Some aspects of evolution and mass transfer in X-ray binaries
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Evolution of Helium Stars: A Self-consistent Determination of the Boundary of a Helium Burning Convective Core

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Summary. A generalization of the Henyey-scheme is given that introduces the mass of the convective core and the density at the outer edge of the convective core boundary as unknowns which have to be solved for simultaneously with the other unknowns. As a result, this boundary is determined in a physically self-consistent way for expanding as well as contracting cores, i.e. during the Henyey iterative cycle, its position becomes consistent with the overall physical structure of the star, including the run of the chemical abundances throughout the star. In earlier investigations on helium star evolution, such a consistency could in general not be achieved. Using this scheme, the evolution of helium stars \( (X = 0, Z = 0.03) \) was followed up to (non-degenerate) carbon ignition for a number of stellar masses: 2.5 \( M_\odot \), 3 \( M_\odot \), 4 \( M_\odot \) and 8 \( M_\odot \). As compared with some earlier investigations, the calculations show a rather large increase in mass of the convective cores during core helium burning. Evolutionary calculations for a 2 \( M_\odot \) helium star show that the critical mass for which a helium star ignites carbon non-degenerately lies near 2\( M_\odot \) (corresponding to a C/O core of 0.99 \( M_\odot \)). The 2 \( M_\odot \) helium star is the only one for which the radius strongly expands after core helium burning.

Key words: helium stars — convective cores — Henyey scheme

Notation

- \( n \) total number of mesh points (including the center)
- \( M \) mass coordinate
- \( r \) radial coordinate
- \( \rho \) density
- \( P \) pressure
- \( T \) temperature
- \( Q \) heat content of the stellar matter
- \( \varepsilon \) rate of nuclear energy generation corrected for neutrino losses
- \( L \) luminosity
- \( Y \) fraction of helium content by mass
- \( t \) time
- \( \varphi \) actual logarithmic temperature gradient \( \frac{\partial \ln T}{\partial n} \) in the model
- \( \varphi_r \) radiative logarithmic temperature gradient
- \( \varphi_a \) adiabatic logarithmic temperature gradient
- \( \Delta, \delta \) evolutionary correction to \( A \): difference in quantity \( A \) between the actual stellar model and the previous one
- \( \delta A \) iterative correction to \( A \): difference in quantity \( A \) between two consecutive Henyey iterations.

1. Introduction

The convective core of a helium-burning helium star increases in mass during most of this evolutionary phase e.g. cf. Paczynski, 1971; Dinger, 1972). This is due to the nuclear burning in the center of the star, which causes the core material to become gradually more opaque and hence more convective. This forces the core boundary to propagate into the radiative helium-rich region around it by convective overshooting; this renders these originally radiative layers convectively unstable and, at the same time, mixes lower opacity helium-rich material into the core. The latter process makes the core material somewhat less convective and thus reduces the overshooting at the core boundary. The position of the convective core boundary (from here on abbreviated by BCC) is determined by the difference between the radiative- and adiabatic temperature gradients, which is a measure of the efficiency of overshooting. This difference \( (\varphi_r - \varphi_a) \), however, depends on the position of the BCC itself, through the amount of low opacity helium-rich material mixed into the convective core. Hence, one can only determine the position of the BCC iteratively, which is, however, hampered by the occurrence of a chemical composition (hence density-) discontinuity at the boundary of the expanding convective core. Usually one adopts for each timestep a fixed amount of core expansion and thus a fixed chemical core composition during the iterative Henyey cycle (e.g. cf. Paczynski, 1971). One then calculates the definitive position of the BCC and definitive chemical composition of the convective core after the Henyey iterations have converged. However, this does not result in physically consistent models, since after convergence of the
Heney iterations has been obtained, the position of the BCC (determined by the criterion for convective stability) is in general not the same as the one assumed for starting the Heney iterations. This method will give rise to a rather large uncertainty in the position of the BCC if there is an extended region near the core boundary where the radiative temperature gradient (i.e. the temperature gradient necessary to transport the energy-flux by radiation only) is nearly equal to the adiabatic temperature gradient. Such a region, corresponding to a minimum in the $V_r$-curve ($V_r$ evaluated as a function of $M$), by adopting a composition equal to that of a completely mixed core) seems to be a general feature of the evolution of helium burning cores (e.g. Paczyński, 1970) and would result in a semi-convective region outside the convective core (Castellani et al., 1971b). Under these circumstances, the determination of the core boundary with the rough method described above can lead to large jumps in the position of the core boundary from one timestep to the next. This is the reason why Paczyński (1971) allowed the convective core to grow with only just one mass zone per timestep. We tried to determine the position of the BCC more consistently by mixing the chemical composition during the Heney iterations (up to the momentary position of the BCC, as resulting from the foregoing iteration), but without special modifications this induced oscillatory instabilities and convergence could not be obtained, as has also been noticed in previous investigations (Iben, 1965). Heney et al. (1964) also mention this kind of oscillatory instability. They tried to make use of a variable mesh point at the convective core boundary, the position of which was determined by a convective stability criterion.

The numerical instabilities, however, remained, and convergence could not be obtained. We have analysed this situation in order to find the cause of these instabilities. We found, after many trials, that they could be resolved if the linearization of the equations (including the stability criterion) is carried out very carefully and as completely as possible.

We generalized the iterative Heney scheme by introducing the mass of the convective core and the density at the outer edge of the composition discontinuity as extra unknowns in the iterative scheme. Next, very important, the chemical composition of the convective core was introduced as an implicit variable, which is coupled to the mass of the convective core, due to mixing. This allows us to linearize all the equations with respect to core composition (especially the convective stability criterion).

To enable the introduction of these extra unknowns (and corresponding extra equations) a more general elimination scheme for the Heney matrix was developed, as is described in the appendix. It appeared that with this method self-consistent models can be obtained after a few Heney iterations.

2. The Equations Describing Stellar Evolution

The hydrostatic evolution of a spherically symmetric star is described by the well known set of non-linear differential equations (e.g. Cox and Giuli, 1968):

$$\frac{d\ln P}{dM} = -\frac{GM}{4\pi r^4 P}$$  \hspace{1cm} (1)

$$\frac{d\ln T}{d\ln P} = \frac{V_r}{V_\ast} \quad \text{or} \quad \frac{V_r}{V_\ast}$$  \hspace{1cm} (2)

$$\frac{d\ln r}{dM} = -\frac{1}{4\pi r^2 Q}$$  \hspace{1cm} (3)

$$\frac{dL}{dM} = \epsilon - \frac{dQ}{dt}.$$  \hspace{1cm} (4)

To solve these differential equations for $T, \varphi, r,$ and $L$ the star is divided into a large number of mass shells, by inserting a grid of mesh points. The differential equations are then transformed into a large number of difference equations, which are expanded between the mesh points of the grid. These difference equations are then linearized and iteratively solved with a Newton-Raphson method (Heney et al., 1964).

The outer boundary condition was obtained by integrating a large number of envelope models for different values of $T$ and $L$ at the surface; the resulting values of $T, \varphi, r,$ and $L$ at a fixed fitting point at the bottom of the envelopes, were stored as a table in the memory of the computer (Paczyński, 1969).

The basic program to compute the evolutionary sequences was kindly provided by Paczyński and has extensively been described in the literature (e.g. Paczyński, 1970).

The inner boundary condition was modified and is described in the appendix. Further modifications of the original program include the special treatment of the convective boundary and the calculations of the nuclear burning rates.

2.1. Determination of the Convective Core Boundary

In the extrapolated stellar model which has to be converged, a new mesh point is inserted at the estimated position of the BCC as described in Section 3. In this boundary mesh point we have six unknowns instead of the usual four, namely $T_c, \varphi_c, V_r, L_c$ and $M_c$. From here on a superscript $i$ or $e$ means that the quantities are evaluated at the inner- or outer edge of the chemical composition discontinuity at the boundary.

All variables have their usual meaning, $M_c$ being the mass of the convective core. Like all other mesh points, the inserted mesh point at the boundary is coupled to its neighbouring mesh points through the difference form of Eqs. (1) to (4). By enabling the inserted mesh point to move through the star during the Heney iterations (correction on $M_c$), the program can search for that position (i.e. that value of $M_c$) in which Eqs. (1) to
(4) and the Eqs. (5) and (6), which make the system mathematically complete again, are simultaneously satisfied. Equation (5) determines the position of the BCC and follows from the equation Castellani et al. (1971a) derived for the (outward) propagation velocity of the BCC due to the overshooting at the core boundary:

\[
(1 - \eta)P^* - P_0 = 0
\]

\[
\eta = \frac{10P_0 \dot{M}_c}{\rho \dot{V}_c} (1 - \mu/\mu^*)
\]

Here \( \eta \) is a measure for the super-adiabaticity of the radiative temperature gradient at the inner edge of the convective boundary, \( \dot{M}_c = (M_c - M_c^{\text{old}})/\Delta t \) and \( \mu \) is the mean molecular weight of the stellar material. In deriving Eqs. (5) and (5a) it was assumed that \( P_{\text{cr}} \approx P_0 \) and that \( \mu/\mu^* \leq 0.99 \). Equation (5) implicitly includes a timescale for convective mixing and gives a lower limit for \( \dot{M}_c \). On the other hand, the Schwarzschild criterion \( P^*_0 - P_0 = 0 \) gives an upper limit for \( \dot{M}_c \), because it implies that an infinitesimal small value of \( P^*_0 - P_0 \) can drive the core expansion (i.e., instantaneous mixing). Calculations show that the momentary overshooting is limited to a negligible small region of only a few \( 10^2 \) cm (\( \eta \sim 10^3 \)).

During core helium burning \( \eta \) appears to be very small (a few times \( 10^{-3} \)), hence Eq. (5) does not differ very much from the classical condition \( P^*_0 - P_0 = 0 \). The latter condition results in a slightly more massive (about 1%) convective core all the time. However, if there is a region with \( P^*_0 - P_0 \sim 0 \) near the core boundary, then Eq. (5) in contrast to the classical condition, prevents the BCC to jump through such a region, because a sudden increase in core mass would require a large value for \( \eta \) at the boundary. This stabilizing effect on the numerical solution was the main reason to use Eq. (5). Equation (6) expresses the continuity of pressure across the chemical composition discontinuity and couples \( q^*_c \) to \( q^*_c \):

\[
P^*_c - P^*_e = 0.
\]

With this scheme for the determination of the convective core boundary, one obtains after every Henyey iteration corrections to all the unknowns, including \( M_c \) and \( q^*_c \), such that the position of the BCC becomes consistent with the overall physical structure of the star. To obtain a rapid convergence of this generalized Henyey scheme, one should carefully linearize the difference equations derived from Eqs. (1) to (6). In Sections 2.2 to 2.4 we give some details of this linearization.

### 2.2 Some General Comments on the Linearization

Assume that \( H_k = (T, q, r, L, M_c) \) is a vector, the components of which are the four unknown physical variables at a mesh point \( k \). Evidently, in linearizing the difference equations resulting from Eqs. (1) to (4) one has to put:

\[
\frac{\partial H_k}{\partial L_k} = (0, 0, 0, 1),
\]

and analogously with respect to the other three unknowns. The difference equations are expressions containing the components of \( H_k \) and of \( H_{k+1} \), and some other physical quantities such as \( V_c, P \) and \( Q \) evaluated at the mesh points \( k \) and \( k + 1 \).

These quantities depend not only on some or all of the components of \( H_k \) or \( H_{k+1} \), but also on the chemical composition at the points \( k \) and \( k + 1 \). In order to obtain rapid convergence of the generalized Henyey iterations, one should completely linearize the set of difference equations, such that during every Henyey iteration the full coupling of all unknowns is felt. This implies that one should also include the dependence on chemical composition into the linearization.

The chemical composition in a mesh point can change by local nuclear burning and/or by convective mixing. To include the linearization of the equations describing local nuclear burning is very easy. However, to enable iterative changes in core composition that are due to an iterative change in the amount of mass that is mixed, is difficult as a consequence of the non-local character of convective mixing. The problem is that the coupling of the composition variables in all mesh points inside the convective core destroys the diagonal structure of the Henyey matrix.

We are at present developing a method that enables the treatment of such complicated couplings without requiring unrealistic amounts of computer time and memory. This method should be applicable to more complex situations such as convective mixing between two boundaries (convective shells) or partial mixing of semi-convective regions.

In this paper, however, we take into account the iterative changes of the physical quantities \( P, V_c, e \) and \( Q \) due to the (iterative) corrections to the chemical composition of the convective core only at the boundary point and its two adjacent mesh points (at both sides). This already resulted in excellent convergence properties of the generalized Henyey iterations. We will now discuss the special features of the linearization at the two edges of the discontinuity in the inserted boundary mesh point.

### 2.3 Linearization of the Equations at the Convective Boundary

Consider the vector \( H_c = (T, q', r_c, L_c, M_c) \) the elements of which are the six unknowns at the inserted mesh point at the boundary. Although the first five elements of \( H_c \) are dependent on \( M_c \), in linearizing the difference equations at the boundary point, one should
put in analogy with Eq. (7): 
\[
\frac{\partial H}{\partial M} = (0, 0, 0, 0, 0, 1). 
\] (8)

To describe the linearization with respect to chemical composition, let us notice that there is only one variable \((Y)\) describing the chemical composition of the stellar material in our program. The thermodynamic quantities are interpolated from tables which have entries \(T, \varrho\) and \(Y\). These tables are calculated for two compositions, a Cox-Stewart mixture \(Y = 0.97, Z = 0.03\) and the Weigert III mixture \(Y = 0, C = O = 0.478\).

At the two edges of the discontinuity in the boundary mesh point the linearization with respect to chemical composition of the equations which contain the quantities \(P, V, \varepsilon\) and \(Q\) is carried out implicitly through the relation:
\[
\delta Y = \frac{\partial Y}{\partial M_c} \delta M_c. 
\] (9)

In Eq. (9) the derivative of the chemical composition with respect to the mass of the convective core is well defined at both sides of the discontinuity at the BCC. One has in every mesh point inside the convective core, including the inner side of the convective core boundary (the local unmixed value for the helium content at a certain mesh point, \(Y_k\), is recalculated after every Henyey iteration):
\[
Y = \frac{1}{2M_c} \sum (Y_{k+1} + Y_k) (M_{k+1} - M_k) 
\] (10)

where the summation index \(k\) runs from 1 to the boundary point \(c\). If the convective core is expanding one has \(\frac{\partial Y}{\partial M_c} = 0\) at the outer edge of the discontinuity at the boundary, unless there is a composition profile present outside the convective core. In case the convective core is shrinking there is no longer a discontinuity at the boundary and hence \(\frac{\partial Y}{\partial M_c} = \left(\frac{\partial Y}{\partial M_c}\right)^c\). The difference form of Eq. (4) delivers a special difficulty in the linearization due to the presence of the time derivative \(\frac{dQ}{dt}\). As usual, we have replaced \(\frac{dQ}{dt}\) by \(\frac{\Delta Q}{\Delta t}\) since the former value is hard to obtain and tends to give rise to mathematical instabilities (Henyey et al., 1964). The error made in this replacement is in general of the same order as the error due to the fact that \(\Delta Q\) is not a total differential and is very small if one takes reasonable timesteps. We have put in every mesh point:
\[
\Delta_t Q = \left(\frac{\Delta Q}{\Delta T}\right)_{e,y} \Delta_t T + \left(\frac{\Delta Q}{\Delta Y}\right)_{e,y} \Delta_t Y. 
\] (11)

The meaning of the last term on the right hand side of Eq. (11) is that a change in composition at constant density results in a change of the number of thermal particles and hence in a change in the heat content \(Q\). At the boundary of the expanding convective core this term is of the same order as the other terms on the right hand side of Eq. (11). This term was first introduced at the boundary of an expanding convective region by Ziolkowski (1972). When linearizing the difference form (11) one obtains coefficients of the type:
\[
\frac{\partial}{\partial Q} \left(\frac{\partial Q}{\partial Y}\right)_{e,y} \Delta_t Q = \left(\frac{\partial Q}{\partial Y}\right)_{e,y} \Delta_t Y + \Delta_t T \left(\frac{\partial Q}{\partial T}\right)_{e,y} \Delta \varepsilon. 
\] (12)

Note that \(\Delta_t Q\) is not of the same order as the small iterative correction \(\delta Q\) (especially in the region of expansion of the convective core one has in general \(\Delta_t Q \gg \delta Q\)) while the second derivative with respect to density is not small compared to the first derivative \(\frac{\partial}{\partial Q} \left(\frac{\partial Q}{\partial Y}\right)_{e,y}\). Hence one cannot neglect the second term on the right hand side of Eq. (12) as part of the cofactor of \(\delta Q\). One obtains analogous expressions for the cofactors of \(\delta T\) and \(\delta M_c\), the latter through the implicit unknown \(\delta Y\). One should also be aware that \(\frac{\partial}{\partial Q} \left(\frac{\partial Q}{\partial T}\right)_{e,y}\) is not equal to \(\frac{\partial}{\partial T} \left(\frac{\partial Q}{\partial Q}\right)\). While evaluating \(\Delta_t Q\) at the boundary one should add correction factors to the evolutionary changes \(\Delta_t T, \Delta_t Y, \) and \(\Delta_t M_c\). Consider for example \(\Delta_t T^{i+1} = T^i + \delta T - T^\text{old}\), where \(i\) gives the iterative cycle number. It is clear that \(T^\text{old}\) should be the temperature (in the previously converged "old" stellar model) at the position of the actual boundary at \(M_c^{i+1} = M_c^i + \delta M_c\) (and not at \(M_c\)). In order to use correct values for \(\Delta_t T\), etc. during every iteration, \(T^\text{old}\) is considered to be a function of \(M_c\) (in contrast to \(T\)). As an example of such neutralizing corrections we give the linearization with respect to the convective core mass \(M_c\) of the first term on the right hand side of equation (11):
\[
\frac{\partial}{\partial M_c} \left(\frac{\partial Q}{\partial T}\right)_{e,y} \Delta_t T = -\left(\frac{\partial Q}{\partial T}\right)_{e,y} \left(\frac{\partial Q}{\partial M_c}\right)_{e,y} \Delta_t Y + \Delta_t T \left(\frac{\partial Q}{\partial Y}\right)_{e,y} \Delta \varepsilon. 
\] (13)

The first term on the right hand side of Eq. (13) gives that part of the cofactor of \(\delta M_c\) that corrects for the effect of evaluating \(\Delta_t Q\) with \(T^\text{old}\) taken at the mass \(M_c\) corresponding to the previous Henyey iteration, instead of taking \(T^\text{old}\) at the mass \(M_c + \delta M_c\). The superscript "old" means that the derivative is taken in the previous stellar model of the evolutionary sequence at a mass corresponding to the new value of \(M_c\). The second term on the right hand side of Eq. (13) results from the implicit dependence of \(\frac{\partial Q}{\partial T}\) on the
chemical composition, according to relation (9). The cofactor of $\delta M_z$ should contain analogous terms to neutralize the mass shift effect of $A_{zq}$ and $A_{zY}$ on the evaluation of $A_{zQ}$. If one does not carefully linearize the equations, for example neglects some of the terms described in this section, serious convergence troubles will arise, as appeared when we carried out test runs in which alternatingly one or some of these terms were neglected.

2.4 Shrinking Convective Core

If the convective core starts shrinking, always a composition profile will be produced and a composition discontinuity at the BCC will no longer occur. This composition discontinuity is now fixed at the point where the convective core reached its maximum size at which position it is burned away by the helium burning shell during the subsequent evolution. At this fixed (in mass) discontinuity we use a simple procedure to introduce the (implicit) unknown density $\varrho^e$ at the outer edge of the discontinuity, next to the unknown density at the inner side. At this special mesh point we complete Eqs. (1) to (4) with the pressure Eq. (6). Linearization of Eq. (6) yields:

$$\delta \varrho^e = \left\{ P^e - P^r + \frac{\partial P^r}{\partial \varrho^r} \delta \varrho^r + \left( \frac{\partial P^r}{\partial T} - \left( \frac{\partial P^r}{\partial T} \right)^{ev} \right) \delta T \right\} \left( \frac{\partial P^r}{\partial \varrho^r} \right)^{ev}.$$  

(14)

It is possible now to transform the cofactors of $\delta \varrho^e$ into cofactors of $\delta \varrho^r$, $\delta T$ and a right hand side term. Having back the standard matrix format, we recover after every Henyey iteration $\delta \varrho^e$ from Eq. (14) with the help of the values of $\delta \varrho^r$, $\delta T$ and the right hand side term. During times that the convective core is shrinking we also take into account the iterative change in composition at the point just outside the convective boundary (due to the correction $\delta M_z$ on the position of the BCC). This resulted in better convergence properties of the generalized Henyey iterations during this phase (near the end of core helium burning). Inclusion of this composition dependence on the position of the BCC for more mesh points was not possible with the elimination scheme used, as was mentioned in Section 2.2.

It is worth noting here that if one calculates with the equation derived by Castellani et al. (1971a) the height $z$ which overshooting convective elements can move beyond the convective boundary, then for all our stellar models $z$ is only a few times 100 cm. Hence there is a real discontinuity in chemical composition.

3. Starting Model for the Henyey Iterations

The starting models on the helium main sequence were calculated with the use of a Schwarzschild fitting technique. This program is the same as the one Paczyński (1971) used to obtain homogeneous starting models. The first extrapolated model was obtained by putting all evolutionary corrections to the structure variables equal to zero and changing the composition by taking a small evolutionary timestep. As usual the following starting models for the Henyey iterations i.e. the values for the four unknowns in every mesh point, were obtained by a linear extrapolation from the two preceding stellar models, except for the region through which the convective core boundary has moved during the last timestep. Suppose that the consecutive boundaries in the two preceding stellar models were BCC 1 and BCC 2. Then the physical structure has changed abruptly in the region between BCC 1 and BCC 2 and a linear extrapolation would give wrong starting conditions there, as is seen from Fig. 1. To obtain better starting values for the four unknowns in the region outside BCC 1 up to the estimated new boundary BCC 3, the higher density core branch was continued by integrating Eqs. (1) to (4) from BCC 1 outwards. A sequence of integrations was made in which each time the convective boundary was assumed to shift one step further outwards. At the same time the nuclearly processed material (with values for $T$, $\varrho$ and $Y$ halfway between the values for the preceding and the extrapolated model) was completely mixed from the center to the new boundary for that integration. After every integration the quantity $(F_z - F_0)$ was evaluated at the new boundary, until a mesh point was reached with $(F_z - F_0) < 0$. The insertion of a movable mesh point at a mass corresponding to $(F_z - F_0) = 0$ terminated this integrating and mixing process. At this inserted mesh point (with six instead of four unknowns) the lower density envelope was coupled to the extrapolated higher density core branch.
3.1. Nucleosynthesis and Neutrino Radiation

The equations governing the change of abundances during helium burning are:

\[ \dot{Y}_p = [zzz]Y_p^3 - [^{12}\text{C}, \alpha ]Y_p Y_e - [^{16}\text{O}, \alpha ]Y_o Y_e \]  
\[ \dot{Y}_e = [zzz]Y_e^3/3 - [^{12}\text{C}, \alpha ]Y_o Y_e \]  
\[ \dot{Y}_o = [^{12}\text{C}, \alpha ]Y_o Y_e - [^{16}\text{O}, \alpha ]Y_o Y_e \]  
\[ \dot{Y}_{he} = [^{16}\text{O}, \alpha ]Y_o Y_e \]  

(15)  
(16)  
(17)  
(18)

The composition variables \( Y_p, Y_e, Y_o \) etc. are defined by \( Y_p = Y/Y_4, Y_e = C/12 \text{ etc.} \) The reaction rates in brackets are calculated with analytical expressions from Fowler et al. (1975). At every mesh point the rates from the previous model and the actual stellar model, e.g. for \( Y_p \) the value

\[ Y_{p}^{i+1} = \frac{(Y_{p}^{i} + Y_{p}^{\text{old}})}{2} \]  

(19)

was taken. Here \( i \) gives the number of Henyey iterations and "old" means that the quantity is to be evaluated in the previous model. The factor (0 to 1) which occurs in the expression for the triple alpha rate was put equal to 0.1. The resulting (local) composition changes were then calculated with the simple equations:

\[ A_i Y = \dot{Y}_A_i \]  

(20)

where the derivatives \( \dot{Y} \) were obtained from Eqs. (15) to (18) in the way described before. The chemical composition of the core was obtained by mixing the nuclear processed matter—according to Eq. (20)—from the center up to the instantaneous position of the movable mesh point at the boundary. As this mixing was performed after every Henyey iteration, \( Y_{p}^{i} \) in Eq. (19) had the mixed value, which was obtained after mixing in the previous \( i^{(0)} \) Henyey iteration, if the mesh point under consideration was inside the convective core. This simple scheme is stable as the helium content of the core goes to zero, at least for \( Y \geq 10^{-4} \).

At the moment that the central helium content became smaller than this value, the helium content was set equal to zero throughout the convective core. The energy generation due to the nuclear reactions involved was calculated in the way described by Fowler et al. (1975).

The energy generation of the \( ^{12}\text{C} + ^{12}\text{C} \) reaction (with an effective \( Q \)-value of 11.7 MeV and electron screening according to Salpeter and van Horn, 1969) was also included in the program.

The energy losses due to neutrino radiation were calculated with the analytical fitting formulae derived by Beaudet et al. (1969).

3.2. Mass Zoning, Timesteps and Mixing Length

The maximal differences allowed for the unknowns between two adjacent mesh points were: \( \Delta \ln T \leq 0.045 \), \( \Delta r/R_{\odot} \leq 0.045 \), \( \Delta L/L_{\text{max}} \leq 0.045 \), \( \Delta Y, \Delta C \) and \( \Delta O \leq 0.045 \). In the region around the BCC these maximal space steps were multiplied by a factor 0.15. These conditions for the space steps resulted in a varying number of mesh points, from about 120 at the helium main sequence to about 1200 during the He-burning-shell phase. The maximal time steps were chosen such that the same conditions were fulfilled by the (guessed) evolutionary changes in every mesh point, except for the changes in chemical composition, which had to be smaller than 0.02 in every mesh point. The mixing length used in the envelope integrations was taken as one pressure scale height.

4. Results

4.1. Behaviour of the Convective Core During the Evolution

As is shown by Fig. 2, the convective cores of the helium stars expand monotonically during most of the time of central helium burning. The convective cores of the 2 \( M_{\odot} \) and 4 \( M_{\odot} \) helium stars keep growing until helium becomes depleted and then start shrinking rapidly. The 3 \( M_{\odot} \) and 8 \( M_{\odot} \) helium stars have more unstable convective cores near the end of helium burning. Here the BCC suddenly jumps inwards, quickly followed by a more gradual inward movement. After some time the convective core suddenly starts reexpanding for a short time. This sudden jump inwards is related to the development of a minimum near the BCC in the \( V_r/V_a \)-curve (evaluated with mixed core composition). Such a situation is schematically described in Fig. 3. It is seen that the minimum in the \( V_r/V_a \)-curve gradually sinks through the line \( V_r/V_a = 1 \) (corresponding to convective neutrality)

\[ \text{Fig. 2: Evolutionary changes in the extent of the convective cores of helium stars during core helium burning.} \]

\[ \text{Fig. 3: Schematic representation of the convective core evolution.} \]
as time goes on. This gives rise to the existence of multiple solutions of the equation \( (V_r - V_c) = 0 \). Earlier test computations with Eq. (5) replaced by this criterion for the convective boundary showed that the points \( S_1 \) as well as \( S_3 \) in Fig. 3 are mathematically stable solutions for the BCC. However, the solution \( S_3 \) gives an inconsistent result because the program completely mixed the core material from the center up to this new BCC (solution \( S_3 \)), in spite of the fact that there is a radiative region inside the BCC (between \( S_1 \) and \( S_2 \)).

During the actual calculations the program started the Henyey iterations with the BCC near \( S_1 \) and it used Eq. (5) to determine the position of the BCC. The Henyey iterations rapidly converged to solution \( S_1 \) in this case. [Notice that Eq. (5) becomes identical to \( (V_r - V_c) = 0 \) if the core is shrinking, because then \( \eta \) is identically zero, due to the fact that \( \mu' = \mu \).] As a result the convective core boundary jumped inward from BCC to \( S_1 \) during one evolutionary timestep. The region outside the new (smaller) convective core, in which a composition profile was produced by the retreating convective shell just inside the former BCC between \( S_2 \) and \( S_3 \). Hence solution \( S_1 \) for the BCC gives a consistent result. In the \( 2M_\od\) helium star a non-transient convective shell appears outside the retreating convective core (solution \( S_1 \)), due to the increase of the relative maximum of the \( V_r/V_c \)-curve between \( S_2 \) and \( S_3 \) for this star. This convective shell was mixed after the Henyey iterations were completed, because the program was not able to apply the movable mesh point technique to a situation with two convective boundaries. Figure 4 shows the resulting log\((V_r/V_c)\)-curves of a number of subsequent converged stellar models during this convective shell phase of the \( 2M_\od \) star. Because during the actual calculations as presented here, the program used Eq. (5) to determine the BCC [and not the condition \( (V_r - V_c) = 0 \)], the value of \( (V_r - V_c) \) at the inner edge of the composition discontinuity was positive and not zero. This is the reason why in Fig. 4 the curves do not end at the "neutral line" (at \( b_1 \) and \( b_2 \)) like they do in Fig. 3.

The time elapsed between the first and the last model shown [appearance and (nearly) disappearance of the shell] is only some 40000 years, or about 1% of the duration of the core helium-burning phase. A test run with a \( 1.5M_\od \) helium star showed the appearance of a minimum in the \( V_r/V_c \)-curve, which intersected the line \( V_r/V_c = 1 \) three times \( (M_5 = .52M_\od, M_{S_2} = .68M_\od \) and \( M_{S_3} = .70M_\od \)) when the helium content \( Y \) was still about .4. Our program [this time using the \( (V_r - V_c) = 0 \) criterion] continued the calculations without convergence troubles, the BCC converging at the \( S_3 \) solution with a rather large amount of mass between \( S_3 \) and \( S_1 \).

Probably one should introduce a semi-convective region outside the convective core in this case, because now the minimum in the \( V_r \)-curve appears while core helium burning is far from being finished. In this case the convection in the core is not yet dying out and due to the continued convective overshooting at the core boundary (now at the position of the minimum in the \( V_r \)-curve) a semi-convective region will be induced (Castellani et al., 1971b). There is, however, uncertainty about the effectiveness of this "induced semi-convection". This uncertainty is related with the (long) time-scale on which the convective elements (released by the convective core as a result of the gradually increasing opacity in this core) move through the convectively neutral region (Saio, 1974).

### 4.2. Possible Physical Reasons for the Unstable Behaviour of the Convective Core Boundary

Near helium depletion in the core, the admixture of helium into the convective core due to (re)expansion into the helium-rich layers outside it, will raise the core
helium content by a non-negligible fraction. This fuel injection will activate the nuclear energy source and the core structure may slightly change. Also, an expansion of the convective core into the helium-rich region results in a small source term for the radiation field at the inner edge of the BCC, due to the gravitational term in the energy balance Eq. (4) (see Eq. (11), Section 2.3).

On the other hand these structural changes may lower the opacity of the core material, reducing the temperature gradient $\rho_r$. If these counteracting effects nearly cancel each other, small changes in the physical parameters could easily reverse the motion of the BCC. It is not clear to what order pure numerical processes are active in this way. Test calculations with smaller timesteps showed exactly the same behaviour of the convective cores for the different stellar masses.

4.3. Carbon Ignition

The evolutionary calculations were terminated at carbon-ignition, i.e., at the moment when the central regions of the stars became convective again due to the large energy supply of the $^{12}\text{C} + ^{12}\text{C}$ reaction, overcoming the large neutrino cooling.

As is seen from Fig. 5, which gives the $\varrho$, $T$ history of the stellar centers, only the $2M_\odot$ helium star moves far into the degenerate region of the $\varrho$, $T$ diagram before carbon will be ignited (the evolutionary calculations were not continued up to that point). For this stellar mass a temperature inversion developed near the center, due to the strong neutrino cooling in the center. As a consequence of the continued (rapid) core contraction combined with the presence of the helium burning shell source outside the C/O-core, the envelope strongly expanded for this star (Fig. 6). This expansion may result in considerable mass loss by Roche-lobe overflow if such a $2M_\odot$ helium star is a member of a close binary system. One should bear in mind that the same process of core contraction comes into action before the next nuclear fuel can ignite (Ne, O and Si). The $2.5M_\odot$ helium star could show the same envelope expansion in such a later evolutionary stage, as the $2M_\odot$ star exhibits before carbon ignition.

Notice that if there would have been some hydrogen in the envelope, the stellar radii would increase relative to the values given in Fig. 6.

The $2M_\odot$ helium star develops a C/O core of 0.99 $M_\odot$ and this value is slightly below the critical core mass of $1M_\odot$ given by Beaudet and Salpeter (1969) for non-degenerate carbon ignition (accepting the existence of the universal Fermi-interaction). The uncertainties in the $^{12}\text{C} + ^{12}\text{C}$ cross-section for hydrostatic carbon burning and the neutrino loss rates are, however, large, both are only theoretically determined without experimental verification.

5. Discussion

5.1. Comparison with other Evolutionary Calculations

Our results confirm the relatively large increase in convective core mass found by Paczyński (1971), as can be observed in Table 1. Divine (1965) found an increase of 9% in convective core mass during core burning for a $6M_\odot$ helium star, which is far less than our results would imply for that stellar mass. Arnett (1972) did not find any core growth at all, but he used a very coarse zoning near the convective core boundary and started with pure helium ($Z=0$). Our calculated evo-
Evolutionary tracks for the different stellar masses (Fig. 7) are also in agreement with the tracks found by Paczynski. Our tracks hook back to the blue for slightly lower effective temperatures in accordance with the slightly smaller masses found for the C/O-cores. Arnett found lower final carbon abundances for the C/O-cores, respectively $C = 0.434$ and $C = 0.358$ for a $4 M_\odot$ and a $8 M_\odot$ helium star.

There is a difference between the reaction rates which are used here and which Arnett used. These reaction rates [3a and $^{12}$C$(\alpha, \gamma)^{16}$O] are more accurate now, due to the experimental work done recently by many investigators (see references in Barnes and Nichols, 1973). There is, however, another and more important difference with Arnett's calculations and that is the expansion of the convective cores during core helium burning. The helium stars in Arnett's calculations had convective cores of lower (constant) mass: $1.72 M_\odot$ and $4.77 M_\odot$ for the $4 M_\odot$ and $8 M_\odot$ helium star, respectively. The larger, expanding convective cores found here, result in a somewhat different $q$, $T$-history of core helium burning. A larger effect is present near the end of core helium burning: then the admixture of helium-rich material into the convective core gives a non-negligible increase of the nuclear fuel supply in the core. At this time nearly the total nuclear energy production is due to the $^{12}$C$(\alpha, \gamma)^{16}$O reaction and this admixture of helium leads to a lower final carbon content. For the $2.5 M_\odot$ helium star the effect of re-expansion of the convective core into the helium-rich region (when the helium content $Y$ is about 0.06) is clearly shown by the curve which gives the change in core helium content during the evolution (Fig. 8). This $Y$-curve exhibits a change in slope near $Y \approx 0.06$, due to the sudden injection of helium into the convective core. This extra helium supply burns away more carbon, until finally the carbon-curve terminates below the oxygen-curve in Fig. 8. The convective core of the $3 M_\odot$ helium star, on the other hand, starts shrinking already for $Y \approx 0.08$ (see Fig. 2) and never reexpands into the helium-rich region. This explains the relatively high final carbon content of the C/O-core for this stellar mass, as is shown in Fig. 9. The irregular behaviour of the convective cores near helium depletion results in irregularities of the otherwise smoothly varying final C/O-ratio as a function of total stellar mass. The re-
results found here (Table 1), seem to indicate that a maximum for this ratio is reached near 3 \( M_\odot \).

Deinzer and Salpeter (1964) found such a maximum in the C/O-ratio for the C/O-cores of helium stars to occur near a stellar mass of 1 \( M_\odot \). A point which needs further investigation is how much the results depend on what kind of interpolation scheme is used for the opacity tables.

5.2. Possible Applications of the Movable Mesh Point Techniques

The method developed here can equally well be applied to the convective cores of hydrogen burning stars, where the precise composition profile left behind by the shrinking convective core is a very important factor in the evolution of the stars during the shell hydrogen burning phase (e.g. Lauterborn et al., 1971). Eventually arising semi-convective regions could be treated with movable mesh points at the two boundaries. Due to the complex couplings which arise here, this will be rather complicated. The application of (a generalization of) our method to the two boundaries of a convective shell with a nuclear energy source in it would be very useful. In such a case the precise extent of the convective shell is important, because it regulates the amount of nuclear fuel which is mixed into the nuclearily active region and at the same time determines where the nuclear ashes are deposited. The self-consistency of our method would simplify the study of the effects of differences in the input physics (e.g. the theory of convective overshooting).

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Appendix

1. The Inner Boundary Condition

At the center of the star a different form is needed for the structure equations. To show the kind of differences which arise, we give the reformulation of Eq. (3). Since \( dq/dr=0 \) at the center, we can use an expansion

\[
q = q_1 \left( 1 - \frac{2}{r_2} r^2 \right)
\]  

(A.1)

between the center and the second mass point at \( r_2 \). We use this in Eq. (3) and integrate from 0 to \( r_2 \):

\[
M_2 = \frac{4}{3} \pi r_2^3 \left( q_1 - q_2 \right)/q_1.
\]

(A.2)

From Eq. (A.1) we obtain \( r = (q_1 - q_2)/q_1 \), and with this substitution Eq. (A.2) yields

\[
M_2 = \frac{4}{3} \pi r_2^3 \left( q_1 + \frac{1}{3} q_2 \right) .
\]

(A.3)

The unequal coefficients for \( q_1 \) and \( q_2 \) are specific for the inner boundary. Equal coefficients follow from the assumption \( \Delta r \ll r \), which is not true at the center. If in Eq. (A.1) more than two parameters are used to express \( q \), then Eq. (A.3) cannot be expressed in the two values \( q_1 \) and \( q_2 \). A third density in this equation would destroy the over all structure of the equations. The other equations at the center are:

\[
P_1 - P_2 = \frac{GM^2}{2r_2} \left( \frac{2}{3} q_1 + \frac{1}{3} q_2 \right) .
\]

(A.4)

\[
L_2/M_2 = \frac{3}{2} \epsilon_1 + \frac{3}{2} \epsilon_2 - \frac{3}{2} \frac{dQ}{dt_1} - \frac{3}{2} \frac{dQ}{dt_2} .
\]

(A.5)

\[
T_1 - T_2 = \frac{1}{2} (F_1 + F_2) (P_1 - P_2) .
\]

(A.6)

The radiative gradient away from the center is obtained by multiplying a tabulated function with \( L_2/M_2 \). In the center the factor \( L/M \) is replaced by the energy production per gram, which is \( \epsilon_1 - (dQ/dt)_1 \). The inner boundary conditions are expressed as two equations in the four unknowns \( q_2 \), \( r_2 \), \( T_2 \) and \( L_2 \). These two equations are obtained from Eqs. (A.3-6) through elimination of \( T_1 \) and \( q_1 \). This elimination proceeds in the most stable way through a pivot selection process. This method to formulate the inner boundary condition improves the numerical stability at the center considerably.

2. Structure of the Elimination Scheme

The linearized system of equations consists of blocks of the form:

\[
H_1 X_{k+1} + J_1 X_k = S_1 .
\]

(A.7)

\[
J_2 X_{k+1} + K_2 X_k = T_2 .
\]

(A.8)

The vector \( X_k \) consists of the corrections \( \delta T \), \( \delta q \), \( \delta r \) and \( \delta L \) at the mass point \( M_k \). The matrices \( H_1 \), \( I \), \( J \) and \( K \) each have two rows in normal cases. Equation (A.7) results from Eqs. (1) and (2), and Eq. (A.8) results from Eqs. (3) and (4). The boundary conditions are:

\[
H_1 X_2 = S_1 .
\]

(A.9)

\[
K_\nu X_\nu = T_\nu .
\]

(A.10)
The two equations of the outer boundary condition (A.10) are obtained from the table of envelopes (see Section 2). The inner boundary condition has been discussed in the previous section. At some points in the star X should contain more than four components (see Section 2.1). Since this disturbs the present form of the elimination scheme, a mass point $M_m$ with more than four variables is removed from the system, and information is stored to retrieve the solution vector $X_m$. This results in an equation of the form of (A.7–8) connecting the points $m-1$ and $m+1$. The extra equations are added as additional rows to $H_{m-1}$, $I_{m-1}$ and $S_{m-1}$. Similarly, the matrices $I_m$, $K_m$, $H_{m-1}$ and $J_{m-1}$ have additional columns. Take matrices and vectors together as follows:

$$
H_m \rightarrow A \quad I_m \rightarrow B \quad H_{m-1} \rightarrow C
$$

$$
I_{m-1} \rightarrow D \quad S_m \rightarrow P \quad S_{m-1} \rightarrow Q.
$$

The equations containing $X_m$ are now:

$$
AX_{m+1} + BX_m = P \quad CX_m + DX_{m-1} = Q.
$$

This set is rearranged to:

$$
X_m = C^{-1}Q - C^{-1}DX_{m-1}
$$

$$
AX_{m+1} - BC^{-1}DX_{m-1} = P - BC^{-1}Q.
$$

Equation (A.12) has the same format as Eq. (A.7–8) and so it fits into the general solution method. Equation (A.11) is used to retrieve $X_m$; to this end $C^{-1}Q$ and $C^{-1}D$ are stored. For the general solution we make $4 \times 4$ matrices $A_k$, $B_k$ and $C_k$ and four-dimensional vectors $P_k$ as follows:

$$
T_{k+1} \rightarrow P_k \quad K_{k+1} \rightarrow A_k \quad J_{k+1} \rightarrow O \quad \rightarrow B_k \quad I_{k+1} \rightarrow C_k.
$$

Here, $O$ stands for a $2 \times 4$ zero-matrix. To solve the complete system in the most efficient way, we calculate and store the sequences $F_k(n-1 \geq k \geq 2)$ and $G_k(n-1 \geq k \geq 1)$ with the equations:

$$
F_{n-1} = A_n^{-1}C_{n-2} \quad F_k = (A_k - B_kF_{k+1})^{-1}C_{k-1}
$$

$$
G_{n-1} = A_n^{-1}P_{n-1} \quad G_k = (A_k - B_kF_{k+1})^{-1}\left\{P_k - B_kG_{k+1}\right\}.
$$

Then the solution follows from

$$
X_2 = G_1 \quad X_{k+1} = G_k - F_kX_k.
$$

Equations (A.13–14) and (A.15) are forward and backward elimination respectively. Most matrix operations involved in the above equations are handled by a highly optimized machine code procedure, which speeds up the program considerably.

At first sight, it seems only an unnecessary complication to make the system of equations blocked tridiagonal by taking blocks of zeros into $B_k$ and $C_k$. However, it turns out that the manipulations are only rearranged throughout the system. The main reason for this formulation is that the zero blocks correspond to coefficients arising from second derivatives with respect to mass in the structure equations. In this way the rearrangement makes it possible, to build into the program diffusion equations for the chemical composition.

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Note added in proof. The expression which was used for the triple alpha rate was taken from a preprint. However, according to the note added in proof in Fowler et al. (1975), the coefficient in the first term of this expression should be changed from 2.49E-08 to 3.00E-08. Computations with this revised rate give very similar results, the final carbon content (mass fraction) of the C/O-cores increased with a few percent.