Some aspects of evolution and mass transfer in X-ray binaries
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ROCHE-LOBE OVERFLOW IN X-RAY BINARIES

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Summary

It is examined whether Roche-lobe overflow can be the main mechanism of mass transfer that powers the low-mass as well as the massive X-ray binaries. Detailed numerical computations of the initial phase of Roche-lobe overflow were performed in order to determine the precise time development of the mass transfer from normal stars with masses ranging from 1.5 \( M_\odot \) up to 20 \( M_\odot \) to compact companions with masses of 1 \( M_\odot \) and 1.5 \( M_\odot \). The numerical calculations were performed with a binary code which enables a numerically accurate determination of the mass transfer rate \( \dot{M} \) by introducing \( \dot{M} \) as an extra Henyey unknown. The binary code includes a simplified hydrodynamical treatment of Roche-lobe overflow. For massive primaries this hydrodynamical treatment appears to result in much longer X-ray lifetimes than obtained in previous investigations. The calculations also include effects of slow, non-synchronous rotation of the contact star and loss of mass and angular momentum from the binary system. It appears that if in massive X-ray binaries (\( M_1 \sim 16 \ M_\odot \)) the mass transfer starts when the primary star is still burning hydrogen in its core, it may take some \( 10^4 \) yrs before \( \dot{M} \) exceeds the critical accretion rate of the neutron star. For Her X-1 and Cen X-3 X-ray lifetimes of the order of \( 10^5 \) and \( 10^4 \) yrs are predicted, respectively. For these two X-ray pulsars the spin-up rates predicted from our calculations - under the assumption that the pulsars are alway spinning at their equilibrium spin rates - agree very well with the observed average spin-up rates. If the companion of SMC X-1 is still burning hydrogen in its core, a lower limit for the spin-up rate can be predicted, which is consistent with the observations.

Key words: Roche-lobe overflow, X-ray binaries, X-ray pulsars.
1. INTRODUCTION

Some ten X-ray sources have been identified with close binary systems. The compact X-ray emitting star in these systems — in several cases a pulsar — is presumably accreting matter which originates from its normal companion star. The accreting gas is strongly accelerated and heated when it falls towards the compact star, causing it to become a strong source of X-rays. The normal components of the X-ray binaries appear to cluster in two distinct mass regions. Either they are massive supergiants (M \textasciitilde 15 to 20 M_☉) like the companions of Vela X-1, SMC X-1 and Cen X-3, or they are dwarf stars with masses of about 1 to 2 M_☉ such as the companions of Her X-1, Sco X-1 and Cyg X-2. The explanation usually suggested for this division into two groups is (cf. Davidson and Ostriker 1973, van den Heuvel 1975) that two different physical processes are responsible for the transfer of matter, i.e. Roche-lobe overflow in the low-mass systems and stellar wind in the massive binaries. Only blue supergiants and Of stars with masses \textasciitilde 15 M_☉ appear to have stellar winds strong enough to turn a compact companion into a bright X-ray source.

According to conventional calculations of Roche-lobe overflow the primary star loses a large fraction of its mass roughly on its Kelvin-Helmholtz timescale (cf. Paczynski 1971). For stars more massive than about 2 M_☉ this would imply an average mass transfer rate in excess of \textasciitilde 10^{-7} M_☉/yr, which is more than the compact star can accrete. In that case the matter is expected to pile up around the X-ray emitting region and to suffocate the X-ray source. On the basis of these conventional calculations it has been concluded that it is unlikely that X-ray sources with companions more massive than 2 M_☉ are powered by Roche-lobe overflow.

However, doubts may be raised regarding the correctness of this simple view since, even for Roche-lobe overflow from massive stars it will take some time before the mass transfer rate increases beyond the
critical value (e.g. Paczynski 1976). During this initial phase the compact X-ray emitting star can still be powered by Roche-lobe overflow.

The calculation of the precise duration of this initial phase requires refinements in the binary evolution code which in general have not been taken into account in previous investigations on X-ray binaries. That the detailed calculation of this duration is most important for a better understanding of the evolution of X-ray binaries is indicated by the characteristics of the X-ray binary HZ Her/Her X-1, one of the best studied X-ray systems. It consists of a normal star of roughly $2 M_\odot$ and a compact star - presumably a magnetized neutron star - of about $1.3 M_\odot$ (Avni and Bahcall 1975). The observed X-ray luminosity of the source ($L_x \sim 1.3 \times 10^{37}$ erg/sec) implies an accretion rate of about $10^{-9} M_\odot$/yr which is two orders of magnitude smaller than expected on the basis of previous calculations of Roche-lobe overflow, as was pointed out by Pratt and Strittmatter (1976). The normal star is expected to have only a weak stellar wind, certainly insufficient to power the observed X-ray source. Also, the self-excited mass transfer due to the evaporative effect of the X-ray flux on the primary's envelope is insufficient, as was shown by numerical hydrodynamical calculations by Alme and Wilson (1974), and confirmed by computations of McCray and Hatchett (1975).

There is also strong observational evidence from optical pulsations that HZ Her fills its Roche lobe (Middleditch and Nelson 1976); also the presence of a large accretion disk in this system (Gerend and Boynton 1976, Petterson 1976) strongly suggests mass transfer by Roche-lobe overflow. Hence, it seems very likely that HZ Her is presently in such an initial stage of Roche-lobe overflow. Also in the case of some of the massive X-ray binaries the observations indicate that the supergiants (e.g. the companions of Vela X-1, Cen X-3 and SMC X-1) are very close to
or filling their Roche lobes (Avni and Bahcall 1977, Hutchings 1976, Hutchings et al. 1977, van Paradijs et al. 1977). In addition, especially in the case of Cen X-3 and SMC X-1 the accretion rates derived from the observed X-ray luminosities are considerably larger than the maximum accretion rates expected from a stellar wind (Lamers et al. 1976). Moreover, optical observations of SMC X-1 suggest the presence of an accretion disk in this system (van Paradijs and Zuiderwijk 1977). Hence, here also beginning Roche-lobe overflow may be taking place - perhaps mixed with stellar wind accretion (cf. Ziolkowski 1976, Conti 1977).

For these reasons, we have performed detailed calculations in order to determine the duration of the initial stage of Roche-lobe overflow for a variety of binary systems. A method is presented which enables a numerically very accurate determination of the mass transfer rate by introducing this transfer rate as an extra Henyey unknown. The corresponding extra Henyey equation is obtained from the relation which links the mass transfer rate with the excess of the stellar radius beyond the Roche lobe radius. The latter relation is a refinement of the expression derived by Jedrzejec (1969).

In chapter 2 we will put forward some refined considerations concerning Roche-lobe overflow and outline our computational method of binary evolution (sections 2.1, 2.2). The effects of non-synchronous (slow) rotation of the contact star as suggested by Pratt and Strittmatter (1976) will be discussed in section (2.3). In section (2.3.4) we give the expressions for the re-synchronization timescale corresponding to different frictional effects in the envelope of the contact star, as used in our evolutionary calculations. The resulting X-ray lifetimes and calculated curves of $\dot{M}$ versus time since the onset of Roche-lobe overflow for the various systems studied, are presented in chapter 3. These results are compared with the observational data on X-ray binaries. In
chapter 4 we compare the observed spin-up rates of Her X-1 and Cen X-3 with the rates predicted on the basis of our computations of beginning Roche-lobe overflow mass transfer. The agreement appears to be surprisingly good. The details of our computational method are presented in appendices I to III.

2. Refined Considerations Concerning Roche-lobe Overflow

2.1. General Assumptions

We have abandoned some of the assumptions usually made in the computations of binary evolution, such as synchronous rotation of the primary (contact star) during mass transfer, negligible spin angular momentum of the primary and conservation of mass and angular momentum of the system. A further, most important refinement is the treatment of the outer boundary condition of the contact star. This treatment, which forms the key part of this investigation, is described in section (2.2) and appendices I, II. The following general assumptions have been retained in our calculations:

1. Roche approximation of the gravitational field, i.e. the gravitational field of the system is approximated by that of two point masses. This is certainly a good approximation for the compact component, but also normal stars are sufficiently centrally condensed for this to be a good approximation.

2. Circular orbits. For most observed X-ray binaries this is a very good approximation.

3. Rigid rotation of the normal star component. Probably this is the most controversial assumption, but very little is known about differential rotation and redistribution of angular momentum in stars (e.g. James and Kahn 1970, 1971).

1) We will use this term loosely, i.e. also when the contact star is not corotating with the orbital revolution.
4. The contact star is assumed to be in hydrostatic equilibrium. This is a very good approximation except in the small region near the neutral point.

5. Synchronous rotation of the primary when it reaches its critical lobe. For the close binary systems studied here, tidal synchronization is efficient enough to erase any initial non-synchronity between the primary's rotation and the orbital motion during the past evolution of the binary. This may not be true for systems in which the primary is in a stage of rapid envelope expansion as is the case during the shell hydrogen burning stage (case B of mass transfer).

6. Negligible rotational momentum of the compact neutron star. The angular momentum stored in the rotation of the neutron star is extremely small compared to the remaining angular momenta in the system, since a neutron star has a very small moment of inertia.

7. Negligible angular momentum of an accretion disk around the compact star. The masses of accretion disks are thought to be so small ($<10^{-6} M_\odot$) that this seems a reasonable assumption (Novikov and Thorne 1974). Very little is known of the structure and time-evolution of accretion disks. We will therefore study the effect of a decrease of the orbital angular momentum during mass transfer in a somewhat ad hoc way (section 3.5.3).

2.2. Outer boundary conditions of the contact star

For reasons of computer economics the binary evolution code deals, as usual, with one dimensional spherically symmetric stellar models. In order to compare the size of the contact star with that of the non-spherical critical lobe one usually defines a critical radius $R^C$, which is the radius of a sphere with the same volume as the lobe.

In computations of binary evolution with mass exchange one
determines the mass transfer rate $\dot{M}$ from the relation between the radius of the contact star $R^*$ and the radius of the critical lobe $R^c$.

Usually one simply demands equality of these two radii, i.e. $R^* = R^c$, during the stage of mass transfer in the binary system (see, e.g. Kippenhahn and Weigert 1967, Paczynski 1971).

In this investigation we intend to follow the beginning of Roche-lobe overflow in detail, so that we are interested in the precise time development of the mass transfer. In order to enable such refined calculations, the single star evolution code as described by Savonije and Takens (1976) was extended to include mass transfer in semi-detached binary systems. With this adjusted code it is possible to calculate evolutionary sequences with mass transfer, for which each individual model of the contact component satisfies the imposed outer boundary conditions to a high degree of accuracy. The special way by which this is achieved is described in appendix II.

The outer boundary conditions themselves were also refined by replacing the simplified condition $R^* = R^c$ by a physically more realistic relation, as follows. In a steady state situation the outward mass flow due to the evolutionary expansion is balanced by the horizontal mass flow, that is the flow along the surface of the contact star toward the region around the neutral point, where it can escape from the star. For a given expansion rate of the envelope, the efficiency of this horizontal flow determines the amount by which $R^*$ exceeds $R^c$, as well as the mass transfer rate $\dot{M}$ towards the companion.

Obviously, the condition $R^* = R^c$ corresponds to an idealized, infinitely efficient horizontal flow. In order to obtain a more realistic boundary condition, we have estimated the mass flux density $\rho v^+\|_\nu$ of this horizontal flow in the vicinity of the neutral point.
by using Bernoulli’s theorem and some results obtained by Lubov and Shu (1975). Following Jedrzejec (Paczynski and Sienkiewicz 1972) the thus derived expression for \( \rho \frac{\partial v}{\partial t} \) is then integrated over the cross-section of the flow in the vicinity of the neutral point. In this way an analytical expression is obtained (A10) which relates the rate of mass transfer \( \dot{M} \) to the excess of the contact star’s radius \( \Delta R = R^* - R^C \).

A consistent determination of the mass transfer rate \( \dot{M} \), however, demands the inclusion of its feedback on the envelope expansion rate of the contact star. This feedback arises because the mass outflow in the envelope (determined by the radius excess \( \Delta R \)) on its turn affects the thermal structure of the contact star and thus \( \Delta R \). The full coupling between \( R^*, R^C \) and \( \dot{M} \) can be taken into account by introducing \( \dot{M} \) as an extra Henyey unknown, to be solved iteratively. The corresponding extra equation follows from the Roche constraint on the contact star:

\[
(1 - \eta) R^* - R^C = 0
\]  

(3)

Such a generalized boundary condition (one usually puts \( \eta = 0 \)) follows from the simplified hydrodynamical treatment of Roche-lobe overflow and allows the contact star to be slightly bigger than its critical lobe. For a given binary system the relative radius excess \( \eta = \Delta R / R^* \) is essentially determined by the mass transfer rate \( \dot{M} \). Its value is calculated from expression (A12a). It appears that the iterative corrections to \( \eta \) can be determined completely from the iterative corrections to the three Henyey unknowns \( L^*, T^* \) and \( \dot{M} \) (see appendices I to III), where \( L^* \) and \( T^* \) are the surface luminosity and surface temperature of the contact star, respectively.

By coupling the generalized condition (3) to the normal single star...
boundary conditions, in a way described in detail in appendix II, \( \dot{M} \) is determined self-consistently during the Henyey iterative cycle. That is, from the corrected values of the three Henyey unknowns \( L^*, T^* \) and \( \dot{M} \) obtained after each Henyey iteration, we recalculate \( n, R^* \) and \( R_C \) for the next iteration; we iterate until all structure equations - including equation (3) - are satisfied to within a prescribed accuracy. In this way all structural variables, including the mass transfer rate \( \dot{M} \), are determined within a few Henyey iterations.

2.3. Effects of non-synchronous slow rotation on the mass transfer rate

2.3.1. Pratt and Strittmatter's mechanism

In order to explain the (low) accretion rate of Her X-1
\( (\dot{M}_{acc} = 10^{-9} M_\odot yr^{-1}) \) in terms of Roche-lobe overflow, Pratt and Strittmatter (1976) - herein after termed PS - suggested that the usual assumption of synchronism between the orbital and rotational motion of the contact star during mass exchange should be abandoned. They argued that the contact component spins down during mass exchange, since the matter ejected near the gravitational saddle point carries a relatively high specific rotational angular momentum. It is implicitly assumed that the gas is essentially corotating with the contact star when it flows through the neutral point region. This seems to be a reasonable approximation at the onset of Roche-lobe overflow, when deviations from synchronous rotation are small.

PS considered binary systems having a primary of 1.5 \( M_\odot \) and secondaries of either 0.5 \( M_\odot \) or 1 \( M_\odot \); the orbital periods were taken smaller than \( \sim 0.7 \). Such systems undergo Roche-lobe overflow when the primary is still burning hydrogen in its core (case A mass transfer). By taking the spin-down of the contact star into account in the evolutionary calculations, PS found that the mass transfer proceeds
very slowly, on roughly a nuclear timescale, in contrast to the "classical" results for which the timescale of mass transfer is roughly the Kelvin-Helmholtz timescale of the star (cf. Paczynski 1971). This seems a promising result, but it does not apply to the binary HZ Her/Her X-1, because this system has a longer binary period of 1.7 days (Avni and Bahcall 1975). Such a longer binary period implies a larger orbital separation and hence a wider Roche lobe around HZ Her. According to evolutionary calculations for such a wider system, the Roche-lobe overflow can only begin during the envelope expansion stage after the exhaustion of hydrogen in the core (case B of mass transfer). We will therefore consider how PS' results are modified when we have case B of mass transfer instead of case A (section 2.3.2).

The stabilizing influence on the process of mass transfer due to the spin-down of the contact star in binary systems results from two effects, i.e., from the decrease of the centrifugal acceleration near the stellar surface and from the conversion of the contact star's rotational momentum into orbital angular momentum (an unknown fraction of the transferred angular momentum may be stored in an accretion disk or may be lost from the system). Calculations show that (for mass ratios \( M_2/M_1 > 0.2 \)) these two effects in general cause the primary's critical lobe to expand slightly during the beginning of the mass transfer stage. Obviously, the stabilizing effect will finally disappear when most of the primary's rotational momentum has been lost. However, in section 2.3.3 we will see that due to dynamical effects in the moving tidal bulge the stabilizing influence disappears long before that (and may even rather soon turn into its opposite).

2.3.2. The timescale for slow mass transfer due to non-synchronous rotation

Without spin-down the critical lobe of the primary would shrink when
mass is transferred. In general, a contact star with a radiative envelope can shrink at practically the same rate as its critical lobe by rapidly transferring all overflowing matter to the companion. This is a self-accelerating process and the star will lose practically its complete envelope on roughly a Kelvin-Helmholtz timescale as mentioned before. This thermal runaway stage is followed by a second, slow phase of mass transfer during which the critical lobe widens again (the widening being due to the reversal of the mass ratio). The increase of the critical lobe in case the PS-mechanism is taken into account invokes a situation rather similar to this second stage of slow mass transfer. When the critical lobe increases in size when mass is exchanged, the mass transfer will no longer be self-accelerating, but is maintained by the continuous expansion of the stellar envelope caused by the evolutionary changes in the internal structure of the star.

Hence, by taking into account the spin-down of the contact star, the initial mass transfer will roughly proceed on the timescale of free envelope expansion. Consequently, the initial slow transfer either proceeds on relatively long timescale related to the nuclear timescale of core hydrogen burning (case A of mass transfer), or - when the star is in the shell hydrogen burning stage of evolution - the initial mass transfer will take place on a (much shorter) timescale related to the thermal timescale of the contracting helium core (case B of mass transfer).

2.3.3. Critical surfaces of non-synchronously rotating binary components

a. General equations.

Pratt and Strittmatter (1976) used an expression for the size of the critical lobe, which is based on the approximation that the effective gravitational potential in the binary system can be regarded as independent of time. However, this approximation is valid only if the deviations from synchronism are small as will be discussed below.
Various authors have derived approximate expressions for the equipotential surfaces of non-synchronously rotating stars in binary systems (cf. Plavec 1958, Limber 1963). We will briefly recapitulate these results in order to clarify the inherent approximations.

Consider a close binary which has as components a normal star of mass $M_1$ and a compact star of mass $M_2$. Suppose that the two stars revolve about their common center of mass along circular orbits with constant angular velocity $\omega$. Introduce a Cartesian coordinate system $(\xi, \eta, \zeta)$ which is at rest in a reference frame corotating with the normal star at constant angular velocity $\omega$. It is assumed that $\omega \parallel \hat{n}$. The center of our coordinate system is taken at the center of mass of star $M_1$, while its positive $\zeta$-axis is directed along $\omega$. We can now express the equation of motion for matter in the binary system as (cf. Kruszewski 1966):

\[
\frac{d^2 \mathbf{r}}{dt^2} + 2 \omega \times \frac{d \mathbf{r}}{dt} = -\frac{1}{\rho} \nabla P + \nabla \psi
\]  

(1)

where the effective gravitational potential $\psi$ is given by:

\[
\psi = \frac{GM_1}{r_1} + \frac{GM_2}{r_2} - \frac{aq}{q+1} \eta^2 \hat{n} + \frac{1}{2} \omega^2 (\xi^2 + \eta^2)
\]  

(2)

In equation (2) $r_1$ and $r_2$ are the distances from the two stellar centers to the point $\mathbf{r}$ under consideration, $a$ is the distance between these two stellar centers, $q = \frac{M_1}{M_2}$ and $r_\hat{n}$ is the projection of $\mathbf{r}$ onto the plane $\zeta = 0$. If the normal star rotates uniformly and synchronously (below break-up velocity) the left-hand side of equation (1) is zero for all mass elements belonging to this star. Under these circumstances equation (1) becomes $\nabla P = \rho \nabla \psi$, which is the equation of hydrostatic equilibrium.

Suppose that star $M_1$ expands beyond the particular equipotential surface on which the neutral point $\mathbf{r}_c = (\xi_1, 0, 0)$ is situated. This neutral point is defined by the condition that the local effective gravity
is zero, i.e. \( (\nabla \psi)_{r_c} = 0 \). In that case the right-hand side of

\[ \text{equation (1)} \]

is no longer zero. The resulting unbalanced pressure

gradient near \( r_c \) will push the gas through the gravitational saddle

point towards the detached component. Since the ejected mass carries a

high specific rotational momentum the contact star will spin down

(PS' mechanism) and will no longer rotate synchronously with the

orbiting neutron star. This makes the potential \( \psi \) time-dependent (the

neutron star moves with respect to our \((\xi, \eta, \zeta)\) system) and the tidal

bulge begins to rotate along the surface of the primary. In principle,

the star is able to adjust its shape to the instantaneous hydrostatic

equilibrium configuration, since the period of free adiabatic pulsations

of a star is always shorter than the orbital period of a companion.

b. The "first approximation".

However, the dynamical effects of the resulting shape-adjusting motions

become increasingly important when the primary continues to slow down.

The latter effect was neglected by PS (1976), i.e., they assumed that

the terms on the left-hand side of equation (1) were small compared to

those on the right-hand side. Following Limber (1963) we will call

this the "first approximation". Limber made a detailed study of the

inherent inconsistency of this approximation. According to this

approximation the critical lobe is given by the limiting surface

\[ \psi = \psi(\hat{r}_c) \]. The mass transfer will start as soon as the star expands

beyond this critical surface, as shown above. However, when the primary's

rotation slows down, the tidal oscillations at its surface become more and

more violent; then at a certain moment, mass elements will be ejected

even from points inside the critical lobe \( \psi = \psi(\hat{r}_c) \). Kruszewki (1966)

derived an expression for the critical surface at which matter is still

just bound to the star. Obviously, this (closed) critical surface

coincides with \( \psi = \psi(\hat{r}_c) \) only when \( \omega = \Omega \) and it lies inside the surface

\[ \psi = \psi(\hat{r}_c) \] as soon as \( (\omega/\Omega) \) becomes smaller than unity. The problem is
that mass elements which are ejected from points just outside this critical equilibrium surface are recaptured by the contact star (Kruszewski 1966). This is due to the time variation of \( \psi \) during the ejection. The actual limiting surface, from which matter is really ejected, will thus be situated somewhere between the two above mentioned critical surfaces.

Thus, when the "first approximation" brakes down it seems only possible to calculate the mass transfer rate with a time-dependent numerical hydrodynamical code. This in order to follow the motion of ejected mass elements in detail. Of course, such a procedure heavily complicates the evolutionary computations and would require very large computer facilities. For these reasons we have restricted ourselves to applying the "first approximation" in our computations. Limber (1963) derived a condition that allows us to check whether the first approximation is still reliable. We have used this criterion in our calculations (section 3.2).

c. Validity of the "first approximation".

It is expected that the "first approximation" is valid only for values of \( (\omega/\Omega) \) that are close to unity. For the type of binaries studied here, the phase velocity of the tidal bulge exceeds the sound velocity at the stellar surface already when \( (\omega/\Omega) \) decreases below a value of about 0.9. Hence, in that case the neutral point rotates at super-sonic velocities with respect to our corotating \( (\xi, \eta, \zeta) \)-system and the simple hydrodynamical analysis of appendix I as well as the "first approximation" will certainly have become invalid. For such a deviation from synchronous rotation the tidal oscillations become so violent that the non-synchronism is expected to result in a very large mass transfer rate. However, it is possible that in close binaries the ejected but recaptured mass elements can efficiently re-synchronize the contact star,
thus preventing appreciable deviations from synchronism. Besides, we know that tidal torques in close binaries also tend to re-synchronize the contact star.

Our calculations show that the "first approximation" is valid only during the very beginning of Roche-lobe overflow, when \( (\omega/\Omega) \) has not yet decreased below a value of 0.99. In the next section we will consider the re-synchronizing effect of tidal torques on the contact star.

2.3.4. Re-synchronization of the contact star - effects of tidal friction

The frictional damping in the surface layers of the contact star will force the tidal bulge to lag behind the orbital motion of the neutron star. If this phase shift can be determined, it is easy to calculate the resulting tidal torque and synchronization rate. According to present opinions, the following frictional mechanisms are important in close binaries:

a) **Eddy viscosity.** Stars with deep convective envelopes (like the sun) are expected to be synchronized very efficiently by turbulent friction in the envelope (Zahn 1966). The estimated timescale for synchronization is:

\[
\tau_s = \frac{1}{6 q^2 k_2} \left( \frac{M R^2}{L} \right)^{1/3} \frac{I}{M R^2} \left( \frac{a}{R} \right)^6 \text{ sec}
\]

(4)

where \( q = \frac{M_1}{M_2} \), \( k_2 \) is the apsidal motion constant of the primary, \( I \) is the moment of inertia of the primary, \( a \) is the orbital separation and the other variables refer to the radius and luminosity of the primary. The synchronization timescale given by equation (4) is so short that the PS-mechanism is not effective when the primary has a deep convective envelope.

b) **Radiative damping of the dynamical tide.** Stars with convective cores and radiative envelopes can be efficiently synchronized during the early phases of core hydrogen burning, when the convective cores have
large dimensions (Zahn 1976). The rotating gravitational field of the orbiting companion gives rise to resonances of the free gravity-modes, yielding a "dynamical tide", which is phase-shifted due to radiative losses near the stellar surface. The efficiency of this process depends very sensitively on the value of the fractional radius of the convective core. Calculations show that for none of the systems studied here this process can compete with the spin-down due to the mass exchange. (For mass exchange according to case B, this mechanism is irrelevant as the primary has no convective core at all in that case.)

c) Shear-turbulent viscosity. Press et al. (1975) and Lecar et al. (1976) have suggested that even fully radiative stars in close binaries can be synchronized quite efficiently. The fluid shear in the tidal flow corresponds to a Reynolds number much beyond the critical value so that a large region of fully developed turbulence may be generated. However, serious doubts on the occurrence of this kind of turbulence have been put forward by Zahn (1976) and Seguin (1976). Seguin, who performed a linear stability analysis of the tidal flow finds that the growth rate of small (linear) instabilities is much smaller than typical tidal frequencies in a close binary, so that the turbulent instabilities will not grow to fully developed turbulence. This will be especially the case when deviations from synchronous rotation are still very small, such as during the initial stage of Roche-lobe overflow. Yet, we have included this mechanism in our calculations in order to study the effect of an efficient (hypothetically) re-synchronizing mechanism.

Press et al. assumed that the eddy turnover time is small enough to result in a turbulent region of the same dimensions as the star. If this region would be confined to the outer layers of the star, \( \tau_s \) in equation (5) should be increased by the square of the length reduction factor. The synchronization timescale given by Press et al. is:
\[ \tau_s = \frac{75}{224} \frac{R_T}{K_\mu |\Omega-\omega|} \frac{I}{M R^2} \left(\frac{a}{R}\right)^9 \left(\frac{M_1}{M_2}\right)^3 \text{ sec} \tag{5} \]

where \( R_T \) is the effective Reynolds number of the tidal flow (assumed to be 20) and \( K_\mu \) is a structural constant defined by Press et al.

Depending on the envelope structure of the contact star either expression (4) or (5) can be included in the equations which govern the variations of the orbital parameters during mass exchange. This is worked out in appendix III.

2.4. Input Physics for the evolutionary code

Apart from the adjustments necessary to treat the occurrence of Roche-lobe overflow, the evolutionary code is identical to that described by Savonije and Takens (1976). This code is a revised version of Paczynski's evolutionary code (eg. Paczynski 1970) and uses the same equation of state for the stellar matter.

Throughout the calculations the Cox-Stewart radiative opacities are applied. The nuclear reaction rates were calculated from the expressions presented by Fowler et al. (1975). The ZAMS chemical composition was adopted as \( X = 0.70, Z = 0.03 \). It was assumed that the equilibrium abundance of \(^{14}\)N during hydrogen burning was always equal to 0.015. The mixing length was adopted to be one pressure scale height.
3. RESULTS

3.1. Calculated rates of mass transfer

With the above described program we have followed the phase of beginning Roche-lobe overflow in binary systems with primaries in the mass range $1.5 \, M_\odot - 20 \, M_\odot$ and with either a $1 \, M_\odot$ or $1.5 \, M_\odot$ secondary (which is considered to be a neutron star). In order to study both case A and case B of mass transfer, for each system at least two different orbital periods were adopted. For all these systems the time development of the mass transfer (Roche-lobe overflow) was calculated under various assumptions regarding the exchange of angular momentum in the binary and the hydrodynamics of the flow. For all systems the initial chemical composition was taken to be $X = 0.70$ and $Z = 0.03$.

The results are shown in figures 1 to 11. The solid curves refer to the calculations in which the hydrodynamics are taken into account, while the dotted curves (fig. 5 and 6) are obtained when the condition $R_* = R_C$ is applied. In the figures the various underlying assumptions concerning exchange of angular momentum are indicated by different symbols, as follows:

- S - "classical" case in which the primary is assumed to rotate synchronously with the orbital motion throughout the mass exchange phase. The orbital angular momentum and the mass of the system are conserved (the rotational angular momenta of the components are neglected). These are the so-called "conservative assumptions" generally used in calculations of close binary evolution (cf. Paczynski 1971).

- $S^*$ - "non-synchronous" treatment; however, due to the assumed very short re-synchronization timescale (e.g. in case expression (4) applies) the contact star remains in synchronous rotation. This case differs from assumptions S as it is not assumed here that the rotational angular momentum of the contact star is negligible. Hence, when the binary period decreases, orbital momentum is converted into rotational
momentum of the contact star, causing a more rapid shrinking of the Roche lobe than in case S.

N - non-synchronous case considered by Pratt and Strittmatter. No tidal re-synchronization is taken into account; here it is assumed that the rotational angular momentum lost by the contact star is directly converted into orbital angular momentum.

N1 - like N, but in this case the spin-up of the contact star due to tidal torques is taken into account according to the expression for the synchronization timescale as given by equation (5). As discussed in section (2.3.3) it is doubtful whether such a strong tidal torque will occur in stars with radiative envelopes.

N2 - in this case it is assumed hypothetically that only the outer layers (with a thickness of $\alpha$ times the stellar radius $R^*$) become turbulent due to the sheared tidal flow at the surface. Hence, expression (5) for the tidal synchronization timescale is multiplied by the factor $\alpha^{-2}$ (Press et al. 1975), where $\alpha = (\xi_1 - R^*)/(\xi_1 R^*)^1$ ($\xi_1$ is defined in section 2.3.3).

L - in this case it is assumed ad hoc that half of the transferred mass leaves the system immediately, carrying a specific angular momentum of twice the specific orbital angular momentum of the neutron star.

Obviously, the curves corresponding to assumptions N and S are the two limiting cases for the development of the real mass transfer rate.

The region where the accretion rates are expected to become rapid enough to suffocate the X-ray source is expected to begin somewhere between the dashed horizontal lines in the figures. The lower dashed line corresponds to the Eddington critical luminosity. Assuming that the X-ray luminosity is given by $L_x = 0.1 c^2 M_{\text{acc}}^2$ (cf. Zel'dovich and Novikov 1971) the corresponding critical accretion rate is
where \( \dot{M} \) is the mass of the accreting object in solar units (Davidson and Ostriker 1973, Shakura and Sunyaev 1973). However, for a magnetized neutron star, the accretion is not likely to be spherically symmetric as is assumed in the derivation of expression (6). This will presumably increase the critical accretion rate, but probably not by more than a factor of \( \sim 3 \) (Rees 1974; Thorne 1974; Basko and Sunyaev 1976). This increased critical rate (3 \( \times \dot{M}_{\text{Edd}} \)) corresponds to the upper dashed line in the figures.

3.2. X-ray lifetimes and validity of the "first approximation"

Consider the curves in the figures (\( \dot{M} \) vs. time since the onset of Roche-lobe overflow) which correspond to calculations including the slow-down of the contact star during mass exchange (N). An arrow mark on such a curve indicates the point where the "first approximation" - as mentioned in section (2.3.3) - becomes no longer justified. In order to find these points, the expression \( \Lambda(\mu;\Omega) \) derived by Limber (1963) was calculated for each evolutionary model; the arrow mark corresponds to \( \Lambda(\mu;\Omega) = 10^{-2} \) which is the limit beyond which - according to Limber - the "first approximation" breaks down. It was found that this value of \( \Lambda \) was in general exceeded already for values of \( (\omega/\Omega) \) below 0.99, i.e. for a very small deviation from synchronous rotation. For the X-ray lifetime \( \tau_x \) of a binary system (containing a compact star) we have taken the time-interval between the onset of Roche-lobe overflow and the instant at which either the mass transfer rate exceeds three times the Eddington critical accretion rate, or at which \( \Lambda(\mu;\Omega) \) becomes larger than \( 10^{-2} \). That is, we have simply assumed that when \( \Lambda > 10^{-2} \) the tidal oscillations induce such a large mass ejection rate from the surface of the contact star that the critical accretion rate of the neutron star
Table 1.

Typical X-ray lifetimes expected for close binary systems undergoing Roche-lobe overflow (Case A and Case B) from a primary of mass $M_1$ onto a compact companion of mass $M_X$.

<table>
<thead>
<tr>
<th>$M_1/M_\odot$</th>
<th>$M_X/M_\odot$</th>
<th>CASE A</th>
<th>CASE B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1</td>
<td>$&gt; 10^8$ yrs</td>
<td>$&lt; 10^2$ yrs</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$2 \times 10^6$ yrs</td>
<td>$10^5$ yrs</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>$3 \times 10^5$ yrs</td>
<td>$4 \times 10^4$ yrs</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>$4 \times 10^4$ yrs</td>
<td>$10^3$ yrs</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>$10^4$ yrs</td>
<td>$&lt; 10^2$ yrs</td>
</tr>
<tr>
<td>20</td>
<td>1.5</td>
<td>$5 \times 10^3$ yrs</td>
<td>$&lt; 10^2$ yrs</td>
</tr>
</tbody>
</table>
is immediately exceeded. This is presumably too pessimistic an estimate, but without detailed hydrodynamical calculations - which are probably beyond present possibilities - no better estimate is feasible. Hence, it is very well possible that in reality when \( \Lambda \) becomes larger than \( 10^{-2} \) the mass transfer rate deviates much less from the drawn curves than we assumed here; our definition for \( \tau_x \) will thus tend to yield a lower limit to the X-ray lifetime of the systems. It follows from the computations that to our order of approximation the relatively short time interval during which the X-ray source is still weak \( (-\dot{M} < 10^{-11} M_\odot/yr) \) can be neglected. The X-ray lifetimes \( \tau_x \) derived from our calculations, which included a hydrodynamical treatment of the overflow and the spin-down of the contact star due to the PS-mechanism, are given in table 1.

We now give a brief description of the time development of \( \dot{M} \) for different values of the mass of the normal (contact) companion of the X-ray source.

3.3. **Low-mass X-ray binaries \( (M_1 < 2 M_\odot) \)**

3.3.1. **Primary of \( 1.5 M_\odot \)**

Case A. Figure 1 shows the time development of the mass transfer rate in a binary system with initial values \( M_1 = 1.5 M_\odot, M_2 = 1 M_\odot \) and \( P_D = 0.68 \) days. Only synchronous rotation of the primary is considered, as main-sequence stars up to a mass of about \( 1.5 M_\odot \) are expected to have deep convective envelopes. In such envelopes the resulting eddy viscosity (section 2.3.4) is thought to yield synchronization times that are short with respect to the mass transfer timescale, so that no deviations from synchronous rotation will occur. Our calculations show that even for a synchronously rotating contact star the transfer rate never exceeds \( 4 \times 10^{-8} M_\odot \text{yr}^{-1} \) and passes its maximum approximately
Table 2.

Evolution of a low-mass binary with initial parameters $M_1 = 1.5 \ M_\odot$, $M_2 = 1 \ M_\odot$ and $P_b = 0.68$ days. It is assumed that the tidal torque on the contact star ($M_1$) is efficient enough to keep this star in synchronous rotation during mass exchange (assumption $S^*$). $X_c$ is the hydrogen abundance (mass fraction), $\Delta L$ is the energy absorption rate in the primary's envelope, $L_c$ is the luminosity (in the same units) at the core boundary. Time is measured from the onset of Roche-lobe overflow in the system. All stellar parameters refer to the primary component.

<table>
<thead>
<tr>
<th>$t/10^7$ (yrs)</th>
<th>$-\dot{M}/M_\odot$</th>
<th>$M_1/M_\odot$</th>
<th>$X_c$</th>
<th>$R^*/R_\odot$</th>
<th>$-\Delta L/L_c$</th>
<th>$P_b$ (days)</th>
<th>$10^3 \log T_{ef}$</th>
<th>$M_{bol}$</th>
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<tr>
<td>0.0</td>
<td>0.0</td>
<td>1.500</td>
<td>.3346</td>
<td>1.820</td>
<td>$&lt;10^{-3}$</td>
<td>0.677</td>
<td>3.826</td>
<td>2.805</td>
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<tr>
<td>0.49</td>
<td>$3.2 \times 10^{-8}$</td>
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<td>0.621</td>
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<td>.3308</td>
<td>1.547</td>
<td>0.16</td>
<td>0.595</td>
<td>3.777</td>
<td>3.648</td>
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<tr>
<td>1.48</td>
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<td>1.217</td>
<td>.3297</td>
<td>1.519</td>
<td>0.07</td>
<td>0.596</td>
<td>3.772</td>
<td>3.735</td>
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<td>1.180</td>
<td>.3244</td>
<td>1.507</td>
<td>$5 \times 10^{-3}$</td>
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<tr>
<td>8.83</td>
<td>$1.7 \times 10^{-10}$</td>
<td>1.175</td>
<td>.3097</td>
<td>1.506</td>
<td>$&lt;10^{-3}$</td>
<td>0.602</td>
<td>3.767</td>
<td>3.808</td>
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</table>
6 × 10^6 years after the onset of overflow. (Hereafter time will be measured from the onset of mass transfer.) After 1.2 × 10^7 years the orbital separation of the binary components begins again to increase; at that time the masses of the two components have passed the point of equality. Especially during the first stage of mass transfer the computed relative radius excess n (expression A12a) appears to be small for this low-mass system. Therefore the contact star is forced to adhere to the Roche geometry and starts to shrink immediately after the onset of mass transfer. Table 2 shows that after approximately 9 × 10^7 years the contact star is still shrinking at a very low rate (the orbital separation, however, increased again from its minimal value of 4.037 R_⊙ to 4.064 R_⊙). During the slow mass exchange the envelope of the contact star is again near thermal equilibrium (L^* = L_{core}) since the size of the Roche lobe stays nearly constant. This slow mass exchange can be maintained during quite a long time (a few times 10^9 years). The computations were terminated, however, after about 10^8 years.

The calculations were performed under the assumption that all transferred matter is accreted onto the neutron star. Thermonuclear burning of this accreted hydrogen-rich material in thin shells near the surface of the neutron star is expected to give rise to thermal pulses on timescales ranging from milliseconds to months, depending on the neutron star model (Hansen and Van Horn 1975).

It is interesting to compare our results with those obtained by PS for a 1.5 M_⊙ non-synchronously (slow) rotating contact star in a system similar to the one studied here. Our computations show that even for a synchronously rotating contact star the system can be a bright X-ray source for more than 10^8 years.

Case B. For longer binary periods the primary will expand and evolve towards the cooler regions on the H-R diagram before the Roche
lobe is filled. In that case the primary can develop a deep convective envelope which is expected to result in a violent mass exchange phase (Paczynski et al. 1969). We have calculated the initial phase of Roche-lobe overflow from a 1.5 $M_\odot$ primary in such a system. The initial mass of the secondary and the initial binary period $P_b$ were adopted as 1 $M_\odot$ and 1.7 days, respectively. This period corresponds to that of the X-ray binary HZ Her/Her X-1. The primary of this system, however, is probably slightly more massive ($\sim 2 M_\odot$).

The resulting time development of the mass transfer rate in the system is shown in Figure 2. This result should not be taken too literally, since the polytropic approximation of the outer (non-adiabatic) convective envelope is rather poor (see appendix I). The general conclusion, however, that systems in which the more massive contact star has a deep convective envelope will have short X-ray lifetimes seems to be justified. According to our calculations the mass transfer rate increases in 100 years from $10^{-11} M_\odot\text{yr}^{-1}$ to approximately $10^{-7} M_\odot\text{yr}^{-1}$.

Inter alia, it should be noted that for binary periods long enough stars with masses even up to about 9 $M_\odot$ will develop deep convective envelopes before filling their Roche lobes.

### 3.3.2 Primary of 2 $M_\odot$

**Case A.** Figure 3 shows the results of our computation for a system with initial parameters $M_1 = 2 M_\odot$, $M_2 = 1 M_\odot$ and $P_b = 0.6$ days. Assuming that no significant tidal torque is acting on the contact component (assumptions N) the rate of mass transfer appears to be very small. The "first approximation" becomes invalid after approximately $2 \times 10^6$ yrs. At that time the rate of mass transfer is still less than $2.5 \times 10^{-10} M_\odot\text{yr}^{-1}$. Our calculations show that even in the synchronous case the X-ray lifetime will be longer than $5 \times 10^5$ years.
Table 3.

Some characteristics of the contact component with initial mass $M_1 = 2 M_\odot$ undergoing Roche-lobe overflow onto a compact secondary of mass $M_x = 1 M_\odot$ in a binary with initial period $P_D = 1.7$ days (assumption N2).

$n$ is the effective polytropic index of the outer envelope which contains 3% of the primary's mass; $S_{\text{phot}}$ and $S_{\text{fit}}$ are the non-dimensional specific entropies of the gas ($S = \ln(T^4/\rho)$) at the photosphere and at the bottom of the outer envelope, respectively; $\Omega$ is the orbital angular velocity; $\omega$ is the spin angular velocity; for $\Lambda \gtrsim 10^{-2}$ the "first approximation" becomes invalid (see text); $R^*$ and $R^c$ are the effective radii of the primary and the critical lobe, respectively. Time is measured from the onset of Roche-lobe overflow.

<table>
<thead>
<tr>
<th>$t/10^5$ (yrs)</th>
<th>$-\dot{M}(M_\odot$/yr)</th>
<th>$n$</th>
<th>$S_{\text{fit}}$</th>
<th>$S_{\text{phot}}$</th>
<th>$R^*/R_\odot$</th>
<th>$R^c/R_\odot$</th>
<th>$1 - \omega/\Omega$</th>
<th>$\Lambda$</th>
<th>log $T_{\text{ef}}$</th>
<th>$M_{\text{bol}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.049</td>
<td>$10^{-11}$</td>
<td>2.301</td>
<td>25.522</td>
<td>30.092</td>
<td>3.7900</td>
<td>3.7899</td>
<td>$1.6 \times 10^{-4}$</td>
<td>$1.7 \times 10^{-4}$</td>
<td>3.8283</td>
<td>1.187</td>
</tr>
<tr>
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<td>2.300</td>
<td>25.523</td>
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<td>3.7908</td>
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<td>$2.1 \times 10^{-3}$</td>
<td>3.8281</td>
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</tr>
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<td>.468</td>
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<td>2.300</td>
<td>25.523</td>
<td>30.086</td>
<td>3.7918</td>
<td>3.7917</td>
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<td>2.299</td>
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<td>30.083</td>
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<td>3.7934</td>
<td>$6.6 \times 10^{-3}$</td>
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<td>$1.7 \times 10^{-2}$</td>
<td>$1.8 \times 10^{-2}$</td>
<td>3.8272</td>
<td>1.193</td>
</tr>
<tr>
<td>2.114</td>
<td>$5.6 \times 10^{-9}$</td>
<td>2.296</td>
<td>25.526</td>
<td>30.067</td>
<td>3.8005</td>
<td>3.8004</td>
<td>$2.1 \times 10^{-2}$</td>
<td>$2.2 \times 10^{-2}$</td>
<td>3.8269</td>
<td>1.195</td>
</tr>
</tbody>
</table>
Case B (Her X-1). The results for a system with the same initial masses but with a binary period of 1.7 days are shown in Figure 4. Such a system resembles closely the X-ray binary HZ Her/Her X-1. In spite of its large distance from the galactic plane, HZ Her is probably a normal population I object (cf. van den Heuvel 1976) so that the initial ZAMS chemical composition used here is presumably adequate. The X-ray lifetime of this binary is expected to be around $10^5$ years, a value much shorter than found by PS. This was already expected from the fact that this binary exchanges mass according to case B instead of case A (section 2.3.2).

The effective temperature of the primary star (neglecting heating by X-rays) is about 6800 K, which would indicate the existence of convection in the outer layers (Baker 1973). This result, however, is rather marginal as for a slight increase of the effective temperature (up to $\sim 7200$ K) the convection disappears. The existence of convection in the envelope is crucial for the magnitude of the tidal torques on HZ Her. Probably there will be some tidal re-synchronization of the primary during the mass exchange.

In chapter (4) we will show that the results obtained under assumptions N2 (i.e. some turbulent friction in the outer layers) are consistent with the observed spin-up rate of the X-ray pulsar. In Table 3 we have listed some details of the structural changes of the contact star's envelope during the mass exchange (N2).

Note, that our refined treatment of the outer boundary condition (appendix I) shows that in this case the stellar radius and critical radius are practically equal during the mass exchange. This is not true for the more massive primaries, as will be discussed in sections (3.4) and (3.5).
3.3.3. **Comparison with observations of low-mass X-ray binaries.**

Our calculations indicate that one expects considerable X-ray lifetimes for the low-mass X-ray binaries which seems to contrast the small number of presently observed low-mass systems. This supports the suggestion by Van den Heuvel (1976) that low-mass X-ray binaries most probably have followed a very rare type of evolution, e.g. that a supernova explosion in a low-mass binary is very likely to disrupt such a system. We are able, however, to understand the absence of low-mass X-ray binaries in which the primary has developed a deep convective envelope - i.e. is a subgiant or a giant. The calculations predict a negligible X-ray lifetime for this type of systems.

To some extent the number of observed low-mass X-ray binaries may be influenced by selection effects. It seems less difficult to optically identify an X-ray source which has an intrinsically bright massive companion, than one with a faint dwarf star companion.

3.4. **Moderately massive X-ray binaries (2 < M_1/M_⊙ < 10)**

For a main-sequence primary with a mass in excess of 1.5 M_⊙ no appreciable convection in the envelope is expected; for such a star the assumptions N or N2 are probably the most realistic ones, i.e. one does not expect a strong tidal re-synchronization during the initial stage of overflow (section 2.3.4).

3.4.1. **Primary of 4 M_⊙**

Case A. The results for a system with initial parameters M_1 = 4 M_⊙, M_2 = 1.5 M_⊙ and P_b = 1 day are shown in Figure 5. X-ray lifetimes of several times 10^5 yrs are expected for such a binary. The results for the case that the hydrodynamics of the overflow is neglected (i.e., if the the simplified condition R^* = R^C is applied) are also shown in Figure 5 (dotted curves). When the hydrodynamics is taken into
Table 4.

Some characteristics of the contact component with initial mass $M_1 = 4 M_\odot$ undergoing Roche-lobe overflow onto a compact secondary of mass $M_2 = 1.5 M_\odot$ in a binary with initial period $P_b = 1$ day (assumption N).

See subscript of table 3 for explanation of used symbols.

<table>
<thead>
<tr>
<th>$t/10^6$ yrs</th>
<th>$-\dot{M}(M_\odot/yr)$</th>
<th>n</th>
<th>$S_{fit}$</th>
<th>$S_{phot}$</th>
<th>$R^*/R_\odot$</th>
<th>$R^C/R_\odot$</th>
<th>$1 - \omega/\Omega$</th>
<th>$\Lambda$</th>
<th>log $T_{ef}$</th>
<th>$M_{bol}$</th>
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<td>4.8x10^{-9}</td>
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<td>25.314</td>
<td>35.378</td>
<td>3.4446</td>
<td>3.4406</td>
<td>4.5x10^{-2}</td>
<td>5.3x10^{-2}</td>
<td>4.1128</td>
<td>-1.451</td>
</tr>
</tbody>
</table>
account the curves (log - A vs. time) are shifted to the right over about $2 \times 10^5$ yrs and much larger X-ray lifetimes result. Contrary to our results for the (lower-mass) system similar to Her X-1 (section 3.3.2) the radius excess $\eta$ is no longer negligible, as is shown in Table 4.

**Case B.** We also followed the initial mass transfer in a system with initial parameters $M_1 = 4 M_\odot$, $M_2 = 1 M_\odot$ and $P_b = 1.7$ days. Such a longer binary period results in a X-ray lifetime of about $5 \times 10^4$ yrs (Figure 6) which is much shorter than for case A. For still longer binary periods the X-ray lifetime appears to be even shorter. This is due to the increase of the envelope expansion rate during the advancing stages of shell hydrogen burning.

### 3.4.2 Primary of $8 M_\odot$

Figure 7 shows the results for a system with initial parameters $M_1 = 8 M_\odot$, $M_2 = 1.0 M_\odot$ and $P_b = 1.5$ days (case A). For this system the expected X-ray lifetime is reduced to approximately $4 \times 10^3$ yrs.

When the adopted binary period is 4 days (case B) the mass transfer is very rapid and the expected X-ray lifetime becomes of the order of $10^3$ yrs (Figure 8).

### 3.4.3 Comparison with observations

Up to now no X-ray binary has been detected for which the mass of the primary is in the range $2 < M_1/M_\odot < 10$. Yet, according to our calculations, we expect X-ray lifetimes of at least a few times $10^4$ yrs (i.e., larger than expected for the massive X-ray binaries) if mass transfer takes place according to case A. Hence, in principle such systems can exist, provided that their binary periods are sufficiently short, i.e. $\lesssim 1.5$ days. Consequently, the usual explanation (see introduction) for the absence of moderately massive X-ray binaries
applies only if all binaries with compact secondaries and with primaries in this mass range have relatively long orbital periods, so that case B applies (as in this case the expected X-ray lifetimes are very short).

This seems not unlikely, as one expects that the supernova explosion, which presumably accompanied the creation of the compact object, enlarges the orbital period. This effect is less important for massive X-ray binaries \( (M_1 > 10 M_\odot) \) as for the latter binaries a much wider range of orbital periods corresponds to case A of mass transfer (section 3.5.3) and as also the relative increase in orbital separation due to the supernova explosion is smaller in massive systems (cf. Suntantyo 1974ab). If this would be the correct explanation for the absence of medium-mass X-ray binaries, one would need two different formation mechanisms to explain the existence of the short-period low-mass and of the massive X-ray binaries. This is quite well possible - as low-mass X-ray binaries might have formed from cataclysmic variable binaries (cf. Gursky 1976; Canal and Schatzmann 1976; Van den Heuvel and Savonije 1977; van den Heuvel 1977).

3.5. Massive X-ray binaries \( (M_1 > 10 M_\odot) \)

Without the more realistic hydrodynamical treatment of the overflow, Roche-lobe overflow in massive X-ray binaries would quench the X-ray source within \( \sim 10^2 \) years, even when the PS-mechanism is taken into account. As mentioned before, the PS-mechanism becomes inefficient for systems with mass ratio \( \frac{M_2}{M_1} \) smaller than about 0.3. This is due to the fact that for lower initial mass ratios the orbital separation diminishes more rapidly per unit transferred mass than for higher mass ratios. This means that the critical lobe no longer tends to widen when mass is transferred. However, when the basic hydrodynamics of the overflow are accounted for, it appears that, initially, the contact star is allowed to expand - even when the critical lobe is shrinking.
Table 5.

Some characteristics of the contact component with initial mass $M_1 = 16 M_\odot$ undergoing Roche-lobe overflow onto a compact secondary of mass $M_x = 1 M_\odot$ in a binary with initial period $P_b = 2.09$ days (assumption N).

See subscript of Table 3 for the explanation of used symbols.

<table>
<thead>
<tr>
<th>$t/10^4$ (yrs)</th>
<th>$\dot{M}(M_\odot$/yr)</th>
<th>$n$</th>
<th>$S_{fit}$</th>
<th>$S_{phot}$</th>
<th>$R^*/R_\odot$</th>
<th>$R^C/R_\odot$</th>
<th>$1 - \omega/\Omega$</th>
<th>$\Lambda$</th>
<th>$\log T_{ef}$</th>
<th>$M_{bol}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.0239</td>
<td>$1.0 \times 10^{-9}$</td>
<td>3.220</td>
<td>27.503</td>
<td>36.299</td>
<td>10.753</td>
<td>10.752</td>
<td>$2.9 \times 10^{-5}$</td>
<td>$6.8 \times 10^{-5}$</td>
<td>4.4013</td>
<td>-6.8078</td>
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<td>$8.7 \times 10^{-9}$</td>
<td>3.220</td>
<td>27.503</td>
<td>36.299</td>
<td>10.755</td>
<td>10.752</td>
<td>$3.4 \times 10^{-4}$</td>
<td>$8.0 \times 10^{-4}$</td>
<td>4.4013</td>
<td>-6.8079</td>
</tr>
<tr>
<td>.252</td>
<td>$1.0 \times 10^{-8}$</td>
<td>3.220</td>
<td>27.503</td>
<td>36.299</td>
<td>10.755</td>
<td>10.752</td>
<td>$4.0 \times 10^{-4}$</td>
<td>$9.2 \times 10^{-4}$</td>
<td>4.4013</td>
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</tr>
<tr>
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<td>3.220</td>
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<td>36.299</td>
<td>10.756</td>
<td>10.752</td>
<td>$5.8 \times 10^{-4}$</td>
<td>$1.3 \times 10^{-3}$</td>
<td>4.4013</td>
<td>-6.8079</td>
</tr>
<tr>
<td>.467</td>
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<td>27.504</td>
<td>36.299</td>
<td>10.758</td>
<td>10.752</td>
<td>$8.4 \times 10^{-4}$</td>
<td>$1.9 \times 10^{-3}$</td>
<td>4.4012</td>
<td>-6.8080</td>
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<td>27.504</td>
<td>36.299</td>
<td>10.761</td>
<td>10.753</td>
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<td>-6.8081</td>
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<tr>
<td>.899</td>
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<td>3.220</td>
<td>27.505</td>
<td>36.299</td>
<td>10.764</td>
<td>10.753</td>
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<td>4.4011</td>
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</tr>
<tr>
<td>1.251</td>
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<td>3.220</td>
<td>27.506</td>
<td>36.300</td>
<td>10.769</td>
<td>10.753</td>
<td>$3.0 \times 10^{-3}$</td>
<td>$7.0 \times 10^{-3}$</td>
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<td>-6.8086</td>
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<tr>
<td>1.766</td>
<td>$1.0 \times 10^{-7}$</td>
<td>3.220</td>
<td>27.508</td>
<td>36.301</td>
<td>10.777</td>
<td>10.753</td>
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<td>$1.2 \times 10^{-2}$</td>
<td>4.4010</td>
<td>-6.8090</td>
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<tr>
<td>2.883</td>
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<td>3.220</td>
<td>27.513</td>
<td>36.302</td>
<td>10.793</td>
<td>10.748</td>
<td>$1.5 \times 10^{-2}$</td>
<td>$3.4 \times 10^{-2}$</td>
<td>4.4007</td>
<td>-6.8095</td>
</tr>
</tbody>
</table>
In this section we will only discuss case A of mass transfer since case B of mass transfer results in very short X-ray lifetimes ($\lesssim 100$ yrs), even when the hydrodynamics are taken into account.

3.5.1. Primary of $16 M_\odot$

First, we consider a system with initial parameters $M_1 = 16 M_\odot$, $M_2 = 1 M_\odot$ and a binary period of 1.07 days. Figure 9 shows that the resulting X-ray lifetime is still of the order of $10^4$ years. For such a rather short orbital period the Roche lobe is filled by the primary still being close to the ZAMS (i.e., $X_c = 0.5$).

Cen X-3. We have also considered a system with the same initial masses ($M_1 = 16 M_\odot$, $M_2 = 1 M_\odot$), but with a binary period of 2.09 days (Figure 10). For such a longer binary period hydrogen is nearly depleted ($X_c = 0.1$) when the primary fills its Roche lobe. (However, see section 3.5.3). Table 5 shows the evolution (from some selected evolutionary models) of the outer envelope of the contact star. It can be observed that, in spite of the non-increasing critical radius, the actual (photospheric) radius of the contact star still increases during the beginning of Roche-lobe overflow. The structure of the outer envelope becomes noticeably affected by the mass outflow not before the transfer rate exceeds $\sim 3 \times 10^{-6} M_\odot$ yr$^{-1}$. For still larger mass transfer rates the star starts to shrink and follows its shrinking critical lobe. If we had imposed the usual, artificial, condition that the contact star should precisely fill its critical lobe, the resulting X-ray lifetime would have been reduced to less than 100 yrs. Such a short X-ray lifetime results because when the expanding star reaches its Roche lobe it can only be made to immediately obey the Roche geometry if its mass loss rate exceeds $\sim 10^{-6.5} M_\odot$ yr$^{-1}$. This is much greater than the critical accretion rate of the neutron star.

In Figure 10 we have also indicated the time development of the
mass transfer rate in case the classical assumptions S are made (section 3.1). For massive systems, however, it is unlikely that the tidal torque is strong enough to keep the contact star in synchronous rotation (see below). One of the classical assumptions (S) is that the orbital angular momentum of the system is conserved, which results in a less steep increase of the mass transfer rate as compared to calculation S*. The calculations show that tidal re-synchronization according to expression (5) is inefficient in keeping the massive contact star in synchronous rotation during the beginning of Roche-lobe overflow. It appeared that all curves N, N1 and N2 merge into one single curve, which is labeled by N in Figure 10. Only the eddy viscosity in a deep convective envelope might result in re-synchronization timescales of the same order as the mass transfer timescales of massive primaries. However, it is well known that massive main-sequence stars are too hot to have such extended convective envelopes. Hence, we conclude that assumptions N are the most realistic. Our results indicate that even a massive X-ray binary like Cen X-3 can be powered by Roche-lobe overflow for a period of the order of \(10^4\) years.

We have also calculated the evolution of a somewhat more massive system with initial parameters \(M_1 = 20 M_\odot\), \(M_2 = 1.5 M_\odot\) and \(P_b = 2.09\) days. The results are shown in Figure 11. It can be observed that also in this case Roche-lobe overflow can power the X-ray source for several thousands of years.

3.5.2. Can most of the massive X-ray binaries be powered by Roche-lobe overflow?

1) enlarged radii due to stellar wind mass loss

Ziolkowski (1976) has pointed out that the longer period binary systems like SMC X-1 will still evolve as case A of mass transfer if one assumes that the primary has already lost a substantial fraction of its mass, presumably by a strong stellar wind during the previous main-sequence
stage. The strong mass loss results in an increased stellar radius. This can also explain the overluminosity which several of these primaries seem to show (see also De Loore et al. 1977). Recent work by Conti (1977) supports these conclusions: Except for 3U0900-40 all primaries of massive X-ray binaries appear to be located in the hydrogen-burning part of the HR diagram, though at much larger luminosities than would normally correspond to their masses.

(ii) enlarged radii due to revised opacities

Even without any previous mass loss most of the massive X-ray binaries might evolve as case A, as Stothers and Chin (1977) have shown that the radii of massive main-sequence stars ($M > 10 \, M_\odot$) may become considerably larger than calculated hitherto. The larger stellar radii result from the adopted larger radiative opacities near the stellar surface, predominantly due to a revised treatment of the ionization of CNO elements. A detailed comparison with the observations is still required to demonstrate the correctness of this interesting suggestion.

If correct, this will have important implications for the evolution of massive close binaries, since practically all massive binaries that were up to now considered to evolve as case B, will instead evolve as case A. This is especially important for the massive X-ray binaries, as in case B these are expected to have very short X-ray lifetimes ($\lesssim 10^2 \, \text{years}$). The evolution of massive X-ray binaries according to case A is in better agreement with the observed (relatively large) number of these systems, as we expect X-ray lifetimes of about $10^4 \, \text{years}$ in that case.

Table 6 lists some characteristics of the five known massive X-ray binaries. According to calculations based on Cox-Stewart's radiative opacities - and without previous mass loss by stellar wind - only Cen X-3 would evolve as case A. Table 7 lists, for several stellar masses, the values of the stellar radii as calculated by Stothers and Chin at the
Table 6.


<table>
<thead>
<tr>
<th>SOURCE</th>
<th>$M_1/M_\odot$</th>
<th>$M_x/M_\odot$</th>
<th>$P_b$(days)</th>
<th>$R^C/R_\odot$</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cyg X-1</td>
<td>15</td>
<td>10</td>
<td>5.60</td>
<td>16</td>
<td>1,2</td>
</tr>
<tr>
<td>Vela X-1</td>
<td>22</td>
<td>1.7</td>
<td>8.95</td>
<td>31</td>
<td>3</td>
</tr>
<tr>
<td>3U1700-37</td>
<td>25</td>
<td>1.4</td>
<td>3.41</td>
<td>18</td>
<td>2</td>
</tr>
<tr>
<td>SMC X-1</td>
<td>15</td>
<td>1</td>
<td>3.89</td>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td>Cen X-3</td>
<td>18</td>
<td>1</td>
<td>2.09</td>
<td>11</td>
<td>1,2</td>
</tr>
</tbody>
</table>

Table 7.

Radii of massive main-sequence stars at the onset of rapid envelope expansion, according to Stothers and Chin (1977).

$S =$ Schwarzschild criterion; $L =$ Ledoux criterion; $Z =$ metal content;

$\alpha_p =$ ratio of mixing length to pressure scale height.

<table>
<thead>
<tr>
<th>CONVECTION CRITERION</th>
<th>$Z$</th>
<th>$\alpha_p$</th>
<th>$M=15M_\odot$</th>
<th>$M=20M_\odot$</th>
<th>$M=30M_\odot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>0.02</td>
<td>1</td>
<td>18 $R_\odot$</td>
<td>28 $R_\odot$</td>
<td>153 $R_\odot$</td>
</tr>
<tr>
<td>$S$</td>
<td>0.02</td>
<td>2</td>
<td>17 $R_\odot$</td>
<td>27 $R_\odot$</td>
<td>152 $R_\odot$</td>
</tr>
<tr>
<td>$S$</td>
<td>0.04</td>
<td>1</td>
<td>29 $R_\odot$</td>
<td>86 $R_\odot$</td>
<td>192 $R_\odot$</td>
</tr>
<tr>
<td>$L$</td>
<td>0.02</td>
<td>1</td>
<td>17 $R_\odot$</td>
<td>30 $R_\odot$</td>
<td>88 $R_\odot$</td>
</tr>
</tbody>
</table>
beginning of the overall contraction stage prior to the end of core hydrogen burning. Adopting a reasonable value of the metal abundance, \( Z = 0.03 \), we find that all five massive X-ray binaries will evolve as case A if Carson's opacities are correct.

From the above discussion we conclude that most and perhaps all of the massive X-ray binaries may be evolving according to case A. Our computations show that such systems can indeed be powered by Roche-lobe overflow for periods exceeding several thousands of years. This conclusion will be discussed further in chapter 4.

3.5.3. Effects of mass and angular momentum loss from the system

Loss of mass and angular momentum from a binary system is expected to influence the X-ray lifetimes appreciably only if the specific angular momentum of the ejected mass is significantly in excess of the value assumed in this paper. We performed a number of computations for which one half of the transferred mass was assumed to be ejected from the system, carrying a specific angular momentum of twice the specific orbital angular momentum of the neutron star. The resulting curves (log \((-\dot{M})\) vs. time) are labeled by the symbol L (Figures 4, 9 and 10). It can be observed that these curves have approximately the same slope as the corresponding curves for which the total mass and angular momentum are conserved. Notice that in the figures the rate of mass transfer is given and not the accretion rate (these rates are no longer equal if mass is lost from the system).
4. EVIDENCE FOR THE OCCURRENCE OF ROCHE-LOBE OVERFLOW IN X-RAY BINARIES

DERIVED FROM THE SPIN-UP RATES OF HER X-1 AND CEN X-3.

In the introduction we mentioned observational evidence for the occurrence of Roche-lobe overflow in several X-ray binaries, notably: Her X-1, Cen X-3 and SMC X-1. Our calculations of Roche-lobe overflow, as presented in chapter 3, indicate that X-ray binaries with short binary periods (Case A) can be powered by Roche-lobe overflow on timescales comparable to the thermal timescale of the envelope of the primary star. This corresponds to some $10^4$ years for a massive system like Cen X-3. In the present chapter we will present evidence suggesting that Roche-lobe overflow is indeed taking place in Her X-1 and in Cen X-3. Also SMC X-1 seems to be a strong candidate for the occurrence of Roche-lobe overflow.

4.1 Observed spin-up rates of the X-ray pulsars.

Most of the binary X-ray sources are pulsars, viz.: Her X-1, Cen X-3, SMC X-1, Vela X-1 and X-Per. All X-ray pulsars appear to be spinning up rapidly, presumably due to the transfer of angular momentum carried by the accreting matter. The fact that all observed X-ray pulsars are spinning up might be influenced by selection effects: a high accretion rate causes them to become observable as bright X-ray sources and at the same time causes a rapid spin-up. Rappaport and Joss (1976) and Mason (1977) have estimated the spin-up rates on the basis of accretion torques acting on the pulsar magnetosphere. For many X-ray pulsars these authors find close agreement between the observed and predicted spin-up rates, by adopting for the pulsar the moment of inertia and the magnetic moment of a typical magnetized neutron star. However, it appears that the X-ray pulsars Her X-1 and Cen X-3 are spinning up at lower rates than predicted.
Probably related to this, these pulsars show significant fluctuations about their average spin-up rates, i.e., they occasionally spin down on timescales of a few months (Her X-1) or a few weeks (Cen X-3). During such occasional spin-downs the X-ray sources become less bright, indicating a smaller accretion rate during such phases (Schreier 1977). For other pulsars these spin-downs have not been observed, however, for many sources the observations are still inconclusive (e.g. SMC X-1). We will discuss this matter further in section 4.3 and first give some results of more detailed calculations of pulsar spin-up by Henry and Schreier (1977).

4.2 Predicted spin-up rates in case of pure accretion torques.

In Table 8 we list some observational data on the X-ray pulsars Her X-1, Cen X-3 and SMC X-1, including the observationally derived spin-up rates. In column 7 the spin-up rates estimated from a pure accretion torque model are listed, i.e., for which it is assumed that no external torques are slowing down the pulsar (Henry and Schreier 1977). In such a model it is assumed that matter in an (essentially) Keplerian orbit near the inner edge of an accretion disk transfers its angular momentum to the pulsar magnetosphere at that point (i.e. at the Alfvén surface). In that case the spin-up rate is given by:

\[ \frac{\dot{P}}{P} = \dot{M}_X R_A^2 \omega_K / (I \omega_N) \]  

In this expression is \( R_A \) the Alfvén radius (depending on the accretion rate \( \dot{M}_X \) and the magnetic field strength of the pulsar); \( I \) and \( \omega_N \) are the pulsar moment of inertia and angular velocity, respectively; \( \omega_K \) is the Keplerian angular velocity at the Alfvén surface.

As can be seen in Table 8 the predicted spin-up rates are much
higher than the observed ones, except for SMC X-1. Apparently, the
spin-up of Her X-1 and Cen X-3 is slowed down by strong dissipative
torques (Pringle and Rees 1972; Davidson and Ostriker 1973; Elsner and
Lamb 1976).

4.3 Equilibrium spin period and the existence of braking torques.
The irregular behaviour of the spin periods of Her X-1 and Cen X-3 and
the results of Henry and Schreier (1977) can be explained by the
existence of a strong braking torque acting at the pulsar magnetosphere.
The position of the Alfvén surface is - for not too rapidly spinning
pulsars- determined by the balance of the ram pressure of the infalling
gas and the magnetic pressure of the magnetosphere (e.g. Lamb et al.
1973). The pulsar is thought to spin at its equilibrium rate if the
infalling gas near the Alfvén surface is just in corotation with the
pulsar magnetosphere. If the pulsar spins at or below this equilibrium
rate, the infalling gas can be accreted; this will cause the pulsar to
spin up. The spin-up will continue until the equilibrium rate is excee-
ded and the centrifugal barrier at the Alfvén surface inhibits further
accretion. Subsequently, the pulsar is expected to spin down to its
equilibrium rate again by transferring its excess spin angular momentum
to the accumulating matter above the Alfvén surface. Most likely, the
near balance of the spin-up accretion torque and the spin-down external
torque explains the observed spin fluctuations in Her X-1 and Cen X-3.

At any given moment one therefore expects them - on the average- to
be spinning near the equilibrium rate which corresponds to the mass
transfer rate at that moment. The fact that the average spin periods
of both sources are gradually decreasing then implies that the average
mass transfer rate from their companions are gradually increasing
as can be seen as follows:
The equilibrium spin period can be expressed numerically (e.g. Wickramasinghe and Whelan 1975) as:

\[ P_{eq} = 1.6 \left( \frac{B_0}{10^{12}} \right)^{6/7} \left( \frac{M_x}{M_\odot} \right)^{-5/6} \left( \frac{\dot{M}_x}{10^{17} \text{gs}^{-1}} \right)^{-3/7} \text{ sec} \] (8)

In expression (8) \( B_0 \) denotes the surface magnetic field of the pulsar (assumed to be dipolar) and \( M_x \) denotes the mass of the pulsar. The precise value of the proportionality factor in the above equation is uncertain, but presumably correct up to a factor of order unity.

By substituting the observationally derived spin periods, accretion rates and masses \( (M_x \sim 1 M_\odot) \) as listed in Table 8 and solving for \( B_0 \), it is found that \( B_0 \) is of the order of \( 10^{12} \) gauss for both Her X-1 and Cen X-3. This is of the same order as the typical value derived for radio pulsars (Ruderman 1971), lending support to the conclusion that these pulsars are spinning near their equilibrium rates.

Differentiating expression (8) logarithmically with respect to time, one obtains an expression which relates the growth rate of \( \dot{M}_x \) to the pulsar spin-up rate:

\[ -\frac{\ddot{p}}{p} = \frac{3}{7} \frac{\dot{M}_x}{\dot{M}_x} \] (9)

The thus derived expression for the spin-up rate is expected to be applicable to pulsars which are spinning - on the average - at their equilibrium rates. The spin-up is independent of the precise value of the specific angular momentum of the infalling gas and is completely determined by the growth rate of the mass transfer (compare eq. 9 and 7).
4.4 Comparison between the spin-up rates predicted from our calculations of Roche-lobe overflow and the observed average spin-up rates of Her X-1, Cen X-3 and SMC X-1.

Expression (9) enables us to check—assuming that the pulsars are indeed spinning near their equilibrium rates—whether our calculations of Roche-lobe overflow are consistent with the observed spin-up rates of the above mentioned X-ray pulsars. From the observations we know for each source the approximate accretion rate $\dot{M}_X$ (from the X-ray brightness and estimated distance of the source). The corresponding growth rates $\dot{M}_X/\dot{M}_X$ can be determined from the calculations presented in chapter 3. The resulting spin-up rates are then determined by substituting these growth rates into expression (9).

It can be seen in Table 8 (column 6) that for Her X-1 and Cen X-3 the thus predicted spin-up rates are close to the observed ones, which is a strong argument in favor of Roche-lobe overflow as the mass transfer mechanism in these X-ray binaries.

For Her X-1 the best agreement between the predicted and observed spin-up rates is obtained by assuming that HZ Her is partly re-synchronized during the initial stage of mass exchange in this binary (curve N2 in Fig.4). This was expected (section 3.3.2) because HZ Her is likely to have a shallow convective layer near the surface. When expression (5) is used for the synchronization time, i.e., when it is assumed that the tidally induced turbulent viscosity extends throughout the contact star, the predicted spin-up rate comes out to be twice as high as observed.

For SMC X-1 it is only possible to determine a lower limit to the spin-up rate since our calculations of Roche-lobe overflow probably become invalid for an accretion rate as high as $10^{-7} M_\odot yr^{-1}$, which is the value inferred for SMC X-1. This is due to the fact that the "first
Table 8.

Parameters of the X-ray pulsars SMC X-1, Centaurus X-3 and Hercules X-1.

The observational data are taken from Rappaport and Joss (1977).

<table>
<thead>
<tr>
<th>SOURCE</th>
<th>(L_X(10^{37} \text{ ergs/s}))</th>
<th>(\dot{M}<em>X(\text{M}</em>\odot/\text{yr}))</th>
<th>(P(\text{sec}))</th>
<th>(\dot{P}/P(\text{yr}^{-1}))</th>
<th>(\dot{P}/P(\text{yr}^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SMC X-1</td>
<td>51</td>
<td>8.9x10^{-8}</td>
<td>0.716</td>
<td>(7.7\pm0.8)x10^{-4}</td>
<td>&gt;10^{-4}(Fig.11,curve N)</td>
</tr>
<tr>
<td>Cen X-3</td>
<td>5</td>
<td>8.9x10^{-9}</td>
<td>4.84</td>
<td>(2.8\pm0.4)x10^{-4}</td>
<td>2\times10^{-4}(Fig.10,curve N)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>5\times10^{-4}(Fig.11,curve N)</td>
</tr>
<tr>
<td>Her X-1</td>
<td>1.4</td>
<td>2.5x10^{-9}</td>
<td>1.24</td>
<td>(3.0\pm0.7)x10^{-6}</td>
<td>1.2\times10^{-6}(Fig.4,curve N)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2.4\times10^{-6}(Fig.4,curve N)</td>
</tr>
</tbody>
</table>

Prediction (1) follows from calculations presented here (Chapter 3);

Prediction (2) is taken from Henry and Schreier (1977) and is based on a pure accretion torque model for the pulsar spin-up.
approximation", which underlies all our calculations, becomes invalid for accretion rate as high as $10^{-7} M_\odot \text{yr}^{-1}$ as can be seen in Figures 9 to 11 and in Table 5. We tentatively assume here that SMC X-1 evolves according to case A (section 3.5.2) and that the X-ray pulsar spins near its equilibrium rate. As shown in column 7 of Table 8 the spin-up rate of SMC X-1 comes out to be close to the observed rate if it is assumed that the braking torque is negligible. Henry and Schreier (1977) explain the different outcomes for Cen X-3 and SMC X-1 by the relative weakness of the magnetic field of SMC X-1. A weaker field would be consistent with the observed high accretion rate, with the wide pulses as well as with the lower pulsed fraction of this source, since a weaker magnetic field is expected to result in a reduced channeling of the accreting matter.

Concluding, it seems very probable that the X-ray binaries Her X-1 and Cen X-3 are in an initial stage of Roche-lobe overflow. The very high "super critical" accretion rate of SMC X-1, however, seems also only realizable if the normal companion is overflowing its Roche lobe.

5. CONCLUDING REMARKS.

As our calculations necessarily contain a number of simplifications, the numerical results should be considered as good order of magnitude estimates, rather than as exact values. Nevertheless, the basic result that Roche-lobe overflow is an important factor in the generation of bright relatively long lived X-ray binaries (including massive systems) seems firmly established. Several points need further study, we mention the most important ones.

First of all one should continue the calculations of mass transfer into the phase where the "first approximation" is no longer valid.
Related to this are the complicated problems of differential rotation and resynchronization of the contact star during mass transfer.

A second point of interest is the existence of a highly anisotropic, intense radiation field (also thought of great importance for the generation of stellar winds) that is expected to affect the process of overflow from massive contact stars. Our assumption that the radiation field is already thermalized in the region just above the critical surface is probably too simplistic.

A further unknown point is how the mechanisms driving the stellar wind and the Roche-lobe overflow interfere with one another. Probably these phenomena cannot be treated separately, because the stellar wind mechanism might be influenced by the fact that the star is a tidally deformed member of a close binary system. Apart from the effect of a reduced surface gravity, the fact that the outer layers of a contact star (including an eventual hot corona) are unstable and flow out through the gravitational saddle point, will certainly influence the properties of the wind.

In massive binaries the distinction between mass loss by "Roche-lobe overflow" and by "stellar wind" from a contact star might well lose its significance. In such a system the mass loss may perhaps only be described in terms of a general highly anisotropic outflow of matter.

There is observational evidence suggesting enhanced and anisotropic mass outflow from massive binary components (Hutchings 1976). Hammerschlag-Henberge (1977) infers from a spectroscopic study of the Of-star HD 153919 (the companion of the X-ray source 3U1700-37) that the Balmer progression is strongly correlated with the binary phase, indicating enhanced outflow from the tidal bulge region.

Our theoretical understanding of all these phenomena is, however, still poor and a unified treatment seems beyond present-day possibilities,
as even the processes which generate the stellar winds in single stars are not yet well understood.

ACKNOWLEDGEMENTS.

I am deeply indebted to Prof. E.P.J. van den Heuvel for suggesting this investigation, his encouragement and for his critical reading of the manuscript. I wish to thank J. van Paradijs for useful discussions. Grants from the Netherlands Organization for the Advancement of Pure Research (Z.W.O.) and from the University of Amsterdam are gratefully acknowledged. The computations were performed on the CDC Cyber 73 of the Academic Computer Centre Amsterdam (SARA).
APPENDIX I

Relation between the radius excess of the contact star and the mass transfer rate

We use the Cartesian coordinate system \((\xi, \eta, \zeta)\) - corotating with the contact star - and definitions of variables as given in section (2.3.3). The neutral point is given by \((\xi_1, 0, 0)\); we can write the effective gravitational potential \(\psi\) for points in the plane \(\xi = \xi_1\) as follows:

\[
\psi(\xi_1, \eta, \zeta) = -\frac{G(M_1 + M_2)}{a} \left\{ \mu (\xi_1^2 + \eta^2 + \zeta^2)^{-\frac{1}{2}} + (1-\mu)((1-\xi_1^2 + \eta^2 + \zeta^2)^{-\frac{1}{2}} \right\} - \frac{1}{2} a^2 \omega^2 \{(\xi_1 + \mu - 1)^2 + \eta^2 \}
\]

where we have introduced \(\mu = M_1/(M_1 + M_2)\). In the vicinity of the neutral point a series expansion of the potential \(\psi\) up to quadratic terms yields for points in the plane \(\xi = \xi_1\):

\[
\psi(\xi_1, \eta, \zeta) = \psi_1 + (a-a_2 \omega^2) \frac{\eta^2}{2} + a \frac{\zeta^2}{2}
\]

where we have introduced:

\[
a = \frac{G(M_1 + M_2)}{a} \left\{ \mu \xi_1^{-3} + (1-\mu)(1-\xi_1)^{-3} \right\} \text{ and } \psi_1 = \psi(\xi_1, 0, 0).
\]

In this approximation the equipotential surfaces intersect the \(\xi = \xi_1\) plane along ellipses with a constant ratio of semi-axes \(l_\eta\) and \(l_\zeta\), given by:

\[
\frac{1}{\eta^2} = \frac{2(\psi - \psi_1)}{(a-a_2 \omega^2)} \quad \text{and} \quad \frac{1}{\zeta^2} = \frac{2}{a} (\psi - \psi_1).
\]

The dimensionalized area \(A\) of such an ellipse can be expressed as:

\[
A = \frac{2\pi (\psi - \psi_1) a^2}{\left\{ a(a-a_2 \omega^2) \right\}^\frac{3}{2}} \quad \text{. (A1)}
\]

Following Jedrzejec (1969) we approximate the structure of the outer envelope of the contact star by a polytropic relation between the pressure \(P\) and density \(\rho\):

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For the equation of state of the gas we will use the same expression as used in the evolutionary code itself, assuming that the radiation field is thermalized:

\[ P = K \rho \frac{1}{n} \quad \text{(A2)} \]

Combination of A2 and A3 yields

\[ K = \beta^{-1} \frac{\mathcal{R}}{\mu} \rho \frac{1}{n} T \quad \text{(A4)} \]

In the vertical direction hydrostatic equilibrium will hold in good approximation:

\[ dP = -\rho d\psi \quad \text{(A5)} \]

Substituting A2 into A5 and integrating one obtains:

\[ \rho^{1/n} = \frac{\psi_0 - \psi}{(n+1)K} + \rho_0^{1/n} \quad \text{(A6)} \]

where \( \psi_0 \) and \( \rho_0 \) are the gravitational potential and density at the photosphere \( (\tau_c = 2/3) \) of the star. As usual in calculations of binary evolution the photosphere is considered to be the stellar surface.

When the star expands beyond its critical lobe the excess matter will flow towards the neutral point where it can escape to the companion (section 2.3.3). The resulting rate of mass transfer can be obtained by integrating the mass flux density \( \rho v \) over the cross section of the flow near the neutral point. A detailed solution of the hydrodynamics of the flow is very complicated, but we can arrive at a reasonably good approximation by using some results from the analysis of the mass flow in semi-detached binaries as given by Lubov and Shu (1975). For a stationary flowing perfect fluid we can apply Bernoulli’s conservation theorem (e.g. Landau and Lifshitz 1966), stating that along a streamline:

\[ \frac{1}{2} v^2 + \frac{5}{2} K \rho^{1/n} + \psi = \text{constant}. \]
We have substituted an expression for the specific enthalpy $w$ of the fluid (to our order of approximation it can be represented by an ideal gas) which follows by rewriting $w = \frac{5}{2} \frac{RT}{\mu}$ with the aid of equation (A4).

To a first approximation the streamlines can be considered to lie on equipotential surfaces (Lubov and Shu). Consider a particular streamline for which $\psi = \psi_p$. Using expression (A6) for the density in the region where the flow velocity $v = 0$, Bernoulli's equation becomes:

$$v^2 + 5 K p^{1/n} = 5K \left[ \frac{\psi_0 - \psi_p}{1/n} + \rho_0 \frac{1/n}{(n+1)K} \right]$$

(A7)

An important result obtained by Lubov and Shu is that the sonic point on matter carrying streamlines must be situated close to the neutral point. They conjecture - but could not prove - that the sonic surface coincides with the Roche lobe around the detached component. For simplicity we will assume here that this sonic surface coincides with the plane $\xi = \xi_1$ through the neutral point and perpendicular to the line connecting the centers of mass of the two stars. By substituting the expression for the (adiabatic) sound velocity

$$v_s^2 = \gamma \frac{RT}{\mu} = \gamma K \rho^{1/n}$$

into equation (A7) (we have put $\gamma = \beta_\gamma$), we arrive at the expression for the density $\rho_s$ at the sonic point on the streamline:

$$\rho_s(\psi_p) = (1 + \gamma^*/5)^{-n} \left[ \frac{\psi_0 - \psi_p}{1/n} + \rho_0 \frac{1/n}{(n+1)K} \right]^n$$

This expression yields a value for the density which is $(1 + \gamma^*/5)^{-n}$ times smaller than the value given by expression (A6). This is, of course, due to the leakage of matter near the neutral point.

Hence, the magnitude of the mass flux density at the intersection
of the equipotential $\psi = \psi_p$ with the plane $\xi = \xi_1$ is given by:

$$\rho_s v_s = (\gamma K)^{1/2} (1 + \gamma/5)^{-n} \left[ \frac{\psi_0 - \psi}{(n+1)K} + \frac{\rho_0}{\rho_0} \right]^{n+1/2}$$

The mass transfer rate is obtained by integrating the expression for $\rho_s v_s$ over the surface of the $\xi = \xi_1$ plane:

$$\dot{M} = \int_{\xi_1} \rho_s v_s dA < \int_{\xi_1} \rho_s v_s dA$$

This integral can be written in terms of the effective potential $\psi$ by applying expression (A1):

$$\dot{M} = \frac{2\pi a^2}{(1 + \gamma/5)^{n+1/2}} \left[ \frac{\gamma K}{a(a^2 - \omega^2)} \right]^{1/2} \left[ \frac{\psi_0 - \psi}{(n+1)K} + \frac{\rho_0}{\rho_0} \right]^{n+1/2} d\psi \quad (A8)$$

where $\psi_1$ and $\psi_0$ are the values for the potential at the neutral point and the stellar surface, respectively.

Obviously, without a finite overflow ($\psi_0 < \psi_1$) no mass transfer takes place. In order to apply relation (A8) in an evolution code, which deals with spherically symmetric stellar models, we follow Jedrzejec and express $(\psi_0 - \psi_1)$ in terms of an averaged radius excess:

$$\psi_0 - \psi_1 = \frac{GM}{R^* - R^C} \quad (A9)$$

where $R^*$ and $R^C$ are the stellar radius and the critical radius, respectively. Substitution of (A9) into (A8) yields:

$$\dot{M} = \frac{2\pi a^2}{(1 + \gamma/5)^{n+1/2}} \left[ \frac{\gamma K}{a(a^2 - \omega^2)} \right]^{1/2} \left[ \frac{\rho_0}{(n+3/2)} + \frac{GM}{R^C} \right]^{n+3/2} \left[ \frac{1 + 3/2n}{n+3/2} \right] \left( \frac{\rho_0}{(n+1)K} \right)^{n+3/2} \quad (A10)$$

where $\eta$ is the relative radius excess: $\eta = (R^* - R^C)/R^*$. For each stellar model (in fact for each Henyey iteration) we determine the constants $n$ and $K$ of equation (A2) from a polytropic fit.
between the photosphere (i.e. $\tau_c = 2/3$) and the bottom of the separately integrated outer envelopes. These outer envelopes contain 3% of the stellar mass (appendix II). For radiative envelopes the value of $K$, which appears in equation (A10), varies by less than a factor of 2 over the fitted 3% region; such a polytropic approximation for the density structure is accurate to within 10%. The situation is worse for convective envelopes of cool stars, due to the large variation of the local effective polytropic index in the ionization zones just below the photosphere. For all stars the extension of the polytropic approximation into the optically thin regions above the photosphere becomes very poor. This is due to the fact that a polytrope cannot represent such an optically thin region by definition: a polytrope does not radiate, because $T$ is zero at its outer boundary. However, this does not affect our results, as above the photosphere the density becomes very small anyhow and the outermost regions do not contribute much to the integrated mass flux density in equation (A10).

Appendix II

The outer boundary conditions of the contact component.

In appendix I a relation was derived between the excess radius and the resulting mass transfer rate, a relation which has to be included in the outer boundary conditions of the stellar models to be calculated. From a mathematical point of view, the inclusion of mass exchange corresponds to the addition of the mass transfer rate $\dot{M}$ as a new Henyey unknown together with an equation to determine $\dot{M}$; this equation follows from the condition on the radius of the contact component (eq. A12, below). We now describe how this equation is incorporated in the outer boundary conditions.

As usual, the outer boundary conditions of the stellar models are
expressed at a point somewhat below the stellar surface to avoid numerical difficulties caused by the presence of ionization zones.

The program constructs an envelope table by integrating a set of eight different envelopes (instead of four as is the case for single stars) from the stellar surface inwards; each envelope is labeled by a somewhat different value for one of the 3 Henyey variables \((L^*, T^*, \dot{M})\), denoting the luminosity and temperature at the surface and the mass transfer rate, respectively. Such a table thus consists of eight sets of four quantities \((X_1^e, X_2^e, X_3^e, X_4^e)\), denoting the temperature, density, radial distance from the stellar center and luminosity, all in the fitting point at the bottom of the integrated outer envelopes. We have to find that particular set \((L^*, T^*, \dot{M})\) for which the corresponding interpolated values at the fitting point satisfy the equations:

\[
X_k^e (L^*, T^*, \dot{M}) = X_k^i \quad \text{with } k = 1, 2, 3, 4. \tag{A11}
\]

Here we have denoted the set of four corresponding Henyey unknowns at the inner edge of the fitting point by \((X_1^i, X_2^i, X_3^i, X_4^i)\). At the same time, for these values of \(L^*, T^*\) and \(\dot{M}\), the stellar radius \(R^*\) and the critical radius \(R^c\) should be related by the outer boundary condition:

\[
(1 - \eta) R^* - R^c = 0 \tag{A12}
\]

where the value of the relative radius excess \(\eta\) follows from the hydrodynamical analysis of the flow. According to expression (A10) \(\eta\) can be expressed as (note that \(\dot{M} < 0\)):

\[
\eta = c_1 \left[ \frac{\rho_0^{1+(3/2n)}}{n+3/2} \right]^{1/n} - c_2 \dot{M}_0^{1/0} \tag{A12a}
\]

where we have introduced:

\[
c_1 = (n+1)K \left( \frac{R^c}{GM} \right) \quad \text{and} \quad c_2 = \frac{n+3/2}{n+1} \frac{(1 + \gamma^*)^{n+3/2}}{2\alpha^2\gamma^*} \left[ \frac{\alpha(\alpha+\omega^2)^{1/2}}{\gamma^*} \right]^{-3/2}
\]
The outer boundary conditions (A11) and (A12) are coupled to the remaining set of difference equations, which follow from the normal set of four non-linear differential equations describing the inner structure of the star. The complete set of difference equations is then iteratively solved with a Newton-Raphson method (Henyey et al. 1964). In this way we determine in particular the set of boundary values \((L^*, T^*, \dot{M})\) for which conditions (A11) and (A12) are satisfied. Before each iteration we linearize these equations with respect to \(L^*, T^*\) and \(\dot{M}\). For equations (A11) we arrive at:

\[
\delta x_k^i - \frac{3x_k^e}{3L^*} \delta L^* + \frac{3x_k^e}{3T^*} \delta T^* + \frac{3x_k^e}{3\dot{M}} = x_k^e - x_k^i
\]  

(A13)

where \(k = 1, 2, 3, 4\). The boundary condition (A12) becomes:

\[
\left[ (1-\eta) \frac{3\dot{R}^*}{3L^*} - R^* \frac{3\eta}{3L^*} \right] \delta L^* + \left[ (1-\eta) \frac{3\dot{x}^*}{3T^*} - R^* \frac{3\eta}{3T^*} \right] \delta T^* - \left[ R^* \frac{3\eta}{3\dot{M}} + \frac{3\dot{R}^*}{3\dot{M}} \right] \delta M =
\]

\[
= R^C - (1-\eta) R^*.
\]  

(A14)

In order to enable the use of the normal single star procedures, the program pre-eliminates the variation \(\delta \dot{M}\) before each Henyey iteration by expressing \(\delta \dot{M}\) in terms of the variations \(\delta L^*\) and \(\delta T^*\) through the related extra Henyey condition (A14). The latter expression for \(\delta M^*\) is then substituted into the four linearized fitting conditions (A13). After each Henyey iteration the correction \(\delta \dot{M}\) can be recovered by substituting the solved values for \(\delta L^*\) and \(\delta T^*\) into equation (A14). In this way also in the case of mass transfer self-consistent stellar models are obtained within only a few Henyey iterations. These models satisfy the structure equations to a very high degree of accuracy; for instance in the converged models the absolute value of the right hand side of equation (A12) is smaller than \(10^{-5}\) (the radii expressed in solar units).
Note

The envelope tables have a grid size of \( \Delta^{10}\log(L^*/L_\odot) = 0.02, \Delta^{10}\log(T^*) = 0.01 \) and \( \Delta^{10}\log(-\dot{M}) = 0.2 \) where \( \dot{M} \) is in \( M_\odot/\text{year} \). The interpolation is linear with respect to the former two variables. When \( 10\log(-\dot{M}) \) is smaller than -9 the interpolation with respect to this variable is made along parabolas which are fitted between \( \dot{M} = 10^{-12} \) and \( \dot{M} = 10^{-9} \, M_\odot \text{yr}^{-1} \). These parabolas ensure an asymptotically correct behaviour when the mass transfer rate becomes vanishingly small.

Appendix III

The critical radius and the variation of the orbital parameters.

For the critical radius \( R^C \) an approximate expression derived by Pratt and Strittmatter (1976) is used:

\[
R^C = a [0.41 - 0.03 \left( \frac{\omega}{\Omega} \right)^2 ] [1 - 0.5 \, 10^{10} \log \frac{M_2}{M_1}] 
\]  

This radius \( R^C \) is the averaged radius of the critical lobe on which the neutral point \((\xi_1,0,0)\) is situated, as discussed in section (2.3). This critical lobe can be considered as the actual limiting volume only if \( \left( \frac{\omega}{\Omega} \right) \) is close to unity.

As a consequence of the PS-mechanism (section 2.3) the orbital parameters \( a, \omega, \Omega \) and \( q = M_2/M_1 \) - which appear in expression (A15) - all depend on the mass transfer rate \( \dot{M} \), as will be shown below. For this reason these parameters can be solved implicitly by the Henyey iterative cycle. We now consider their variation as a function of the amount of mass transferred.

Let us denote by \(-\Delta M_1\) the transferred mass during the fixed evolutionary timestep \( \Delta t \), hence, \( \Delta M_1 = \dot{M}\Delta t \). Suppose that a fraction \( f \) of this mass leaves the system immediately, carrying a specific angular momentum equal to twice the specific orbital angular momentum of the neutron...
The orbital variations then follow from:

\[ \Delta M_2 = (f-1) \Delta M_1 \quad (A16) \]

\[ \Delta J_1 = \omega L_1^2 \Delta M_1 + I(\Omega-\omega)(1-e^{-\Delta t/\tau_S}) \quad (A17) \]

\[ \Delta J_0 = \left[ 2f\Omega \left( \frac{a}{I+q} \right)^2 - \omega L_1^2 \right] \Delta M_1 - I(\Omega-\omega)(1-e^{-\Delta t/\tau_S}) \quad (A18) \]

\[ \Delta \omega = \frac{\omega}{I} \left( \xi_1^2 - \frac{\Delta I}{\Delta M_1} \right) \Delta M_1 \quad (A19) \]

where \( M_2 \) is the mass of the neutron star, \( J_1 \) is the spin angular momentum of the primary, \( J_0 \) is the orbital angular momentum of the binary system and \( I \) is the primary's moment of inertia. The synchronization timescale \( \tau_S \) depends on the physical circumstances and is determined according to expression (4) or (5). The new values for the binary separation \( a \) and the orbital angular velocity \( \Omega \) then follow from:

\[ a = \frac{M_1 + M_2}{2GM_1^2M_2^2} \quad J_0^2 \quad (A20) \]

\[ \Omega = \frac{GM_1M_2}{aJ_0} \quad (A21) \]

where the parameters on the right hand side of equations (A20) and (A21) have been corrected according to equations (A16) - (A19).

\( R^C \) is recalculated after each Henyey iteration with the corrected orbital parameters \( a, \omega, \Omega \) and \( q \), and with the corrected value of \( \dot{M} \).

The derivative \( \frac{dR^C}{d\dot{M}} \), which appears in equation (A14), is a rather complicated expression but its approximate value can be determined through the expressions (A16) - (A21).
FIGURE 1. The $10^\log$ of the mass transfer rate versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 1.5 \ M_\odot$ and binary period $P_b = 0.68$ days. The label $S^w$ indicates synchronous rotation of the contact star and conservation of the total mass and angular momentum during mass exchange. In all figures the two dashed horizontal lines correspond to one and three times the Eddington critical accretion rate, respectively.

FIGURE 2. The $10^\log$ of the mass transfer rate versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 1.5 \ M_\odot$, $M_x = 1.5 \ M_\odot$ and binary period $P_b = 1.7$ days. See subscript of Fig. 1.

FIGURE 3. The rate of mass transfer versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 2 \ M_\odot$, $M_x = 1 \ M_\odot$ and binary period $P_b = 0.6$ days. The mass transfer rate calculated under assumption N appears to be very low, so that the corresponding curve falls outside the range of the figure. See subscript of Fig. 1 and section (3.1) for the explanation of the curve labels.

FIGURE 4. The rate of mass transfer versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 2 \ M_\odot$, $M_x = 1 \ M_\odot$ and $P_b = 1.7$ days. This system is representative for the X-ray binary Her X-1. The arrow marks indicate the brake down of the 'first approximation'. Curve N2 is probably the most realistic for this system (see text). According to this curve the present X-ray age of Her X-1, as deduced from its observed X-ray luminosity, is approximately $10^5$ yrs.

FIGURE 5. The rate of mass transfer versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 4 \ M_\odot$, $M_x = 1.5 \ M_\odot$ and $P_b = 1$ day. See section (3.1) for explanation of the curve labels. The arrow marks on the curves indicate the brake down of the 'first approximation' (see text). The dotted curves in Figures 5 and 6 correspond to calculations in which the contact star is forced to precisely fill its critical lobe during mass transfer, i.e., for which the hydrodynamics of the overflow are neglected. The solid curve N is probably the most realistic one for massive systems.
FIGURE 6. The rate of mass transfer versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 4 M_\odot$, $M_x = 1 M_\odot$ and $P_b = 1.7$ days. Further, see subscript of Fig. 5.

FIGURE 7. The rate of mass transfer versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 8 M_\odot$, $M_x = 1.5 M_\odot$ and $P_b = 1.5$ days. Further, see subscript of Fig. 5.

FIGURE 8. The rate of mass transfer versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 8 M_\odot$, $M_x = 1.5 M_\odot$ and $P_b = 4$ days. Further, see subscript of Fig. 5.

FIGURE 9. The rate of mass transfer versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 16 M_\odot$, $M_x = 1 M_\odot$ and $P_b = 1.07$ days. The curves labeled N, N1 and N2 all merge into one single curve which is labeled N. The curve degeneracy is due to the short evolutionary timescale of the massive primary as compared to the tidal resynchronization timescales. Further, see subscript of Fig. 5.

FIGURE 10. The rate of mass transfer versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 16 M_\odot$, $M_x = 1 M_\odot$ and $P_b = 2.09$ days. This system is thought to be rather similar to the X-ray binary Cen X-3. Further, see subscript of Fig. 9.

FIGURE 11. The rate of mass transfer versus time since the onset of Roche-lobe overflow for a system with initial parameters $M_1 = 20 M_\odot$, $M_x = 1.5 M_\odot$ and $P_b = 2.09$ days. This system has the same binary period as Cen X-3, but probably the primary star in Cen X-3 is slightly less massive than adopted here. Further, see subscript of Fig. 9.
**Figure 1**

\[
\log(-\dot{M} / \dot{M}_0 \text{yr}^{-1})
\]

- \(q = 1\)
- \(M_1 = 1.5 M_\odot\)
- \(M_2 = 1 M_\odot\)
- \(P = 0.68\)
- **CASE A**

\[
M_1 = 1.5 M_\odot
M_2 = 1 M_\odot
P = 0.68
\]

**Figure 2**

\[
\log(-\dot{M} / \dot{M}_0 \text{yr}^{-1})
\]

- \(S^*\)
- \(M_1 = 1.5 M_\odot\)
- \(M_2 = 1 M_\odot\)
- \(P = 1.7\)
- **CASE B**

- \(M_1 = 1.5 M_\odot\)
- \(M_2 = 1 M_\odot\)
- \(P = 0.68\)
- **CASE A**

\[
M_1 = 2 M_\odot
M_2 = 1 M_\odot
P = 0.6
\]

**Figure 3**

- \(S^*\)
- \(N_1\)
- \(M_1 = 2 M_\odot\)
- \(M_2 = 1 M_\odot\)
- \(P = 0.6\)
- **CASE A**

\[
M_1 = 2 M_\odot
M_2 = 1 M_\odot
P = 0.6
\]

- \(S^*\)
- \(N_1\)
- \(N_2\)
- **CASE B**

- \(M_1 = 2 M_\odot\)
- \(M_2 = 1 M_\odot\)
- \(P = 1.7\)
FIGURE 5

$M_1 = 4M_\odot$
$M_2 = 1.5M_\odot$
$P = 1.0$
CASE A

FIGURE 6

$M_1 = 4M_\odot$
$M_2 = 1M_\odot$
$P = 1.7$
CASE B

FIGURE 7

$M_1 = 8M_\odot$
$M_2 = 1M_\odot$
$P = 1.5$
CASE A

FIGURE 8

$M_1 = 8M_\odot$
$M_2 = 1.5M_\odot$
$P = 4.0$
CASE B
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