Advanced evolution of helium stars and massive close binaries
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IV. DISCUSSION AND COMPARISON WITH RESULTS OF OTHER AUTHORS.

Abstract.

We investigate the differences found in stellar evolution calculations due to the computational method and due to the input physics. Furthermore, we consider the influence of these differences and of the lower mass limit for black hole formation on bulk yields of stellar nucleosynthesis.

1. Introduction.

In the following we examine some of the discrepancies found among the results of the large variety of stellar evolution computations which were undertaken since the late sixties with the aim to study the evolution of the C-O cores with masses close to the Chandrasekhar limiting mass. The general outcome of these investigations for single hydrogen stars has been described in three papers (Paczynski (1970a), Nomoto (1984a), and Müller and Arnett (1984), see section IIIC). In order to terminate with a neutron star (or black hole) the progenitor of the neutron star (or black hole) should build up a C-O core mass larger than the Chandrasekhar limiting mass. Paczynski (1970a) showed that single hydrogen stars more massive than $7 - 10 \ M_\odot$ are the likely candidates for such an evolution. Nomoto (1980, 1984a, b) inferred that in the lower part of this mass range ($8 \pm 1 - 10 \pm 1 \ M_\odot$), the stars ignite carbon and undergo an electron-capture supernova explosion afterwards, and that at higher masses the stars ignite carbon as well as neon before the core collapses. Nomoto's conclusions are based on computations of the core-helium burning phase in which the mass of the convective helium-burning core does not grow by convective overshoot. However, in Paczynski's calculations the convective helium-burning core does grow in mass by convective overshoot at the boundary of the convective helium-burning core. Also in our investigations we find that the mass of the convective helium-burning core grows. Hence, the lower mass limits for helium cores of (initially) hydrogen-rich stars required for terminating with a neutron star as derived by Nomoto ($2.5 - 2.7 \ M_\odot$, and $2.0 - 2.5 \ M_\odot$, for collapse triggered by photo-disintegration reactions and electron captures, respectively) are likely to be too high as will be shown below. The masses of the C-O cores of hydrogen stars less massive than $7 - 8 \ M_\odot$ (but more massive than $3 - 4 \ M_\odot$) grow towards a value of $1.39 \ M_\odot$ before carbon ignites.
under very degenerate conditions. Müller and Arnett (1984) argue that the precise outcome of this ignition is at present very uncertain (i.e., either a detonation or a deflagration of the core in which the burning region moves at supersonic and subsonic speed, respectively). These latter authors showed that the ignition of carbon in hydrodynamic layers has to be treated at least in two dimensions. Müller and Arnett find in their explicit numerical two-dimensional hydrodynamic calculations that 4 - 8 $M_\odot$ hydrogen stars will most likely disrupt by combustion and will give rise to a supernova explosion without leaving a neutron star (or pulsar) as supernova remnant.

In the following we will discuss the evolution of hydrogen stars more massive than 7 (- 8) $M_\odot$ and the evolution of helium and C-O stars which most likely do yield a neutron star as supernova remnant. Finally, we discuss the bulk yields of stellar nucleosynthesis from intermediate-mass and massive stars as follows from our stellar evolution calculations combined with those of others.

Before explaining some discrepancies we summarize four points on which disagreement exists in stellar evolution computations. First of all, Schwarzschild (1970) has pointed out that a semi-convective region above the convective helium-burning core has not been found by all investigators of the horizontal branch phase. Secondly, the occurrence of the second blue loop in the HR diagram of helium-shell burning stars is not found with every evolutionary code, see Alcock and Paczynski (1978). Thirdly, so far not all investigators agree about the amount of growth of the convective helium- and carbon-burning core masses in hydrogen stars in the mass range 0.9 - 20 $M_\odot$ and in helium stars with masses in the range 0.5 - 16 $M_\odot$. Finally, different results are obtained by different authors for the numbers of convective helium- and carbon-burning shells which occur during the evolution.

2. Discrepancies due to the computational method and input physics.

Paczynski (1970b) has argued that the differences between the results of different authors are mainly due to the evolutionary code which is used (i.e., computational method as well as input physics). Indeed, Becker (1981) has demonstrated that some discrepancies concerning the occurrence of the second blue loop are due to the sensitivity of the results to the input parameters (such as initial mass, choice of the mixing-length parameter, initial composition, heavy element content, and adopted opacities) and to whether or not semi-convection, convective overshoot (see Huang and Weigert (1983) and
references therein for a detailed study) and rotation are included. Also,
Fricke et al. (1971) have demonstrated that the computer code of Kippenhahn et
al. (1967) generated different results if time step was chosen differently:
small time steps caused many erratic blue loops in the calculations of a 9 M\(_{\odot}\)
hydrogen star as carried out by Hofmeister (1967) and larger time steps caused
the occurrence of only two blue loops in the 9 M\(_{\odot}\) calculations of Kippenhahn
et al. (1970), with the same input physics and computational method (see also
the calculations for a 9 M\(_{\odot}\) hydrogen star of Meyer-Hofmeister (1972)).

3. Increasing mass of the convective helium-burning core.
   a. in helium stars

   Even after the appearance of the papers of Castellani et al. (1971a, b),
Paczynski (1971a), Dinger (1972), Robertson and Faulkner (1972), Savonije and
Takens (1976), and Stothers and Chin (1977) it is not generally known that the
convective helium-burning core of a helium star or a hydrogen star increases
in mass during the core-helium burning phase. The growth of the convective
core mass is due to the occurrence of a convectively unstable zone outside the
core when the helium content of the core decreases, cf. Castellani et al.
(1971a). The occurrence of this zone is connected with the growing disconti-
nuity in composition at the boundary of the convective helium-burning core
during this evolutionary phase. This discontinuity itself gives rise to the
problem of a consistent determination of the boundary of the convective core,
see Schwarzschild (1958, chapter V, § 20, p. 167) and see also Sweigart and
Renzini (1979) for a recent discussion. In order to allow the convective core
to grow one has to include the possibility of convective overshoot in the
program. Despite the evidence of overshooting, Nomoto (1980a, b, 1981, 1984a,
b, c, e), Delgado and Thomas (1981), and Arnett (1972a) found not much if any
growth of the convective helium-burning core, because they did not include
the possibility for the growth into their codes (special techniques should be
employed to treat the problem of the consistent determination of the boundary
of the convective core and to avoid computational instabilities, see Savonije
and Takens (1976)). This leads to considerable discrepancies between their
results for the post helium-burning evolutionary phases and ours.

   b. dependencies on computational method - reason for increase in core mass
When the implicit mesh point technique (see Chapter II) was not used in
our program the convective overshoot was accounted for by inserting new mesh
points near the boundary of the convective core. In that way we obtain the
same increase in mass extent of the convective helium-burning cores as with
the implicit mesh point technique (which gives a self-consistent determination
of the boundary of the convective core). In order to obtain core-mass growth
it is essential that the grid of mesh points is frequently rezoned near the
boundary of the convective core during helium burning: test calculations with
a 2.0 $M_\odot$ helium star showed that the convective helium-burning core grows in
mass independent of whether analytic fits to radiative opacity tables were
used or interpolations in those tables were done. To this end we considered
the fits used by Arnett (1972a), by Iben (1975, 1976), and by Van der Linden
(1982), as well as the interpolations in the opacity tables as described in
Chapter II. Among the published analytical fits (of Arnett, Iben, Van der
Linden, and Stothers and Chin (1977), and references in these papers) there
are differences in the dependencies on $\rho$, $T$ and composition (i.e. on $\mu_\odot^{-1}$
$= \sum \frac{Z_i^2}{X_i/A_i}$, see Chapter II). Van der Linden used a very accurate fit and
we can show with the help of its simple dependencies that the convective
helium-burning core has to increase in mass as helium burns to carbon and
oxygen (assuming that convective overshoot leads to almost instantaneous
and complete mixing in the overshoot zones). This is because in the core the
abundance of helium decreases and the abundances of carbon and oxygen
increase, which results into an increase of $\mu_\odot^{-1}$ and of the radiative opacity,
$\kappa_{\text{rad}} ([\kappa_{\text{rad}} - \kappa_{\text{es}}] \cdot \mu_\odot^{-1}$ for $Y > 0.5$; $\kappa_{\text{rad}} \cdot \mu_\odot^{-1}$ for $Y < 0.5$; $\kappa_{\text{es}}$).
Therefore, the radiative opacity in (the outer) part of the convective core
keeps growing gradually during most of the core-helium burning phase. Hence,
the radiative gradient at the convective side of the boundary of the
convective core becomes larger during this evolutionary phase:

$$\nabla_{\text{rad}}(r) = \frac{3\kappa \rho L_r \Gamma_r}{16 \pi G T_r^4 n_r}; \text{ during most of the core-helium burning phase}$$

the changes in $\rho$, $L$, $T$, and $n$ contribute less to the change in $\nabla_{\text{rad}}$ than
the change of $\kappa_{\text{rad}}$ [the contribution of the conductive opacity $\kappa_{\text{cond}}$ to $\kappa$ is
negligible, thus $\kappa$ is equal to $\kappa_{\text{rad}}$; $\kappa_{\text{es}}$ is the dominant component to $\kappa_{\text{rad}}$ for
$Y > 0.5$, but $\kappa_{\text{es}}$ is continuous across the compositional discontinuity at the
convective boundary]. Also overshooting becomes gradually more effective at
the boundary as the compositional discontinuity grows (cf. Castellani et al.
(1971a) and Savonije and Takens (1976)) and in order to maintain convective neutrality at the convective side of the boundary this boundary is forced to enter into the radiative helium layer above it. The position of the boundary of the convective core must be determined iteratively, because it depends on the amount of helium-rich fuel which is mixed into the convective helium-burning core, cf. Savonije and Takens. Only at the end of the core-helium burning phase overshooting becomes less effective. At that time the temperature in the convective core increases faster than before. Since the radiative opacity depends strongly on $T \ [K_{\text{rad}} \sim T^{-2.665}]$ in Van der Linden's fit for $Y < 0.5$ the radiative opacity and the radiative gradient ($\nabla_{\text{rad}} = T^{-6.665}$) will ultimately decrease at the end of core-helium burning. Therefore, the growth of the convective core will stop and the core will become radiative.

c. Test calculations - fine and coarse zoning - different opacities

Having explained the physical reason for the core growth we return to our test calculations for a $2 M_\odot$ helium star. Indeed, coarse zoning may be a cause of the fixed extent in mass of the convective helium-burning cores in the calculations of Arnett (1972a). However, also fine zoning may lead to a fixed convective core mass, as a result of the employed type of computational technique. For example, with very fine, but fixed zoning at the boundary of the convective core and with Arnett's analytical fit to the radiative opacity we do not obtain much growth of the convective core mass. That this result is not correct can be illustrated with Figures 27a and 27b which schematically depict the run of $\nabla_{\text{ad}}$ and $\nabla_{\text{rad}}$ near the boundary of the convective core at a helium abundance $Y$ of 0.655 and of 0.171, respectively. In our code the Schwarzschild criterion for convective stability is used. Hence, one way to determine the boundary of the convective region is to use only this criterion in each mesh point. In Figure 27a this criterion gives a boundary near the composition discontinuity between the outermost convective mesh point of the core (point b) and the first radiative mesh point in the grid outside the core (point a). The exact place of the boundary would be given by point c at the discontinuity with this method. If the grid is not rezoned, the boundary of the convective core remains almost fixed by the first radiative mesh point at the radiative side of the boundary (point a). At $Y = 0.171$ the boundary is still at the same mesh point as at the time when $Y$ was 0.655 (see Figures 27).
Only at the very beginning of convective helium burning in the core, when the radiative side is semi-convective we obtain a small core-mass growth with a fixed grid. This radiative mesh point which is initially semi-convective, gets a gradually smaller radiative gradient in our test calculations. Probably, this is due to the strong dependency of $\kappa_{\text{rad}}$ on $T$ and to the small changes in $\rho$ and $\mu_{\text{He}}^{-1}$ (and $P$ and $L$) at the radiative side of the boundary where nuclear burning of helium is negligible. Consequently, the radiative opacity at the

Figs. 27a and 27b.—$\nabla_{\text{rad}}$ and $\nabla_{\text{ad}}$ near the composition discontinuity (marked by the vertical dashed line) as a function of mass, without convective overshoot at the discontinuity, in a $2.0\ M_\odot$ helium star with $Z = 0.03$. Point a is the first radiative mesh point outside the convective helium-burning core, point b is the last convective mesh point inside the convective core, and point c marks the exact place of the composition discontinuity. Since in point c holds: $\nabla_{\text{rad}} > \nabla_{\text{ad}}$, at the convective side of the boundary of the convective core, there will be convective overshoot, causing the composition discontinuity to move outwards in mass. The true boundary of the convective core is given by point a' where — after mixing — the boundary is precisely convectively neutral, i.e $\nabla_{\text{rad}} = \nabla_{\text{ad}}$, at the convective side of the boundary.
radiative side of the boundary \( (k_{\text{rad}} \sim \mu_s \rho_\text{p}^{-1} 0.667 T^{-2.665}) \), very roughly also for \( Y > 0.5 \) or for a Kramers' type of opacity) decreases and the extent in mass of the convective helium-burning core is fixed if the grid of zones is not altered. With this method we obtain results which agree with those of Nomoto (1981, 1984a, e) for a 2.4 \( M_\odot \) helium star and those of Arnett (1972a) for a 2 \( M_\odot \) helium star. Test calculations with the fits of Iben and with interpolation in opacity tables give similar results. However, this computational method is physically not correct (assuming almost instantaneous and complete mixing by convective overshoot), because the boundary of the convective region should be given by the point at the convective side of the boundary where \( \nu_{\text{ad}} \) exactly equals \( \nu_{\text{rad}} \). (This convectively neutral point has to be determined iteratively as matter will be mixed in from outside the convective core, see before). Hence, the correct boundary (point \( a' \)) will lie considerably beyond point \( a \) and its place is determined by the amount of fuel which is mixed inwards (in mass) by overshooting, see Figure 27a and the explanation given in the figure caption. In Figure 27b point \( a' \) lies very far from the position given by the incorrect method, leading to a very large correction to the extent of the convective core (see also Savonije and Takens, 1976).

d. semi-convection

The above picture is, however, not complete, because a semi-convective zone may form at the boundary of the convective helium-burning core, cf. Castellani et al. (1971a). The proper treatment of such a semi-convective region is not very clear and even the introduction of a semi-convective zone in the computations is weakly justified (Saio, 1974). The latter author showed that the time scale for convective mixing in a semi-convective zone is long with respect to the time scale of evolution in helium stars. The effect of this (induced) semi-convective zone on the surrounding radiative layer is small: the overshooting is negligibly small at the boundary of the semi-convective region. If this is considered only the semi-convective region would move outwards on a very large time scale (see Saio for details). However, the adjacent convective zone will move outwards on a much shorter time scale, because overshooting is much more efficient at the boundary of this zone. Therefore we expect that the induced semi-convective zone will advance outwards on the same (fast) time scale as the adjacent convective zone. Hence
the semi-convective zone will not act as a buffer for the growing convective core. Yet, a detailed and convincing analysis of the time scale of complete convective mixing is not available.

The test calculations for a 2.0 M\(_\odot\) helium star (Z = 0.03) show that if rezoning is applied frequently or if the implicit mesh point technique is used the convective helium-burning core increases in mass independent of the way the radiative opacities are calculated (i.e. from analytical fits or from interpolation in tables). However, when the convective helium-burning core is fixed in mass, semi-convection occurs only at the very beginning of core-helium burning. At core-helium exhaustion we found semi-convective zones to arise if the analytical fit of Arnett was used or if interpolation in the opacity tables was used to derive the radiative opacities for the constant convective core-mass calculations. The analytical fits of Iben to the radiative opacities of Cox and Stewart's (1970) tables (if used in these calculations), however, do not give rise to semi-convective zones in the helium-burning shell, but they were close to doing so. On the other hand, if the time step was chosen too large the semi-convective zones would have been missed. Summarizing we conclude that the occurrence of semi-convection above the convective helium-burning core is dependent on the stellar mass (see Figures 17 - 22 in section IIIB), but if the convective core is fixed in mass semi-convection is likely to occur at the very beginning of core-helium burning and at the very beginning of helium-shell burning. However, the latter is sensitive to the radiative opacities used and to the time step applied.

4. Increase of the convective helium-burning core mass obtained by other authors.

a. in helium stars

The groups (or persons) who found the convective helium-burning core mass of helium stars to grow used various Henyey type of programs. Most of these authors assumed the convective region to expand until the convective side of its boundary is convectively neutral (e.g. Paczynski (1971a), Savonije and Takens (1976), Stothers and Chin (1977)). Divine (1965a) and Savonije and Takens calculated the new composition at each iteration. Divine found a 10% growth in the convective helium-burning core mass in a 6 M\(_\odot\) pure helium star, but no growth in a 0.5 M\(_\odot\) and in a 1 M\(_\odot\) pure helium star. Paczynski allowed the convective core to expand in mass by one mesh point per time step. In that
way he obtained a core-mass growth in all helium stars in the range 0.5 - 16 $M_\odot$ (with $Z = 0.03$). Savonije and Takens (1976) calculated the evolution of 2 - 8 $M_\odot$ helium stars (with $Y = 0.97$ and $Z = 0.03$) till the end of the helium shell-burning phase or up to carbon ignition. In all their stars the convective helium-burning core grew in mass. Stothers and Chin (1977) treated the growth of the mass of the convective helium-burning cores of 4 - 15 $M_\odot$ helium stars ($Z = 0.02$ and $Z = 0.04$) with special care and derived a 10% growth in core mass in all cases, which is smaller than in the calculations of Paczynski and of Savonije and Takens. This small 10% growth can be understood as being due to the strong temperature dependence of their analytical fit to the Carson's opacities ($\kappa_{\text{rad}} \propto T^{-3.2}$), which is different from the fits used by Van der Linden (1982, for $Z = 0.03$) and from the temperature dependence of the Cox and Stewart's (1969) and Cox et al. (1965) opacities as used by Savonije and Takens and by Paczynski. Due to the strong $T$-dependence $\kappa_{\text{rad}}$ and $\nu_{\text{rad}}$ in the core are smaller in Stothers and Chin's work than in the calculations of Paczynski and of Savonije and Takens. Hence the latter three authors obtained a larger increase in mass of the convective helium-burning core than Stothers and Chin.

b. in hydrogen stars

Not only for helium stars (with $Z = 0.00 - 0.04$) but also for hydrogen stars the convective core-helium burning phase has been calculated with various evolutionary codes. With a model fitting technique Hayashi and Cameron (1962) studied this phase in a 15.6 $M_\odot$ hydrogen star. In their calculations the mass of the convective helium-burning core grew by a factor 2. With a revised program of Kippenhahn et al. (1967) Robertson and Faulkner (1972, and references to these authors in this paper) found an increase of the masses of the convective helium-burning cores of 0.9 $M_\odot$, 1.5 $M_\odot$, and 5 $M_\odot$ hydrogen stars. These authors treated the semi-convection outside the convective core with special care. After every iteration their program corrects the chemical abundances, and the (semi-) convective boundaries are allowed to extend until the convective sides of the boundaries are convectively neutral. Eggleton (1972) computed the evolution of a 14 $M_\odot$ hydrogen star with his evolutionary code. Ziolkowski (1972) and Endal (1975b) calculated the helium-burning phases of a 15 $M_\odot$ hydrogen star with Paczynski's program. Lamb et al. (1976) do find the convective helium-burning core of a 15 $M_\odot$ hydrogen star to grow in mass.
without a special treatment of semi-convection. Their result is confirmed by the calculations of Brunish and Truran (1982) with the same program, but with revised input physics. The convective helium-burning core increases in mass also if mass loss occurs (see Brunish and Truran) and even more than without mass loss [see Maeder (1981c), who obtained only a small increase in mass of the convective helium-burning core in case of no mass loss in 9 to 60 $M_\odot$ hydrogen stars, see lateron]. Eggleton (1972) pointed out that in massive stars of \~{} 15 $M_\odot$ the main source of opacity at the boundary of the convective helium-burning core is "almost entirely electron scattering". Hence, the cause of the growth in mass of the convective helium-burning core in massive hydrogen stars is thought to be different from that in helium stars of 0.5 - 10 $M_\odot$ in which a Kramers' type of opacity is the reason for the convective core-mass growth; however, only a slight discontinuity in the total radiative opacity caused by a Kramers' type of opacity may cause the convective core mass to grow during convective core-helium burning. Sweigart and Renzini (1979) give references to work in which the convective helium-burning cores of low- and intermediate-mass hydrogen stars are allowed to grow in mass extent during the horizontal branch evolution, see also Giannone and Rossi (1981). In most of this work semi-convection was specially treated. Recently, Ramadurai (1984) computed the evolution of a 16 $M_\odot$ hydrogen star up to carbon-shell burning with Eggleton's code. Ramadurai obtained a growth of the convective helium-burning core mass, but no convective helium- and hydrogen-burning shells were found after core-hydrogen burning (which disagrees with all previous work). Tornambé (1984) obtained the growth of mass of the convective helium-burning core of a 10 $M_\odot$ hydrogen star for various low metal contents ($Z = 0$ to $Z = 4 \times 10^{-4}$) with a program in which semi-convection was not specially treated. Therefore, we can conclude safely that the convective helium-burning cores of hydrogen stars will grow in mass independent of their metal content, their mass, the occurrence of semi-convection, and of mass loss, but only under the assumption of almost instantaneous complete convective mixing in the overshooted regions. In addition, they will also grow due to the increase of the total core-helium mass by hydrogen-shell burning.

5. (Semi-) convective helium shells - different opacities.

The occurrence of (semi-) convective helium shells in helium stars just after core-helium exhaustion was not found by Nomoto (1980a, b, 1981, 1982a,
b, 1984a, b, e), Nomoto et al. (1983), and Sugimoto (1970c, d, 1971). However, Arnett (1972b) and Delgado and Thomas (1981) do obtain such shells. All of these authors found no growth of the mass of the convective helium-burning core. As partly mentioned before, in our test calculations of a 2 $M_\odot$ helium star and of a 4 $M_\odot$ helium star in which the convective helium-burning cores were not allowed to grow in mass by the computational method, we obtain (semi-) convective shells if the radiative opacities derived from an analytical fit of Arnett (1972a) are used or if these opacities are interpolated from Cox and Stewart's (1969) opacities. With the analytical fits of Iben to the Cox and Stewart's (1970) opacities we do not obtain (semi-) convective shells in the 2.0 $M_\odot$ star just after convective core-helium burning. As Nomoto used these fits too we conclude that the apparent disagreement with Delgado and Thomas or Arnett is just due to the radiative opacities used and not due to the use of hydrogen-rich or helium-rich envelopes. However, our mass-loss calculations for the 2.5 $M_\odot$ helium star show that the extent of the (carbon-burning) convective shell may depend on the surface boundary condition used (see section IIIA). Hence, this condition may play a crucial role in the evolution of helium stars with helium envelopes [as in the calculations of Arnett, Delgado and Thomas, and ours], and with hydrogen envelopes [as in the work of Nomoto and Sugimoto]. However, Ramadurai (1984) found no convective helium- and hydrogen-burning shells after core-hydrogen burning using the Cox and Stewart's opacities or the analytical fit of Iben to these opacities. This result disagrees with all previous calculations of hydrogen stars of 14 - 16 $M_\odot$ (also with those of Eggleton (1972) whose code was used). Lamb et al. (1976), on the other hand, obtained a convective helium-burning shell in a 15 $M_\odot$ as well as a 25 $M_\odot$ hydrogen star which sustained longer than found in any other calculation of the evolution of such massive stars. These results prove that a yet undetected reason must exist for the disagreement, possibly it is coarseness of the grid of mesh points (the choice of a too large time step may be a reason for missing convective regions). However, the occurrence of convective helium-burning shells (not semi-convective shells) may lead to a smaller total C-O core mass for a given stellar mass in the subsequent evolution than in case these convective shells do not appear [independent of whether convective overshoot is included during convective core-helium burning or not], see also section IIIC and below.
6. Convective carbon-burning shells.

Apart from the calculations of Sugimoto (1970c, d, 1971) and Rakavy et al. (1967) all investigations of the post-core-carbon-burning stage do show the occurrence of at least one convective carbon-burning shell (see e.g. our Figures 4 and 5 in section IIIA and Figures 17 - 22 in section IIIB). Arnett (1972b) pointed out that the carbon reaction rate caused some of the differences between the result of the above-mentioned authors and others. The number of convective carbon-burning shells which develop in the phase D-E is very uncertain, but at least two of such shells are formed (see Figure 18 of section IIIB). The different result of Rakavy et al. can be due to the choice of the initial model which was incorrect as argued by Ikeuchi et al. (1971) or due to coarse mass zoning (Arnett, 1974b). Note that Nomoto (1980a, b, 1981, 1984a, b, c, e) obtained convective carbon-burning shells with the (revised) program of Sugimoto, but with different input physics.

7. Mass of the convective carbon-burning core.

Ramadurai and earlier Eggleton (1978, see Ramadurai, 1984) obtained with the evolutionary code of Eggleton (1973) a mass of the convective carbon-burning core of 1.5 $M_\odot$ for 14 - 16 $M_\odot$ hydrogen stars. The 14 - 16 $M_\odot$ hydrogen stars have convective helium-burning core masses larger than ~ 2 $M_\odot$ and the total helium core mass is ~ 4 $M_\odot$. [Why Ramadurai (1984) and Eggleton obtained such a large convective carbon-burning core mass and other authors obtained core masses of only 0.50 ± 0.25 $M_\odot$ for 14 - 16 $M_\odot$ hydrogen stars is puzzling (see Ramadurai for references)]. The results of Arnett (1972b) for 4 - 32 $M_\odot$ helium stars suggest that the extent of the convective carbon-burning core mass decreases with increasing mass of the helium star. Our results for 2 - 4 $M_\odot$ helium stars indicate that a maximum convective carbon-burning core mass of ~ 1.03 $M_\odot$ is reached by helium stars of 3 - 4 $M_\odot$. Delgado and Thomas (1981) obtained a convective carbon-burning core mass of ~ 1 $M_\odot$ for a 4 $M_\odot$ helium star (with mass loss, and with non-growing masses of the convective helium-burning cores), which agrees with our results. [We cannot reproduce the results of Ramadurai who derived a convective helium-burning core mass of ~ 2.5 $M_\odot$. Also we cannot support the solution of Ramadurai to the problem (i.e. changing the method by which the composition equation is solved), because we obtain no large discrepancies with nearly the same method. A comparison of our
equation of state with that in Eggleton's code yields differences within the accuracy of the equation of state for the \( \rho - T \) regime of helium- and carbon-burning cores and shells for helium stars in the mass range \( 2.0 - 4.0 \, M_\odot \). Thus, the formation of a large convective C-O core in the calculations of Ramadurai and Eggleton is not due to the equation of state used (see also Ramadurai, 1984); see paragraph 10 for comments on possible differences due to the opacities used.). Further, we comment that the mass of the convective helium-burning core of a \( 4 \, M_\odot \) helium star can be spuriously enlarged if the mass zoning is too coarse near the convective side of the boundary and if together with this new mesh points are inserted (with a wrong composition and density due to a wrong interpolation) and old mesh points are deleted. In this way the mass of the convective helium-burning core of our \( 4 \, M_\odot \) helium star became a factor 1.5 too large (for the erroneous sequence; mixing was not complete in the erroneously annexed zone).

8. Comparison with others.

In view of the above discrepancies we compare in Table 4 the results for \( 2 - 3 \, M_\odot \) helium stars with helium or hydrogen-rich envelopes obtained by different investigators (as listed in column 1). Column 2 of the table gives the mass of the helium star or the total helium core mass of a hydrogen star, column 3 lists the mass of the convective helium-burning core or the initial and final extent in mass of this core. The mass of the C-O core, as given in column 4, is taken to be the extent in mass of the C-O core at the time of carbon ignition (convectively as defined in paragraph 9a of this chapter) either in the centre or in a shell. Columns 5 and 6 list, when available, the mass fraction of \( ^{12}C \) at helium exhaustion in the core and the mass of the convective carbon-burning core, respectively. The values of the central temperatures and densities at ignition of carbon (in the centre or off-centre) are listed in columns 7 and 8, respectively. In column 9 we list whether central or off-centre ignition of carbon occurs or if carbon does not ignite before the C-O core has grown to \( 1.4 \, M_\odot \). The latter ignition is expected to occur under very degenerate conditions at high densities \( \sim 10^9 \, g \, cm^{-3} \), however, see section IIIIC). The superscripts (a) and (b) in column (2) denote whether or not neutrino-Bremsstrahlung emission was omitted. If we do not know whether such neutrino-emission was included, we used no superscript. In Table 4 we report the results of test calculations with a \( 2 \, M_\odot \) helium star (without
neutrino-Bremsstrahlung emission).

Table 4. Comparison of some characteristic parameters of helium stars and helium cores of 2 - 3 $M_\odot$ during the helium- and carbon-burning phases (detailed explanation in the text).

<table>
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<th>references</th>
<th>mass of the He star or core of the He-burning core</th>
<th>mass of the C-O core</th>
<th>mass fraction of $^{12}\text{C}$ in the He-burning core</th>
<th>mass of the conv. C-O core</th>
<th>ignition of carbon at $^{12}\text{C}$-burning $T_C$</th>
<th>ignition type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sugimoto (1970d)</td>
<td>3.5($a$)</td>
<td>1.25</td>
<td>1.7</td>
<td>.5</td>
<td>.7</td>
<td>4.54</td>
</tr>
<tr>
<td>Paczynski (1971a)</td>
<td>2.6($a$)</td>
<td>.66 ± .03</td>
<td>.52</td>
<td>-</td>
<td>-</td>
<td>central</td>
</tr>
<tr>
<td>Arnett (1972a, b)</td>
<td>2.2-2.6($b$)</td>
<td>.99</td>
<td>.99</td>
<td>.466</td>
<td>-</td>
<td>off-centre</td>
</tr>
<tr>
<td>Savonije and Takens (1976)</td>
<td>2.5($a$)</td>
<td>.90 ± 1.2</td>
<td>1.23</td>
<td>.463</td>
<td>-</td>
<td>central</td>
</tr>
<tr>
<td>Nomoto (1984a, b, c, e)</td>
<td>2.2-2.6($b$)</td>
<td>.88</td>
<td>1.15</td>
<td>.67</td>
<td>&lt;1.35</td>
<td>off-centre</td>
</tr>
<tr>
<td>Nomoto (1984a, b, c, e)</td>
<td>2.4-2.6($b$)</td>
<td>1.02</td>
<td>1.23</td>
<td>.67</td>
<td>&lt;1.35</td>
<td>off-centre</td>
</tr>
<tr>
<td>Nomoto (1984a, b, c, e)</td>
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<td>1.10</td>
<td>1.37</td>
<td>.67</td>
<td>&lt;1.35</td>
<td>off-centre</td>
</tr>
<tr>
<td>Delgado and Thomas (1981)</td>
<td>2.0</td>
<td>.62</td>
<td>.94</td>
<td>-</td>
<td>-</td>
<td>no ignition</td>
</tr>
<tr>
<td>Delgado and Thomas (1981)</td>
<td>2.7</td>
<td>.98</td>
<td>1.20</td>
<td>-</td>
<td>-</td>
<td>off-centre</td>
</tr>
<tr>
<td>Woosley et al. (1984)</td>
<td>2.7($a$)</td>
<td>.73</td>
<td>1.10</td>
<td>.60</td>
<td>-</td>
<td>off-centre</td>
</tr>
<tr>
<td>this paper</td>
<td>2.6($a$)</td>
<td>.66 -.98</td>
<td>1.16</td>
<td>.55</td>
<td>.66</td>
<td>5.66</td>
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<td>1.48</td>
<td>.49</td>
<td>.74</td>
<td>6.13</td>
</tr>
</tbody>
</table>

(a) without neutrino-Bremsstrahlung emission
(b) with neutrino-Bremsstrahlung emission.

9a. Minimum masses for central and off-centre ignition of carbon, neon, and oxygen.

In the following we discuss the values found in different investigations for the minimum mass, $M_{\text{min}}$, for a C-O core to ignite carbon in the centre or in a shell. Several definitions for the ignition of carbon exist in the literature. Among them are: carbon ignites if the energy losses by neutrino-emission outweigh the energy production of the carbon-burning reactions (assuming e.g. equal abundances of $^{12}\text{C}$ and $^{16}\text{O}$), or carbon ignites if the carbon-burning reaction rate becomes non-negligible, or carbon ignites in a layer when carbon burning is strong enough to make that layer convectively unstable. The latter definition is adopted in Table 4 and in the following.

With an explicit method Murai et al. (1968) and Ikeuchi et al. (1971) found $M_{\text{min}}$ for ignition of carbon in a shell to be 1.06 $M_\odot$ for stars consisting of 50% $^{12}\text{C}$ and 50% $^{16}\text{O}$. Kutter and Savedoff (1969) obtained a value
of $1.08 \, M_\odot$ for $M_{\text{min}}$ to ignite in a shell with pure carbon stars. Beaudet and Salpeter (1969) derived values for $M_{\text{min}}$ in the range $0.97 - 1.06 \, M_\odot$ depending on the adopted reaction rates. Iben's (1972, 1973) $9 \, M_\odot$ hydrogen star ignited carbon in a shell at $m_\star \sim 0.50 \, M_\odot$. Boozer et al. (1973) estimated the lower limit for stars consisting of $50\%^{12}\text{C}$ and $50\%^{16}\text{O}$ to ignite carbon in a shell to be $1.03 \, M_\odot$ and the lower limit for these stars to ignite carbon in the centre to be $1.15 \, M_\odot$. The results of Boozer et al. are confirmed by those of Joss et al. (1973) who used helium-rich envelopes for these stars. Kutter and Savedoff, however, obtained ignition of carbon in the centre if the pure carbon stars were more massive than $1.35 \, M_\odot$. Barkat et al. (1974) studied the evolution of an $8 \, M_\odot$ hydrogen star using the same program as Boozer et al. and Joss et al. with updated physics. When the C-O core mass of the $8 \, M_\odot$ star was about $1.25 \, M_\odot$, carbon ignited in the centre. Strittmatter et al. (1970) pointed out, however, that this program (a version of the Rakaev et al. program (1967)) complicates the test for convective instability in a convective zone. Also, Strittmatter et al. argued that Kutter and Savedoff used an incorrect equation of energy conservation (see e.g. Kutter and Savedoff, 1967) which explains the deviating results of these last-mentioned authors, cf. Sugimoto (1971). However, Barkat et al. most probably derived such a low mass for carbon ignition in the centre, because they chose a low $^{14}\text{N}$ abundance ($0.0014$) for the CNO cycle and a low $Z$ value ($0.01$) in the calculations of the $8 \, M_\odot$ star, see e.g. Havazelet and Barkat (1979). Moreover, Barkat et al. excluded the electron conductivity (but it is likely that they included the neutrino-Bremsstrahlung emission rates).

Lamb and Van Horn (1975, including neutrino-Bremsstrahlung emission and using the revised evolutionary code of Kutter and Savedoff (1969)) found that a $1 \, M_\odot$ pure carbon star did not ignite carbon. In their calculations revised opacities were used (which were used first by Iben (1972) and later by other investigators). The results of Savonije and Takes (1976) for a $2 \, M_\odot$ helium star indicate that this star ignites carbon off-centre as found previously by Paczynski (1971a). The mass of the C-O core of the $2 \, M_\odot$ helium star was at least $0.99 \, M_\odot$ (see Savonije and Takes) at helium exhaustion in the core. The $2.5 \, M_\odot$ helium star of Savonije and Takes ignited carbon in the centre when the C-O core mass was at least $1.23 \, M_\odot$.

Ergma and Vilhu (1978) reported that the $1.15 \, M_\odot$ C-O core of a $9 \, M_\odot$ hydrogen star ignited carbon in a shell. Nomoto (1980a, b, 1982a, 1984a, e) finds that carbon is ignited off-centre in the C-O cores of $2.2 - 2.4 \, M_\odot$ and
2.4 - 2.6 $M_\odot$ helium stars ($Y = 0.98$ and $Z = 0.02$) with hydrogen-rich envelopes ($X = 0.602$ and $Z = 0.02$; due to hydrogen-shell burning the helium core grows in mass by about 0.2 $M_\odot$ from core-helium burning up to about core-carbon burning in the calculations of Nomoto). The masses of the C-O cores of these stars at carbon ignition are 1.20 $M_\odot$ and 1.25 $M_\odot$, respectively. The masses of the C-O cores are affected by the $^{12}$C and $^{16}$O abundances and by the neutrino loss rates adopted. The effects of Coulomb interaction are non-negligible in ~ 2.0 - 2.6 $M_\odot$ helium stars (the exact mass range depends also on the adopted neutrino losses). Nomoto (1982a, b, 1984a, and references therein) does give (almost) all the details of the input physics to the evolutionary code employed. We conclude that only the reaction network used may be different from ours, i.e. we treat the abundance of $^{23}$Na in another way (but also as a nearly "steady state" abundance). Maeder (1981b, c), Maeder and Mermilliod (1981) and Maeder and Lequeux (1982) describe the evolution of a 9 $M_\odot$ hydrogen star (with $Z = 0.03$) up to the end of the carbon-burning stage. Maeder and co-authors used physics updated similarly to that of ours (see Maeder, 1981a). The 9 $M_\odot$ star ignited carbon in the centre independent of whether or not mass loss was taken into account. The Schwarzschild criterion for convective mixing was used by Maeder and collaborators (and convective overshoot at the convective boundary of the helium-burning core was not included). The $^{12}$C and $^{16}$O abundances by mass at ignition of carbon obtained by these authors were 0.629 and 0.357, respectively. Probably, the high carbon abundance in the core causes carbon to ignite in the centre; these abundances agree with our results for a 2.0 $M_\odot$ helium star without convective overshoot (see section IIIB). The discrepancy with the 9 $M_\odot$ hydrogen stars of Ergma and Vilhu is probably due to the use of different triple-alpha and $^{12}$C($\alpha, \gamma$)$^{16}$O reaction rates which gives different $^{12}$C and $^{16}$O abundances at the end of core-helium burning [see e.g. Lamb et al. (1976) and section IIIB]. The mass of the C-O core cannot be inferred from the papers of Maeder and co-authors. The mass of the convective helium-burning core was at maximum extent ~ 0.60 $M_\odot$ without mass loss and ~ 0.68 with mass loss in the 9 $M_\odot$ hydrogen star, which implies that the evolution of this star should be similar to that of a ~ 2 $M_\odot$ helium star (with $Z = 0.03$). However, the mass of the convective helium-burning core of the 9 $M_\odot$ hydrogen star does not grow as much as in our corresponding 2 $M_\odot$ helium star. The mass of the convective helium-burning core of the 9 $M_\odot$ star grows due to the increase of the total interior helium content by hydrogen shell-burning, but it may also grow due to convective overshoot at the boundary of convective
helium-burning core. Probably, the latter effect is not properly included in the calculations of Maeder and co-workers. This leads to a smaller C-O core mass at the end of core-carbon burning (~ 1.31 \( M_\odot \) in case of no mass loss, see Maeder, 1981c) than as expected from our calculations for a 2.0 \( M_\odot \) helium star. The central Ne, O and Mg abundances at the end of core-carbon burning (0.40, 0.21, and 0.36, respectively) are also much different from those in our work (see e.g. Table 6 in section IIIB). This is due to the high C and low O abundances obtained after core-helium burning, which in turn is the result of the small extent in mass of the convective helium-burning core: in section IIIB we found that the smaller the mass of the convective helium-burning core the higher the C and the lower the O abundance will be for a 2 \( M_\odot \) helium star. Maeder (1981b) used the minimum reaction network of Arnett (1974c) in which the proton channel of the carbon-burning reaction is included, but in which the \(^{24}\text{Mg}\) production is underestimated and the \(^{20}\text{Ne}\) production is overestimated, because the reaction \( ^{23}\text{Na} + ^{1}\text{H} \rightarrow ^{24}\text{Mg} + \gamma \) is omitted in that network, see section IIIB. However, the under- and over-production is only slightly as argued in the following. For the 15 \( M_\odot \) and 30 \( M_\odot \) hydrogen stars without mass loss Maeder (1981c) obtains central abundances of \(^{20}\text{Ne}\) and \(^{16}\text{O}\) at the end of carbon burning in the core comparable to those of ours in that stage. On the other hand the \(^{12}\text{C}\) and \(^{16}\text{O}\) abundances at the beginning of core-carbon burning in these stars are the same as in our 2.0 - 4.0 \( M_\odot \) helium stars. Hence, the minimum reaction network for carbon-burning of Arnett yields abundances comparable to those from our more detailed network.

Delgado and Thomas (1981) obtain a C-O core mass of 0.94 \( M_\odot \) for their 2 \( M_\odot \) helium star, which is too small to ignite carbon before the densities have increased to \( \sim 10^9 \text{ g cm}^{-3} \). This result agrees with the calculations for a 2 \( M_\odot \) helium star as done by Arnett (1972a, b), see Table 4. The 2.7 \( M_\odot \) helium star of Delgado and Thomas ignited carbon in a shell when the mass of the C-O core reached 1.20 \( M_\odot \). We may assume that these last-mentioned authors updated the input physics in the program of Kippenhahn et al. (1967). Hence, the agreement between the results of Nomoto and Delgado and Thomas as far as the minimum masses for ignition of carbon are concerned is most probably due to the almost constant mass of the convective helium-burning core obtained by these authors. The lack of agreement between the results of these authors and those of Savonije and Takens, of Paczynski (1971a), and of ourselves can be attributed to the neglect of overshooting in the core-helium burning phase by Delgado and Thomas and by Nomoto, see Table 4 and section IIIB.
9b. Minimum mass of the C-O core for off-centre carbon ignition.

At first glance there is essentially no agreement among the results obtained by different authors for hydrogen stars of \(7 - 10\ M_\odot\) during the helium- and carbon-burning stages. Part of this discrepancy is due to the different metal abundances \(Z\) that were used as was shown by Alcock and Paczynski (1978) and partly it is due to the use of different CNO cycle abundances \(Z_{\text{CNO}}\) as was argued by Havazelet and Barkat (1979, see also Bodenheimer et al., 1965). According to Becker and Iben (1980) \(M_{\text{min}}\) of the C-O core for ignition of carbon in a shell under non- or weakly degenerate conditions is \(1.005\) to \(1.060\ M_\odot\), a value which depends on \(Z\) and \(Y\); Iben and Tutukov (1984c) obtained \(\sim 1.08\ M_\odot\) for a pure C-O star as estimated from results of Becker and Iben (1979). Nonetheless, even with the same \(Z\) value, Maeder and co-authors (see above) and Ergma and Vilhu (1978) do find in a 9 \(M_\odot\) hydrogen star central and off-centre ignition of carbon, respectively. Hence, the input physics (e.g. reaction rates and corrections for neutral current effects) and computational technique must cause some of the differences found. From Becker and Iben (1980) we infer that \(M_{\text{min}}\) reaches its lower limit for high \(Z\) values \((Z > 0.03)\) and its upper limit for low \(Z\) values \((Z < 0.02)\), cf. Alcock and Paczynski. Biermann and Kippenhahn (1971) derived roughly that carbon does not ignite in C-O cores of helium stars with helium envelopes if the C-O core mass is smaller than \(0.95\ M_\odot\), but ignites only under degenerate conditions if the C-O core mass has grown to \(\sim 1.44\ M_\odot\).

Recently, Tornambé (1984) explored the evolution of low-metal \(10\ M_\odot\) hydrogen stars \((Z = 0, 10^{-8}, 10^{-6}, 10^{-5}\) and \(4 \times 10^{-4}\)) from hydrogen burning to carbon ignition. Tornambé found all \(M_{\text{min}}\) values to be larger than in Becker and Iben's, and Alcock and Paczynski's computations; also he found an increase of \(M_{\text{min}}\) with decreasing \(Z\); however, for \(Z < 10^{-8}\), \(M_{\text{min}}\) decreases again. As a result the original mass of the hydrogen star which ignites carbon in a shell is at minimum \(8\ M_\odot\) for \(Z = 0.03\) and at maximum \(\sim 9.5\ M_\odot\) for \(Z = 10^{-8}\). In the calculations of Tornambé the convective helium-burning core grows in mass during the core-helium burning phase. Hence, the results of Tornambé are a consistent extension of the results of Alcock and Paczynski towards low metal content. However, Tornambé (and Alcock and Paczynski) used neutrino loss rates uncorrected for neutral current effects. Moreover, Tornambé omitted the neutrino-Bremsstrahlung emission just like Boozer et al. (1973), Joss et al. (1973), Paczynski (1971a), Savonije and Takens (1976), Arnett (1972a, b),
Murai et al. (1968), Ikeuchi et al. (1971, 1972), Kutter and Savedoff (1969), Sugimoto (1971), and Rakavy et al. (1967). Since Tornambe and Boozer et al. used the same neutrino loss rates and obtained different values for the minimum mass of the C-O core that is required to ignite carbon in the centre (> 1.27 M_☉ and > 1.15 M_☉, respectively), we conclude that the minimum masses are sensitive to the other input physics used. First of all the different chemical abundances prior to carbon ignition can account for differences, because the reaction rate for carbon burning depends on \( \chi_C^2 \). However, the agreement between the evolutionary tracks of these authors becomes worse in the electron degenerate regime (\( \log \rho > 6.4 \)) which may be due to the possible omission of the correction for relativistic and electron degeneracy effects on the photon-electron scattering opacity, \( \kappa_{es} \), by Tornambe or due to the different conductive opacities adopted. On the other hand, Boozer et al. corrected the electron-scattering opacity \( \kappa_{es} \) with the formulae of Beaudet and Salpeter (1969). Particularly, the empirical fit of these latter authors to the factor by which \( \kappa_{es} \) has to be corrected for electron degeneracy effects, can be larger by 20% than the fit as adopted here in the relevant range (\( T < 10^9 \) K, \( \rho < 10^7 \) g cm\(^{-3}\)). Moreover, this correction can be of the same order of magnitude as the correction for the relativistic effects. Sampson (1961) derived nearly the same corrections for the electron degeneracy effect as we adopted (i.e. within 10%). We note also that Tornambe did not include the electron conductivities corrected for relativistic electron degeneracy effects as given by Canuto (for an extra correction see Iben (1975)).

9c. Minimum mass of the C-O core for central ignition.

Despite their differences in origin we conclude that C-O stars, pure carbon stars, and helium stars with either helium-rich or hydrogen-rich envelopes have about the same minimum masses for off-centre ignition of carbon as stars with extended hydrogen envelopes. The minimum mass for central ignition of carbon is equally uncertain for all these stars. The lower limit to the C-O core mass for central ignition is \( < 1.15 \) M_☉ as obtained by Boozer et al. for stars composed of 50% ^12C and 50% ^16O, and \( > 1.25 \) M_☉ (but \( < 1.37 \) M_☉) as inferred from the calculations of Nomoto (1980a, b, 1981, 1982a, b, 1984a, e) for helium stars (with \( Y = 0.98 \) and \( Z = 0.02 \)) with hydrogen-rich envelopes, or even \( > 1.27 \) M_☉ for a 10 M_☉ hydrogen star with \( Z = 0 \) (Tornambe, 1984). The results of Delgado and Thomas (1981) lead to a lower limit \( > 1.20 \)
$M_\odot$ (but $<1.50\, M_\odot$) for C-O cores of helium stars with helium envelopes. Test calculations for a $2\, M_\odot$ helium star with a helium-rich envelope ($Y = 0.97$ and $Z = 0.03$) show that this lower limit is very sensitive to the input physics. For example: to the factors by which the neutrino emission rates have to be corrected for neutral current effects, to corrections to the equation of state of ions, and to the neutrino-Bremsstrahlung emission rate, see Table 4. From our calculations we deduce that the lower limit for central ignition of carbon under non- or weakly electron degenerate conditions is larger than $1.195\, M_\odot$, but smaller than $1.315$ (with $Y = 0.97$ and $Z = 0.03$ in the initial helium star). This result agrees with the calculations of Delgado and Thomas and of Nomoto. However, if neutrino-Bremsstrahlung emission is omitted the lower limit is about $1.16\, M_\odot$, which agrees with the results of Boozer et al. (who omitted neutrino-Bremsstrahlung).

10. **Uncertainty in the mass extent of the convective carbon-burning core and of the C-O core; the roles of the opacity, the neutrino-emission rate, the nuclear reaction rates, and the neglect of overshooting.**

Ramadurai (1984) investigated the sensitivity of the mass of the convective carbon-burning core to the neutrino-emission rates adopted. If these rates are corrected for neutral current effects the mass of the convective carbon-burning core of a $16\, M_\odot$ hydrogen star is increased by less than 1%. However, Ramadurai did not correct the neutrino-Bremstrahlung emission rate. We find the mass of the C-O core of $2.0 - 2.2\, M_\odot$ helium stars to be uncertain by 2%: the corrected neutrino-loss rates give a 2% larger C-O core mass than the uncorrected ones (including neutrino-Bremsstrahlung).

Ramadurai considered also the role of the opacity in the evolution of the $16\, M_\odot$ hydrogen star. Ramudurai calculated tracks with the radiative and conductive electron opacities from Paczynski's tables and from analytical fits by Iben (1975). Differences in opacities between the tables and these fits occur mainly for the conductive opacity in the relativistic regime ($\rho > 10^6\, g\, cm^{-3}$). In Paczynski's tables the opacities of Canuto (1970) are not included (or these opacities are not correctly programmed as in the version presented to Savonije and Takens (1976)). Iben (1975) corrected for an error in the conductive electron opacities obtained by Canuto. We found that the expression for $Z_a$ in equation A26 of Iben (1975) should be multiplied by $\mu_a^{2/3}$ to give a better fit. However, in the calculations of Ramudurai the central density was
always less than $10^6 \text{ g cm}^{-3}$. Finally, Iben and Renzini (1982a, b) give the range of temperatures ($T > 10^7 \text{ K}$) for which the fits to the radiative opacities of carbon-rich mixtures are made, see also Iben (1976). Since the opacities for the C-O core are calculated (not extrapolated) well within the allowed range of the fits of Iben, we ascribe the large difference found by Ramadurai for the mass of the convective C-O core derived from these tables and these fits (1.497 $M_\odot$ and 1.354 $M_\odot$, respectively) to an error in the programming of Iben's formulae or to the omission of the correction to the formula for $\kappa_{ee}$ as given by Iben (1976).

Nomoto (1980a, b, 1982a, b, 1984a, e) used Iben's analytical fits to the opacities and similar electron scattering opacities as used by us. Apart from the reaction network used all of the input physics of Nomoto is almost similar to that of ours. Nomoto neglected convective overshoot at the boundary of the convective helium-burning core. In that way Nomoto obtained different abundances for carbon and oxygen at helium exhaustion in the core of 2.0 - 2.6 $M_\odot$ helium stars. Test calculations with a 2.0 $M_\odot$ helium star give 10% higher carbon abundances if overshooting is neglected (i.e. if the convective helium-burning core mass is fixed). The low temperatures and densities at ignition of carbon (columns 7 and 8 of Table 4) found by Nomoto indicate a higher central abundance of carbon as a result of core-helium burning in the calculations of Nomoto than in ours. Also the neglect of overshooting implies that in Nomoto's work the original mass of the helium star has to be larger than in our work in order to give the same C-O core mass. Inspection of the figures of the evolution of the interior regions as given in Nomoto's papers reveals that the C-O core mass grows more (after helium exhaustion in the core) in the calculations of Nomoto than in ours. Nevertheless, we find good agreement between Nomoto's and our results on the carbon-ignition temperatures and densities (in case neutrino-Bremsstrahlung emission is included).

The agreement of our results with those of Savonije and Takens is also good in case we excluded neutrino-Bremsstrahlung. Notice that the 2.6 - 2.79 $M_\odot$ helium star of Nomoto (1984a, e) has a quite similar evolution of the interior regions as our 2.2 $M_\odot$ helium star: we obtain a C-O core mass > 1.475 $M_\odot$ just prior to neon ignition and Nomoto finds a value of 1.45 $M_\odot$. However, the time scale of carbon burning in the convective carbon-burning core of the 2.4 $M_\odot$ helium star of Nomoto agrees better with that of our 2.2 $M_\odot$ helium star. The differences in time scales as found here (see section IIIB, paragraph 5iii, and in Table 1 of Lamb et al. (1976)) is caused by the input physics adopted
(especially the neutrino-Bremsstrahlung emission rate and the nuclear reaction rates for carbon burning, cf. Lamb et al.).

Although the convective helium-burning cores grow in mass in the calculations of Tornambe (1984) for a 10 $M_\odot$ hydrogen star with low $Z$ content, their results differ from ours. The main reason of the discrepancy is the omission of neutrino-Bremsstrahlung by Tornambe; an additional contribution comes from the use of different opacities. Tornambe obtained low abundances of carbon ($\sim 0.3 - 0.4$) at helium exhaustion in the core. In order to ignite carbon with these abundances the temperature and densities in the core have to be higher than in the calculations in which carbon is more abundant. Hence, carbon ignition occurs later in their calculations; at that time the C-O mass is larger.

Recently Woosley et al. (1984) calculated the evolution of a 2 $M_\odot$ pure helium star from helium burning up to the deflagrative burning from the centre outwards and the forced explosion. However, the ignition of neon in a shell as obtained by Woosley et al. is not correct: Woosley et al. [see Weaver et al. (1984) for the input physics] included strong screening effects for the neon-disintegration reaction rate, but the effect of screening is negligible for this rate, cf. Reeves (1965) and Nomoto (1984d). The characteristic parameters of this 2 $M_\odot$ star are listed in Table 4. Woosley et al. found off-centre ignition of carbon at $m_T \sim 0.35 M_\odot$ at a central temperature of $4 \times 10^8$ K and at temperatures and densities in the shell of $7 \times 10^8$ K and $3 \times 10^5 \text{ g cm}^{-3}$, respectively. The $^{12}$C and $^{16}$O abundances after core-helium burning were .60 and .40, respectively. The radiative O-Ne-Mg core (formed after the convective carbon-burning shell moved inwards to the centre) had central abundances of $^{16}$O, $^{20}$Ne, and $^{24}$Mg of .15, .60, and .25, respectively. These abundances of the C-O and O-Ne-Mg cores are the highest found so far in the literature for a 2 $M_\odot$ pure helium star. They are derived with approximate reaction networks for carbon and neon burning (see Weaver et al., 1978). With our approximate networks we obtain abundances of $^{12}$C and $^{16}$O which are close to those of Woosley et al., the difference is partly due to our non-zero metal content ($Z = 0.03$) and mainly due to including convective overshoot in our calculations (see section IIIB). If we compare our $^{16}$O, $^{20}$Ne, and $^{24}$Mg abundances with those of Woosley et al. at central carbon depletion (see Table 6 of section IIIB) we find reasonable agreement (i.e. also a high central $^{20}$Ne content). Therefore we expect that our approximate reaction network for carbon burning yields realistic abundances even though the $^{23}$Na abundance is grossly underestimated.
11. Minimum core masses for ignition of neon and oxygen.

In order to ignite neon, the C-O core mass has to be larger than 1.37 $M_\odot$ according to Boozer et al. (1973) and according to the calculations for pure neon stars of Nomoto (1984a). This value may be rather uncertain, because Boozer et al. derived it from computations for C-O stars in which neutrino-Bremsstrahlung was neglected and Nomoto derived it with this neutrino emission included. Arnett (1974a) estimated that the neon-rich core has to be larger than $\sim 1.30 M_\odot$ in order to ignite neon (either in a shell or in the centre, assuming a polytropic structure of the star). Furthermore, Arnett (1978b) and Boozer et al. estimated that the oxygen-rich core mass has to be larger than 1.39 $M_\odot$ in order to ignite oxygen. The results for a 2 $M_\odot$ pure helium star of Woosley et al. (1984, who also neglected neutrino-Bremsstrahlung) indicate that the O-Ne-Mg core mass has to be $\geq 1.3 M_\odot$ and $\leq 1.43 M_\odot$ in order to ignite neon and oxygen in a shell. However, these latter results are based on an erroneously enhanced neon-disintegration rate (Nomoto, 1984b). Nomoto (1980a, b, 1982a, b, 1984a, e) found that a 2.6 - 2.8 $M_\odot$ helium star ignites neon off-centre, and that a 2.4 - 2.6 $M_\odot$ helium star with a C-O core of 1.34 $M_\odot$ does not ignite neon. The C-O core mass in the 2.6 - 2.8 $M_\odot$ helium star was $\sim 1.45 M_\odot$ at neon ignition. Since Nomoto used (corrected) neutrino-Bremsstrahlung emission rates we infer that the lower mass limit to the C-O core to ignite neon is in the range 1.34 - 1.45 $M_\odot$. Nomoto (1984a) found that the mass of the O-Ne-Mg core (limited by the shell of maximum energy generation by carbon-burning) of the 2.6 - 2.8 $M_\odot$ helium star has to be larger than 1.37 $M_\odot$ in order to ignite neon. Furthermore, Nomoto (1984a) argued that the lower mass limit for the C-O cores to ignite neon is the same as for a pure neon star. This is due to the steepness of the density gradient near the helium-burning shell which limits the C-O core and near the outer boundary of the neon star. Apparently in contrast, Ikeuchi et al. (1972) found that the minimum masses for an O-Ne-Mg core to ignite neon and oxygen are as low as 1.10 $M_\odot$ and 1.15 $M_\odot$, respectively. However, these values are derived from combining polytropic star models of index 3 and evolutionary models of stars composed of 50% $^{12}$C and 50% $^{16}$O with the smallest C-O stellar mass being 1.5 $M_\odot$. Furthermore, the C-O burning shell in the latter star was at a mass coordinate of 1.39 $M_\odot$ at the time of neon ignition (see figure 5 of Ikeuchi et al., 1972). Hence, the density gradient should be very sharp at that coordinate.

The results of Nomoto concerning the minimum mass for neon ignition are
confirmed by our calculations of a 2.2 $M_\odot$ helium star, which ignites neon off-centre (at $m_r \sim 0.75 M_\odot$) after the 0-Ne-Mg core mass has become larger than 1.37 $M_\odot$ (see Figure 18 of section IIIB).

The minimum masses for ignition of neon and oxygen are less than the (Newtonian) Chandrasekhar limiting masses for pure neon and oxygen stars (i.e. less than 1.4181 $M_\odot$ and $\sim 1.41 M_\odot$, cf. Ibanez Cabanell, 1984). The critical masses in the Einsteinian theory of gravity for pure neon stars is about 1.37 $M_\odot$ and for pure oxygen stars 1.3796 $M_\odot$ (Ibanez Cabanell).

12. On bulk yields of stellar nucleosynthesis.
   a. from intermediate-mass stars

Arnett (1978a) estimated the amount of synthesized matter of hydrogen stars more massive than $\sim 10 M_\odot$ using stellar evolution calculations of pure helium stars with masses in the range 3 to 48 $M_\odot$. These helium core masses have to be related (by a transformation) to the initial main-sequence mass in order to couple them to the initial mass function and to obtain the death rate of the main-sequence stars. Arnett (1978a) derived a transformation which gives low initial main-sequence masses for a given total helium core mass and which may be appropriate for single hydrogen stars only. Such a transformation can be affected in many ways, for example by stellar wind mass-loss effects or by mass exchange in binaries, see above and Arnett (1978a, 1972a), Maeder (1981b, c), and see section IIIC. In the following we adopt that helium stars with masses in the range $2 - 4 M_\odot$ have descended from hydrogen stars which are 4 times more massive. This transformation gives probably an upper limit to the initial single hydrogen-star mass. Arnett (1978a) finds that the limiting mass for the helium star required for reaching collapse is $\sim 3.5 M_\odot$ (i.e. $\sim 11 M_\odot$ for the hydrogen-star mass according to Arnett, 1978a). The low-metal content calculations for a 10 $M_\odot$ hydrogen star by Tornambe (1984) do indicate a limiting mass for collapse of single hydrogen stars which is in the mass range 8 to 9.5 $M_\odot$, as these stars will ignite carbon in a shell. The value of 9.5 $M_\odot$ is the upper mass limit attained at $Z = 10^{-8}$. This limiting mass will be even smaller if the neutrino losses are properly included in the evolutionary calculations. Hence, single hydrogen stars with masses as low as 9.5 $M_\odot$ are expected to undergo core collapse. If, furthermore, the effect of the increasing mass of the convective helium-burning core is included, the bulk yield of chemical elements by nucleosynthesis in intermediate-mass stars is

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markedly different from that obtained by Arnett (1978a).

Especially in the mass range $10 - 15 \, M_\odot$ we expect more carbon and oxygen to be formed than predicted by Arnett. However, the helium abundance in the ejectae of the supernova remnants might be even lower, i.e. if the effects of penetration of the hydrogen shell into the helium layer in single hydrogen stars (which do increase the helium abundance) are neglected, see Maeder (1981b). Hence, the effects of the increased convective helium-burning core mass and of the penetration of the hydrogen shell, on the helium abundance do not allow us to predict the helium abundance in the ejectae. Although Weaver et al. (1978), Thielemann and Arnett [1984, also for an increased rate of the reaction $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$], and we (see e.g. Table 6 in section IIIB) obtain enhanced abundances for Ne, Na, Mg, and Si in the intermediate-mass stars relative to the massive stars, the result of lowering the lower limit to core collapse will be an underproduction in the bulk yield of $^{20}\text{Ne}$ relative to that of Arnett (1978a). This may remove the difference between the solar abundances of chemical elements and those predicted by nucleosynthesis from massive stars for various initial mass functions (for main-sequence star masses $> 10 \, M_\odot$).

Hence, possibly explosive reprocessing may not be required to explain that difference. On the other hand, the enhancement of Ne, Na, Mg, and Si may be simply an effect of the uncertainties in the carbon- and sodium-burning reaction rates and may be removed by a small increase of the temperature of the carbon-burning shell according to Thielemann and Arnett (1984). We stress also that the bulk yield will be crucially determined by the transformation of helium core mass to initial main-sequence star mass (see Arnett, 1978a). The yield of intermediate-mass nuclei by nucleosynthesis in explosive oxygen burning is most probably not determined by the poorly known rate of the reaction $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$, except for the ratio of carbon to oxygen [see e.g. the discussion following the paper of Woosley et al. (1984) and below, although increasing that rate can give almost solar abundances in the pre-explosive yield of intermediate-mass nuclei (Thielemann and Arnett, 1984)].

b. from massive stars

The oxygen content in the ejectae of exploding massive stars is very large (i.e. $> 6 \, M_\odot$ for a 35 $M_\odot$ hydrogen star and up to 30 $M_\odot$ for a 95 $M_\odot$ hydrogen star, see Arnett (1978a) and also Weaver et al. (1978) and Woosley et al. (1984)). The remnant core mass is small, of order $M_{\odot}$, and at most $\leq 2.2$
for a 75 $M_\odot$ hydrogen star (see Arnett, 1978a). Hence, if the lowest progenitor mass for a black hole is 40 ($\pm$ 5) $M_\odot$ [see section IIID, Van den Heuvel and Habets (1984) and Schild and Maeder (1985)] and if these black holes are indeed as massive as $\sim$ 10 $M_\odot$ or more, then the production of intermediate-mass nuclei (and particularly oxygen) by nucleosynthesis from massive stars is much less than predicted by Arnett (1978a). Thus, the discrepancy with solar abundances would be even stronger. However, a higher upper limit to the progenitor mass of a neutron star, of 60 to 80 $M_\odot$ (as may be possible, see Van den Heuvel and Habets, 1984), will only slightly alter the yield. This is due to the dominant contribution to the bulk yield by stars of $\sim$ 30 $M_\odot$ (Arnett 1978a, Maeder, 1981b) which have abundance ratios close to solar [however, see also Weaver et al. (1978) and Woosley and Weaver (1982b)]. This yield is reduced for the heavy elements if mass loss by stellar winds is included (Maeder, 1981c). According to Twarog and Wheeler (1982) the observed stellar oxygen yields can only be matched by the yields of stellar nucleosynthesis in the ejectae of supernova explosions of massive stars if all stars more massive than 25 $M_\odot$ contribute less to the yield, see also Nomoto et al. (1984a). It is a matter of debate if a change in the rates of the reactions

$$^{12}C(\alpha, \gamma)\,^1^6O \ [\text{Arnett and Thielemann (1984a, b), Thielemann and Arnett (1984)]}$$

or

$$^{16}O(\alpha, \gamma)\,^{20}Ne \ (\text{Nomoto et al., 1984a})$$

can lead to a higher production of intermediate-mass nuclei and to a lower oxygen abundance, see Woosley et al. (1984) and see above. On the other hand, if all hydrogen stars more massive than 25 $M_\odot$ produce black holes and do not contribute to the stellar yield, this would give a black hole formation rate $< 4 \times 10^{-6}$ kpc$^{-2}$ yr$^{-1}$ in the solar neighbourhood. This rate is one to two orders of magnitude smaller than the Galactic supernova rate (Tammann, 1982) and the pulsar birthrate [Manchester and Taylor (1977), but revised in Lyne et al. (1984)], see also Van den Heuvel and Habets (1984). (We assumed a mean life-time of $\sim 4.5 \times 10^6$ years for stars more massive than 25 $M_\odot$ and we adopted from Garmany et al. (1982) that these stars produce 280 to 327 0 and WR progenitor stars of black holes within 2.5 kpc of the Sun).

c. combined bulk yield from stellar nucleosynthesis in Types I and II supernovae

Nomoto et al. (1984a) suggest that Type I supernovae may produce a significant amount of intermediate-mass nuclei [which are underproduced in the
present Type II models of supernova explosions in massive stars, see e.g. Woosley and Weaver (1982b, 1984), but not in pre-explosive models (Thielemann and Arnett, 1984) if the rate of the reaction $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$ is increased with respect to the rate used by Woosley and Weaver. A good match with the observed stellar yield derived from standard supernova rates cannot be obtained (Nomoto et al., 1984a). This is due to the oxygen overproduction in massive stars, to the overproduction of Ca and Fe in Type I supernovae, and to the (now uncertain) underproduction of intermediate-mass nuclei in Type II explosions. Furthermore, the canonical Type I and II supernovae models cannot satisfactorily explain the observed nucleosynthesis or spectra of supernovae (see Woosley et al., 1984). Nonetheless, if the oxygen production by massive stars is diminished (see paragraph 12b above) and if the yield of Ca and Fe by Type I supernovae is reduced, reasonable agreement between observed yields and bulk yields by nucleosynthesis can be obtained (Nomoto et al., 1984a). Nomoto et al. argue that the Type I supernova rate can be lower than the "standard" supernova rate (Tammann, 1982), if the model for the formation of Type I supernovae works only in a restricted parameter space: e.g., if the Urca process is included the model requires rapid accretion of matter onto the white dwarf progenitor like in double white dwarf binaries [see e.g. Webbink, (1979) and (1984), and Iben and Tutukov (1984a, c)]. In closing, we can confirm the results of Nomoto et al. (1984a) by concluding that the bulk yield of stellar nucleosynthesis as obtained from present stellar evolution calculations and from present statistics of observed radio pulsars and supernovae can not be predicted accurately due to uncertainties inherent to stellar evolution theory, stellar nucleosynthesis, and pulsar and supernova statistics. However, the observed yield can be matched with an expected yield if these uncertainties are tuned properly.