

ON RELATIONS AMONG 1-CYCLES ON CUBIC HYPERSURFACES

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Abstract

In this paper we give two explicit relations among 1-cycles modulo rational equivalence on a smooth cubic hypersurface X . Such a relation is given in terms of a (pair of) curve(s) and its secant lines. As the first application, we reprove Paranjape's theorem that $\mathrm{CH}_1(X)$ is always generated by lines and that it is isomorphic to \mathbb{Z} if the dimension of X is at least 5. Another application is to the intermediate jacobian of a cubic threefold X . To be more precise, we show that the intermediate jacobian of X is naturally isomorphic to the Prym–Tjurin variety constructed from the curve parameterizing all lines meeting a given rational curve on X . The incidence correspondences play an important role in this study. We also give a description of the Abel–Jacobi map for 1-cycles in this setting.

1. Introduction

Let $X \subset \mathbb{P}_k^n$, where $n \geq 3$, be a smooth cubic hypersurface in projective space over an algebraically closed field k . In this paper, we are going to study relations among 1-cycles on X . We explain the main theorem (Theorem 4.2) of this paper in the following special situations. Let $C \subset X$ be a general smooth curve on X . Then there are finitely many lines, $E_i \subset X$, meeting C in two points. These lines will be called the secant lines of C . The first relation we get is

$$(2e - 3)C + \sum E_i = \text{const.}$$

in $\mathrm{CH}_1(X)$, where $e = \deg(C)$. A second relation is about a pair of curves on X . A simple version goes as follows: Let C_1 and C_2 be a pair of general smooth curves on X . Then there are finitely many lines, $E_i \subset X$, meeting both C_1 and C_2 . These are called secant lines of the pair (C_1, C_2) . Then our

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second relation reads

$$2e_2C_1 + 2e_1C_2 + \sum E_i = \text{const.}$$

in $\text{CH}_1(X)$, where $e_1 = \deg(C_1)$ and $e_2 = \deg(C_2)$. In all cases, the right-hand side is a multiple of the class of the restriction of a linear \mathbb{P}^2 to X . Note that in these relations the dimension of X does not appear in the coefficients and they hold in all characteristics. As an application, we reprove the following theorem of K. H. Paranjape, see [Par, Proposition 4.2].

Theorem 1.1. *Let X/k be a smooth cubic hypersurface as above. Then the Chow group $\text{CH}_1(X)$ of 1-cycles on X is generated by lines. When $\dim X \geq 5$, we have $\text{CH}_1(X) \cong \mathbb{Z}$.*

If $X \subset \mathbb{P}_k^4$ is a smooth cubic threefold with $\text{char}(k) \neq 2$, then $\text{CH}_1(X)$ is well understood. Let $A_1(X) \subset \text{CH}_1(X)$ be the subgroup of 1-cycles algebraically equivalent to zero modulo rational equivalence. Then $A_1(X)$ is isomorphic to the intermediate jacobian $J(X)$ of X . A precise definition of the intermediate jacobian will be given in section 5, Definition 5.4. Roughly speaking, $J(X)$ is the universal abelian variety with a principal polarization into which $A_1(X)$ maps. One important feature is that for any algebraic family $Z \rightarrow T$ of 1-cycles on X , there is an induced Abel–Jacobi map $\Psi_T : T \rightarrow J(X)$, which is a morphism between varieties. Associated to Ψ_T , we also have $\psi_T : \text{Alb}(T) \rightarrow J(X)$, which is a homomorphism of abelian varieties and is also called the Abel–Jacobi map. There are two ways to realize $J(X)$ from the geometry of X . If we use $F = F(X)$ to denote the Fano surface of lines on X , then the Albanese variety $\text{Alb}(F)$ of F carries a natural principal polarization, and the Abel–Jacobi map induces an isomorphism between $\text{Alb}(F)$ and $J(X)$ as principally polarized abelian varieties. A second realization goes as follows. Let $l \subset X$ be a general line on X . Then all lines meeting l are parameterized by a smooth curve $\tilde{\Delta}$ of genus 11. The curve $\tilde{\Delta}$ carries a natural fixed-point-free involution whose quotient is a smooth plane quintic Δ . Then the associated Prym variety $\text{Pr}(\tilde{\Delta}/\Delta)$ is naturally isomorphic to $J(X)$.

As another application of the above natural relations, we give a third realization of the intermediate jacobian $J(X)$ as a Prym–Tjurin variety. We refer to [Tju], [BM], and [Kan] for the basic facts about Prym–Tjurin varieties. Let $C \subset X$ be a general smooth rational curve. We use \tilde{C} to denote the normalization of the curve parameterizing all lines meeting C . Then let $D_U = \{([l_1], [l_2]) \in \tilde{C} \times \tilde{C}\}$ such that l_1 and l_2 are not secant lines of C and that l_1 and l_2 meet transversally in a point not on C . We take D to be the closure of D_U in $\tilde{C} \times \tilde{C}$, which can be called the incidence correspondence on \tilde{C} . Then D defines an endomorphism i of $\tilde{J} = J(\tilde{C})$. Our next result is the following.

Theorem 1.2. *Let X/k be a smooth cubic threefold, and assume that $\text{char}(k) \neq 2$. Let C, \tilde{C}, \tilde{J} , and i be as above. Then the following are true.*

(a) *The endomorphism i satisfies the following quadratic relation*

$$(i - 1)(i + q - 1) = 0,$$

where $q = 2 \deg(C)$.

(b) *Assume that $\text{char}(k) \nmid q$. Then the Prym–Tjurin variety $P = \text{Pr}(\tilde{C}, i) = \text{Im}(i - 1) \subset \tilde{J}$ carries a natural principal polarization whose theta divisor Ξ satisfies*

$$\Theta_{\tilde{J}}|_P \equiv q\Xi.$$

(c) *Assume that $\text{char}(k) \nmid q$. Then the Abel–Jacobi map $\psi = \psi_{\tilde{C}} : \tilde{J} \rightarrow J(X)$ factors through $i - 1$ and gives an isomorphism*

$$u_C : (\text{Pr}(\tilde{C}, i), \Xi) \rightarrow (J(X), \Theta_{J(X)}).$$

Remark 1.3. This is a simplified version of Theorem 6.4. When C is a general line on X , the above construction recovers the Prym realization of $J(X)$.

We next study how the above construction varies when we change the curve C . Let C_1 and C_2 be two general smooth rational curves on X . Let $D_{21} = \{([l_1], [l_2]) \in \tilde{C}_1 \times \tilde{C}_2\}$ such that l_1 meets l_2 . We view D_{21} as a homomorphism from \tilde{J}_1 to \tilde{J}_2 .

Proposition 1.4. *Assume that $\text{char}(k) \nmid q_1q_2$. Then the following are true:*

(a) *The image of D_{21} is $\text{Pr}(\tilde{C}_2, i_2)$. The homomorphism D_{21} factors through $i_1 - 1$ and gives an isomorphism*

$$t_{21} : (\text{Pr}(\tilde{C}_1, i_1), \Xi_1) \rightarrow (\text{Pr}(\tilde{C}_2, i_2), \Xi_2).$$

(b) *The isomorphism t_{21} is compatible with u_{C_i} , namely $u_{C_2} \circ t_{21} = u_{C_1}$.*

We also get a description for the Abel–Jacobi map for a family of 1-cycles. Fix a general smooth rational curve $C \subset X$ such that $\text{char}(k) \nmid q$. Let $\mathcal{C} \subset X \times T$ be a family of curves on X parameterized by T . Assume that for general $t \in T$, there are finitely many secant lines L_i of the pair (C, \mathcal{C}_t) . Then the rule $t \mapsto \sum [L_i]$ defines a morphism

$$\Psi_{T, \tilde{J}} : T \rightarrow \tilde{J}.$$

Proposition 1.5. *Let notations and assumptions be as above. Then the image of $\Psi_{T, \tilde{J}}$ is contained in $\text{Pr}(\tilde{C}, i) \subset \tilde{J}$. The composition $u_C \circ \Psi_{T, \tilde{J}}$ is identified with the Abel–Jacobi map associated to \mathcal{C}/T .*

We summarize the structure of the paper. In section 2, we review the theory of secant bundles on symmetric products. We state the results in the form that we are going to use and all proofs are included for completeness. Section 3 is

devoted to the construction and basic properties of residue surfaces associated to a curve or a pair of curves. Those surfaces play an important role in later proofs in section 4. In section 4, we state and prove our main theorem on relations among 1-cycles. We also derive Paranjape's theorem as a corollary. Section 5 is a review on basic results on the Fano scheme of lines and the intermediate jacobian of a cubic threefold. In section 6, we show how one can realize the intermediate jacobian of a cubic threefold as a Prym–Tjurin variety described above.

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Notation and conventions:

1. $G(r, V)$ is the grassmannian rank r quotient of V ; \mathcal{E}_r is the canonical rank r quotient bundle of V on $G(r, V)$;
2. $\mathbb{P}(V) = \text{Proj}(\text{Sym}^* V) = G(1, V)$;
3. $G(r_1, r_2, V)$ is the flag variety of successive quotients of V with ranks $r_2 > r_1$;
4. $N(e, g) = \frac{5e(e-3)}{2} + 6 - 6g$;
5. We use \equiv to denote numerical equivalence;
6. A general curve on X means that it comes from a dense open subset of the corresponding component of the Hilbert scheme of curves on X ;
7. For two algebraic cycles D_1 and D_2 , an equation $D_1 = D_2$ can mean equality either as cycles or modulo rational equivalence;
8. For $P, Q \in \mathbb{P}(V)$ distinct, we use \overline{PQ} to denote the line through both P and Q .

2. Secant bundles on symmetric products

Most of the results in this section are special cases of that of [Matt]. We state the results in the form we need and include the proofs for completeness. Let C be a smooth projective curve over an algebraically closed field k . We use $C^{(2)}$ to denote the symmetric product of C . Let $\pi : C \times C \rightarrow C^{(2)}$ be the canonical double cover which ramifies along the diagonal $\Delta_C \subset C \times C$. For any invertible sheaf \mathcal{L} on C , we define the *symmetrization* $\mathcal{E}(\mathcal{L})$ of \mathcal{L} to be $\pi_* p_1^* \mathcal{L}$ on $C^{(2)}$, where $p_1 : C \times C \rightarrow C$ is the projection to the first factor. For example, if $\mathcal{L} = \omega_C$, then $\mathcal{E}(\mathcal{L}) \cong \Omega_{C^{(2)}}^1$.

To compute the Chern classes of $\mathcal{E}(\mathcal{L})$, we describe several special cycles on $C^{(2)}$. Let $x \in C$ be a closed point. We define

$$(1) \quad D_x = \pi_*(\{x\} \times C) \subset C^{(2)}.$$

If $\mathbf{a} = \sum x_i$ is an effective divisor on C with $x_i \neq x_j$ for different i and j , we write

$$(2) \quad D_{\mathbf{a}} = \sum D_{x_i}, \quad \mathbf{a}^{[2]} = \sum_{i < j} \pi_*(x_i, x_j), \quad \delta_* \mathbf{a} = \sum \pi_*(x_i, x_i).$$

Here $\delta = \pi \circ \Delta_C : C \rightarrow C^{(2)}$. Note that the definition of $D_{\mathbf{a}}$ can be extended linearly to all divisors. The projection formula implies

$$\begin{aligned} D_x \cdot D_y &= \pi_*(x \times C \cdot \pi^* \pi_*(y \times C)) \\ &= \pi_*(x \times C \cdot (y \times C + C \times y)) \\ &= \pi_*(x, y) \end{aligned}$$

for any $x, y \in C$. Then we easily get

$$(3) \quad (D_{\mathbf{a}})^2 = \delta_* \mathbf{a} + 2 \mathbf{a}^{[2]}.$$

Let $\Delta_0 = c_1(\pi_* \mathcal{O}_{C \times C})$ in $\text{CH}_1(C^{(2)})$.

Assume that \mathcal{L} is very ample, and for some $V \subset H^0(C, \mathcal{L})$, it defines a closed immersion $i : C \rightarrow \mathbb{P}(V)$. Note that any two points x and y on C define a line $\overline{i(x) i(y)}$ (the tangent line at $i(x)$ if $x = y$). This defines a natural morphism $\varphi : C^{(2)} \rightarrow G(2, V)$. Let $V \otimes \mathcal{O}_{G(2, V)} \rightarrow \mathcal{E}_2$ be the universal rank 2 quotient bundle.

Proposition 2.1. *We have an isomorphism $\mathcal{E}(\mathcal{L}) \cong \varphi^* \mathcal{E}_2$.*

Proof. If we view $G(2, V)$ as the space of lines in $\mathbb{P}(V)$, then by pulling back the universal family of lines, we get

$$\begin{array}{ccc} \mathbb{P}(\varphi^* \mathcal{E}_2) & \xrightarrow{f} & \mathbb{P}(V) \\ \tilde{\pi} \downarrow & & \\ C^{(2)} & & \end{array}$$

and we have $\varphi^* \mathcal{E}_2 \cong \tilde{\pi}_* f^* \mathcal{O}_{\mathbb{P}(V)}(1)$. The fiber $\tilde{\pi}^{-1}(t)$ is the line $\varphi(t) \in G(2, V)$, and f is the inclusion of the line into $\mathbb{P}(V)$. Note that there are two distinguished points x and y on the fiber $\tilde{\pi}^{-1}(t) = \overline{i(x) i(y)}$, where $t \in C^{(2)}$ is the point $\pi(x, y)$. Those distinguished points form a divisor D on $\mathbb{P}(\varphi^*(\mathcal{E}_2))$. By choosing an isomorphism $D \cong C \times C$, we can assume that $f|_D = i \circ p_1$ and $\tilde{\pi}|_D = \pi$. Then $f^* \mathcal{O}_{\mathbb{P}(V)}(1)|_D = p_1^* i^* \mathcal{O}_{\mathbb{P}(V)}(1) = p_1^* \mathcal{L}$. Consider the following

short exact sequence

$$0 \longrightarrow f^* \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \mathcal{O}_{\mathbb{P}(\varphi^* \mathcal{E}_2)}(-D) \longrightarrow f^* \mathcal{O}_{\mathbb{P}(V)}(1) \longrightarrow f^* \mathcal{O}_{\mathbb{P}(V)}(1)|_D \longrightarrow 0.$$

Apply $R\pi_*$ and note that as the left term restricts to $\mathcal{O}_{\mathbb{P}^1}(-1)$ on each fiber of $\tilde{\pi}$, we get $\varphi^* \mathcal{E}_2 = \tilde{\pi}_* f^* \mathcal{O}_{\mathbb{P}(V)}(1) \cong \pi_* \mathcal{D}_1^* \mathcal{L} = \mathcal{E}(\mathcal{L})$. \square

Corollary 2.2. *We have the equality $c_2(\mathcal{E}(\mathcal{L})) = \mathbf{a}^{[2]}$ in $\text{CH}_0(C^{(2)})$, where \mathbf{a} is a general element of the complete linear system $|\mathcal{L}|$.*

Proof. It is known from Schubert calculus (see [Ful, 14.7]) that $c_2(\mathcal{E}_2)$ is represented by the cycle defined by the space of all lines that are contained in a hyperplane $H \subset \mathbb{P}(V)$. When H is chosen to be general, then $\mathbf{a} = i^* H$ is an element of $|\mathcal{L}|$ such that $\mathbf{a}^{[2]}$ is defined. Then we have $c_2(\mathcal{E}(\mathcal{L})) = c_2(\varphi^* \mathcal{E}_2) = \varphi^* c_2(\mathcal{E}_2) = \mathbf{a}^{[2]}$. \square

For very ample \mathcal{L} , we take a general section $s \in H^0(C, \mathcal{L})$ and write $\mathbf{a} = \text{div}(s) = \sum_{i=1}^d x_i$, where $d = \text{deg } \mathcal{L}$. The section s defines a short exact sequence

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{s} \mathcal{L} \longrightarrow \bigoplus_{i=1}^d \kappa(x_i) \longrightarrow 0.$$

By pulling the sequence back to $C \times C$ via p_1^* and then applying $R\pi_*$, we get

$$0 \longrightarrow \pi_*(\mathcal{O}_{C \times C}) \longrightarrow \mathcal{E}(\mathcal{L}) \longrightarrow \bigoplus_{i=1}^d \mathcal{O}_{D_{x_i}} \longrightarrow 0.$$

Since $\pi_* \mathcal{O}_{C \times C}$ admits a nowhere-vanishing section “1”, the second Chern class of $\pi_* \mathcal{O}_{C \times C}$ is zero. Hence we have $c(\pi_* \mathcal{O}_{C \times C}) = 1 + \Delta_0$. By taking total Chern classes in the above exact sequence and noting that $c(\mathcal{O}_{D_i}) = \frac{1}{1-D_i}$, we get

$$\begin{aligned} c(\mathcal{E}(\mathcal{L})) &= (1 + \Delta_0) \prod_{i=1}^d \left(\frac{1}{1 - D_{x_i}} \right) \\ &= (1 + \Delta_0) \prod_{i=1}^d (1 + D_{x_i} + D_{x_i}^2) \\ &= (1 + \Delta_0) \left(1 + \sum_{i=1}^d D_{x_i} + \sum_{i=1}^d D_{x_i}^2 + \sum_{1 \leq i < j \leq d} D_{x_i} \cdot D_{x_j} \right) \\ &= 1 + (D_{\mathbf{a}} + \Delta_0) + (\Delta_0 \cdot D_{\mathbf{a}} + \delta_* \mathbf{a} + \mathbf{a}^{[2]}). \end{aligned}$$

Hence we get

$$(4) \quad c_1(\mathcal{E}(\mathcal{L})) = D_{\mathbf{a}} + \Delta_0, \quad c_2(\mathcal{E}(\mathcal{L})) = \mathbf{a}^{[2]} + \Delta_0 \cdot D_{\mathbf{a}} + \delta_* \mathbf{a}.$$

Comparing this with Corollary 2.2, we have

$$(5) \quad \Delta_0 \cdot D_{\mathbf{a}} = -\delta_* \mathbf{a}.$$

Now let \mathcal{L}_1 and \mathcal{L}_2 be two very ample line bundles on C with degrees d_1 and d_2 , respectively. Pick a general section $s_i \in H^0(C, \mathcal{L}_i)$ where $i = 1, 2$. Set

$$\mathbf{a}_1 = \operatorname{div}(s_1) = \sum_{i=1}^{d_1} x_i, \quad \mathbf{a}_2 = \operatorname{div}(s_2) = \sum_{i=1}^{d_2} y_i.$$

As before, we can use s_2 to construct the short exact sequence,

$$0 \rightarrow \mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \xrightarrow{s_2} \mathcal{L}_1 \longrightarrow \bigoplus_{i=1}^{d_2} \kappa(y_i) \rightarrow 0.$$

Similarly, after applying $R\pi_*$, we get a short exact sequence on $C^{(2)}$,

$$0 \rightarrow \mathcal{E}(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}) \longrightarrow \mathcal{E}(\mathcal{L}_1) \longrightarrow \bigoplus_{i=1}^{d_2} \mathcal{O}_{D_{y_i}} \rightarrow 0.$$

One easily computes the Chern classes as

$$\begin{aligned} c(\mathcal{E}(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1})) &= (1 + D_{\mathbf{a}_1} + \Delta_0 + \mathbf{a}_1^{[2]}) \prod_{i=1}^{d_2} (1 - D_{y_i}) \\ &= (1 + D_{\mathbf{a}_1} + \Delta_0 + \mathbf{a}_1^{[2]})(1 - D_{\mathbf{a}_2} + \mathbf{a}_2^{[2]}) \\ &= 1 + (\Delta_0 + D_{\mathbf{a}_1} - D_{\mathbf{a}_2}) \\ &\quad + (\mathbf{a}_1^{[2]} + \mathbf{a}_2^{[2]} - \Delta_0 \cdot D_{\mathbf{a}_2} - D_{\mathbf{a}_1} \cdot D_{\mathbf{a}_2}). \end{aligned}$$

We summarize the above discussion into the following:

Theorem 2.3. *Let C be a smooth projective curve, and let \mathcal{L} be an invertible sheaf. Let $\mathbf{a}_1, \mathbf{a}_2$ and \mathbf{a} be effective divisors on C such that each is a sum of distinct points with multiplicity 1. Let $\mathcal{E} = \mathcal{E}(\mathcal{L})$ be the symmetrization of \mathcal{L} . Then the following are true.*

(a) *If $\mathcal{L} \cong \mathcal{O}_C(\mathbf{a}_1 - \mathbf{a}_2)$, then the following equalities hold in the Chow ring of $C^{(2)}$:*

$$(6) \quad c_1(\mathcal{E}) = \Delta_0 + D_{\mathbf{a}_1} - D_{\mathbf{a}_2},$$

$$(7) \quad c_2(\mathcal{E}) = \mathbf{a}_1^{[2]} + \mathbf{a}_2^{[2]} + \delta_* \mathbf{a}_2 - D_{\mathbf{a}_1} \cdot D_{\mathbf{a}_2}.$$

(b) *The following identities hold in the Chow ring of $C^{(2)}$:*

$$(8) \quad D_{\mathbf{a}} \cdot D_{\mathbf{a}} = \delta_* \mathbf{a} + 2 \mathbf{a}^{[2]},$$

$$(9) \quad D_{\mathbf{a}} \cdot \Delta_0 = -\delta_* \mathbf{a},$$

$$(10) \quad 2 \Delta_0 = -\pi_*(\Delta_C),$$

$$(11) \quad 2\Delta_0 \cdot \Delta_0 = -\delta_* K_C.$$

Proof. Only the last two equalities need to be proved. By construction we have the exact sequence

$$0 \rightarrow \mathcal{O}_{C \times C} \longrightarrow \pi^* \pi_* \mathcal{O}_{C \times C} \longrightarrow \mathcal{O}_{C \times C}(-\Delta_C) \rightarrow 0.$$

This implies that $\pi^*\Delta_0 = c_1(\pi^*\pi_*\mathcal{O}_{C\times C}) = -\Delta_C$. Hence $-\pi_*\Delta_C = \pi_*\pi^*\Delta_0 = 2\Delta_0$ and by the projection formula we have $2\Delta_0 \cdot \Delta_0 = -\pi_*\Delta_C \cdot \Delta_0 = \pi_*(\Delta_C \cdot \Delta_C) = -\delta_*K_C$. \square

3. Residue surfaces

Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a smooth cubic hypersurface over an algebraically closed field k . In this section we will construct so-called residue surfaces associated to a curve or a pair of curves on X . Let $V = H^0(X, \mathcal{O}_X(1))$.

Let $C \subset X$ be a complete at-worst nodal curve on X . Note that C might be disconnected. If $x \in C$ is a nodal point, let Π_x be the plane spanned by the tangent directions of the two branches at x .

Definition 3.1. A line l on X is called a *secant line* of C if: (first type) it meets C in two smooth points (which might be infinitesimally close to each other); or (second type) it passes through a node x and lies in the plane Π_x , and also it is not a component of C ; or (third type) it is a component of C for some node $x \in l$, and we have $\Pi_x \cdot X = 2l + l'$ for some line l' .

3.1. Residue surface associated to a single curve. Throughout this subsection, we make the following assumption.

Assumption 3.2. *The curve $C \subset X$ is smooth irreducible of degree $e \geq 2$ and genus g . Furthermore, C has only finitely many secant lines.*

We want to produce a surface on X which is birational to the symmetric product of C . First we fix some notation. Set $\mathcal{L} = \mathcal{O}_X(1)|_C$ and $\Sigma = C^{(2)}$. As in the previous section, we have a morphism $\varphi : \Sigma \rightarrow G(2, V)$ with $\mathcal{E} \cong \varphi^*\mathcal{E}_2$, where $\mathcal{E} = \mathcal{E}(\mathcal{L})$ is the symmetrization of \mathcal{L} and \mathcal{E}_2 is the canonical rank 2 quotient bundle on $G(2, V)$. Set $P = \mathbb{P}(\mathcal{E})$, then we have the following diagram

$$\begin{array}{ccc} D & \hookrightarrow & P \xrightarrow{f} \mathbb{P}(V) \\ & & \downarrow \tilde{\pi} \\ & & \Sigma \end{array}$$

such that a fiber of $\tilde{\pi}$ maps to a line on $\mathbb{P}(V)$ and $D \cong C \times C$ with $f|_D = p_1$ and $\pi = \tilde{\pi}|_D$ is the canonical morphism from $C \times C$ to $C^{(2)}$. See the previous section for more details. Let $\mathfrak{a} \in |\mathcal{L}|$ be a general hyperplane section, and let $\Delta_0 = c_1(\pi_*\mathcal{O}_{C\times C})$ as before. Set $\xi = c_1(f^*\mathcal{O}(1))$. By Theorem 2.3, it is easy to get

$$(12) \quad K_{P/\Sigma} = c_1(\omega_{\tilde{\pi}}) = \tilde{\pi}^*c_1(\mathcal{E}) - 2\xi = \tilde{\pi}^*(D_{\mathfrak{a}} + \Delta_0) - 2\xi.$$

Apply $R\tilde{\pi}_*$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_P(-D) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

and we get the short exact sequence

$$0 \longrightarrow \tilde{\pi}_*\mathcal{O}_P = \mathcal{O}_\Sigma \longrightarrow \pi_*\mathcal{O}_D \longrightarrow R^1\tilde{\pi}_*\mathcal{O}_P(-D) \longrightarrow 0.$$

By relative duality, we have

$$R^1\tilde{\pi}_*\mathcal{O}_P(-D) \cong (\tilde{\pi}_*\mathcal{O}(D + K_{P/\Sigma}))^* \cong (\tilde{\pi}_*\mathcal{O}(D - 2\xi + \tilde{\pi}^*D_\alpha + \tilde{\pi}^*\Delta_0))^*.$$

Hence, by taking the first Chern classes, we get

$$\Delta_0 = c_1(\pi_*\mathcal{O}_D) = c_1(R^1\tilde{\pi}_*\mathcal{O}_P(-D)) = -c_1(\tilde{\pi}_*\mathcal{O}(D - 2\xi + \tilde{\pi}^*D_\alpha + \tilde{\pi}^*\Delta_0)).$$

Note that $D - 2\xi$ is the pullback of some class from Σ , say $D - 2\xi = \tilde{\pi}^*D'$. Then the projection formula implies

$$\tilde{\pi}_*\mathcal{O}(D - 2\xi + \tilde{\pi}^*D_\alpha + \tilde{\pi}^*\Delta_0) \cong \mathcal{O}(D' + D_\alpha + \Delta_0) \otimes \tilde{\pi}_*\mathcal{O}_P = \mathcal{O}(D' + D_\alpha + \Delta_0).$$

Combining the above two identities, we have

$$(13) \quad D = 2\xi - \tilde{\pi}^*(D_\alpha + 2\Delta_0).$$

Let $s \in H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(3)) = \text{Sym}^3 V$ be the degree 3 homogeneous polynomial whose zero defines $X \subset \mathbb{P}(V) = \mathbb{P}^4$. Then

$$\text{div}(f^*s) = D + S$$

for some surface S in P .

Definition 3.3. The surface S together with the morphism $\phi = f|_S : S \rightarrow X$ is called the *residue surface* associated to $C \subset X$.

Noting that the class of $D + S$ is 3ξ , we combine this with (13) and easily get

$$(14) \quad S = \xi + \tilde{\pi}^*(D_\alpha + 2\Delta_0).$$

The following lemma was proved in [HRS, Lemma 4.2]; here we give a different proof.

Lemma 3.4. *Counting with multiplicities, there are $N(e, g) = \frac{5e(e-3)}{2} + 6 - 6g$ secant lines of C .*

Proof. Let $\sigma = \tilde{\pi}|_S : S \rightarrow \Sigma$. As a divisor on P , the surface S defines an invertible sheaf $\mathcal{O}_P(S)$ together with a section s_0 . Then s_0 can be viewed as a section of $\tilde{\pi}_*\mathcal{O}_P(S)$, and the zero locus of this section is exactly the scheme B_C of lines on X that meet C in two points. To compute the length of B_C ,

it suffices to compute $c_2(\tilde{\pi}_*\mathcal{O}_P(S))$. From (14), we know that $\tilde{\pi}_*\mathcal{O}_P(S) = \mathcal{E} \otimes \mathcal{O}_\Sigma(D_a + 2\Delta_0)$. It follows that

$$\begin{aligned} c_2(\tilde{\pi}_*\mathcal{O}_P(S)) &= c_1(\mathcal{E}) \cdot (D_a + 2\Delta_0) + c_2(\mathcal{E}) + (D_a + 2\Delta_0)^2 \\ &= (D_a + \Delta_0)(D_a + 2\Delta_0) + \mathbf{a}^{[2]} + (D_a + 2\Delta_0)^2 \\ &= 2D_a^2 + 7D_a \cdot \Delta_0 + 6\Delta_0^2 + \mathbf{a}^{[2]} \\ &= 5\mathbf{a}^{[2]} - 5\delta_*\mathbf{a} - 3\delta_*K_C \\ &\equiv 5e(e - 1)/2 - 5e + 6(1 - g) = N(e, g). \end{aligned}$$

Here \equiv denotes numerical equivalence. □

Definition 3.5. We define the *multiplicity* $e(C; l)$ of a secant line l to be the length of B_C at the point $[l]$. The curve $C \subset X$ is *well positioned* if it has exactly $N(e, g)$ distinct secant lines or equivalently if all secant lines are of multiplicity 1.

Proposition 3.6. *Assume that C is neither a line nor a conic. Then the following are true.*

(1) *The surface S is smooth if and only if C is well positioned. If C is well positioned, then S is the blow-up of Σ at $N(e, g)$ points.*

(2) *Assume that C is well positioned. Then the following equalities hold in $\text{CH}_1(S)$:*

$$(15) \quad \xi|_S = 2\sigma^*D_a + 3\sigma^*\Delta_0 - \sum_{i=1}^{N(e,g)} E_i,$$

$$(16) \quad D|_S = 3\sigma^*D_a + 4\sigma^*\Delta_0 - 2\sum_{i=1}^{N(e,g)} E_i,$$

$$(17) \quad S|_S = 3\sigma^*D_a + 5\sigma^*\Delta_0 - \sum_{i=1}^{N(e,g)} E_i,$$

where $\sigma : S \rightarrow \Sigma$ is the blow-up and E_i are the exceptional curves.

Proof. (1) $\sigma = \tilde{\pi}|_S : S \rightarrow \Sigma$ is an isomorphism away from the exceptional locus E_i , and each E_i corresponds to a line on X that meets C in two points. Let E be one of those E_i 's that meets C in points P_1 and P_2 . Let (x, y) be a set of local parameters of Σ at $\sigma(E)$. We use $[T_0 : T_1]$ to denote the homogeneous coordinates of E with $P_1 = [0 : 1]$ and $P_2 = [\lambda : 1]$. Then we have

$$f^*s = Q(T_0, T_1) \cdot (u(x, y)T_0 + v(x, y)T_1)$$

in a neighborhood of E , where $Q(T_0, T_1)$ is a quadratic form in T_0 and T_1 with

$$Q|_{(x,y)=(0,0)} = T_0(T_0 - \lambda T_1).$$

Hence the local equation for S is $uT_0 + vT_1 = 0$. Thus S is smooth along E if and only if (u, v) generate the maximal idea of $\sigma(E)$ in Σ . The last condition is the same as $\tilde{\pi}_*(f^*s)$ and has a simple zero at $\sigma(E)$ as a section of $\tilde{\pi}_*(\mathcal{O}_P(S))$. Hence S is smooth if and only if C is well positioned.

(2) By adjunction formula, we have

$$K_S = [K_P + S]|_S = (\tilde{\pi}^*K_\Sigma + K_{P/\Sigma} + [S])|_S = \sigma^*K_\Sigma + \sigma^*(2D_a + 3\Delta_0) - \xi|_S.$$

On the other hand S is the blow-up of Σ , and we have

$$K_S = \sigma^*K_\Sigma + \sum E_i.$$

Comparing the above two identities, we get

$$\xi|_S = \sigma^*(2D_a + 3\Delta_0) - \sum E_i.$$

Hence, $D|_S$ and $S|_S$ can be computed easily using (13) and (14). □

3.2. Residue surface associated to a pair of curves. In this section we fix two at-worst nodal complete curves $i_1 : C_1 \hookrightarrow X$ of degree e_1 and genus g_1 and $i_2 : C_2 \hookrightarrow X$ of degree e_2 and genus g_2 . Let $C = C_1 \cup C_2 \subset X$ be the union. We allow C_1 and C_2 to meet each other transversally at x_1, \dots, x_r , which are smooth points on each curve. If C_1 and C_2 do meet, we use $\Pi_i = \Pi_{x_i}$ to denote the plane spanned by the tangent directions of C_1 and C_2 at the point x_i .

Definition 3.7. A line $l \subset X$ is called a *secant line* of the pair (C_1, C_2) if l is a secant line of $C_1 \cup C_2$ but not of C_1 or of C_2 .

We fix the following assumptions for the remainder of this subsection.

Assumption 3.8. *Both C_1 and C_2 are smooth irreducible. There are only finitely many secant lines of the pair (C_1, C_2) . If one of the curves is a line, then they do not meet each other.*

Let $\mathcal{L}_1 = \mathcal{O}_X(1)|_{C_1}$ and $\mathcal{L}_2 = \mathcal{O}_X(1)|_{C_2}$. By sending $(x, y) \in C_1 \times C_2$ to the line $\overline{i_1(x)i_2(y)} \subset \mathbb{P}(V)$, we get a rational map $\varphi_0 : C_1 \times C_2 \dashrightarrow G(2, V)$. The point $x_i \in C_1 \cap C_2$ determines $\bar{x}_i \in C_1 \times C_2$, $i = 1, \dots, r$, which forms the locus where φ_0 is not defined. Let Σ be the blow-up of $C_1 \times C_2$ at the points \bar{x}_i , and let F_i be the corresponding exceptional curves, $i = 1, \dots, r$. Then φ_0 extends to a morphism

$$\varphi : \Sigma \rightarrow G(2, V)$$

such that $\varphi|_{F_i}$ parameterizes all lines through x_i lying on the plane Π_i . This can be easily seen from the local description of the map φ_0 . Let t_α be the local parameter of C_α at x_i , $\alpha = 1, 2$. In homogeneous coordinates, the line $\overline{i_1(t_1)i_2(t_2)}$ can be parameterized by $(1 - \lambda)u_1(t_1) + \lambda u_2(t_2)$, where λ is a local coordinate of the line. Note that $u_1(0) = u_2(0) = \vec{a}$ as vectors of V^* . Hence,

$u_1 = \vec{a} + t_1 v_1(t_1)$ and $u_2 = \vec{a} + t_2 v_2(t_2)$. Thus the above parametrization can be written as

$$u_1 + \lambda(t_1 v_1(t_1) - t_2 v_2(t_2)).$$

A neighborhood of F_i has two charts. On the chart with coordinates $(x = t_1, y = t_2/t_1)$ we define $\varphi(x, y)$ to be the line parameterized by

$$u_1 + \lambda(v_1(x) - y v_2(xy)),$$

and on the chart with coordinates $(x' = t_1/t_2, y' = t_2)$, we set $\varphi(x', y')$ to be the line parameterized by

$$u_1 + \lambda(x' v_1(x' y') - v_2(y')).$$

This gives the morphism φ as an extension of φ_0 .

Let $\mathcal{E} = \varphi^* \mathcal{E}_2$. The definition of φ implies that on $\Sigma - \bigcup F_i$ the sheaf \mathcal{E} is isomorphic to $p_1^* \mathcal{L}_1 \oplus p_2^* \mathcal{L}_2$. We also note that $\varphi(F_i)$ is a line (with respect to the Plücker embedding) and hence $c_1(\mathcal{E}) \cdot F_i = 1$. Combining the above two facts, we have

$$(18) \quad c_1(\mathcal{E}) = \mathbf{a}_1 \times C_2 + C_1 \times \mathbf{a}_2 - \sum_{i=1}^r F_i.$$

The Schubert calculus tells us that $c_2(\mathcal{E}_2)$ is represented by the Schubert cycle parameterizing all lines contained in a hyperplane. This gives

$$(19) \quad c_2(\mathcal{E}) = \mathbf{a}_1 \times \mathbf{a}_2.$$

Here $\mathbf{a}_1 \in |\mathcal{L}_1|$ and $\mathbf{a}_2 \in |\mathcal{L}_2|$ are general elements from the corresponding complete linear systems. We are also viewing $\mathbf{a}_1 \times C_2$, $C_1 \times \mathbf{a}_2$ and $\mathbf{a}_1 \times \mathbf{a}_2$ as cycles on Σ . Let

$$\begin{array}{ccc} P & \xrightarrow{f} & \mathbb{P}(V) \\ \tilde{\pi} \downarrow & & \\ \Sigma & & \end{array}$$

be the total family of lines over Σ . The morphism $\tilde{\pi}$ admits two distinguished sections $D_1 \subset P$ and $D_2 \subset P$ corresponding to points on C_1 and C_2 , respectively. It is easy to see that D_1 and D_2 meet above F_i . Let $D = D_1 + D_2$. If $s \in \text{Sym}^3 V$ is the homogeneous polynomial that defines X , then $f^* s$ can be viewed as a section of $\mathcal{O}_P(3) = f^* \mathcal{O}_{\mathbb{P}(V)}(3)$, whose zero defines

$$\text{div}(f^* s) = D + S$$

for some divisor $S \subset P$.

Definition 3.9. The surface S together with the morphism $\phi = f|_S : S \rightarrow X$ is called the *residue surface* associated to the pair of curves C_1 and C_2 .

In order to determine the divisor classes of D and S , let $\xi = c_1(\mathcal{O}_P(1))$, where $\mathcal{O}_P(1) = f^*\mathcal{O}_{\mathbb{P}(V)}(1)$. Let $\pi = \tilde{\pi}|_D : D \rightarrow \Sigma$. Consider the following short exact sequence

$$0 \rightarrow \mathcal{O}_P(-D) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_D \rightarrow 0.$$

By applying $R\tilde{\pi}_*$ to the above sequence, we get

$$0 \rightarrow \mathcal{O}_\Sigma = \tilde{\pi}_*\mathcal{O}_P \longrightarrow \pi_*\mathcal{O}_D \longrightarrow R^1\tilde{\pi}_*\mathcal{O}_P(-D) \rightarrow 0.$$

By duality, $R^1\tilde{\pi}_*\mathcal{O}_P(-D) = (\tilde{\pi}_*\mathcal{O}(K_{P/\Sigma} + D))^* = (\tilde{\pi}_*\mathcal{O}_P(-2\xi + D + \tilde{\pi}^*c_1(\mathcal{E})))^*$. Hence, we get

$$D = 2\xi - \tilde{\pi}^*c_1(\mathcal{E}) - \tilde{\pi}^*c_1(\tilde{\pi}_*\mathcal{O}_D).$$

Note that we have the following short exact sequence on D ,

$$0 \rightarrow \mathcal{O}_{D_1}(-\sum F_i) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D_2} \rightarrow 0.$$

Here we view F_i as divisors on D_1 by identifying D_1 with Σ . By taking direct images, we have

$$0 \rightarrow \mathcal{O}_\Sigma(-\sum F_i) \longrightarrow \pi_*\mathcal{O}_D \longrightarrow \mathcal{O}_\Sigma \rightarrow 0,$$

and it follows that $c_1(\pi_*\mathcal{O}_D) = -\sum F_i$. This gives the class of D by

$$(20) \quad D = 2\xi - \tilde{\pi}^*(\mathbf{a}_1 \times C_2 + C_1 \times \mathbf{a}_2) + 2 \sum_{i=1}^r \tilde{\pi}^*F_i.$$

Since $D + S = 3\xi$, we get

$$(21) \quad S = \xi + \tilde{\pi}^*(\mathbf{a}_1 \times C_2 + C_1 \times \mathbf{a}_2) - 2 \sum_{i=1}^r \tilde{\pi}^*F_i.$$

Lemma 3.10. *Counted with multiplicity, there are $5e_1e_2 - 6r$ secant lines of the pair (C_1, C_2) , where r is the number of points in which the two curves meet each other.*

Proof. The surface S gives rise to a section s_0 of $\mathcal{O}_P(S)$. Then s_0 can also be viewed as a section of $\tilde{\pi}_*\mathcal{O}_P(S)$, whose zero locus defines the scheme B_{C_1, C_2} of secant lines of (C_1, C_2) . By (21), we easily see that

$$\tilde{\pi}_*\mathcal{O}(S) = \mathcal{E} \otimes p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2 \otimes \mathcal{O}_\Sigma(-\sum F_i).$$

Hence one easily computes

$$c_2(\tilde{\pi}_*\mathcal{O}_P(S)) = 5\mathbf{a}_1 \times \mathbf{a}_2 + 6(\sum F_i)^2 \equiv 5e_1e_2 - 6r.$$

This implies that the length of B_{C_1, C_2} is $5e_1e_2 - 6r$. □

Definition 3.11. For each secant line l of (C_1, C_2) we define the *multiplicity*, $e(C_1, C_2; l)$, of l to be the length of B_{C_1, C_2} at the point $[l]$. The pair (C_1, C_2) is called *well positioned* if it has exactly $5e_1e_2 - 6r$ distinct secant lines; namely, all the secant lines are of multiplicity 1.

Proposition 3.12. *Let C_1 and C_2 be two smooth complete curves on X as above. Then the following are true.*

(1) *The residue surface S associated to (C_1, C_2) is smooth if and only if the pair (C_1, C_2) is well positioned. In that case, S is a blow-up of Σ at $5e_1e_2 - 6r$ points.*

(2) *Assume that (C_1, C_2) is a well-positioned pair. The following equalities hold in $\text{CH}_1(S)$:*

$$(22) \quad \xi|_S = 2\sigma^*(\mathbf{a}_1 \times C_2 + C_1 \times \mathbf{a}_2) - 3 \sum_{i=1}^r \sigma^* F_i - \sum_{i=1}^{5e_1e_2-6r} E_i,$$

$$(23) \quad D|_S = 3\sigma^*(\mathbf{a}_1 \times C_2 + C_1 \times \mathbf{a}_2) - 4 \sum_{i=1}^r \sigma^* F_i - 2 \sum_{i=1}^{5e_1e_2-6r} E_i,$$

$$(24) \quad S|_S = 3\sigma^*(\mathbf{a}_1 \times C_2 + C_1 \times \mathbf{a}_2) - 5 \sum_{i=1}^r \sigma^* F_i - \sum_{i=1}^{5e_1e_2-6r} E_i,$$

where $\sigma = \tilde{\pi}|_S : S \rightarrow \Sigma$ is the blow-up and E_i are the exceptional divisors of σ .

Proof. The statement (1) follows from direct local computation as before. For (2), note that under the assumption, S is a smooth surface. By the adjunction formula,

$$K_S = (K_P + S)|_S = \sigma^* K_\Sigma - \xi|_S + 2\sigma^*(\mathbf{a}_1 \times C_2 + C_1 \times \mathbf{a}_2) - 3 \sum_{i=1}^r \sigma^* F_i.$$

On the other hand the blow-up gives

$$K_S = \sigma^* K_\Sigma + \sum_{i=1}^{5e_1e_2-6r} E_i.$$

We easily get $\xi|_S$ by comparing the above two equalities. The other equalities follow easily. □

4. Relations among 1-cycles

In this section we assume that $X \subset \mathbb{P}(V) = \mathbb{P}_k^n$ is a smooth cubic hypersurface. We want to investigate the cycle classes represented by a curve on X .

Let $\text{Hilb}_{e,g}(X)$ be the Hilbert scheme of degree e genus g curves on X . Let $\mathcal{H}^{e,g} = \mathcal{H}^{e,g}(X)$ be the normalization of the reduced Hilbert scheme $(\text{Hilb}_{e,g}(X))_{red}$. We use $Z \subset X \times \mathcal{H}^{e,g}$ to denote the universal family over $\mathcal{H}^{e,g}$. Note that Z is nothing but the pullback of the universal family over $\text{Hilb}_{e,g}(X)$. Let $\mathcal{U}^{e,g} \subset \mathcal{H}^{e,g}$ be the open subscheme of all points $[C]$ whose corresponding curve C is well positioned. When $\mathcal{U}^{e,g} \neq \emptyset$, we define

$$Y_0 = \{(x, [C]) \in X \times \mathcal{U}^{e,g} : x \in l \text{ for some secant line } l \text{ of } C\}$$

and take $Y = \overline{Y_0} \subset X \times \mathcal{H}^{e,g}$. We need to generalize the concept of secant lines.

Definition 4.1. Let $[C] \in \mathcal{H}^{e,g}$ be a closed point. Suppose that the fiber $Y_{[C]} = \bigcup \tilde{E}_i$ is purely one dimensional. Let E_i be \tilde{E}_i with the reduced structure. Then we call E_i a *generalized secant line* of C with *multiplicity* $e(C; E_i)$ being the length of \tilde{E}_i at its generic point. For a pair (C_1, C_2) of curves on X , a *generalized secant line* l of the pair is defined to be a generalized secant line of $C_1 \cup C_2$ with multiplicity $e(C_1, C_2; l) = e(C_1 \cup C_2; l) - e(C_1; l) - e(C_2; l)$. If the above multiplicity is 0, we do not count l as a generalized secant line of the pair.

Let $\mathcal{T}^{e,g} \subset \mathcal{H}^{e,g}$ be the open subset of all curves (including reducible ones) that have finitely many generalized secant lines. Then we have $\mathcal{U}^{e,g} \subset \mathcal{T}^{e,g}$. We use $\mathcal{U}_{[C]}^{e,g} \subset \mathcal{T}_{[C]}^{e,g} \subset \mathcal{H}_{[C]}^{e,g}$ to denote the corresponding subsets of the component containing the point $[C]$.

Theorem 4.2. Let $X \subset \mathbb{P}_k^n$ be a smooth cubic hypersurface of dimension at least 3. Let h denote the class of a hyperplane section on X .

(a) Let $[C] \in \mathcal{H}^{e,g}(X)$ be a connected curve of degree e and genus g on X . Assume the following: (a1) C is not a line; and (a2) C has finitely many (generalized) secant lines $E_i, i = 1, \dots, m$, with multiplicity $a_i = e(C; E_i)$. Then we have

$$(25) \quad (2e - 3)C + \sum_{i=1}^m a_i E_i = \left(\frac{1}{2}(e - 1)(3e - 4) - 2g\right)h^{n-2}$$

in $\text{CH}_1(X)$.

(b) Let C_1 and C_2 be two connected curves on X with degrees e_1 and e_2 , respectively. Assume the following: (b1) C_1 and C_2 only meet each other transversally at r smooth points x_1, \dots, x_r ; (b2) any component of C_1 or C_2 containing any of the x_i 's is not a line; and (b3) (C_1, C_2) has finitely many (generalized) secant lines $E_i, i = 1, \dots, m$, with multiplicities a_i . Then we have

$$(26) \quad 2e_2 C_1 + 2e_1 C_2 + \sum_{i=1}^m a_i E_i = (3e_1 e_2 - 2r)h^{n-2}$$

in $\text{CH}_1(X)$.

(c) Let $L \subset X$ be a line, and let $C \subset X$ be an irreducible curve of degree e on X . Assume the following: (c1) L meet C transversally in a point x ; (c2) C has finitely many (generalized) secant lines; and (c3) the pair (L, C) has finitely many (generalized) secant lines E_i with multiplicity a_i , $i = 1, \dots, m$. Then

$$(27) \quad (2e - 1)L + 2C + \sum_{i=1}^m a_i E_i = (3e - 2)h^{n-2}$$

in $\text{CH}_1(X)$.

Corollary 4.3. For smooth cubic hypersurfaces of dimension at least 2, the Chow group $\text{CH}_1(X)$ is generated by the classes of all lines on X . If $\dim X \geq 3$, then algebraic equivalence and homological equivalence are the same for 1-cycles on X . For $\dim X \geq 5$, we have $\text{CH}_1(X) \cong \mathbb{Z}$.

Proof of Theorem 4.2. First, we will prove both statements (a) and (b) in a similar way, under the additional assumption that the curve C (resp. C_1 and C_2) is (are) smooth and C (resp. the pair (C_1, C_2)) is well positioned. Consider the following situation. Let Σ be a surface together with a morphism $\varphi : \Sigma \rightarrow G(2, V)$ and a diagram

$$\begin{array}{ccccc} D \cup S & \xrightarrow{\varphi'} & Y & \xrightarrow{p'} & X \\ \downarrow i' & & \downarrow i_Y & & \downarrow i_X \\ P & \xrightarrow{\tilde{\varphi}} & G(1, 2, V) & \xrightarrow{p} & \mathbb{P}(V) \\ \downarrow \tilde{\pi} & & \downarrow \pi_0 & & \\ \Sigma & \xrightarrow{\varphi} & G(2, V) & & \end{array}$$

where all the squares are fiber products. Let $\phi : S \rightarrow X$ and $\phi' : D \rightarrow X$ be the natural morphisms. Assume that the image of D in X is at most one dimensional and the image of S in X is two dimensional. Then we compute

$$\phi_* S = p'_*(D + S) = p'_* i_Y^* \tilde{\varphi}_* P = i_X^* p_* \tilde{\varphi}_* P = (p_* \tilde{\varphi}_* P)|_X.$$

Note that $p_* \tilde{\varphi}_* P = a[\mathbb{P}^3]$ in $\text{CH}_3(\mathbb{P}(V))$ for some integer $a > 0$. It follows that $\phi_* S = a h^{n-3}$. Let $\xi = c_1(\mathcal{O}_P(1))$, where $\mathcal{O}_P(1) = \tilde{\varphi}^* p^* \mathcal{O}_{\mathbb{P}(V)}(1)$. Then by the projection formula, we get

$$(28) \quad \phi_*(\xi|_S) = \phi_* \phi^* h = h \cdot \phi_* S = a h^{n-2}.$$

Namely, $\phi_* \xi|_S$ is an integral multiple of h^{n-2} . Let \mathfrak{A} be a divisor on Σ . By construction,

$$\varphi'_*(\tilde{\pi}^* \mathfrak{A}|_D + \tilde{\pi}^* \mathfrak{A}|_S) = \tilde{\varphi}_* \tilde{\pi}^* \mathfrak{A}|_Y = \pi^* \varphi_* \mathfrak{A}|_Y.$$

By applying p'_* , we get that

$$(29) \quad \phi'_*(\tilde{\pi}^*\mathfrak{A}|_D) + \phi_*(\tilde{\pi}^*\mathfrak{A}|_S) = p'_*i_Y^*\pi^*\varphi_*\mathfrak{A} = (p_*\pi_0^*\varphi_*\mathfrak{A})|_X = b[\mathbb{P}^2]|_X = bh^{n-2}$$

for some integer b .

To prove Theorem 4.2(a), we take Σ to be $C^{(2)}$ and S to be the associated residue surface. We know that $D \cong C \times C$ and $\phi' = p_1$. We take $\mathfrak{A} = D_{\mathfrak{a}}$ in (29), where $\mathfrak{a} \subset C$ a general hyperplane section of C . Recall that $D_{\mathfrak{a}} = (\tilde{\pi}|_D)_*(\mathfrak{a} \times C)$ is defined in (2). Note that $\phi'_*(\tilde{\pi}^*D_{\mathfrak{a}}|_D) = \phi'_*(\mathfrak{a} \times C + C \times \mathfrak{a}) = e[C]$. We know that

$$\phi_*(\tilde{\pi}^*D_{\mathfrak{a}}|_S) = -e[C] + (\dots)h^{n-2}.$$

Similarly, in (29) if we take $\mathfrak{A} = \Delta_0$ (see section 2 for the definition of Δ_0) and note that $\phi'_*(\tilde{\pi}^*\Delta_0|_D) = -[C]$, we get

$$\phi_*(\tilde{\pi}^*\Delta_0|_S) = [C] + (\dots)h^{n-2}.$$

Putting all these into the equality for $\xi|_S$ in Proposition 3.6, we have

$$(2e - 3)[C] + \sum E_i = -\phi_*(\xi|_S) + (\dots)h^{n-2} = \text{multiple of } h^{n-2}.$$

We know that the degree of h^{n-2} is 3, hence by comparing the total degrees of the two sides of the above equality, we see that the right-hand side is $(1/2(e - 1)(3e - 4) - 2g)h^{n-2}$.

For (b), we take Σ to be the blow-up of $C_1 \times C_2$ at the points where the two curves meet. We use the notation from Proposition 3.12. First we notice that σ^*F_i maps to a singular plane cubic whose class is h^{n-2} . This is because the image of σ^*F_i in X is the intersection of Π_i and X . Then taking $\mathfrak{A} = \mathfrak{a}_1 \times C_2 + C_1 \times \mathfrak{a}_2$ in (29) and noting that $\phi'_*(\tilde{\pi}^*\mathfrak{A}|_D) = e_2[C_1] + e_1[C_2]$, we get

$$\phi_*(\sigma^*\mathfrak{A}) = -e_2[C_1] - e_1[C_2] + (\dots)h^{n-2}.$$

Putting this relation into the equality for $\xi|_S$ in Proposition 3.12, we get

$$2e_2[C_1] + 2e_1[C_2] + \sum E_i = (\dots)h^2.$$

By comparing the degrees, we know that the right-hand side is $(3e_1e_2 - 2r)h^{n-2}$.

Note that if the statement of (b) holds for each pair $(C_{1,i}, C_{2,j})$ where $C_{1,i}$ is a component of C_1 and $C_{2,j}$ is a component of C_2 , then (b) holds for (C_1, C_2) . This means that we proved (b) when all components of C_1 and C_2 are smooth and the components are pairwise well positioned.

To prove the theorem in full generality, we need the following.

Lemma 4.4. *Let B/k be a smooth projective curve. Let $y \in Z_{p+1}(X \times B)$ be an algebraic cycle of relative dimension p (over B). Let $\gamma : B(k) \rightarrow \text{CH}_p(X)$ be the map induced by y . Assume that there exists a Zariski open $U \subset B$ such that γ is constant on U . Then γ is constant on B .*

The proof of this lemma is easy since any closed point of B is rationally equivalent to a linear combination of points of U .

Observation I: If $[C] \in \mathcal{T}^{e,g}$ is a smooth point and $\mathcal{U}_{[C]}^{e,g} \neq \emptyset$, then (a) holds for C .

Set $z = (2e - 3)Z + Y \in \text{CH}^{n-2}(X \times \mathcal{H}^{e,g})$. We can pick a smooth curve B that maps to $\mathcal{H}_{[C]}^{e,g}$ and connects $[C]$ with a general point. Then we apply Lemma 4.4 with y being the pullback of z . Since (a) holds for a general point of B , then the lemma implies that it holds for all points of B , whose images are smooth points of $\mathcal{T}_{[C]}^{e,g}$. Completely similarly, we have the following:

Observation II: Using notation as in (b), let $[C_1] \in \mathcal{F}_{C_1, C_2}$ be a smooth point, where $\mathcal{F}_{C_1, C_2} \subset \mathcal{H}_{[C_1]}^{e,g}$ is the subscheme of curves meeting C_2 in r smooth points. If (b) holds for (C'_1, C_2) for a general point $[C'_1] \in \mathcal{F}_{C_1, C_2}$, then (b) holds for (C_1, C_2) .

We first prove (b) in its full generality. *Step 1:* We assume that C_2 has smooth components. Without loss of generality, we assume that C_2 is smooth irreducible. We can attach sufficiently many very free rational curves C'_i to C_1 at smooth points and get $C' = C_1 \cup (\bigcup_i C'_i)$ of degree e' and genus $g' = g_1$, such that (i) the deformation of C' in X , fixing the points where the curve meets C_2 , is unobstructed (see [Kol, II.7]); (ii) the pairs (C'_i, C_2) are all well positioned; (iii) for a general point $[\tilde{C}] \in \mathcal{F}_{C', C_2}$, the pair (\tilde{C}, C_2) is well positioned. Note that (i) implies that $[C'] \in \mathcal{F}_{C', C_2}$ is a smooth point. Furthermore, (ii) and (iii) imply that (b) holds for (C'_i, C_2) and (\tilde{C}, C_2) . By Observation II, we know that (b) holds for (C', C_2) . Then by linearity, we know that (b) holds for (C_1, C_2) . *Step 2:* This is the general case. We can repeat the above argument by attaching a very free rational curve to one of the curves.

So for (a) with C being general, the only case we need to verify is when $\mathcal{U}^{e,g} = \emptyset$ or $[C] \in \mathcal{H}^{e,g}$ is not a smooth point. In either case we can attach very free rational curves C_i to C at smooth points and get a curve $C' = C \cup (\bigcup_i C_i)$. We may assume that the following conditions holds: (i) The deformation of the curve C' in X is unobstructed and hence gives a smooth point $[C'] \in \mathcal{H}^{e', g'}$. (ii) The pair (C, C_i) has finitely many generalized secant lines $E_{i,s}$ of multiplicity $a_{i,s}$. (iii) Each pair (C_i, C_j) is well positioned with secant lines $L_{i,j,s}$. (iv) Each curve C_i is well positioned with secant lines $L_{i,s}$. (v) A general point $[\tilde{C}] \in \mathcal{H}_{[C']}^{e', g'}$ corresponds to a well-positioned curve $\tilde{C} \subset X$.

By Observation I, (i) and (v) imply that (a) holds for C' , i.e.,

$$(2e + 2 \sum_i e_i - 3)(C + \sum_i C_i) + \sum_i \sum_s L_{i,s} + \sum_{i,j} \sum_s L_{ij,s} + \sum_i \sum_s a_{i,s} E_{i,s} + \sum_{i=1}^m a_i E_i$$

is a multiple of h^{n-2} . By (iv), we have

$$(2e_i - 3)C_i + \sum_s L_{i,s} = (\dots)h^{n-2}.$$

By (ii), we have

$$2e_i C + 2e C_i + \sum_s a_{i,s} E_{i,s} = (\dots)h^{n-2}.$$

By (iii), we have

$$2e_i C_j + 2e_j C_i + \sum_s L_{ij,s} = (\dots)h^{n-2}, \quad i < j.$$

Let i (and j) run over all possible choices and sum up the above three equations, then take the difference of that with the first big sum above. We get (a) for the curve C .

To prove (c), let L_i be the (generalized) secant lines of C , and let b_i be the corresponding multiplicities. Then by (a) for $L \cup C$, we have

$$(2e - 1)(L + C) + \sum b_i L_i + \sum a_i E_i = (\dots)h^{n-2}.$$

By (a) for the curve C , we get

$$(2e - 3)C + \sum b_i L_i = (\dots)h^{n-2}.$$

Then we take the difference of the above two equations and get (c). □

Proof of Corollary 4.3. Let $C \subset X$ be an irreducible curve on X . First assume that C is smooth and has finitely many secant lines. Then (a) implies that $(2e - 3)C$ is an integral combination of lines. We choose a line L such that (C, L) has finitely many secant lines, then (b) implies that $2C$ is an integral combination of lines. Hence C itself is an integral combination of lines in $\text{CH}_1(X)$. If C has infinitely many secant lines, then we attach sufficiently many very free rational curves C_i to C and get $C' = C \cup (\bigcup_i C_i)$ such that (i) the class each C_i can be written as integral combination of lines; (ii) the deformation of C' in X is unobstructed; (iii) C' can be smoothed out to get a complete family $\pi : S \rightarrow B$ to a smooth complete curve B ; (iv) S_{b_0} is the curve C' and a general fiber S_b is well positioned on X . By associating the secant lines of S_b to the point b , we get a rational map $\varphi_0 : B \dashrightarrow \text{Sym}^{n_1}(F)$, where $F = F(X)$ is the Fano scheme of lines on X . Pick a general line L

on X . By associating the secant lines of (L, S_b) to b , we have a rational map $\phi_0 : B \dashrightarrow \text{Sym}^{n_2}(F)$. Both of the two maps can be extended to the whole curve B and get morphisms φ and ϕ . Assume that $\varphi(b_0) = \sum a_i[L_i]$ and $\phi(b_0) = \sum a'_i[E_i]$. Then the Lemma 4.4 implies that the following equalities hold in the Chow group:

$$(2e' - 3)C' + \sum a_i L_i = (\dots)h^{n-2}$$

and

$$2C' + (2e' - 1)L + \sum b_i E_i = (\dots)h^{n-2}.$$

Hence C' is an integral combination of lines. This implies that C itself is also.

When $n \geq 4$, the scheme, $F = F(X)$, of lines on X is smooth irreducible. This shows that algebraic equivalence agrees with homological equivalence for 1-cycles on X . When $n \geq 6$, the scheme $F = F(X)$ is a smooth Fano variety; see Theorem 5.1. This implies that F is rationally connected; see [KMM] and [Cam]. Hence all lines on X are rationally equivalent to each other. \square

5. Fano scheme of lines and intermediate jacobian of a cubic threefold

In this section we first review some geometry of the Fano scheme of lines on a cubic hypersurface and then collect some known results about intermediate jacobians of a cubic threefold.

5.1. General theory on Fano schemes. Let $X \subset \mathbb{P}_k^n$ be a smooth cubic hypersurface over an algebraically closed field. Let $F = F(X)$ be the Fano scheme of lines on X . Let $V = H^0(\mathbb{P}^n, \mathcal{O}(1))$. By definition, $F(X)$ can be viewed as a subscheme of $G(2, V)$, the grassmannian that parameterizes all lines on \mathbb{P}^n . Let

$$(30) \quad \begin{array}{ccc} G(1, 2, V) & \xrightarrow{f} & G(1, V) = \mathbb{P}^n \\ \pi \downarrow & & \\ G(2, V) & & \end{array}$$

be the universal family of lines on $\mathbb{P}^n = \mathbb{P}(V)$. Let $s \in \text{Sym}^3 V$ be the homogeneous polynomial defining X . Then s can also be viewed as a section \tilde{s} of $\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(3) = \text{Sym}^3(\mathcal{E}_2)$, where \mathcal{E}_2 is the universal rank-2 quotient bundle on $G(2, V)$. Then $F \subset G(2, V)$ is exactly the zero locus of \tilde{s} .

Theorem 5.1 ([AK]). *Notations as above, then the following are true.*
 (a) F is smooth irreducible of dimension $2n - 6$.

(b) The dualizing sheaf $\omega_F \cong \mathcal{O}_F(5-n)$, here $\mathcal{O}_F(1)$ comes from the Plücker embedding of $G(2, V)$. In particular, if $n \geq 6$, then F is a Fano variety.

5.2. Intermediate jacobian of a cubic threefold. In this subsection we fix $X \subset \mathbb{P}^4_k = \mathbb{P}(V)$ as a smooth cubic threefold, and we assume that $\text{char}(k) \neq 2$. Let $F = F(X)$ the Fano surface of lines on X . Let

$$(31) \quad \begin{array}{ccc} P & \xrightarrow{f} & X \\ \pi \downarrow & & \\ F & & \end{array}$$

be the universal family of lines on X . Let $l \subset X$ be a line on X , and let $[l]$ be the corresponding point on F . Let $\mathcal{N}_{l/X}$ be the normal bundle. Then we have the short exact sequence

$$0 \rightarrow \mathcal{N}_{l/X} \rightarrow \mathcal{N}_{l/\mathbb{P}^4} \rightarrow \mathcal{N}_{X/\mathbb{P}^4}|_l \rightarrow 0.$$

Since $\mathcal{N}_{l/\mathbb{P}^4} \cong \mathcal{O}(1)^{\oplus 3}$ and $\mathcal{N}_{X/\mathbb{P}^4}|_l \cong \mathcal{O}(3)$, we get $\mathcal{N}_{l/X} \cong \mathcal{O}^2$ or $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. The line l is of *first type* if $\mathcal{N}_{l/X} \cong \mathcal{O}^2$; otherwise, l is of *second type*. Let $D_0 \subset F$ be the locus whose points are all lines of second type. The condition for l to be of second type is equivalent to the existence of a plane $\Pi \subset \mathbb{P}^4$ such that $\Pi \cdot X = 2l + l'$ for some other line $l' \subset X$; see Lemma 1.14 of [Mu1]. A point $x \in X$ is called an *Eckardt point* if there are infinitely many lines through x . We know that there are at most finitely many of such points; see the discussion on p. 315 of [CG].

Proposition 5.2 ([CG], [Mu1], [AK]). *The following are true:*

- (a) We have a canonical isomorphism $\Omega_F^1 \cong \mathcal{E}_2|_F$.
- (b) D_0 is a smooth curve on F whose divisor class is $2K_F$.
- (c) The ramification divisor of f in diagram (31) is given by $R = \pi^{-1}D_0 \subset P$. If we write $B = f(R) \subset X$, then B is linearly equivalent to $30h$, where h is a hyperplane section of X .
- (d) We have the following equalities:

$$\begin{aligned} h^0(F, \mathcal{O}_F) &= 1, & h^1(F, \mathcal{O}_F) &= 5, & h^2(F, \mathcal{O}_F) &= 10, \\ h^0(F, \omega_F) &= 10, & h^1(F, \omega_F) &= 5, & h^2(F, \omega_F) &= 1. \end{aligned}$$

Proof. (a) is proved in [AK, Theorem 1.10]; (d) is proved in [AK, Proposition 1.15]. For (b), we note that smoothness of D_0 is Corollary 1.9 of [Mu1] and the class of D_0 is computed in [CG, Proposition 10.21]. If $[l] \in D_0$, then $H^0(l, \mathcal{N}_{l/X})$ does not generate $\mathcal{N}_{l/X}$. This implies that infinitesimally l only moves in the $\mathcal{O}(1)$ direction of $\mathcal{N}_{l/X}$. Hence f ramifies along $\pi^{-1}([l])$. If $[l] \notin D_0$, then $\mathcal{N}_{l/X}$ is globally generated and f is étale along $\pi^{-1}([l])$. Since X has Picard number one, to know the divisor class of B , we only need

to compute the intersection number $B \cdot l$ for a general line $l \subset X$. By the projection formula,

$$B \cdot l = (f_* \pi^* D_0) \cdot l = D_0 \cdot (\pi_* f^* l) = 2K_F \cdot D_l,$$

where $D_l \subset F$ is the divisor of all lines meeting l . It is known (see [CG, Section 10]) that K_F is numerically equivalent to $3D_l$ and $(D_l)^2 = 5$. Hence we have $B \cdot l = 30$. This proves (c). □

Definition 5.3. Let T/k be a variety, then the group of *divisorial correspondence*, $\text{Corr}(T)$ is defined to be

$$\text{Corr}(T) = \frac{\text{Div}(T \times T)}{p_1^* \text{Div}(T) + p_2^* \text{Div}(T)}.$$

For the Fano surface F , there is a natural divisorial correspondence $I \in \text{Corr}(F)$, called *incidence correspondence*. To be more precise, we have the universal line $P \subset X \times F$. Let p and q be the two projections from $X \times F \times F$ to $X \times F$ and r be projection to $F \times F$. Then

$$I = r_*(p^* P \cdot q^* P - P_0),$$

where $P_0 = \{(x, [l], [l]) : x \in l\}$. Equivalently, I is the closure of all pairs (l_1, l_2) of distinct lines such that l_1 meets l_2 . Then I induces a homomorphism $\lambda_I : \text{Alb}(F) \rightarrow \text{Pic}_F^0$.

Let $A_1(X)$ be the group of algebraic 1-cycles on X that are algebraically equivalent to 0 modulo rational equivalence. Let $\varphi : A_1(X) \rightarrow A$ be a homomorphism from $A_1(X)$ to an abelian variety A . The homomorphism φ is said to be *regular* if for any algebraic family $Z \subset X \times T$ of curves on X parameterized by T , the rule $t \mapsto \varphi([Z_t] - [Z_{t_0}])$ induces a morphism from T to A . The homomorphism φ is said to be *universal* if for any regular homomorphism $\varphi' : A_1(X) \rightarrow A'$ to an abelian variety A' , there is a unique homomorphism $\psi : A \rightarrow A'$ of abelian varieties such that $\varphi' = \psi \circ \varphi$. By the main result of [Mu3], there is a universal regular homomorphism $\varphi_0 : A_1(X) \rightarrow J(X)$ from $A_1(X)$ to an abelian variety $J(X)$. In [Mu2], $J(X)$ is naturally realized as a Prym variety. To be more precise, let $l \subset X$ be a general line on X , and let $\tilde{\Delta}$ be the smooth curve of genus 11 that parameterizes all lines on X meeting l . We use S_l to denote the surface swept out by lines parameterized by $\tilde{\Delta}$. There is natural involution σ on $\tilde{\Delta}$. Here $\sigma([l_1])$ is defined to be the residue line $[l_2]$ of $l \cup l_1$. Let $\Delta = \tilde{\Delta}/\sigma$ be the quotient curve which is a smooth plane quintic, hence of genus 6. The involution σ induces an involution $i : J(\tilde{\Delta}) \rightarrow J(\tilde{\Delta})$. The Prym variety associated to $\tilde{\Delta}/\Delta$ is defined to be $\text{Pr}(\tilde{\Delta}/\Delta) = \text{Im}(1 - i) \subset J(\tilde{\Delta})$. Theorem 5 of [Mu2] says that $J(X) \cong \text{Pr}(\tilde{\Delta}/\Delta)$. There is a natural principal polarization on $\text{Pr}(\tilde{\Delta}/\Delta)$ whose theta divisor Ξ is half of the restriction of the theta divisor from $J(\tilde{\Delta})$;

see part iii of [Mum]. This induces a principal polarization on $J(X)$ whose theta divisor will be denoted by Θ . It is shown in Section IV of [Mu2] that the principal polarization Θ essentially comes from the Poincaré duality on $H^3(X)$. Here $H^3(X) = H^3(X, \mathbb{Q}_\ell)$ is the ℓ -adic cohomology with $\ell \neq \text{char}(k)$.

Definition 5.4. We define the *intermediate jacobian* of X to be $J(X)$ together with the principal polarization Θ .

Remark 5.5. When $k = \mathbb{C}$, the above definition agrees with the classical definition using Hodge theory; see [CG], and see [Mu3] for a proof of this fact.

Suppose that we are given a family $Z \rightarrow T$ of algebraic 1-cycles on X . We have a natural homomorphism $\Psi_T : A_0(T) \rightarrow A_1(X)$ by sending a point of T to the class of the 1-cycle it represents. Since $\varphi_0 : A_1(X) \rightarrow J(X)$ is regular, the composition $\varphi_0 \circ \Psi_T : T \rightarrow J(X)$ is a morphism, and hence it induces a homomorphism

$$\psi_T : \text{Alb}(T) \rightarrow J(X)$$

of abelian varieties. Both Ψ_T and ψ_T will be called the *Abel–Jacobi map*. In particular, we have the Abel–Jacobi maps ψ_F and Ψ_F associated to the Fano surface F .

Theorem 5.6 ([CG], [Mu2], [Bea1], [Bea2]). *Let X/k be a smooth cubic threefold, and let $\text{char}(k) \neq 2$. Let $F = F(X)$ be the Fano surface of lines on X . Then the following are true.*

- (a) *The incidence correspondence $I \in \text{Corr}(F)$ defines a principal polarization on $\text{Alb}(F)$. Namely, $\lambda_I : \text{Alb}(F) \rightarrow \text{Pic}_F^0$ is an isomorphism.*
- (b) *The homomorphism $\varphi_0 : A_1(X) \rightarrow J(X)$ is an isomorphism of abelian groups.*
- (c) *The homomorphism Ψ_F induces a isomorphism*

$$\psi_F : \text{Alb}(F) \rightarrow J(X)$$

as principally polarized abelian varieties.

- (d) *X is completely determined by $J(X)$ together with the polarization.*

Proof. For (a), see Theorem 8 in [Mu2]. We use the realization of $J(X)$ as $\text{Pr}(\tilde{\Delta}/\Delta)$. Let X_l be the blow-up of X along the general line $l \subset X$. Then the projection from l defines a morphism $p : X_l \rightarrow \mathbb{P}^2$. This makes X_l into a conic bundle over \mathbb{P}^2 and $\tilde{\Delta}/\Delta$ is the associated double cover. Then by Théorème 3.1 of [Bea1], we get $\text{Pr}(\tilde{\Delta}/\Delta) \cong A_1(X_l)$. Since $l \cong \mathbb{P}^1$, we also have $A_1(X_l) \cong A_1(X)$. Hence (b) follows easily. For (c), it is known that ψ_F is an isomorphism; see Theorem 7 of [Mu2]. Hence we only need to track the polarizations. Fix $s_0 \in F$, then we have the Albanese morphism $\alpha : F \rightarrow \text{Alb}(F)$ which sends s_0 to 0. Let $f = \psi_F \circ \alpha$. Then by Lemma 7 of [Mu2], we know that $(f \times f)^* \tilde{\Theta} \equiv I$, where $\tilde{\Theta} \subset J(X) \times J(X)$ is the divisorial correspondence on $J(X)$ that induces the principal polarization.

By passing to the Albanese variety, the above fact says that the polarization on $J(X)$ pulls back to the polarization λ_Y . The statement (d), also known as the Torelli theorem, was first obtained in [CG] for $k = \mathbb{C}$ and then generalized to the general case in [Bea2]. \square

6. Intermediate jacobian of a cubic threefold as Prym–Tjurin variety

In this section, we use the relations among 1-cycles to study the intermediate jacobian of a smooth cubic threefold X . To be more precise, we show that the intermediate jacobian is naturally isomorphic to the Prym–Tjurin variety constructed from curves on X . Throughout this section we fix $X \subset \mathbb{P}_k^4$ to be a smooth cubic threefold over an algebraically closed field. Let $F = F(X)$ be the Fano surface of lines on X , and let $I \subset F \times F$ be the incidence correspondence.

Definition 6.1. Let Γ be a smooth curve which might be reducible. Let $J = J(\Gamma)$ be jacobian of Γ . Let $i : J \rightarrow J$ be an endomorphism which is induced by a correspondence. Assume that i satisfies the quadratic equation

$$(32) \quad (i - 1)(i + q - 1) = 0$$

for some integer $q \geq 1$. Then we define the *Prym–Tjurin variety* associated to i as

$$(33) \quad \text{Pr}(C, i) = \text{Im}(1 - i) \subset J(C).$$

Now let C be a possibly reducible smooth curve on the cubic threefold X . Assume that C has only finitely many secant lines. Let \tilde{C} be the normalization of the curve that parameterizes all lines meeting C . Hence we have a natural morphism $\eta : \tilde{C} \rightarrow F$. Let $D(I) = (\eta \times \eta)^* I$ be the pullback of the incidence correspondence of F . Let $i_0 : \tilde{J} = J(\tilde{C}) \rightarrow \tilde{J}$ be the endomorphism induced by $D(I)$. Let $[L] \in \tilde{C}$ be a general point, and let L_i be all the secant lines of the pair (L, C) . Let $U \subset \tilde{C}$ be the dense open subset of lines L such that (L, C) has finitely many secant lines L_i . The association

$$[L] \mapsto \sum [L_i]$$

defines a correspondence D_U on the dense open subset U of \tilde{C} . Let $D \subset \tilde{C} \times \tilde{C}$ be the closure of D_U .

Definition 6.2. A smooth complete curve $C \subset X$ is *admissible* if the following conditions hold: (i) C has only finitely many secant lines; (ii) if a

line L meets C , it meets C transversally; (iii) C meets the divisor $B \subset X$ (see Proposition 5.2) transversally in smooth points of B .

The following lemma will be needed in later proofs.

Lemma 6.3. *Let $x \in X$ be a general point, and let $L_j, j = 1, \dots, 6$, be the six lines on X passing through x . Then*

$$\sum_{j=1}^6 L_j = 2h^2$$

in $\text{CH}_1(X)$, where h is the hyperplane class on X .

Proof. Fix a line $L = L_6$ passing through x . Let $E_t, t \in T$, be a one-dimensional family of lines on X such that $E_{t_0} = L_5$. By choosing E_t general enough, we may assume that (L, E_t) is well positioned for t close to but different from t_0 . Let $L_{t,j}, j = 0, \dots, 4$, be the secant lines of $(L, E_t), t \neq t_0$. By (b) of Theorem 4.2, we have

$$2E_t + 2L + \sum_{j=0}^4 L_{t,j} = 3h^2.$$

Now let $t \rightarrow t_0$, and assume that $L_{t,0}$ specializes to the residue line L_0 of $L_5 \cup L_6$ and that $L_{t,j}$ specializes to L_j for $j = 1, 2, 3, 4$. By taking the limit (see §11.1 of [Ful]), the above identity gives

$$2L_5 + 2L_6 + L_0 + \sum_{j=1}^4 L_j = 3h^2.$$

Since $L_0 + L_5 + L_6 = h^2$, the lemma follows. □

Theorem 6.4. *Assume that $\text{char}(k) \neq 2$. Let $C = \bigcup C_s \subset X$ be an admissible smooth curve of degree e . Assume that all components of C are rational. Let \tilde{C} be the normalization of the curve parameterizing lines meeting C as above. Let $i = i_0 + 1 \in \text{End}(\tilde{J})$. Then the following are true.*

- (a) *The endomorphism i satisfies the quadratic relation (32) with $q = 2e$.*
- (b) *The correspondence $D \in \text{Corr}(\tilde{C})$ is symmetric, effective, and without fixed point. The endomorphism i is induced by D .*
- (c) *Assume that $\text{char}(k) \nmid q$. Then the associated Prym–Tjurin variety $\text{Pr}(\tilde{C}, i)$ carries a natural principal polarization whose theta divisor Ξ satisfies the equation*

$$\tilde{\Theta}|_{\text{Pr}(\tilde{C}, i)} \equiv q\Xi,$$

where $\tilde{\Theta}$ is the theta divisor of \tilde{J} .

- (d) *Assume that $\text{char}(k) \nmid q$. Then the Abel–Jacobi map $\psi_{\tilde{J}} : \tilde{J} \rightarrow J(X)$ factors through $\text{Pr}(\tilde{C}, i)$ and induces an isomorphism*

$$u_C : \text{Pr}(\tilde{C}, i) \rightarrow J(X)$$

as principally polarized abelian varieties.

Proof. To prove (a), we consider the following commutative diagram.

$$(34) \quad \begin{array}{ccccc} \tilde{J} & \xrightarrow{i_0+q} & \tilde{J} & \xrightarrow{\eta_*} & \text{Alb}(F) \\ & & \downarrow i_0 & & \downarrow \lambda_I \\ & & \tilde{J} & \xleftarrow{\eta^*} & \text{Pic}_F^0 \\ & & & & \nearrow \psi_F \\ & & & & J(X) \end{array}$$

Here the map $J(X) \rightarrow \text{Pic}_F^0$ is induced by the map $A_1(X) \rightarrow \text{Pic}(F)$, which sends a curve on X to the divisor (on F) of all lines meeting the curve. Let $[L], [L'] \in \tilde{C}$ be two general points from the same component such that the pair (C, L) (resp. (C, L')) is well positioned. Let E_j (resp. E'_j), $j = 1, \dots, 5e - 5$, be all the secant lines of the pair (C, L) (resp. (C, L')). Let L_j (resp. L'_j), $j = 1, \dots, 5$, be the other five lines on X passing through $C \cap L$ (resp. $C \cap L'$). Then by (27) (and also (26) if C is disconnected), we have

$$(2e - 1)L + \sum_{j=1}^{5e-5} E_j = (2e - 1)L' + \sum_{j=1}^{5e-5} E'_j.$$

Note that $i_0([L]) = D(I)([L]) = \sum_{j=1}^5 [L_j] + \sum [E_j]$ and that $L + \sum_{j=1}^5 L_j = 2h^2$; see Lemma 6.3. Then we get

$$\begin{aligned} & \psi_F \circ \eta_* \circ (i_0 + q)([L] - [L']) \\ &= 2e(L - L') + (\sum L_j - \sum L'_j) + (\sum E_j - \sum E'_j) \\ &= 2e(L - L') + (L' - L) + (1 - 2e)(L - L') \\ &= 0. \end{aligned}$$

Since ψ_F is isomorphism, this implies that

$$(35) \quad \eta_* \circ (i_0 + q) = 0.$$

Hence

$$i_0(i_0 + q) = \eta^* \circ \lambda_I \circ \eta_* \circ (i_0 + q) = 0,$$

which is the same as the quadratic equation (32). By construction, we have

$$\begin{aligned} (D(I) - D)([L] - [L']) &= \sum [L_j] - \sum [L'_j] \\ &= -([L] - [L']) + ([L] + \sum [L_j]) - ([L'] + \sum [L'_j]). \end{aligned}$$

We also know that $([L] + \sum [L_j]) - ([L'] + \sum [L'_j])$ is the pullback of $x - x'$ on C , where $x = L \cap C$ and $x' = L' \cap C$. Since C has rational components, the

class of $x - x'$ is trivial. Hence we get

$$(D(I) - D)([L] - [L']) = -([L] - [L']).$$

This implies that i is induced by D . By definition, D is symmetric and effective. We still need to show that D is fixed point free. If $[L] \in \tilde{C}$ is a fixed point of D , let x be the point of L meeting C . This is well defined even if L is a secant line of C . This is because when L a secant line of C , the point $[L]$ of $f^{-1}C$ is a nodal point; here f is the morphism in (31). Hence $[L]$ gives rise to two points $[L]$ and $[L']$ of the normalization \tilde{C} corresponding to the choice of a point of $L \cap C$. Let Π be the plane spanned by the tangent direction of C at x and the line L . The condition $[L] \in D([L])$ implies that $\Pi \cdot X = 2L + L'$. This implies that $L \subset B$ and C is tangent to B at the point x . But this is not allowed since C is admissible. The statement (c) follows from (b) by applying [Kan, Theorem 3.1]. Furthermore, (c) implies that $\ker(i - 1)$ is connected and equal to $\text{im}(i + q - 1)$; see Lemma 7.7 of [BM]. Hence in diagram (34), the morphism η_* factors through $\text{Pr}(\tilde{C}, i)$ via i_0 . In this way we get $u'_C : \text{Pr}(\tilde{C}, i) \rightarrow \text{Alb}(F)$ such that $u'_C \circ i_0 = \eta_*$. We take $u_C = \psi_F \circ u'_C$. Hence we only need to prove that u'_C induces an isomorphism between principally polarized abelian varieties. To do this, let $\psi : \tilde{C} \rightarrow A = \text{Alb}(F)$ be the natural morphism (determined up to a choice of a point on \tilde{C}). After identifying A with $J(X)$, we have

$$(36) \quad \begin{aligned} \psi(D([L])) &= \sum \psi([L_i]) = \sum L_i \\ &= (1 - 2e)L - 2C + \text{const.} = (1 - q)\psi([L]) + \text{const.} \end{aligned}$$

Also notice that η_* is an epimorphism (since $\eta(\tilde{C})$ is ample on F). By a theorem of Welters (see Theorem 5.4 of [Kan]), this condition together with (35) implies

$$(37) \quad \psi_*(\tilde{C}) \equiv (q/24)\theta^4,$$

where θ is a theta divisor of A . Then by Theorem 5.6 of [Kan], we know that (A, θ) is a direct summand of $(\text{Pr}(\tilde{C}, i), \Xi)$. But we know that $\eta^* \circ \lambda_I$ is surjective, hence u'_C has to be an isomorphism. \square

Now let C_1 and C_2 be two smooth admissible rational curves on X which do not meet. Let $C = C_1 \cup C_2$ be the disjoint union of two admissible curves, and assume that C is again admissible. We use the a subscription for notations of the corresponding curve. So we have $\tilde{C}_1, \tilde{C}_2, \tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$ and, correspondingly, $\tilde{J}_1, \tilde{J}_2, \tilde{J} = \tilde{J}_1 \oplus \tilde{J}_2$. Assume that $\text{char}(k) \nmid q_1 q_2$. The correspondence D on \tilde{C} can be written as

$$D = \begin{pmatrix} D_1 & D_{12} \\ D_{21} & D_2 \end{pmatrix}.$$

Let $[L] \in \tilde{C}_1$, then $D_{21}([L]) = \sum [L_j]$ where L_j are all secant lines of (C_2, L) . We also see that ${}^t D_{12} = D_{21}$.

Proposition 6.5. *Let notations and assumptions be as above. Then the following hold.*

(a) *The induced morphism $D_{21} : \tilde{J}_1 \rightarrow \tilde{J}_2$ fits into the commutative diagram*

$$\begin{CD} \tilde{J}_1 @>D_{21}>> \tilde{J}_2 \\ @V{i_1+q-1}VV @VV{i_2-1}V \\ \tilde{J}_1 @>>-D_{21}> \tilde{J}_2 \end{CD}$$

(b) *The following equalities hold:*

$$D_{21}(i_1 + q_1 - 1) = (i_2 + q_2 - 1)D_{21} = 0.$$

Also we have $\text{Im}(D_{21}) = \text{Pr}(\tilde{C}_2, i_2) \subset \tilde{J}_2$, and the endomorphism D_{21} factors through $i_1 - 1 : \tilde{J}_1 \rightarrow \text{Pr}(\tilde{C}_1, i_1)$ and induces an isomorphism

$$t_{21} : (\text{Pr}(\tilde{C}_1, i_1), \Xi_1) \rightarrow (\text{Pr}(\tilde{C}_2, i_2), \Xi_2)$$

of principally polarized abelian varieties.

(c) *The isomorphism t_{21} is compatible with u_{C_1} and u_{C_2} , namely $u_{C_2} \circ t_{21} = u_{C_1}$.*

Proof. By (a) of Theorem 6.4, we know that $(D - 1)(D + q - 1) = 0$ as an endomorphism of \tilde{J} . This can be written in the matrix form,

$$(38) \quad \begin{pmatrix} q_2(i_1 - 1) + D_{12}D_{21} & (i_1 - 1)D_{12} + D_{12}(i_2 + q - 1) \\ D_{21}(i_1 + q - 1) + (i_2 - 1)D_{21} & D_{21}D_{12} + q_1(i_2 - 1) \end{pmatrix} = 0.$$

In particular, we have $D_{21}(i_1 + q - 1) + (i_2 - 1)D_{21} = 0$. This is the same as the commutativity of the diagram in (a). If we write $P_1 = \text{Pr}(\tilde{C}_1, i_1)$ and $Q_1 = \text{Im}(i_1 + q - 1) \subset \tilde{J}_1$, then $\tilde{J}_1 = P_1 + Q_1$ and $i_1|_{P_1} = 1 - q_1, i_1|_{Q_1} = 1$; see [BM]. Hence the morphism $i_1 + q - 1$ is surjective. This implies that $\text{Im}(D_{21}) \subset P_2 = \text{Pr}(\tilde{C}_2, i_2)$, which implies $(i_2 + q_2 - 1)D_{21} = 0$. Hence, we get the commutative diagram

$$\begin{CD} \tilde{J}_1 @>D_{21}>> P_2 \\ @V{i_1+q-1}VV @VV{-q_2}V \\ \tilde{J}_1 @>>-D_{21}> P_2 \end{CD}$$

Thus $q_2 D_{21}(\alpha) = D_{21}(i_1 + q - 1)(\alpha) = D_{21}(q\alpha)$, i.e., $q_1 D_{21}(\alpha) = 0$, for all $\alpha \in Q_1$. This implies that $D_{21}(i_1 + q_1 - 1) = 0$. Hence D_{21} factors through P_1 and gives the morphism $t_{21} : P_1 \rightarrow P_2$, i.e., $t_{21} \circ (i_1 - 1) = D_{21}$. One

easily verifies that $D_{21}|_{P_1} = -q_1 t_{21}$. We have similar identities for D_{12} and t_{12} . Note that the equations of the diagonal entries of (38) lead to

$$(D_{12}D_{21})|_{P_1} = q_1 q_2, \quad (D_{21}D_{12})|_{P_2} = q_1 q_2.$$

These relations give $t_{21}t_{12} = 1$ and $t_{12}t_{21} = 1$. The compatibility of t_{21} and $u_i = u_{C_i}$ is an easy diagram chasing from

$$\begin{array}{ccccc} \tilde{J}_1 & \xrightarrow{D_{21}} & P_2 & \xrightarrow{j_2} & \tilde{J}_2 \\ i_1 - 1 \downarrow & \nearrow t_{21} & \downarrow u_2 & & \downarrow \psi_{\tilde{J}_2} \\ P_1 & \xrightarrow{u_1} & J(X) & \xrightarrow{-q_2} & J(X) \end{array}$$

Note that $u_1 \circ (i_1 - 1) = \psi_{\tilde{J}_1}$ and the up left triangle, the right square, and the outside square are all commutative. Then one easily sees that $q_2(u_2 \circ t_{21} - u_1) = 0$, which implies $u_2 \circ t_{21} = u_1$. \square

Corollary 6.6. *Let $C \subset X$ be a smooth admissible rational curve, and let $\text{char}(k) \nmid q$. Let L and L' be two lines without meeting C . Let L_i (resp. L'_i) be all the secant lines of the pair (L, C) (resp. (L', C)). Then*

$$\sum [L_i] - \sum [L'_i] \in \text{Pr}(\tilde{C}, i) \subset \tilde{J}.$$

Proof. Choose another smooth admissible rational curve C_1 that meets both L and L' transversally in a single point. Take $C_2 = C$, then the left hand side is just $D_{21}([L] - [L'])$. \square

Fix a smooth admissible rational curve $C \subset X$ such that $\text{char}(k) \nmid q$. The above corollary allows us to define a map

$$v'_C : A_1(X) \rightarrow \text{Pr}(\tilde{C}, i), \quad L - L' \mapsto \sum [L_i] - \sum [L'_i].$$

Corollary 6.7. *The above map induces an morphism $v_C : J(X) \rightarrow \text{Pr}(\tilde{C}, i) = P$, which is the same as u_C^{-1} .*

Proof. We have the following commutative diagram.

$$\begin{array}{ccccc} J(X) & \xrightarrow{v_C} & P & \xrightarrow{j} & \tilde{J} & \xrightarrow{\psi_{\tilde{J}}} & J(X) \\ -q \downarrow & & \searrow -q & & \downarrow & \nearrow u_C & \\ J(X) & \xrightarrow{v_C} & P & & P & & \end{array}$$

Then we have $-qu_C \circ v_C = \psi_{\tilde{J}} \circ j \circ v_C = -q$ on $J(X)$. This implies that $u_C \circ v_C = 1$. \square

Now we study the Abel–Jacobi map associated to a family of curves on X . Let $\mathcal{C} \subset (X \times T)$ be a family of curves on X parameterized by T . The Abel–Jacobi map ψ_T associated to this family is given by sending t to the class of \mathcal{C}_t . Assume that for a general point $t \in T$, the curve \mathcal{C}_t does not meet

C and the pair (\mathcal{C}_t, C) has finitely many (generalized) secant lines. The rule associating all the (generalized) secant lines of (\mathcal{C}_t, C) to the point t defines a correspondence

$$\Psi_{T,C} : T \dashrightarrow \tilde{C}.$$

Up to the choice of a basepoint of T , one gets a morphism

$$\Psi_{T,\tilde{J}} : T \rightarrow \tilde{J}.$$

Proposition 6.8. *Let T and $\Psi_{T,\tilde{J}}$ be as above, then the following are true.*

(a) *The image of $\Psi_{T,\tilde{J}}$ is in $P = \text{Pr}(\tilde{C}, i) \subset \tilde{J}$.*

(b) *The composition $T \xrightarrow{\Psi_{T,\tilde{J}}} P \xrightarrow{u_C} J(X)$ is identified with the Abel–Jacobi map ψ_T .*

Proof. By definition, $\Psi_{T,\tilde{J}}(t) = v'_C(\mathcal{C}_t)$. Then the proposition follows easily. \square

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