeq(-1)^{l/2}\binom{l}{\frac{1}{2}l}2^{-l}\sum_{p=0}^{l/2}\frac{(-\frac{1}{2}l)_p(\frac{1}{2}l+\frac{1}{2})_p}{p\,!(\frac{1}{2})_p}\,p(\log\eta-2\log 2+T_p+\tfrac{1}{2}). \tag{B.9}$$

We can perform the p-summation with the help of (3.10)–(3.14). Then the final result reads

$$F_l(\eta)\simeq\tfrac{1}{4}l(l+1)(\log\eta-2\log 2+\tfrac{1}{2}S_{l/2}+T_{l/2}). \tag{B.10}$$

Appendix C

Derivation of the asymptotic expressions for the integrals occurring in the modified Boltzmann generating function

In section 4 we encountered the integrals

$$J^l(\epsilon) = \int_0^\infty \mathrm{d}\xi \int_0^1 \mathrm{d}\eta \frac{1}{\xi^2\eta^2(1-\eta)} \exp\left(-\frac{\epsilon}{\xi\eta} - \xi\right) \times i_l[(\tfrac{1}{2}\xi)^{1/2}(1-\eta)^{1/2}x]\, i_l[(\tfrac{1}{2}\xi)^{1/2}y]\, F_l(\eta), \tag{C.1}$$

for even $l \geqq 2$, with $F_l(\eta)$ given in (4.5). Substitution of the series (2.6) for i_l leads to

$$J^l(\epsilon) = \frac{\pi}{4} \sum_{m,n=0}^\infty \frac{c^l_{mn}}{(l+m+n-1)!} J^l_{mn}(\epsilon)(\tfrac{1}{8}x^2)^{l/2+m}(\tfrac{1}{8}y^2)^{l/2+n}. \tag{C.2}$$

The coefficients c^l_{mn} have been defined in (3.16), while J^l_{mn} stands for the integral

$$J^l_{mn}(\epsilon) = \int_0^\infty \mathrm{d}\xi \int_0^1 \mathrm{d}\eta\, \xi^{l+m+n-2}(1-\eta)^{l/2+m-1}\eta^{-2} \exp\left(-\frac{\epsilon}{\xi\eta} - \xi\right) F_l(\eta). \tag{C.3}$$

To obtain the dominant contributions to J^l_{mn} for small ϵ we may proceed as in appendix A of paper I. First we remark that it is sufficient to consider only the leading terms in the asymptotic expansion of $F_l(\eta)$ around $\eta = 0$; they have been given in (4.6). If the factors $(1-\eta)$ in (C.3) are accordingly replaced by 1 one gets upon introducing the variable $\eta' = \epsilon/(\eta\xi)$ instead of η:

$$J^l_{mn}(\epsilon) \simeq \tfrac{1}{4} l(l+1)\epsilon^{-1} \int_0^\infty \mathrm{d}\xi \int_{\epsilon/\xi}^\infty \mathrm{d}\eta'\, \xi^{l+m+n-1} \mathrm{e}^{-\eta'-\xi} \times (-\log\xi - \log\eta' + \log\epsilon - 2\log 2 + \tfrac{1}{2}S_{l/2} + T_{l/2}). \tag{C.4}$$

The integral over η' is easily performed for all terms between the parentheses except $\log\eta'$. The contribution of the latter may be rewritten by carrying out two partial integrations, first with respect to η' and then, by means of the identity

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sum_{k=0}^{l} \frac{l!}{k!}\xi^k \mathrm{e}^{-\xi}\right) = -\xi^l \mathrm{e}^{-\xi}. \tag{C.5}$$

with respect to ξ. As a result we obtain from (C.4)

$$J^l_{mn}(\epsilon) \simeq \tfrac{1}{4} l(l+1)\epsilon^{-1} \int_0^\infty \mathrm{d}\xi \exp\left(-\frac{\epsilon}{\xi} - \xi\right) \times \left[\xi^{l+m+n-1}(-2\log 2 + \tfrac{1}{2}S_{l/2} + T_{l/2}) - \sum_{k=0}^{l+m+n-1} \frac{(l+m+n-1)!}{k!}\xi^{k-1}\right]. \tag{C.6}$$

We now split the integration domain into the intervals $I_1 = [0, \epsilon^{1/2}]$ and $I_2 = [\epsilon^{1/2}, \infty)$. In the first interval $\exp(-\xi)$ can be replaced by unity. With the variable $\xi' = \epsilon/\xi$ the dominant terms in the contribution from I_1 become

$$\begin{aligned} J^{l}_{mn,1}(\epsilon) &\simeq -\tfrac{1}{4}\, l(l+1)(l+m+n-1)!\,\epsilon^{-1} \int\limits_{\epsilon^{1/2}}^{\infty} \frac{d\xi'}{\xi'}\, e^{-\xi'} \\ &\simeq \tfrac{1}{4}\, l(l+1)(l+m+n-1)!\,\epsilon^{-1}(\tfrac{1}{2}\log\epsilon + \gamma). \end{aligned} \tag{C.7}$$

Here we used the identity (I.A.8). In the second interval I_2 one may omit the factor $\exp(-\epsilon/\xi)$. Integration then yields

$$\begin{aligned} J^{l}_{mn,2}(\epsilon) \simeq \tfrac{1}{4}\, l(l+1)(l+m+n-1)!(\tfrac{1}{2}\log\epsilon + \gamma - 2\log 2 \\ - S_{l+m+n-1} + \tfrac{1}{2} S_{l/2} + T_{l/2}). \end{aligned} \tag{C.8}$$

Adding (C.7) and (C.8) we arrive at the asymptotic expression for $J^{l}_{mn}(\epsilon)$ in the neighbourhood of $\epsilon = 0$:

$$\begin{aligned} J^{l}_{mn}(\epsilon) \simeq \tfrac{1}{4}\, l(l+1)(l+m+n-1)!\,\epsilon^{-1}(\log\epsilon + 2\gamma - 2\log 2 \\ - S_{l+m+n-1} + \tfrac{1}{2} S_{l/2} + T_{l/2}). \end{aligned} \tag{C.9}$$

The omitted terms are at most of order $\epsilon^{1/2}$ as compared to the leading contributions.

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