Some problems in classical mechanics and relativistic astrophysics
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Cosmological tests of general relativity

General relativity theory could be tested on a cosmological scale by measuring the Hubble constant and the deceleration parameter, if, in addition, everything would be known about the matter filling the universe. If, on the other hand, nothing would be presupposed about the matter content of the universe, general relativity could not be tested by measuring any number of time derivatives of the scale factor. But upon making the assumption of a universe filled with a non-interacting mixture of non-relativistic matter and radiation we can in principle test general relativity by measuring the first five derivatives of the scale factor. In the following, some general relations are presented using this assumption.

A reasonable model for the universe in a not too early stage of its development is the Robertson-Walker metric

$$ds^2 = -dt^2 + R^2(t) \left[ (dx^2) / (1 - kr^2) + r^2d\Omega^2 \right]$$

(1)

describing a homogeneous isotropic universe.

From general relativity theory it follows that the scale factor $R(t)$ obeys the Friedmann equation

$$\frac{\dot{R}}{R} + k = \frac{8\pi G}{3} \rho + \frac{\lambda}{3} R$$

(2)

and conservation of energy and momentum is expressed by

$$\frac{d}{dt} (\rho R^2) = -3 \rho R^2$$

(3)

Here and throughout this paper we put $c = 1$; $\lambda$ stands for the cosmological constant, while $\rho$ and $\rho$ denote the density and pressure of the matter content of the universe.

To test these equations we need observations. By measuring the luminosity distance or the angular diameter distance of remote objects as a function of redshift $z$, we can in principle determine the Hubble constant $H = \dot{R}/R$, the deceleration parameter $q = -\ddot{R}/R^2$, and an indefinite number of similar dimensionless parameters constructed from higher time derivatives. We shall use here, following Harrison

$$Q = \frac{\dot{R}^2}{R^2} \quad X = \frac{\ddot{R} R^3}{R^2} \quad Y = \frac{R^2 \dddot{R}}{R^2}$$

(4)

(all evaluated at the present time, with signs which make them positive for vanishing $\lambda$, as will be shown). Without using Friedmann’s equation (2), however, knowledge of any number of those parameters will not suffice to obtain the value of the present curvature $K = k/R^2$ (ref. 3). But even equations (2) and (3) are not sufficient without additional information about the equation of state, as can be seen as follows: by differentiating equation (2) $N$ times we obtain $(N + 2)$ equations for $(N + 2)$ unknowns: $\rho$ and its first $N$ derivatives and $\lambda$.

A plausible form for the equation of state is obtained by assuming that at present the universe contains a non-interacting mixture of two forms of matter, non-relativistic matter with density $\rho_m$ and ultra-relativistic matter with density $\rho_r$. Then

$$\rho = \rho_m + \rho_r$$

$$\rho = \frac{1}{3} \rho_r$$

(5)

For the non-relativistic matter density, particle number conservation gives

$$\rho_m = \Lambda / R^2$$

(6)

and thus with (3)

$$\rho_r = B / R^4$$

(7)

where $\Lambda$ and $B$ are constant.

Now, carrying out three successive time differentiations of equation (2) leads with (2) to four equations for the newly introduced time derivative parameters. Solving these for the quantities $K, \lambda, \rho_m$ and $\rho_r$ gives

$$K = \frac{1}{3} (X + 4Q + Q_2 - 2H^2)$$

(8)

$$\lambda = \frac{1}{4} (X + 6Q + Q_6 - 6H^2)$$

(9)

$$\rho_m = \frac{1}{4\pi G} (-X + 3Q + Q_3 + 3g) H^4$$

(10)

$$\rho_r = \frac{3}{32\pi G} (X^2 - 2Q - Q_2 - 2Q_2 + 6Q^2 - 6Q - 6Q) H^4$$

(11)

Differentiating (2) a fourth time finally leads to a relation between the four parameters $Q, X$ and $Y$ which should hold if general relativity is valid

$$Y = 7X + 3Q_2 - 6Q - 3Q^2 - 7Q_2 - 3Q^4 - 6Q$$

(12)

This condition provides (in principle) a full test of the theory.

If less than five parameters measuring the time derivatives of the scale factor are known—as certainly will be the case in practice—still conditional relations can be derived between the various parameters. From equations (10) and (11) it follows that, since $\rho_m > 0$ and $\rho_r > 0$

$$Q_2 + 2(2Q + 2Q_2) < X < Q_2 + 3(Q + q)$$

(13)

a condition valid if only $\rho, Q$ and $X$ are known.

If $X$ is not available either we introduce a new parameter $\alpha$, still to be able to express the four quantities $K, \lambda, \rho_m$ and $\rho_r$ explicitly. For $\alpha$ we take

$$\alpha = \rho_r / \rho_m + \rho_r \quad 0 < \alpha < 1$$

(14)

and we then obtain

$$K = [-1 + (3 + \alpha) / (3 + 5\alpha) Q - (4\alpha) / (3 + 5\alpha)] q H^2$$

(15)

$$\rho = \Lambda / 8\pi G [6(3 + 5\alpha) / (3 + 5\alpha)] q H^2$$

(16)

$$\lambda = 3 \{ (1 + \alpha) / (3 + 5\alpha) Q - (2 + 4\alpha) / (3 + 5\alpha) q \} H^4$$

(17)

Equation (16) gives us, as $\rho > 0$, the condition

$$Q + q > 0$$

(18)

If we only know $H$ and $q$—a situation which is at present strain for—it is regrettable no longer possible to write down even conditional equations, unless observational evidence of a different nature, for example, regarding the age of particular objects in the universe, is taken into account.

Figure 1 illustrates a classification, based on equations (15) and (16), of the structural and evolutionary properties of the universe, as depending on the values of $q$ and $Q$, but with arbitrary $\lambda$. We can distinguish between domains of positive and negative $p$ (the latter physically unrealistic), of $k = \pm 1$ (separated by a line $k = 0$), and the presence or absence of an initial and a final singularity (only with a final singularity the universe, at present expanding, has a finite lifetime). Depending
Where the status of the universe depends on $a$, this is indicated by a double label as explained in the caption. Comparable diagrams have been published earlier by Stabell and Refsdal
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(As a referee commented, Fig. 3 of ref. 4, which is a $\sigma$-$q$ diagram, is topologically very similar to Fig. 1 of this paper, as $\sigma = 4\pi G \rho / 3H^2 \sim Q + q$.)

We can further simplify equations (8)–(11) by introducing two transformed time-derivative parameters

$$\tilde{Q} = Q + q$$
$$\tilde{X} = X - Qq$$

Then the right-hand side of equations (8)–(11) can be seen to become linear expressions in the new parameters, where equation (12) transforms into

$$Y = 7\tilde{X} + 3Xq - 6\tilde{Q} + (\tilde{Q} - q)^3$$

The conditional relations (13) and (18) become very simple

$$2\tilde{Q} < \tilde{X} < 3\tilde{Q}$$
$$\tilde{\tilde{Q}} > 0$$

We conclude that surprisingly simple relations can be derived, on the basis of rather general assumptions regarding the structure and matter content of the universe, between the 'physical' parameters of the universe $\rho_m, \rho_r$ and $\lambda$, together with $K$, and the 'geometrical' parameters involving the first four time derivatives of the scale factor. Between the first five time derivatives a strict relation should hold if general relativity applies. This relation forms, in principle, a definite test for the theory. If less than five parameters are known, but more than two, conditional relations can be derived.

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