Primordial high-frequency perturbations in cosmology
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Summary

There exist some important problems related to cosmology and elementary particle physics. One of them is the development of singularities. Other problems arise through the introduction of new topics such as spontaneous symmetry breaking. Our analysis will be focused on the influence of a scalar field on the evolution of a spherically symmetric background and an inhomogeneous (Einstein-Rosen) background. The interaction of gravitational waves with the scalar field and background metric and the formation of trapped surfaces is investigated. An outstanding problem of the multiple-scale method is how to obtain an asymptotic wave solution. We investigate the higher order equations of the coupled gravity-scalar field. The modification of the behaviour of solitary wave solutions on the Einstein-Rosen metric is stipulated. A numerical solution is presented.
3.1 Introduction

The phenomenon of the non-linear wave propagation plays an important role in various areas of relativistic astrophysics and cosmology. The resulting steepening of a finite amplitude wave, stiffening of signals in hydrodynamics (which is linked up with the apparition of shocks) for example, could be investigated by considering a family of exact solutions of the relativistic hydrodynamic equations, called the simple waves (Taub, 1978; Whitham, 1974). These solutions were essentially the non-linear analogue of the plane waves of linearized theory. For application to realistic problems it is necessary to consider more general wavefronts, allowing for non-planar wavefront geometry, an arbitrary background spacetime, the self-gravity of the wave, the interaction among wave modes. A promising way of dealing with an arbitrary background spacetime and at the same time allowing for arbitrary wavefront geometry, is to resort to asymptotic waves. The mathematical formulation in the linear and generic case was given by Garding, Kotake and Leray (1964) and later reformulated by Choquet-Bruhat (1969). A fundamental problem for these asymptotic solutions to be physically relevant is for them to be true asymptotic solutions in the usual sense, when the expansion parameter is small. In general some conditions of boundedness will be necessary. However, in the non-linear case some additional restrictions, sometimes severe, on the expansions are necessary. The investigation will be concentrated on the stability conditions for partial non-linear hyperbolic differential equations. No such stability theory of non-linear hyperbolic differential equations exists in literature. However, some progress has been made for the linear case, using the progressive wave expansion (Eckhoff, 1981). A problem of the method is that it breaks down at caustics (Choquet-Bruhat, 1969; Ludwig, 1966). At a caustic, the amplitude has a singularity and the asymptotic expansion is no longer valid. However, there could be some way out (Ludwig, 1966).

There are two classes of wave motion, the hyperbolic type and the dispersive type. The theory of hyperbolic equations, with its attendant discussion of characteristics, shock formation in non-linear problems, wave-wave interaction, etc, is quite different from that of the dispersive problem. However, both problems can be handled by the multiple-scale method.
The study of dispersive waves starts with periodic traveling wave-trains solutions \( f(\theta, u) \), with \( \theta = kx - \omega t \) and \( u \) a constant amplitude, arising as a constant of integration. Then one can develop a wider class of solutions, which describe the slow modulation of this wave-train by allowing the former constants, wavenumber \( \theta_x = k \), frequency \( \theta_t = -\omega \) and amplitude \( u \) to be slowly varying functions of position \( x \) and time \( t \). Normally, one can develop more general solutions from periodic wave trains by considering the Fourier synthesis. A formal general solution

\[
f = \int_{-\infty}^{\infty} F(k) e^{i(kx-W(k)t)} \, dk \tag{1}
\]

is obtained from \( f(\theta, u) \), where \( \omega = W(k) \) is a specific solution of the dispersion relation \( G(\omega, k) = 0 \). The complete solution will be the sum of terms like equation (1) with one integral for each of the solutions \( \omega = W(k) \). \( F(k) \) would be determined from appropriate initial or boundary conditions. These Fourier integrals give exact solutions, but their content is hard to see, and we are looking for ways to avoid them for later extension to non-linear problems. Equations for \( k, \omega \) and \( u \) can be obtained as follows. First, we have the conservation of wave number

\[
k_t + \omega_x = 0. \tag{2}
\]

Second, when we substitute the solution ansatz into the partial differential equation, the leading order part is a non-linear ordinary differential equation in \( \theta \) for \( f(\theta) \). We know this has periodic solutions because of our assumption that the original partial differential equation admits periodic traveling wave solutions. The imposition of a fixed periodicity gives an algebraic relation between \( \omega, k \) and \( u \), the dispersion relation. Because these parameters vary slowly, there are \( O(\varepsilon) \) terms, containing first derivatives of \( k, \omega \) and \( u \) with respect to \( x \) and \( t \), left over in the partial differential equation. The condition that the next iterate, which satisfies a linear ordinary differential equation in \( \theta \) with coefficients depending on \( f \) and its derivatives, is also periodic, imposes a solvability condition on the equation. This condition is a first order partial differential equation in \( k, \omega \) and \( u \) and expresses the conservation of the wave action.
Thus we have three equations, one algebraic and two differential, for the three unknowns $k$, $\omega$ and $u$. The great power of the approximation is that it is not a small amplitude theory. However what happens in the small amplitude limit? The Whitham theory does not produce, for example, the non-linear Schrödinger equation. The difficulty is that when the amplitude is finite, the solvability condition at order $\varepsilon$ changes in the sense that the null space of the linear operator acting on the first iterate of the solution is only half the size of what it would be if $u$ were small. This results in only one equation, the conservation of wave action, which is an equation for the phase of the wave and corresponds to the imaginary part of the non-linear Schrödinger equation which gives the evolution of the phase of the complex amplitude. The amplitude $u$ is already fixed by the dispersion relation. What happens when $u$ is small? In that case it turns out that the dispersion relation must be augmented in order to satisfy the extra solvability requirements. The extra terms contain derivatives of $u$ and so what was originally an algebraic relation giving $u$ as function of $\omega$ and $k$, now become a differential equation for $u$. This equation can be compared with the amplitude part of the non-linear Schrödinger equation.

To make the Whitham theory more uniformly valid, one can reformulate the theory by using an averaged Lagrangian (Whitham, 1965). Suppose that the field equations are derived from a variational principle

$$ I = \int L(y^n, \partial_x y^n) \, dx. $$

If we expand the dependent variables (as functions of the five independent variables $x^\mu$ and $\theta$)

$$ y^n(x^\mu, \theta) = \sum_o^n y^n_o(x^\mu, \theta), $$

we will obtain a formal expansion

$$ L = L^0 + \varepsilon L^1 + \varepsilon^2 L^2 + \ldots, $$

where $\varepsilon$ is a formal expansion parameter, related to $\omega$ of §1.3 and §1.9. In due course we shall want to reorder some of these terms in order to get a consistent scheme of successive approx-
imations, since we shall make $v_1^a$ rapidly oscillating, and the process of differentiating and averaging may modify the scaling of the contributions in the series. In all cases, such reordering will amount to "demoting" terms further down the series. Clearly $L^0$ is independent of $v_1^a$ and its derivatives, $L^1$ is linear in them, $L^2$ quadratic etc, while $v_0^a$ can be involved in any of these terms. If the waves do not react so strongly on the background then the procedure begins with Hamilton's principle

$$\int L^0 d^4x \quad \text{stationary.} \quad (6)$$

By definition this ensures that

$$\int eL^1 d^4x = 0, \quad (7)$$

for all $v_1^a$ which leave $v_1^a$ satisfying the same boundary conditions as $v_0^a$. To improve on equation (6), and obtain equations governing the waves, we consider

$$\int (L^0 + e^2L^2 + e^3L^3 + \ldots) d^4x \quad \text{stationary.} \quad (8)$$

In this variational principle, the background and wave variables may be treated as true independent variables, provided they conform to the approximation scheme laid down. We may then construct Euler-Lagrange equations in them separately. Owing to equation (6) the background variables already satisfy Hamilton's principle with error $O(e^2)$. The waves may be regarded as governed by an action integral

$$e^2 \int (L^2 + eL^3 + \ldots) d^4x, \quad (9)$$

in which we vary $v_1^a$ regarding $v_0^a$ as prescribed. We are looking for oscillating solutions with a slowly varying amplitude, and we want to construct an average lagrangian density. Clearly $L^2$ will give the linear theory of the oscillations, $L^3$ describes interaction among the waves, and so on. An improved description of the background will be obtained by considering the action integral

$$\int (L^0 + e^2L^2) d^4x, \quad (10)$$
in which we vary $\psi_0^a$ regarding $\psi_1^a$ as prescribed. This correction to equation (6) gives the reaction of the waves onto the background, and we actually use not $L^2$ itself but an averaged version of it. It might appear that this correction will in turn vitiate equation (7). However, the averaging procedure ensures that the $L^1$ term is negligible to a higher order than the above would suggest. Owing to the general nature of a variational principle, it is open to us to make further changes of variables whereby $\psi_0^a$ and $\psi_1^a$ are superseded by new variables quite independently. The equations for the background fields and the waves have to be solved simultaneously. We can look for solutions of the form of wave trains

$$\psi^a = A^a(x^u)\exp\left\{\frac{i\theta(x^u)}{\epsilon}\right\}, \quad (11)$$

where $x^u = \epsilon x^u$ and $\theta = \epsilon^{-1}\phi(x^u)$ a phase function.

It is only through the phase $\theta$ that $\psi^a$ varies on the fast scale. This is the essence of the idea of a wavetrain. The amplitude $A$ may vary on the slow scale $x^u$. The action integral

$$I_{\text{wave}} = \int L^2(\psi_1^a, \partial_{\mu} \psi_1^a, \epsilon x^u) d\theta, \quad (12)$$

can be evaluated approximately by setting up a coordinate system in which locally $\theta$ is one of the coordinates. The integration over $\theta$ may then be carried out approximately by conversion to an integral over an averaged lagrangian

$$\overline{I} = \frac{1}{t_0} \int L^2(\psi_1^a, \partial_{\mu} \psi_1^a, \epsilon x^u) d\theta, \quad (13)$$

where $t$ is the period of $\psi_1^a$. Whitham (1974) made the observation that the Euler-Lagrange equations are the Euler-Lagrange equations of the five-dimensional variational principle

$$I = \int L(\psi^a, \partial_{\mu} \psi^a, \epsilon x^u) d^4x d\theta. \quad (14)$$

So the formalism can be set up more general, where $L^0$ too can depend on $\epsilon$. We have

$$I_0 = \int L^0 d^4x, \quad (15)$$
where $L^0$ arises from the average of the lowest order term. One proves (Bretherton, 1968) that

$$ I_0 = \int L^0(\lambda^\mu) d^4x + O(\epsilon^2) . \quad (16) $$

The averaged lagrangian technique does produce the results of the two-timing method. However, it does not provide the propagation equations, nor can it be expected to, for those equations involve the explicit dependence of the second order perturbations in $\theta$. Such a dependence will have been averaged over in the average lagrangian technique (Taub, 1980).

In §3.2 we will outline the more general procedure of the multiple-scale method (two-timing), by following Choquet-Bruhat (1969). In §3.3 we applied the method to the coupled gravity-Higgs field and in §3.4 we extend the theory. In §3.5 we apply the procedure to the inhomogeneous background metric.

Many questions will remain open, especially in the case that the equations are hyperbolic. Principal among these questions are those related to the long-time behavior. Do shocks evolve and what do they mean? How do the characteristics evolve and do solitary waves emerge (in an initial value problem)?

**References**

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3.2 Multiple-scale analysis

Multiple-scale analysis is a very general collection of perturbation techniques that embodies the ideas of both boundary-layer theory and WKB theory. Multiple-scale analysis is particularly useful for constructing uniformly valid approximations to solutions of perturbation problems. In this section we will outline the method of Choquet-Bruhat (1969).

Let us consider, for example, the quasi-linear partial differential equation of first order of the form (compare with equation (75) of §1.9)

\[ L^j(u) = a^j_l(x,u) \frac{\partial u^i}{\partial x^\lambda} + b^j(x,u) \]  

(17)

\((i,j=1,\ldots,N; \lambda=0,\ldots,l-1)\), with \(l\) the dimension of the manifold \(X\).

The \(u = \{u^1,\ldots,u^N\}\) will determine by \(u+Lu\) a non-linear map of the space of series \(\{u^k(x)\}\) of \(N\) differentiable functions contained in a poly-cylinder \(P\)

\[ |u^k(x) - u^k_0(x)| \leq R^k \quad (u^k_0 \text{ known}) \]  

(18)

onto the space of series of \(N\) functions \(L^j(u)\) given by equation (17). The \(a^j_l\) and \(b^j\) are functions on \(C^\infty\), analytic in \(u\) on \(P\)

\[ a^j_l(x,u) = a^j_l(x,u_0) + (u^h - u^h_0)a^j_l_{1h} + \frac{1}{2}(u^h - u^h_0)(u^k - u^k_0)a^j_l_{1hk} + \ldots \]  

(19)

\[ b^j(x,u) = b^j(x,u_0) + (u^h - u^h_0)b^j_{1h} + \ldots \]  

(20)

where

\[ a^j_{1h} = \frac{2a^j_{1u}}{2u^h} \]  

(21)

When we introduce the abbreviations
then we have

\[
\frac{\partial u^i}{\partial \xi} = \sum_{q=0}^\infty \omega^{\mu-q} \frac{\partial u^i}{\partial \lambda} \phi + \sum_{q=0}^\infty \omega^{\mu-q} \phi_q \quad (23)
\]

Substituting the expansion for \( u \) (equation (73) of §1.8) into equation (17), we obtain for the coefficients \( F^j_q \) (for \( \mu=0 \), see for example Choquet-Bruhat, 1976)

\[
F^j_{-1} = a^j_{10} u^i_0 \phi^i, \\
F^j_q = a^j_{10} (u^i_{q+1} + a^j_{10} \phi^i) + a^j_{10} h^i_{1h} \phi^i + b^j_{1h}, \quad \ldots, \\
F^j_q = a^j_{10} (u^i_{q+2} + a^j_{10} \phi^i) + a^j_{10} h^i_{1h} (u^i_{q+1} + a^j_{10} \phi^i) + u^i_{q+1} (u^i_{q+2} + a^j_{10} \phi^i) + b^j_{1h} \phi^i + \ldots
\]

When we consider the special case, where \( u^i_0 \) is a given function of equation (17) independent of \( \xi \)

\[
\dot{u}^i_0 = 0, a^j_{10} \phi^i + b^j_0 = 0, \quad (27)
\]

we obtain from equations (24), (25) and (26)

\[
F^j_{-1} = 0, \quad (28)
\]
\[ F^j = a_{10}^{j \lambda} \phi^i u^i_1 = 0, \quad (29) \]
\[ F^j = a_{10}^{j \lambda} \phi^i u^i_1 + a_{10}^{j \lambda} \phi^i u^i_1 + a_{10}^{j \lambda} \phi^i u^i_1 + (a_{10}^{j \lambda} \phi^i u^i_1 + b^j_1) u^i_1 = 0, \quad \ldots \ldots, \quad (30) \]
\[ F^j = a_{10}^{j \lambda} \phi^i u^i q + 1 + a_{10}^{j \lambda} \phi^i u^i q + a_{10}^{j \lambda} \phi^i u^i q + (a_{10}^{j \lambda} \phi^i u^i q + b^j_1) u^i q = 0. \quad (31) \]

In equation (31) \( f_i^j \) depends only on \( u^i_1, \phi^i \) and \( \partial u^i_1 \) (\( p < q \)).

From equation (29) we obtain (with \( u^i_1 \neq 0 \))

\[ A(x, \phi_x) \equiv \det(a_{10}^{j \lambda}) = 0. \quad (32) \]

So \( \phi \) is a solution of the approximate characteristic equation. The coefficients \( a_{10}^{j \lambda} = a_{10}^{j \lambda}(x, \phi_x) \) are given by the solution \( u^i_1(x) \).

The gradient \( \phi_x \) is a root (simple or not) of the polynomial of components \( p_\lambda \) of the covariant vector \( p \), defined by

\[ A(x, p) = \det(a_{10}^{j \lambda} p_\lambda). \quad (33) \]

For a simple root we have

\[ (A^\lambda(x, p))_{p=\phi_x} \neq 0, \quad \forall x \in X, \quad (34) \]

where \( A^\lambda(x, p) = A_\partial p_\lambda \). So \( A \) is of rank \( n-1 \) for \( p=\phi_x \).

From equation (29) we can deduce

\[ u^i_1(x, \xi) = V_1(x, \xi) h^i(x), \quad (35) \]

where \( h^i(x) \) is a solution of the system of linear homogeneous equations

\[ a_{10}^{j \lambda} \phi^i h^i = 0 \quad (\phi \text{ known}) \quad (36) \]
and \( V_1 \) an arbitrary function. From equation (35) we obtain

\[
U_1(x, \xi) = U_1(x, \xi) h_i^i(x) + v^i_1(x),
\]

where \( U_1(x, \xi) \) is the primitive of \( V_1 \) and \( v^i_1 \) an arbitrary function of \( x \). The equations \( F_1^j = 0 \) are \( N \) linear equations for the \( N \) unknown \( u_1^i \). These \( u_1^i \) exist if and only if

\[
\bar{R}_j g^j_1 = 0,
\]

where

\[
g^j_1 = a^{j\lambda}_0 u_1^i + a^{j\lambda}_{0 \xi} u_{1\xi}^i + \frac{\partial}{\partial x} a^{j\lambda}_{0 \xi} u_1^i + (a^{j\lambda}_{1\lambda} u_1^i + b^j_1) u^h_1
\]

and \( \bar{R}_j \) a solution of the transposed system

\[
\bar{R}_j a_{0 \lambda}^j = 0.
\]

Further, we have

\[
h^i \bar{R}_j = k A^i_j(x, \phi_x),
\]

where \( A^i_j(x, p) \) is the minor of \( a^{j\lambda}_{0 \lambda} \) in the determinant \( A(x, p) \) and \( k \) a function of \( x \). With the help of

\[
A^\lambda(x, p) = \frac{A(x, p)}{\partial x} a^{j\lambda}_{0 \lambda} (a^{j\lambda}_{1\lambda} P_\lambda),
\]

and

\[
h^i \bar{R}_j a_{0 \lambda}^j = k A^\lambda(x, \phi_x),
\]

we can derive from equation (38)

\[
k A^\lambda_{1\lambda} U_1(x, \xi) + a(x) U_1(x, \xi) \bar{U}_1(x, \xi) + \beta(x) U_1(x, \xi) + \gamma_1(x) \bar{U}_1(x, \xi) + \delta_1(x) = 0,
\]

(44)
where

\[ \alpha(x) = \alpha^{j\lambda}_{m\lambda} \phi h^{i_1} h^{m_1}, \]  

\[ \beta(x) = \beta^{j\lambda}_{m\lambda} \phi h^{i_1} (a^{j\lambda}_{m\lambda} a^{i_1}_{m\lambda} + b_{m}^{j}) h^{m}. \]  

\[ \gamma_1(x) = \gamma^{j\lambda}_{m\lambda} \phi h^{i_1} h^{m_1}. \]  

and

\[ \delta_1(x) = \delta^{j\lambda}_{m\lambda} \phi h^{i_1} (a^{j\lambda}_{m\lambda} a^{i_1}_{m\lambda} + b_{m}^{j}) v^{m}. \]  

In equation (44) we observe that \( A^\lambda \partial_\lambda \) represents the derivative along the rays, corresponding to the wavefront \( \phi(x) = \text{const.} \)  

From equation (44) we obtain

\[ kA^\lambda \frac{\partial U_1}{\partial \lambda} + (aU_1 + \gamma_1) \frac{\partial U_1}{\partial \xi} = -(\beta U_1 + \delta_1). \]  

When we introduce the total derivative \( \frac{d}{dt} = aU_1 + \gamma_1 / \partial \xi + kA^\lambda \partial / \partial \lambda \), then

\[ \frac{dU_1}{dt} = (aU_1 + \gamma_1) \frac{\partial U_1}{\partial \xi} + kA^\lambda \frac{\partial U_1}{\partial \lambda} = -(\beta U_1 + \delta_1). \]  

The direction derivative is defined by

\[ \frac{dx^\lambda}{dt} = kA^\lambda \frac{\partial U_1}{aU_1 + \gamma_1}. \]  

The lines defined by equation (51) are the corresponding rays on the wave surface \( \phi(x) = \text{const.} \) passing through the point \( y \):

\[ x = x(t, y), \quad x(0, y) = y. \]  

So we have
\[
\frac{dx^1}{k\alpha_1} = \frac{d\xi}{a U_1 + \gamma_1} = \frac{dt}{\delta U_1 + \delta_1}.
\]

(53)

The solution \(U_1\) corresponding to the initial manifold

\[(\xi) : s(y) = 0, \quad (U_1(x, \xi))_{x=y} = W_1(y, \eta) \quad \quad \xi = \eta\]

is generated by the bi-characteristic curves raised up in \(\Gamma\).

This means, obtained by elimination of \(t, y\) and \(\eta\) in the solution of equation (53):

\[x=x(t, y) \quad , \quad U_1=U_1(t, y, \eta) \quad , \quad \xi = \xi(t, y, \eta)\]

and satisfying

\[x(o, y)=y, \quad U_1(o, y, \eta)=W_1(y, \eta), \quad \xi(o, y, \eta)=\eta \quad (s(y)=0).\]

(55)

The functions \(U_1\) and \(\xi\) are now obtained by quadrature

\[U_1 = \exp(-/8 [x(t, y)] d\tau)\{W_1(y, \eta)+/\delta_1 [x(t, y)] \exp(8d\eta d\tau)\},\]

(56)

\[\xi = \eta + /\alpha [x(t, y)] U_1(t, y, \eta) + /\gamma_1 [x(t, y)] d\tau .\]

(57)

With the choice \(v_1^t=0\), we obtain

\[U_1 = W_1(y, \eta) \exp(-/8d\tau),\]

(58)

\[\xi = \eta + W_1(y, \eta) /\alpha \exp(-/8d\eta) d\tau .\]

(59)

We conclude that

\[\psi = /\alpha \exp(-/8d\tau) d\tau\]

(60)
is a monotonic function in t, when a does not change sign. So 
\( \frac{\partial \xi}{\partial \eta} \) will be annihilated for certain time \( t_0 \) (dependent of \( y \)), when \( \frac{\partial W_1}{\partial \eta} \) is of opposite sign with respect to \( a \). This fact has to do with the existence of non-linear shocks (For \( \frac{\partial \xi}{\partial \eta} \) a continuous function in \( t \), we can obtain for small \( t \) a differentiable function \( \eta = \eta(t,y,\xi) \) from equation (57) for \( |t| \leq t_0 \) when 
\( \frac{\partial \xi}{\partial \eta} \neq 0 \).

We have now constructed the first term \( u^1_1(x,\xi) \) of the asymptotic wave

\[
u^1_1(x,\omega) = W_1(y(x),\eta(t(x),y(x),\omega_\phi(x)))\phi(t(x),y(x))h^4(x),
\]

where

\[
\phi = \exp\left(-\int_{0}^{t} \beta \mathrm{d}t\right).
\]

This term is, just as in the linear case, proportional to the righthand eigenvector \( h^i \) of the matrix \( A(x,\phi) \) (eigenvalue zero). The term \( \phi \) depends on the initial wave surface and on the given equation (and in our case on \( u^1_0 \)). \( \phi \) is determined by integration along the corresponding rays. \( W_1 \) (form factor) depends on the initial situation. This function is in the linear case constant along the rays. In the non-linear case there will be a distortion of the signal. From equation (43) one observes that

\[
h^i_{n_j}A_{jlm}^i = \frac{\partial A}{\partial u_m}u = u_o, p = \phi \_x
\]

and one can prove that expression (63) is proportional \( \frac{\partial}{\partial u^m} \), the derivative of the propagation velocity corresponding to the wave \( \phi(x) = \text{const} \). When

\[
\frac{\partial A}{\partial u_m}u = u_o, p = \phi \_x \n
\]

we have \( a = 0 \). This means that along the characteristics there is no distortion of the signals. In general, however, we have

\[
\eta \eta - W_1(y,\xi)\Psi(t,y).
\]
In order to obtain the successive terms $u^i_q(x, \xi)$ $(q<2)$, one can proceed as follows. One can construct $u^i_2$ satisfying $F^i_1=0$:

$$u^i_2(x, \xi)=V^i_2(x, \xi)h^i(x)+V^i_2(x, \xi), \quad (66)$$

where $V^i_2(x, \xi)$ is an arbitrary function and $V^i_2(x, \xi)$ a solution of

$$a^{j3}_0 \Phi^i_2+q^i_1=0. \quad (67)$$

When $U^i_2(x, \xi)$ and $U^i_2(x, \xi)$ are the primitives of $V^i_2$ and $V^i_2$ with respect to $\xi$ respectively, then

$$u^i_2(x, \xi)=U^i_2(x, \xi)h^i(x)+U^i_2(x, \xi). \quad (68)$$

In general we will have

$$u^i_q(x, \xi)=U^i_q(x, \xi)h^i(x)+U^i_q(x, \xi), \quad (69)$$

where $\tilde{u}^i_q$ is a solution of

$$a^{j3}_0 \Phi^i_q+q^i_{q-1}=0. \quad (70)$$

For $u^i_p$ and $u^i_q$ $(p<q-1)$ we also have

$$F^j_q=0. \quad (71)$$

The equation $F^j_q=0$ can be written as

$$a^{j3}_0 \Phi^i_q+q^i_{q}=0. \quad (72)$$

The $u^i_{q+1}$ can be obtained if and only if

$$h^i_{j}g^j_q=0. \quad (73)$$
This equation is a partial differential equation of first order in $u_q$:

$$kA^q_3 U_q + a(U_1 U_q + U_0 U_q) + bU_q + \gamma_1 q + \delta q = 0. \quad (74)$$

One can proceed just as in the $U_1$ case. The projection on $V_n$ of the bi-characteristic curves are still the corresponding rays on the wave surface $\phi(x) = \text{const}$. One again obtains the functions

$$x = x(t, y), \quad \xi = \xi(t, y, n), \quad (75)$$

from which one concludes that $y$ is regular on an initial manifold and transverse on the rays $s(y) = 0$ (t small); $t$, $y$ and $n$ are regular functions in $x$ and $\xi$. $U_q$ can be obtained by quadrature from equation (74). We now say that

$$u^i = u_0^i(x) + \frac{1}{\omega} u_1^i(x, \omega^i) \quad (76)$$

is an approximate wave of order 0 when

$$u_1^i = U_1(x, \xi) h^i, \quad (77)$$

where $U_1$ is an arbitrary function and bounded in $x$ and $\xi$, just as its first derivatives. Further,

$$u^i = u_0^i(x) + \frac{1}{\omega} u_1^i(x, \omega^i) + \frac{1}{\omega^2} u_2^i(x, \omega^i) \quad (78)$$

will be an approximate wave of order 1 when $u_1^i$ is of the form of equation (77), $U_1$ given by equations (58) and (59) and $W_1$ and its first derivatives bounded with respect to $y$ and $n$. Further, $u_2^i$ will be a solution of $F_1^i = 0$, so satisfies equations (66) and (67); $u_2^i$ as well as its derivatives again bounded. The formal series

$$u^i = \sum_{p=0}^{q+1} a \frac{u^i}{p} (x, \xi) \quad (79)$$
will be an approximate wave of order $q$, if $u^i_p$ and its first derivatives are bounded. Suppose that $u^i_p(p \leq q)$ satisfies this condition. Then $u^i_{p+1}$ will also satisfy this condition if equation (72) has a solution with bounded primitive. For some equations, it will be impossible to construct such a solution with real values. An approximate wave of arbitrary order (with complex values) will be obtained by taking

$$W^i_1(y, n) = a(y)e^{in},$$

where $a(y)$ will be bounded as well as its derivative to all orders. Then

$$\xi = n + a(y)e^{in}y(t, y),$$

$$U^i_1(x, \xi) = a[y(x)]\exp[in[t(x), y(x), \xi]]\delta[t(x), y(x)].$$

By integration we obtain

$$\int U^2_1d\xi = \int a^2(y)e^{2in}\delta^2(t, y)d\xi$$

$$= \frac{a^2}{2i}e^{2in(t, y, \xi)} + \frac{a^2}{3}e^{3in(t, y, \xi)}.$$

It is then observed that in general (in the non-linear case) there will be no approximate wave, because terms proportional to $\xi$ will evolve. So additional restrictions on the expansion must be made.

references

3.3 HIGH-FREQUENCY PERTURBATIONS AND GRAVITATIONAL COLLAPSE IN GRAVITY THEORY COUPLED WITH A HIGGS FIELD

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ABSTRACT

We present an approximate wave solution of order two of Einstein’s theory of gravitation coupled with a Higgs field, using the multiple-scale method. The characteristic (eikonal) equation as well as propagation and back-reaction equations are investigated in the Lorentz-gauge on a spherically symmetric background. From the constraint equations it is found that the initial high-frequency perturbation has a significant influence on the formation of trapped surfaces. A numerical solution of the equations is presented using a finite-element collocation method.

Subject headings: cosmology — gravitation

I. INTRODUCTION

In this paper we consider high-frequency perturbations in gravity theory coupled with a Higgs field. There is a very powerful method for studying high-frequency waves. It originates from the WKB method (Choquet-Bruhat 1969), which was used in the construction of approximate solutions of the Schrödinger equation. The idea of the method is based on the expansion of the gravitational and Higgs field in power series of the ratio $\epsilon = L / \lambda$, where $\lambda$ is the characteristic wavelength of the perturbations and $L$ the characteristic dimension of the background. One of the advantages of the method is that one can keep track of the different orders of approximation. The method is in general applicable to nonlinear partial differential equations. In § II we outline this method and apply the method to the coupled gravity-Higgs field.

An intriguing outcome of this coupled theory is cosmological inflation (Brandenberger 1985): during the epoch of inflation the universe was in a metastable phase, a false vacuum which eventually decayed into a phase of lower energy density. In this false vacuum state, the universe may have experienced a relative long period of exponential expansion (de Sitte r phase), or inflation. The attractiveness of this inflation is that it would solve many of the outstanding puzzles of the standard big bang model. The original inflationary model was proposed by Guth (1981), and the new inflationary model by Albrecht and Steinhardt (1982) and Linde (1982). This model is based on grand unified theories with Coleman-Weinberg-type spontaneous symmetry breaking. In § III we apply the results of the high-frequency approximation to a spherically symmetric spacetime and deduce a system of partial differential equations, describing the back-reaction and propagation of the high-frequency perturbations.

In cosmological models, gravitational effects can cause catastrophic gravitational collapse. Gravitational collapse is one of the most interesting problems in cosmology. In order to incorporate in a model this nonlinear effect, one has to go beyond linearized theory. The high-frequency approximation is very suitable for investigating this effect: one can simply push the approximation to higher orders. Further, one will also uncover problems such as shock waves, persistent waves, and wave-wave scattering. In § IV we investigate numerically the behavior of the formation of trapped surfaces. The presence of a trapped surface is a very useful criterion that provides a sufficient indication that a region of spacetime lies within a black hole. The equations are solved with the help of a finite element collocation method, and we compare our results with those obtained recently by Chao (1984).

II. THE MULTIPLE-SCALE METHOD

Multiple-scale analysis is an approximation which embodies, e.g., both boundary-layer theory and WKB theory. This method has been applied to a large variety of physical problems. Its mathematical aspects are discussed by Garding, Kotake, and Leray (1964). Because of the appearance of two rates of variation, a fast one and a slow one, the method is also called the two-timing or high-frequency approximation. High-frequency perturbations in general relativity are studied extensively in the literature. See, for example, Choquet-Bruhat (1969a, b, 1976), Choquet-Bruhat and Taub (1977), MacCallum and Taub (1972), and Taub (1980). Application to anisotropic Bianchi IX cosmological models are given by Slagter (1983, 1984). In this section we apply the multiple-scale method to a gravitational field $\phi^a$, in interaction with a Higgs field $\phi$. The action of this system is assumed to be

$$ S = \int \sqrt{-g} \left[ \left( \frac{1}{16\pi G} + \frac{1}{2} \phi^2 \right) R + \frac{1}{2} \tilde{\psi} \tilde{\phi}^2 - V(\phi) \right] d^4x, $$

where $\tilde{g}$ is a dimensionless parameter, $G$ Newton’s gravitational constant, $R$ the curvature scalar, $\psi$ the determinant of the covariant metric tensor, and $V(\phi)$ the effective potential of the Higgs field. The action principle gives the following equation of motion:

$$ \psi^{-1} \nabla_{\mu} \nabla_{\nu} \phi - f R \phi + \frac{dV}{d\phi} = 0, $$

$$ \left( \frac{1}{8\pi G} + j \omega^2 \right) R_{\alpha \beta} + \tilde{\psi} \tilde{\phi}^2 \phi - \phi_{\alpha \beta} \psi + f \left( \nabla_{\nu} \phi, \phi^2 + \frac{1}{2} \phi_{\alpha \beta} \nabla^\nu \phi \right) = 0. $$
where $\mathcal{V}$ is the covariant derivative with respect to $\tilde{g}_{\mu\nu}$. In the two-timing approximation one obtains solutions which have the form of a sum of so-called background fields ($\phi, \tilde{g}_{\mu\nu}$) and small, rapidly changing fields. More precisely, we look for approximate solutions of equations (2) and (3) which have the form

$$
\phi(x^\alpha, \xi) = \tilde{\phi}(x^\alpha) + \epsilon \phi_0(x^\alpha, \xi) + \epsilon^2 \phi_2(x^\alpha, \xi) + \cdots
$$

and

$$
\mathcal{V}(x^\alpha, \xi) = \tilde{\mathcal{V}}(x^\alpha, \xi) + \epsilon \mathcal{V}_0(x^\alpha, \xi) + \epsilon^2 \mathcal{V}_2(x^\alpha, \xi) + \cdots
$$

where $\epsilon$ is a small parameter. Furthermore,

$$
\xi = \epsilon^{-1} \pi(x^\alpha),
$$

where $\pi$ is a scalar function called the phase. The occurrence of the small parameter $\epsilon$ in equation (6) is responsible for the rapid changes of the field $h_{\mu\nu}, k_{\mu\nu}, \psi, \chi, \ldots$. Alternatives for equation (6) are discussed by Choquet-Bruhat (1976). Furthermore, we assume that the fields $h_{\mu\nu}, k_{\mu\nu}, \psi, \chi, \ldots$ are periodic functions of $\xi$. For a function $F = F(x^\alpha, \xi)$ one has

$$
\frac{\partial F}{\partial \xi} = F_{,\xi} + \epsilon^{-1} k_{,\xi} F,
$$

where we introduce the abbreviations

$$
F_{,\xi} \equiv \frac{\partial F}{\partial \xi} \bigg|_{\text{fixed}}, \quad F_{,\xi}^\prime \equiv \frac{\partial F}{\partial \xi} \bigg|_{\text{approximated}},
$$

and

$$
k_{,\xi} \equiv \frac{\partial \pi}{\partial \xi}.
$$

We now substitute the expansions of $\phi, \tilde{\phi}$ into the equations of motion (2) and (3). Subsequently the coefficients of the various powers of $\epsilon$ are put equal to zero. This procedure gives a system of partial differential equations for the fields $\tilde{g}_{\mu\nu}, h_{\mu\nu}, \psi, \chi, \ldots$ and the phase $\pi$. The independent variables are the five variables $x^\alpha$ and $\xi$. For a solution $\pi = \pi(x^\alpha)$, the approximate solutions of equations (2) and (3) are obtained by eliminating $\xi = \epsilon^{-1} \pi(x^\alpha)$ in equations (4) and (5). The expansion of the Ricci tensor reads

$$
R_{\mu\nu} = \epsilon^{-1} R_{\mu\nu}^{(1)} + R_{\mu\nu} + R_{\mu\nu}^{(2)} + \epsilon R_{\mu\nu}^{(3)} + \cdots,
$$

where

$$
R_{\mu\nu}^{(1)} = \frac{1}{2} [k_{,\mu} h_{,\nu} + k_{,\nu} h_{,\mu} - k_{,\mu} h_{,\nu} - k_{,\nu} h_{,\mu}],
$$

and

$$
R_{\mu\nu}^{(2)} = \Gamma_{\mu\lambda}^{\alpha} R_{\lambda\nu} + \Gamma_{\nu\lambda}^{\alpha} R_{\lambda\mu} - \Gamma_{\nu\mu}^{\alpha} R_{\lambda\lambda} - \Gamma_{\mu\nu}^{\alpha} R_{\lambda\lambda} - R_{\mu\nu},
$$

and

$$
R_{\mu\nu}^{(3)} = \Gamma_{\mu\lambda}^{\alpha} R_{\lambda\nu} - \Gamma_{\nu\lambda}^{\alpha} R_{\lambda\mu} + \Gamma_{\nu\mu}^{\alpha} R_{\lambda\lambda} + \Gamma_{\mu\nu}^{\alpha} R_{\lambda\lambda} - R_{\mu\nu},
$$

where

$$
B_{\mu\nu}^\alpha = \frac{1}{2} k_{,\mu} h_{,\nu}^\alpha + h_{,\mu}^\alpha k_{,\nu} - k_{,\mu} h_{,\nu}^\alpha + k_{,\nu} h_{,\mu}^\alpha,
$$

and $\Gamma$ denotes the Christoffel symbol of the metric $\tilde{g}_{\mu\nu}$. Indices are raised and lowered by $\tilde{g}^{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. Substitution of the expansions of $\phi, \tilde{\phi}$, and $R_{\mu\nu}$ into equations (2) and (3) yields, collecting terms of the order $\epsilon^{-1}$,

$$
\mathcal{F}(\tilde{\phi}, \phi^\prime) R_{\mu\nu}^{(1)} = k_{,\mu} \tilde{\phi},
$$

and

$$
R_{\mu\nu}^{(1)} = \frac{-8 \pi G \tilde{\phi}}{1 + 8 \pi G \tilde{\phi}} (\tilde{g}_{\mu\nu} k_{,\mu} k_{,\nu} + 2 k_{,\mu} k_{,\nu} \tilde{\phi}),
$$

where $\tilde{g}_{\mu\nu}$ are the Einstein field equations in the two-timing approximation.
respectively. From equation (15) and the contraction of equation (16) with $\gamma^\mu$, there follows

$$[1 + 8\pi G/(1 + \tilde{G}^2)] k^\mu \psi = 0 \ .$$

(17)

In the generic case this implies

$$\gamma^\mu k_\mu = 0 \ ,$$

(18)

which is just the eikonal equation

$$\gamma^\nu \frac{\partial \phi}{\partial x^\nu} \frac{\partial \phi}{\partial x^\nu} = 0 \ .$$

(19)

Substitution of the expansions of $g_{\nu\nu}$, $\phi$, and $R_{\alpha\nu}$ into equation (2) and collecting terms of the order $\psi^0$ gives

$$\gamma^\nu \nabla_\nu \phi + \Omega_j \nabla_\nu \phi + \frac{dP}{d\phi} = \kappa_k k^\nu \psi - \gamma^\nu [\nabla_\nu (k, \psi) + k \psi, + 2\Omega_j \phi, k, \psi] \ .$$

(20)

where $\phi_\mu = \phi_{\mu\nu}$ and $\psi$ is the covariant derivative with respect to $\Gamma$. Furthermore, the following abbreviations are used:

$$\Omega_j = \frac{8\pi G\phi(1 + \tilde{G})}{1 + 8\pi G\phi(1 + \tilde{G})} \ .$$

(21)

and

$$\frac{dP}{d\phi} = \frac{(1 + 8\pi G\phi^2)^2 V d\phi - 32\pi G\phi V}{1 + 8\pi G\phi^2(1 + \tilde{G})} \ .$$

(22)

Analogously, there follows from equation (3) the equation of motion for the background metric $\tilde{g}_{\alpha\nu}$. In the Lorentz–de Donder gauge

$$\nabla^2 (\tilde{h}_{\mu\nu} - \frac{1}{4} h g_{\mu\nu}) = 0 \ .$$

(23)

it reads

$$R_{\alpha\nu} = -\frac{1}{2} \left[ (\nabla_\nu h_{\alpha\nu} + 2 \kappa \nabla_\nu \phi_{\alpha\nu} - k, B_\alpha - k, B_\alpha + 2 \kappa \left( h_{\alpha\nu} - \frac{1}{2} h h_{\alpha\nu} \right) B_{\alpha\nu} \right]$$

$$+ \frac{8\pi G}{1 + 8\pi G\phi^2} \left\{ -k, k, N + 2 \kappa \phi B_{\alpha\nu} \phi_{\alpha\nu} - (1 + 2 \kappa) \phi_{\alpha\nu} - \phi_\alpha \phi_{\alpha\nu} + \phi_{\alpha\nu} V \right\}$$

$$+ 2 \kappa (h_{\alpha\nu} + \phi_{\alpha\nu} + \phi \delta_\alpha \phi_{\nu} + \phi \delta_{\nu} \phi_{\alpha} + \phi_{\alpha\nu} \phi_\mu + 2 \phi_{\alpha\nu} k, \phi) \ .$$

(24)

which contains the abbreviations

$$N = \frac{(1 + 8\pi G\phi^2 + 32\pi^2 G^2 \phi) \psi^2 + 2(1 - 8\pi G\phi^2)(\phi \psi^2)}{1 + 8\pi G\phi^2} - \phi \psi^2 + (1 + 8\pi G\phi^2)$$

$$\times \left[ \frac{1}{4} \left( h_{\alpha\nu} - \frac{1}{2} h h_{\alpha\nu} \right) \right]$$

(25)

and

$$B_{\alpha\nu} = k, \delta h_{\alpha\nu} - (h_{\alpha\nu} - \frac{1}{2} h h_{\alpha\nu}) \ .$$

(26)

From equations (4) and (23) there follow, collecting again equal powers of $\psi$,

$$k, (h_{\alpha\nu} - \frac{1}{2} h h_{\alpha\nu}) = 0 \ .$$

(27)

and

$$\nabla_\mu (\phi_{\alpha\nu} - \frac{1}{2} h h_{\alpha\nu}) - k, h \phi_{\alpha\nu} - \frac{1}{2} k, h h_{\alpha\nu} = 0 \ .$$

(28)

The perturbations $h_{\alpha\nu}$ and $\psi$ are periodic functions of $\psi$. Let their period be $t$. Hence the right-hand side of equation (20) is also a periodic function of $\psi$ with period $t$. The left-hand side of equation (20) is independent of $\psi$. We now integrate both sides of equation (20) with respect to $\psi$ over a period $t$. From the left-hand side of equation (20), this results merely in a multiplication by $t$. The integral of the term in square brackets on the right-hand side of equation (20) vanishes because $\int_0^t \psi dt^2 = 0$. Furthermore, one easily verifies that

$$k, k, \int_0^t \phi_{\alpha\nu} dt^2 = 0 \ .$$

(29)
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by partial integration, and \(a_p k_6 \bar{\psi} = 0\). The latter relation is a consequence of equations (27) and (18). Hence,

\[
\bar{\psi}^a \nabla_a \phi_1 + \Omega_0 \bar{\psi} \phi_1 + \frac{dP}{d\theta} = 0.
\]  

(30)

From equations (20) and (30), there follows

\[
k_p \bar{\psi} \phi_1 - \bar{\psi} \{ \nabla \phi_1 k_6 \psi + k_6 \bar{\psi} \} - 2 \Omega_0 \phi_1 k \bar{\psi} = 0.
\]  

(31)

Integration of both sides of equation (24) with respect to \(\xi\) over a period \(T\) gives analogously

\[
R_m = -\frac{8\pi G}{1 + 8\pi G/G_0^2} \left[ \{1 + 2\bar{\psi} \phi_1 + \bar{\psi} \phi_1 \} V + \bar{\psi} \{ \nabla \phi_1 \bar{\psi} + \bar{\psi} \phi_1 \bar{\psi} \} - \frac{k_p}{4} \int_0^T Ed\xi \right],
\]  

(32)

where

\[
E(x^a, \xi) = \frac{8\pi G(1 + 8\pi G/G_0^2 + 32\pi G^2 G_0^2) \bar{\psi}^2 + \bar{\psi} \left( \frac{k_p^2 \bar{\psi}^2 - \frac{1}{2} \bar{\psi}^2}{1 + 8\pi G/G_0^2} \right)}{1 + 8\pi G/G_0^2}.
\]  

(33)

For a further discussion and interpretation of this quantity see Choquet-Bruhat (1982). Equation (32) is the Einstein equation for the background field \(\bar{\psi} \), where the energy-momentum tensor is the sum of the energy-momentum tensor of the background Higgs field \(\phi\) and the back-reaction term

\[
S_{\psi} = \frac{k_p}{4} \int_0^T Ed\xi
\]  

(34)

From equations (24) and (32) there follows the propagation equation for \(h_{\psi}\):

\[
-\frac{1}{2} \left( \nabla_x \bar{\phi}_{\psi} + 2k^2 \nabla_x h_{\psi} - k_x B_x - k_y B_y - 2k_0 \left( \frac{k_p^2}{2} \bar{\psi}^2 \right) \right) + \frac{8\pi G}{1 + 8\pi G/G_0^2} \left\{ \left[ 1 + 2\phi \phi_1 - 2\phi \phi_1 \right] \bar{\psi}^2 + \bar{\psi} \left( \frac{k_p^2 \bar{\psi}^2 - \frac{1}{2} \bar{\psi}^2}{1 + 8\pi G/G_0^2} \right) \right\}
\]  

(35)

This is an integro-differential equation for \(h_{\psi}\); however, it also contains the higher order corrections \(k_p\) and \(\xi\).

III. APPLICATION TO A SPHERICALLY SYMMETRIC SPACETIME

We can try to find approximate wave solutions of the gravitational field, \(c h_{\psi} + e^2 k_{\psi}\), and the Higgs field, \(\phi + e^2 \psi\), in the case of a spherical symmetric background

\[
ds^2 = e^{2\kappa t} c^2 [-dt^2 + dr^2] + e^{2\kappa \psi} [d\psi^2 + \sin^2 \theta d\psi^2].
\]  

(36)

For a propagation vector \(k_{\psi}\), satisfying (18), we choose

\[
k_{\psi} = k_0 \delta r, r(1, \pm 1, 0, 0).
\]  

(37)

Its divergence is given by

\[
\nabla_\psi k_{\psi} = 2e^{-2\kappa x} \left[ \partial \bar{\psi} \bar{\psi} - \partial \psi \psi \right].
\]  

(38)

From the gauge conditions (see eq. [26]) we obtain for \(h_{\psi}\), with \(h_{0\psi} = 0\),

\[
h_{\psi} = \left( \begin{array}{ccc}
0 & 0 & \partial \\
0 & 0 & 0 \\
\partial & h_{12} & h_{13} \\
h_{23} & h_{33} & -h_{22} \sin^2 \theta
\end{array} \right)
\]  

(39)

From the \(0-0\) component of equation (16) and equation (11) together with equation (39), we find that

\[
\psi = 0.
\]  

(40)

The solution of this equation is linear in \(\zeta\), but it is also assumed to be periodic in \(\zeta\); thus it is independent of \(\zeta\):

\[
\phi = 0.
\]  

(41)

Hence \(\phi\) drops out of the foregoing equations.

From the \(0-0\), \(1-1\), \(0-1\), \(0-2\), \(0-3\), and \(1-3\) components of equation (35), one obtains respectively

\[
\left( \begin{array}{c}
k_{01} \\
k_{22} \\
k_{33}
\end{array} \right) = \left( \begin{array}{c}
\pm \frac{1}{2} \left( k_{11} + k_{00} \right) \\
\frac{2}{\sin^2 \theta} \\
\frac{2}{\sin^2 \theta}
\end{array} \right).
\]  

(42)

\[
\int_0^T Ed\xi \frac{16\pi G e^{2\psi}}{1 + 8\pi G/G_0^2} N.
\]  

(43)
\[ E = \frac{1}{2} e^{-4\epsilon \left( h_{32}^2 + k_{32}^2 \sin^2 \theta \right)} \] (48)

and \( N \),
\[ N = \frac{1 + 8\pi G/\phi^2}{8\pi G} \left[ \frac{1}{2} h_{32} \cdot h_{32} - \frac{1}{2} \left( h_{32} \cdot h_{32} \right) - \frac{16\pi G \phi}{1 + 8\pi G/\phi^2} \chi \right]. \] (49)

If we impose on \( k_{\nu} \) the conditions \( k_{\nu} = 0 \) \((a, b = 2, 3)\) and \( k_{\nu} = 0 \), we obtain (Choquet-Bruhat 1969b) from the equations (42), (43), (44), (45), and (49):
\[ k_{11} = 0, \] (50)
\[ k_{12} = \frac{1}{k_{0}} \left[ \partial_{\alpha} h_{32} + \partial_{\alpha} h_{32} \right] e^{2\omega - 2\phi}, \] (51)
\[ k_{13} = \frac{1}{k_{0}} \left[ \partial_{\alpha} h_{32} - \partial_{\alpha} h_{32} \right] e^{2\omega - 2\phi}. \] (52)

and
\[ \chi = \frac{1}{1 + 8\pi G/\phi^2} \left[ \frac{1}{2} e^{-4\epsilon \left( h_{32}^2 + k_{32}^2 \sin^2 \theta \right)} + \frac{1}{r_{0}^2} E d\xi + e^{-4\epsilon \left( h_{32}^2 + k_{32}^2 \sin^2 \theta \right)} \right]. \] (53)

The \((0-0), (1-1), (2-2), \) and \((0-1)\) components of equation (32) gives, after some algebra,
\[ \partial_{\alpha}^2 - \partial_{\alpha}^2 = \partial_{\alpha}^2 + (\partial_{\alpha}^2 \phi^2 + \partial_{\alpha}^2 \phi^2) = e^{2\omega - 2\phi} + \frac{4\pi G}{1 + 8\pi G/\phi^2} \times \left\{ -2\phi e^{2\phi} \frac{dP}{d\phi} + (1 - 2\gamma) \phi (\partial_{\alpha} \phi^2 - (\partial_{\alpha} \phi)^2) + 2\phi (\partial_{\alpha} \phi^2 - (\partial_{\alpha} \phi)^2) \right\}. \] (54)
\[ \partial_{\alpha}^2 - \partial_{\alpha}^2 = \partial_{\alpha}^2 + 2\partial_{\alpha} \phi^2 - 2\partial_{\alpha} \phi^2 = -e^{2\omega - 2\phi} + \frac{8\pi G}{1 + 8\pi G/\phi^2} \times \left\{ -f \phi e^{2\phi} \frac{dP}{d\phi} + f (1 - \Omega) \phi \phi^2 - (\partial_{\alpha} \phi)^2 + 2\phi (\partial_{\alpha} \phi^2 - (\partial_{\alpha} \phi)^2) \right\}. \] (55)

and the constraint equations
\[ \partial_{\alpha} \phi^2 + \partial_{\alpha} \phi^2 + \partial_{\alpha} \phi^2 + \partial_{\alpha} \phi^2 + \frac{4\pi G}{1 + 8\pi G/\phi^2} \times \left\{ [1 + 2f] \phi \phi^2 + 2\phi \phi^2, \phi \phi^2 - 2\phi \phi^2, \phi \phi^2 - 2\phi \phi^2, \phi \phi^2 \right\} \] \[ \times \left[ \frac{1}{2} (\phi \phi^2 + (\phi \phi^2) \phi \phi^2 - (\phi \phi^2) \phi \phi^2 - (\phi \phi^2) \phi \phi^2 - (\phi \phi^2) \phi \phi^2 \right] + \frac{k^2}{r} \int_0^r E d\xi = 0 \] (56)

and
\[ 2\partial_{\alpha} \phi^2 + \partial_{\alpha} \phi^2 + 3\partial_{\alpha} \phi^2 + 2\partial_{\alpha} \phi^2 - 2\partial_{\alpha} \phi^2 = e^{2\omega - 2\phi} + \frac{8\pi G}{1 + 8\pi G/\phi^2} \times \left\{ \left[ \frac{1}{2} (\phi \phi^2 + (\phi \phi^2) \phi \phi^2 - (\phi \phi^2) \phi \phi^2 - (\phi \phi^2) \phi \phi^2 \right] + \frac{k^2}{r} \int_0^r E d\xi = 0 \right\}. \] (57)
Equation (30) reads, in the present case,
\[
\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial \phi^2} + 2 \beta(\phi, \phi) \phi - \partial_\phi \delta \phi + \Omega_f (\phi, \phi) - \ell \frac{d \phi}{d \phi} = e^{2 \phi} \Phi \frac{d \phi}{d t} = 0. \tag{58}
\]
We conclude that the high-frequency perturbations only occurs in the constraint equations. The equations (46), (54), (55), and (58) represent a system of partial differential equations of the hyperbolic type and are solved numerically in the next section. Let us now choose for instance
\[
k_{22} = \frac{h_{22}}{\cos \theta} = f(r, t) \sin \sin \xi. \tag{59}
\]
Then, from equations (51), (52), and (53), we obtain the second-order high-frequency perturbations the expressions
\[
k_{12} = \pm \frac{3}{2} \frac{e^{2a - 3b} \phi}{k_0} \cos \theta \cos \xi, \tag{60}
\]
\[
k_{13} = \pm \frac{1}{2} \frac{e^{2a - 3b} \phi}{k_0} \cos \theta - \sin^2 \theta \cos \xi, \tag{61}
\]
and
\[
x = \frac{3(1 + 8 \eta G(\phi^2))}{256 \eta G(\phi^2)} \frac{\hat{\phi}^2 e^{-4\phi}}{\cos 2\xi}. \tag{62}
\]
In our specific choice of gauge (see eq. [23]), we notice that the local energy density (see eq. [48]) does not contain $\phi$. So we do not find a result comparable to that of Choquet-Bruhat and Taub (1977) in the case of a charged scalar field: a high-frequency scalar field can create through an electromagnetic field a high-frequency gravitational field. In our case the coupling is more complicated, through the second-order term $\chi$. So it will be of interest to introduce a third-order correction term in the expansion of equation (4) (wave-wave scattering). In order to say something about the outgoing energy of the perturbations, one usually admits the "radiative" coordinates $k_n = \phi_n$ (Choquet-Bruhat 1969b). Then one obtains for the mass loss by the radiation
\[
\lim_{r \to \infty} \frac{1}{r^2} \int_{-\infty}^{\infty} Ed\xi = \frac{1}{4} \frac{e^{-4\phi}}{e^{2a - 3b}}. \tag{63}
\]
If we could transform the equations into the radiative coordinates, then expression (63) will be equal to
\[
\frac{1}{r^2} \frac{dM}{dt}, \tag{64}
\]
where $M$ represents the well-known parameter in the Schwarzschild metric
\[
ds^2 = -\left(1 - \frac{2MC}{r}\right) dt^2 + \left(1 - \frac{2MC}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{65}
\]
\textbf{IV. NUMERICAL CALCULATIONS}

Some quantitative properties of the system of partial differential equations of § III can be investigated for a specific choice of $V(\phi)$ and the initial conditions. One of the features the model could have is a trapped surface, i.e., a closed two-dimensional surface for which both inward- and outward-pointing normal null geodesics are convergent. There is a well-known theorem of Penrose (1965) that states that a spacetime cannot be null-geodesically complete if there is a closed trapped surface and if $R_n k^n \geq 0$ for all null vectors $k_n$. If this condition is satisfied, one proves with the help of Raychaudhuri's equation that null geodesics orthogonal to two-surfaces in general will develop conjugate points where $\nabla_n k^n = \infty$ (roughly, infinitesimal near geodesics in the family intersect each other). The presence of a trapped surface is a very useful criterion that provides a sufficient indication that a region of spacetime lies within a blackhole. Further, there will be connection between the problem of primordial black hole formation and the problem of primordial high-frequency perturbations. Numerical calculations on the formation of trapped surfaces in an inflationary model were done by Chao (1984) but in a quite different approach from ours. He follows the evolution of the divergence of the inward and outward geodesic normals (see eq. [37]) and finds that a sufficiently large fluctuation must have caused trapped surfaces, and he predicts that the expansion of the background spacetime must considerably increase the fluctuation threshold for black hole formation. Here we are dealing with a coupled system of back-reaction and propagation equations of the high-frequency fields, and it is by no means clear whether the nonlinear effects will hasten or abort collapse. In order to investigate the influence of the high-frequency perturbation $k_n$ on the formation of trapped surfaces, we consider as an example for the initial intrinsic geometry the flat metric $\mathcal{Z}(r, \phi) = \beta(r, \phi) = 0$ and $\phi(0, r) = 0$. Further, for the initial Higgs field we choose
\[
\phi(0, r) = \begin{cases} 
0.3 & \text{for } r < 0.4, \\
0.5 & \text{for } 0.4 < r < 1.0, \\
1.0 & \text{for } r > 1.0.
\end{cases} \tag{66}
\]
and for the initial high-frequency perturbation

$$\delta(0, r) = 4e^{-r^2} \sin 2\pi r.$$  

(67)

Then, from the constraint equations (56) and (57) we obtain for the initial extrinsic metric (with $8\pi G = 3$)

$$\bar{c}_i = \pm \left[ e^{-r^2} + \frac{\pi e^{-r^2}}{\pi^2 + 1} \left( \sin 2\pi r - \frac{1}{\pi} \cos 2\pi r \right) \right]$$

(68)

and

$$\bar{c}_{\alpha\beta} = \frac{1}{\bar{c}_i} \left[ \frac{1}{2} \bar{g}_{\alpha\beta} - 1 - (\bar{c}_i, \beta)^2 - \frac{3V}{1 + 3/\bar{g}_{\alpha\beta}^2} \right].$$

(69)

From equation (68) we can see that the behavior of the initial $\bar{c}_i, \beta$ is strongly dependent on the initial high-frequency perturbation $\delta$. If we choose for the effective potential the Coleman-Weinberg potential (with the global minimum at $\phi = 1$)

$$V(\phi) = 2\phi^4(\ln \phi^2 - 0.5) + 1,$$

(70)

then we are in a position to solve the equations numerically. For the computer program we used a finite element collocation procedure (Vichnevetsky and Bowles 1982). This method is based on piecewise polynomials for the discretization of the spatial variable. The collocation procedure reduces the partial differential equations to a semidiscrete system which then depends only on the time variable. The time integration is accomplished by use of a standard method for treating time-dependent differential equations (we used the ACM algorithm 540 of IMSL). For $f = 0$ we plotted $\phi$ (Fig. 1), $\tilde{h}$ (Fig. 2), $\tilde{c}_i, \beta + \tilde{c}_i, \beta$ (Fig. 3), and $\tilde{c}_i, \beta - \tilde{c}_i, \beta$ (Fig. 4). From Figure 2 we conclude that the wave behavior of the high-frequency perturbation shows a kind of persistence.

Figs. 5 and 6 we plot $\tilde{c}_i, \beta + \tilde{c}_i, \beta$ and $\tilde{c}_i, \beta - \tilde{c}_i, \beta$ for $f = -1/6$.

The behavior of $\phi$ (see Fig. 1) resembles the solution found by Bril and Hartle (1964) several years ago. In Figures 5 and 6 we plot $\tilde{c}_i, \beta + \tilde{c}_i, \beta$ and $\tilde{c}_i, \beta - \tilde{c}_i, \beta$ for $f = -1/6$.

V. CONCLUSIONS

We investigated high-frequency perturbations in gravity theory coupled with a Higgs field. The phenomenon of back-reaction of the perturbations on the background fields occurs.

![Diagram](image-url)

Fig. 1. The evolution of the Higgs field $f = 0$. We started at $r = 0.13$ and we took four time steps. The space gridding was 50.
In the Lorentz gauge we find an approximate wave solution of the form
\[ g_{\mu\nu} = \delta_{\mu\nu} + \epsilon h_{\mu\nu} + \epsilon^2 k_{\mu\nu} \]  
(72)
and
\[ \phi = \phi_0 + \epsilon \chi. \]
(73)

Because of the appearance of the high-frequency component \( h_{\mu\nu} \) in the dynamical equations of the background fields, the formation of trapped surfaces will depend on the initial situation of these high-frequency perturbations (The existence of trapped surfaces could imply that singularities will necessarily develop).

In order to draw some reliable conclusions concerning primordial black hole formation from numerical solutions, one has to choose some other types of high-frequency perturbations. However, after comparison with the results of Chao (1984), it is already
evident that in our approximation the numerical behavior of \( \partial, \beta + \partial, \beta \), and hence the behavior of the trapped surfaces, is quite different. Starting with \( \partial, \beta + \partial, \beta \) everywhere negative, it will not diverge to minus infinity within a finite time. Clearly, the formation of trapped surfaces will depend on the initial \( h_u \).

Some attention must be focused on the "null-convergence condition" \( R_u, k^* k^* \geq 0 \), in particular for \( h_u \) of equation (37). From equation (32) we obtain for this condition

\[
-\frac{8\pi Ge^{-\delta k^*_j}}{1 + 8\pi G f \delta^2} \left[ (1 + 2f)(\partial_\phi \phi)^2 + (\partial_\beta \beta)^2 + 2\partial_\phi \phi \partial_\beta \beta - 2(\partial_\phi \phi \partial_\beta \beta + \partial_\beta \beta \partial_\phi \phi) + 2\partial_\phi \phi \partial_\beta \beta + \partial_\beta \beta \partial_\phi \phi \right] \geq 0 .
\]  

(74)

For suitable \( f \) this expression can become 0 or greater. Hence the expression must be evaluated during the numerical calculations to check if it remains 0 or greater (when a trapped surface occurs).

![Fig. 4. The evolution of the inward null-geodesic congruence for \( f = 0 \)](image)

![Fig. 5. As in Fig. 3, for \( f = -\frac{1}{2} \). Clearly, the evolution of \( \gamma, \beta + \gamma, \beta \) depends on \( f \).](image)
It will be of interest to extend the analysis above to metrics other than spherically symmetric. Then the back-reaction will be more profound. Further, it will be worth pushing the expansions to higher orders. The results thus obtained could shed light on the interaction of the second-order term $\chi$ of the Higgs field expansion with the gravitational waves.

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3.4 Higher-order equations of the coupled gravity-scalar theory

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A summary of this work was presented at the 11th International Conference on General Relativity and Gravitation, July 6-12, 1986 in Stockholm.

Abstract

Using the multiple-scale method, the second order equations of the high-frequency gravitational perturbations are obtained as well as the equations for the first order term $\phi$ of the scalar field. Due to the fact that, in the special gauge we used, $\phi$ is independent of the fast scale $\xi$, we obtain a coupled system of the background field $\tilde{\phi}$ and the first order field $\phi$. The pseudo-potential will be of the form $V(\tilde{\phi}, \phi) = V(\tilde{\phi}) + m^2 \phi^2$, where $m$ depends on the couplings parameter of the scalar field with gravity and $V(\tilde{\phi})$. In the case of a Higgs potential the stability of the theory is investigated.

3.4.1 Introduction

In §3.3 we applied the multiple-scale method to the coupled system of gravitation and the Higgs field and investigated the approximate wave solution to first order on a spherical symmetric background. It was found that the initial high-frequency gravitational perturbation has a significant influence on the formation of trapped surfaces. In figures 1 and 2 we plotted a characteristic solution of the system of equations of §3.3. For two different initial gravitational perturbations, i.e., $\hat{h}=e^{-r}$ and $\hat{h}=re^{1-r}$ we followed the evolution of the divergence of the inward and outward null-geodesic congruence normal to the two-surface $t,r=$constant. From these pictures we can conclude that the apparent horizon (black-hole horizon) threshold increases for initial $\hat{h}$ located closer to $r=0$. The influence of the scalar field perturbation $\psi$ on the location and motion of the apparent and event horizon can be determined when the equation for $\psi$ is obtained. In the Lorentz-gauge, the first order scalar field perturbation $\psi$ drops out of the equations (the dot means differentiation with respect to the fast scale $\xi$). In order to obtain the dynamical equation for $\psi$, we must go beyond first order. In §3.4.2 we will investigate the second order equations. These equations also exhibit the wave-wave interaction of the high-frequency gravitational perturbations. Further, the stability of the scalar field against the fluctuations can be examined. The conditions for this stability are quite different from those obtained in the usual linear approximation (Hosotani, 1985).

In §3.4.3 we will examine the induced potential of the $\psi$-particle.

3.4.2 The second order equations

The $\mathcal{O}(\epsilon)$ equations can be obtained from equations (2) and (3) of §3.3.2. They yield equations for $m_{\mu\nu}$ and $\psi$ in the expansion

$$g_{\mu\nu}(x^\sigma,\xi)=\tilde{g}_{\mu\nu}(x^\sigma)+\epsilon h_{\mu\nu}(x^\sigma,\xi)+\epsilon^2 k_{\mu\nu}(x^\sigma,\xi)+\epsilon^3 m_{\mu\nu}(x^\sigma,\xi)+... \quad (1)$$
Figure 1. Plot of the metric component $\beta$, the gravitational perturbation $\hat{h}$, and the divergence of the outward and inward null geodesic congruences normal to the two-surface $t,r=$ const.
Figure 2. As figure 1, but now with a different initial gravitational perturbation. Clearly, the location of the trapped surface depends on the initial \( \hat{h} \).
and

\[ \phi(x^\sigma, \xi) = \tilde{\phi}(x^\sigma) + \epsilon \psi(x^\sigma, \xi) + \epsilon^2 \chi(x^\sigma, \xi) + \epsilon^3 \omega(x^\sigma, \xi) + \ldots \quad (2) \]

The equation for \( \psi \) then becomes

\[ \bar{\nabla}_{\alpha} \phi + 2 \Delta \phi \bar{a}_{\alpha} \phi + \epsilon \psi + \frac{dV^1}{d\phi} = h^{\alpha \beta} (\bar{\nabla}_{\alpha} \phi + k_{\alpha \beta} \chi) + \Delta h^{\alpha \beta \gamma} \bar{a}_{\alpha} \phi \bar{a}_{\beta} \phi \]

\[ + \chi \dot{\psi} + \chi \bar{\nabla}_{\alpha} \bar{\nabla} \phi (k_{\alpha \beta} - k_{\alpha \hat{\beta}} \hat{\phi} - 2 \Delta \phi \bar{a}_{\alpha} \phi \bar{a}_{\beta} \phi) + h^{\alpha \beta \gamma} \bar{a}_{\alpha} \phi \bar{a}_{\beta} \phi + \chi \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} \phi \bar{a}_{\alpha} \phi \bar{a}_{\beta} \phi \quad (3) \]

where \( \bar{\nabla} \) means covariant differentiation with respect to \( \bar{g}_{\mu \nu} \),

\[ \Omega = \frac{8 \pi G f \bar{\psi} (1+6f)}{1+8 \pi G f \bar{\psi}^2 (1+6f)} \quad (4) \]

\[ \Sigma = \frac{1-8 \pi G f \bar{\psi}^2}{1+8 \pi G f \bar{\psi}^2} (1+6f) \bar{\psi} \Sigma \left((1+6f) \bar{a}_{\alpha} \phi \bar{a}_{\alpha} \phi + 12 f \bar{\nabla}_{\alpha} \bar{\nabla}_{\mu} \phi \bar{a}_{\alpha} \phi - 4 \bar{\nabla} \right), \quad (5) \]

and

\[ \frac{dV^1}{d\phi} = \frac{1}{1+8 \pi G f \bar{\psi}^2 (1+6f)} \left((1+8 \pi G f \bar{\psi}^2) \frac{dV^1}{d\phi} - 32 \pi G f \bar{\psi} V^1 \right). \quad (6) \]

Further, \( B_{\mu \nu} \) and \( A_{\mu \nu} \) represent the first and second order terms in the expansion of the christoffel symbol. \( Y \) is an expression linear in \( \dot{\phi} \) and \( \ddot{\phi} \). \( \bar{\nabla} \) and \( V^1 \) are the first two terms in the expansion of the potential.

When we apply the equations to the spherically symmetric situation of \( \S 3.3 \), we obtain after integration with respect to \( \xi \) over a period \( \tau \)

\[ \bar{\nabla}_{\alpha} \phi + 2 \Delta \phi \bar{a}_{\alpha} \phi + \frac{dW}{d\phi} = 0 \quad (7) \]

where
Due to the periodicity of $h_{\nu \nu}$, the only contribution to the right hand side of equation (3) will eventually arise from a term proportional to

$$\frac{r}{h^{ab}} \theta^{a} \alpha^{b} \delta \xi .$$

In most cases, this term also disappears (see §3.3). The $\mathcal{O}(\varepsilon)$ term of the Einstein equation yields

$$R_{(1)} = \frac{-128 \pi^{2} G^{2} f_{\psi}^{2}}{(1+8 \pi G f_{\psi}^{2})^{2}} \left\{ -(1+2f) \bar{\psi}_{,\mu \nu} \bar{\psi} + g_{\mu \nu} \bar{\psi} - 2f \bar{\psi} \left( k_{,\mu \nu} \psi + \bar{\psi} \bar{\psi} \right) \right\}$$

$$- B_{\mu \nu} \lambda \bar{\psi} \chi - f \bar{\psi}_{,\mu \nu} \left( \partial_{\lambda} \bar{\psi}_{,\lambda} + \bar{\psi}_{,\lambda} \bar{\psi}_{,\lambda} - \partial_{\lambda} \bar{\psi}_{,\lambda} \right)$$

$$+ \frac{8 \pi G}{1+8 \pi G f_{\psi}^{2}} \left\{ -(1+2f) \left( \bar{\psi}_{,\mu \nu} \bar{\psi}_{,\lambda} + \bar{\psi}_{,\mu \nu} \bar{\psi}_{,\lambda} + \bar{\psi}_{,\mu \nu} \bar{\psi}_{,\lambda} \right) \right\}$$

$$+ \bar{\psi}_{,\mu \nu} \psi^{1} + h_{\mu \nu} \bar{\psi} - 2f \left( \bar{\psi}_{,\mu \nu} \bar{\psi} + \bar{\psi}_{,\mu \nu} \bar{\psi} \right) + k_{,\mu \nu} \psi + k_{,\mu \nu} \bar{\psi}$$

$$+ \psi_{,\mu \nu} \psi + \bar{\psi}_{,\mu \nu} \bar{\psi} + k_{,\mu \nu} \psi + k_{,\mu \nu} \bar{\psi} - \partial_{\lambda} \bar{\psi} \left( \partial_{\lambda} \psi + \partial_{\lambda} \bar{\psi} \right)$$

$$- (\partial_{\lambda} \psi + k_{,\mu \nu} \bar{\psi} - \partial_{\lambda} \psi - k_{,\mu \nu} \bar{\psi}) - \bar{\psi}_{,\mu \nu} \psi + \bar{\psi}_{,\mu \nu} \bar{\psi} + \bar{\psi}_{,\mu \nu} \psi$$

$$+ 2 \partial_{\lambda} \partial_{\rho} \psi + 2 \partial_{\lambda} \partial_{\rho} \bar{\psi} + 2 \partial_{\lambda} \partial_{\rho} \psi - \partial_{\lambda} \partial_{\rho} \psi - \partial_{\lambda} \partial_{\rho} \bar{\psi} - \partial_{\lambda} \partial_{\rho} \bar{\psi}$$

$$+ \bar{\psi}_{,\mu \nu} \psi + \bar{\psi}_{,\mu \nu} \bar{\psi} + \bar{\psi}_{,\mu \nu} \psi - \partial_{\lambda} \partial_{\rho} \psi - \partial_{\lambda} \partial_{\rho} \bar{\psi} + h_{\mu \nu} \bar{\psi}$$

$$+ Z \left( \bar{\psi}, \psi \right),$$

where $Z$ is an expression linear in $\bar{\psi}$ and $\psi$. Further,
\[ 4R_{\nu}^{(1)} = g^{\rho\sigma} \left( \frac{g_{\nu\rho} - g_{\rho\nu}}{} + \frac{g_{\lambda\nu}}{\mu} + \frac{g_{\lambda\mu}}{\nu} - \frac{g_{\lambda\mu}}{\rho} - \frac{k_{\lambda\rho}}{\mu} + k_{\rho\lambda} \right) \left( \frac{g_{\nu\rho} - g_{\rho\nu}}{} - \frac{g_{\lambda\mu}}{\nu} - \frac{g_{\lambda\mu}}{\tau} + \frac{g_{\lambda\mu}}{\rho} \right) \left( \frac{g_{\nu\rho} - g_{\rho\nu}}{} + \frac{g_{\lambda\mu}}{\nu} - \frac{g_{\lambda\mu}}{\tau} + \frac{g_{\lambda\mu}}{\rho} \right) \]
From equation (10) we can derive the equation for \( m_{\mu \nu} \). After some tedious algebra, we obtain for the (0-2) component (for example):

\[
\begin{align*}
\tilde{m}_{\nu 2}^{\mu} & = \frac{2}{k_\alpha} e^{2a - 2\theta} \{ \cos \theta \sin \xi (2 \hat{\alpha} - \frac{3}{2} \hat{\alpha} - \hat{\alpha}) \\
& + \cos \theta \sin \xi (4 \hat{\beta} - 3 \hat{\beta}) \} + \frac{1}{2} k_\alpha e^{-2\theta} \cos \xi \sin \xi \times (1 + \cos^2 \theta) \cos \theta \sin \theta 
\end{align*}
\]

(13)

In the situation of §3.3, we will obtain from equation (13), after integration twice with respect to \( \xi \), a bounded expression for \( m_{02} - m_{12} \). In the derivation, we used the solutions for \( k_{\mu \nu} \) and \( h_{\nu \mu} \) given in §3.3. In general, however, we will obtain some restrictions on \( h_{\mu \nu} \) and \( k_{\mu \nu} \) so that \( m_{\mu \nu} \) remains bounded. The following terms must then be bounded.

\[
1^{\mu \nu} = k^{\mu \nu} - h^{\alpha \nu} h^{\mu}_{\alpha}. 
\]
Further, we see from equation (13) that $m_{12} = m_0$ is not regular for all $\theta$; for $\theta = 0, \pi$ the solution is divergent.

3.4.3 Stability of the scalar field

In the linear approximation (Hosotani, 1985) the stability of the theory can be examined. For the semi-stable de Sitter spacetime, primordial inflation can be caused due to the potential

$$V = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4,$$

and stability is guaranteed under special conditions on $f, \lambda, G$ and $m$. In the case of the pseudo-potential $\tilde{V}$ (see §3.3, equation (22)), we can define $\phi_c = -(8 \pi Gf(1+6f))^{-1}$ and for $-1/6 < f < 0$, the potential is sketched in figure 3.

![Figure 3. $\tilde{V}$ in the semistable de Sitter model ($-1/6 < f < 0$).](image)
There are two types of local minima, $\phi^2 = 0$ and $\phi^2 = \phi_o^2$. The de Sitter spacetime ($\phi^2 = \phi_o^2$) is expected to be only semistable, decaying into the Minkowski spacetime ($\phi^2 = 0$). This is due to the fact that the effective gravitational constant in the de Sitter spacetime, $G^{-1}_{dS} = G^{-1} + 8\pi\phi_o^2$, is negative and so the spacetime is unstable at the quantum level. Another interesting case is $\phi = \phi_o + \sqrt{m^2}\phi^2$. This potential yields an unacceptably large cosmological constant $\Lambda_o = 8\pi G\phi_o^2$ when $G = M_{Pl}^{-2}$.

This problem can be overcome by starting with an extremely tiny bare gravitational constant. Under certain conditions, non-vanishing expectation value of the scalar field $\phi = \phi_o$ reduces the vacuum energy density to an extremely tiny value and increases $G$ to the observed value $M_{Pl}^{-2}$ simultaneously. The cosmological problem is then not a problem of how to reduce the effective vacuum energy density, but rather that of how to get a large effective gravitational constant ($G_{obs} \gg R/V_o$).

We will apply the equations of §3.4.2 to the effective potential

$$V(\phi) = V_o + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\phi^4.$$  \tag{18}

We obtain from equation (7) with $\psi = \psi(r,t)P_\ell(\cos\theta)$

$$\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2} + 2\partial_r(\partial_r \psi - \partial_t \partial_r \psi) + 2\Delta(\partial_r \partial_t \psi - \partial_t \partial_r \psi) + \frac{dW}{d\phi} = 0,$$  \tag{19}

where

$$W = \frac{1}{2}m^2\phi^2 + W_o$$  \tag{20}

and

$$\frac{-2}{m^2} \equiv \frac{d^2}{d\phi^2} \left[ \frac{(\phi_C^2(1+6f) - \phi^2)(1+6f)}{(\phi_C^2(1+6f) - \phi^2)(\phi_C^2 - \phi^2)} \right] \frac{\phi^2}{\phi_C^2(1+6f) - \phi^2} - \frac{\phi^2}{\phi^2(\phi_C^2 - \phi^2)} \right] \frac{\phi^2}{\phi_C^2 - \phi^2} \left[ \frac{\phi^2}{\phi_C^2(1+6f) - \phi^2} \right]$$

$$- \frac{12f^2}{(1+6f)(\phi_C^2 - \phi^2)} \left[ \frac{(\phi_C^2(1+6f) - \phi^2)(1+6f)}{(\phi_C^2 - \phi^2)(\phi_C^2 - \phi^2)} \right] \left[ \phi^2 + \frac{1}{6}\phi_C^3 \right] + 4\phi V - 4V.$$
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\[
\begin{align*}
\frac{m^2}{(1+6f)^2} \left( \frac{\dot{\phi}^2}{\phi^2} (1+6f) + 3\dot{\phi}^2 \right) + \frac{3\dot{\phi}^2}{6} \left( \frac{\ddot{\phi}}{\phi} \right) (1+6f) + \ddot{\phi}^2 \\
(1+6f) (\ddot{\phi}^2 - \dot{\phi}^2) 
\end{align*}
\]

In order to have spontaneous symmetry breaking, we must take \( m^2 < 0 \) (when radiative corrections are taken into account, spontaneous symmetry breaking occurs even if \( m^2 = 0 \); mass will then be developed by the Higgs mechanism (Huang, 1982)). There are, however, some problems with the Higgs field. First, \( m^2 \) must be \(< 0 \) (or finetuned to zero in the case of one-loop corrections) to obtain the desired spontaneous symmetry breaking. Further, as we already mentioned, the vacuum energy density associated with ground state value has immense consequences in cosmology. It is tens of orders of magnitude too large to be consistent with observational limits on the cosmological constant.

Equation (19) represents a non-linear wave equation submitted to the potential \( W \). Suppose that \(-1/6 < f < 0 \) and \( \phi = \phi_0 \). Then, under certain conditions on \( \lambda \) and \( m^2 \), \( m^2 > 0 \) and stability to second order would be guaranteed. If the universe came through some kind of fluctuation in the metastable situation \( \phi = \phi_0 \) and underwent an exponential expansion for a while, it could make a transition to the Minkowski spacetime \( \phi = 0 \) to settle down (Linde, 1983). However, the sign of \( W \) could change. This will depend on the several parameters in the theory.

Because of the complicated inhomogeneous perturbations, a numerical solution will be necessary. Variation of the parameters could lead to new insight.

There is another possibility to obtain the spontaneous symmetry breaking without the reversed sign of the squared mass and the large cosmological constant. Consider a free scalar field. Then the equations of Einstein (with in the righthand side also the term containing \( E \) of equation (33) of §3.3) we could obtain an inflationary period from \( \int \dot{\phi}^2 \, d\xi \) for some gauge condition.
3.4.4 Conclusion

It is generally believed that inflation—the rapid expansion of the universe from a tiny size to an enormous size—can only be due to a repulsive vacuum energy term arising from a phase transition. We investigated high-frequency perturbations in the coupled system of gravity and scalar field. The back-reaction of the perturbations show some new features. It is found that additional restrictions on the first and second order perturbations are necessary in order to obtain bounded solutions for the third order correction to the gravitational field and the first order perturbation of the scalar field. The mass of the \( \psi \)-particle will depend on the background scalar field \( \psi \), its potential and the coupling parameter. The propagation of \( \psi \) will not drag along the gravitational perturbation as usually is found. In the case of a Higgs potential, the stability during the transition from the \( \psi^2<\phi^2 \) region to the \( \psi^2>\phi^2 \) region is not evident. Numerical investigation would be necessary. The formation of primordial black holes will depend on the initial high-frequency perturbations. Further, it would be of interest to investigate in our model the conditions for inflation without the restrictive bounds on the coupling constant \( \lambda \), \( m^2 \) and the cosmological constant.

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3.5 High-frequency perturbations on the Einstein-Rosen metric

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Abstract

In recent years, interest has grown in new techniques for generating vacuum solutions of the Einstein field equations, based on the method of inverse scattering. These techniques (applied on the Einstein-Rosen metric) aim to produce entire families of new solutions starting from known ones and to interpret the generated solutions in terms of "perturbations" of the original ones. In our approximation to the non-vacuum situation, the equations become quite different and it is hard to proceed in the way as done for the vacuum case. The numerical solution shows a different result from that usually found. A smooth initial $h_{uv}$ tends to break up and the inhomogeneity increases. The stability of the transverse part of the metric is disturbed. Study of the late-time behaviour could be of interest in connection with singularities.

3.5.1 Introduction

After considering high-frequency perturbations on a spherically symmetric background, we can now extend the approximation to the inhomogeneous Einstein-Rosen metric. The investigations on an Einstein-Rosen metric have a long history and originate from the work of Einstein and Rosen (1937). A substantial part of the present knowledge on gravitational waves grew from the intensive investigations of the original Einstein-Rosen spacetime (see for example Weber, 1961 and Caves et al., 1980). In spite of their relative simplicity, these spacetimes involve a wide variety of physical situations, some of which might involve gravitational waves. Proposed originally to describe the linearly polarized cylindrical gravitational waves, the Einstein-Rosen metric was extended to the case of two polarization waves in cylindrically and plane-symmetric spacetimes (Thorne, 1965). A number of non-vacuum solutions were found (Melvin, 1965).

They describe the interaction between the gravitational waves and matter. The plane-symmetric spacetime were later generalized to include collisions between gravitational waves (Szekeres, 1972). It was found that such collisions necessarily involve singularities, either in the past or in the future (Tipler, 1980).

Later, the Einstein-Rosen spacetimes were reformulated and studied in relation to the problem of primordial gravitational waves. These primordial gravitational waves might follow different scenarios. One of these is to consider them as fluctuations on a spatial homogeneous background, which could have originated from the graviton production processes near the Planck era, $t=10^{-43}$s (Hu and Parker, 1978; Parker, 1983).

In the other scenario, the influence of gravitational waves on the evolution of the universe increases going back in time. The initial situation of the universe was highly irregular, i.e., inhomogeneous and anisotropic. At later times the model would evolve into a homogeneous isotropic model, where the chaotic behaviour near the initial singularity has been transformed into gravitational waves (Adams et al., 1982). The description of these chaotic stages of evolution is complicated, especially when a scalar field is present.
In the homogeneous models of the Bianchi types I-VII the initial stages of evolution were found to behave like the Kasner vacuum solutions and for models of types VIII and IX the initial regime is much more complicated. We saw in section 2 that the evolution can be decomposed into eras during which there exist oscillations of expansion along two axes and monotonic expansion along the third one. One can generalize these results to inhomogeneous models (Lifshitz et al., 1970). It turns out that each era in these models can be approximately described by a generalized wave solution of the Einstein-Rosen type, with the spatial singularity replaced by the initial singularity at $t=0$. There are other exact solutions representing universes with gravitational waves (Gowdy, 1974; Berger, 1975; Misner, 1973). The influence of a scalar field on the early stages was also investigated (Belinskii and Khalatnikov, 1973; Bekenstein, 1975). However, the stability in these models was not investigated. The specific features found in the cosmological models with a scalar field require further research.

Normally, the interaction between scalar and gravitational field is introduced in a non-consistent way, and it is doubtful if the results obtained in this way can be fully trusted. The result that the inhomogeneities are caused by standing scalar and linearly polarized gravitational waves is maybe even oversimplified. One can do better by considering the non-linear approximation method on the inhomogeneous background metric. One of the theoretically predicted consequences of primordial irregularities is the emergence of a cosmological gravitational radiation background. Any success in determining them could yield vital information about the early universe (see also section 2). They could be observable by a variety of new detection techniques, for example by Doppler tracking of interplanetary spacecraft (Mashhoon and Grishchuck, 1980) or by scrutinizing the time noise in pulsars (Romani and Taylor, 1983) or by monitoring perturbations to planetary orbits (Mashhoon, Carr and Hu, 1981).

There are, however, some problems in the model. One of them is the development of singularities in the "far" region (in addition to the initial singularity). We should like to have a flattening mechanism for the initial spatial curvature associated with the Bianchi type models in the Einstein-Rosen class. Furthermore, on introducing the scalar field in the model, one will be interested in the influence on the solitonic gravitational wave solutions,
found in the vacuum case by Belinskii and Zakharov (1980). Especially the stability of the solutions has to be investigated. The soliton technique provides a procedure to solve the Einstein equations in vacuum when there are two commuting Killing vectors. This situation includes Bianchi types I to VII and their inhomogeneous generalizations (in this context, the connection with the oscillatory behaviour of the intermediate Bianchi VII situation in the Mixmaster model, will be of interest).

The main ideas of the soliton technique are the so-called pole-trajectories and particular solutions of the Einstein equations which serve as "seeds". It is possible to construct cosmological models which look like gravitational waves at later times (in the sense that they evolve towards homogeneity in a wave like manner), but which behave like very inhomogeneous "graviti-solitons" at early times. Since these solitons propagate with approximately the speed of light and carry energy in some sense, they are naturally interpreted as incipient gravitational waves. Yet there are some problems with this interpretation. The global gravitational wave energy in this way is poorly defined. A step forwards was the formulation of the peeling-off theorem for these class of inhomogeneous cosmologies, and the determination of the Petrov type of the radiation (Carmeli and Feinstein, 1984). In these models, discontinuities in the first derivative of the metric on the lightcone occur, interpreted as gravitational shockwaves (Belinskii and Francaviglia, 1982).

In §3.5.2 we will apply the high-frequency approximation on the Einstein-Rosen non-diagonal metric and investigate the stability of the solitary solutions when a non-minimal coupled scalar field is present. In the linear case (Charach and Malin, 1979) the equations describing the transverse scale expansion and longitudinal scale evolution will remain, as in the vacuum case, uncoupled. In the approximation we will follow they will not remain uncoupled. So this feature will make hard to generate solutions from the vacuum models. In §3.5.3 some numerical results are presented.
3.5.2 The Einstein-Rosen metric

The general Einstein-Rosen metric is characterized by

\[ ds^2 = e^{2a}(dr^2 - dt^2) + \gamma_{ab}dx^adx^b, \]  

(22)

where \( a \) and \( \gamma_{ab} \) depend on \( t \) and \( r \) only and \( a, b = 2, 3 \).

The spacetime admits an abelian group of isometries \( G_2 \) with corresponding Killing vectors \( \zeta^\mu = \delta^\mu_1 \) and \( \xi^\mu = \delta^\mu_2 \). The metric can be diagonalized when one of the Killing vectors is orthogonal to the hypersurface, obtained by dragging the \((t,r)\)-hypersurface along the direction of the other Killing vector. The surface area of transitivity \( G \) is then given by

\[ G = \sqrt{(\zeta_1 \cdot \zeta_2 - \xi_1 \cdot \xi_2)\gamma_{12}} = \sqrt{\det \gamma_{ab}}. \]  

(23)

The local behaviour of the spacetimes considered is defined by the gradient of \( G \), i.e., \( G_{,\mu} \) which can be spacelike, null or timelike. The globally spacelike case corresponds to the cylindrical spacetime. Particularly, when \( G_{,\mu} \) is spacelike, the metric is globally diagonalizable. If we write

\[ ds^2 = e^{2a}(dr^2 - dt^2) + e^{2\beta}(e^{2\psi}dx^2 + e^{-2\psi}dy^2), \]  

(24)

where \( a \), \( \beta \) and \( \psi \) are functions of \( r \) and \( t \) only, then \( \psi \) describes what is usually called the "+"-polarization of gravitational waves, \( e^{2\beta} \) describes the transverse scale expansion created by the energy density of the waves and \( e^{2a} \) the longitudinal scale evolution created by collisions and backscattering of the waves. The wave fronts are formed by the homogeneous two-surface spanned by \( \partial/\partial x \) and \( \partial/\partial y \). If one drops the \( r \)-dependence in equation (24), the line element then describes a homogeneous Bianchi I cosmology. Conversely, the line element of equation (24) can be obtained from homogeneous Bianchi I line element by choosing a propagation direction (\( r \)-axis), writing the line element in the form of equation (24) with a \((dr^2 - dt^2)\) term (re-defining time) and then inserting explicit \( r \)-dependence into the functions.
This breaks the Bianchi I symmetry in the chosen direction, but preserves it in the transverse direction. The resulting metric may then be said to represent an inhomogeneous universe in which a plane gravitational wave of a single polarization propagates in a Bianchi I background. Due to the presence of the abelian subgroup $G_2$ in the Bianchi models of types I to VII, the Einstein-Rosen metrics include them. The $G_2$ permitted by equation (22) may be either the full group of motions or a subgroup of a more general symmetry group. One can have i) spatially homogeneous isotropic (FRW) models, ii) spatially homogeneous (Bianchi) models and iii) spatially inhomogeneous anisotropic models. Spatially homogeneous isotropic models are invariant under a six-parameter isometry group $G_6$ which includes the three-dimensional rotation group and a group $G_3$ acting simply transitively on the spacelike three-dimensional hypersurface of constant curvature. All these models belong to the generalized Einstein-Rosen spacetimes. Spatially homogeneous models are either Bianchi or Kantowski-Sachs universes. The Bianchi models admit three-dimensional isometry groups $G_3$ acting simply transitively on three-dimensional spacelike hypersurfaces. The models I to VII are invariant under $G_3$ which contain the abelian $G_2$ as a subgroup and, therefore, belong to the generalized Einstein-Rosen spacetimes. The symmetry groups of the Bianchi VIII and IX do not, in general include $G_2$. However, rotationally invariant models will do (Carmeli, Charach and Feinstein, 1983). The FRW models correspond to the isotropic modifications of the Bianchi models $I(VII_0)$, $V(VII_1)$ and IX. They describe, respectively, the isotropic models with vanishing, negative and positive spatial curvature. The KS model admits four-dimensional symmetry group $G_4$, which act multiply transitively on the three-dimensional spacelike hypersurfaces. Although the $G_2$ is a symmetry group acting on two-dimensional spacelike hypersurfaces, the three-dimensional hypersurfaces are not hypersurfaces of transitivity. Therefore, the generalized Einstein-Rosen spacetimes, which do not belong to the families above, are referred to inhomogeneous models. As particular cases this family includes various self-similar models and models with the symmetry group $G_3$ acting multiply transitively on two-dimensional spacelike hypersurfaces (Tabensky and Taub, 1973). It can be shown without loss of generality that we can take
Because of their cosmological relevance we will consider the metric

\[ ds^2 = e^{2\alpha}(dr^2 - dt^2) + e^{2\psi}dx^2 + 4t\chi dxdy + t^2e^{-2\psi}(1+4\chi^2)dy^2. \]  

(26)

The physical behaviour of the cosmological models described by the metric of equation (26) may be best understood by examining the equations of §3.2 with respect to the null tetrad (Stachel, 1966)

\[ k^\mu = e^{-\alpha}(1,\gamma,0,0), \]  

(27)

\[ n^\mu = \frac{e^{-\alpha}}{2}(1,^+1,0,0), \]  

(28)

\[ m^\mu = \frac{1}{\sqrt{2}}(e^{-\psi}\delta^\mu_x + i(\delta^\mu_y + 2te^{-2\psi}\chi\delta^\mu_x) \frac{e^\psi}{t}). \]  

(29)

From equation (27) we can calculate some characteristic quantities, such as the divergence

\[ \theta \equiv \nabla_\mu k^\mu = e^{-\alpha}(\partial_\lambda a + \partial_\lambda a) + \frac{e^{-\alpha}}{t}, \]  

(30)

the shear

\[ |\sigma|^2 \equiv \frac{1}{2}(\nabla_\mu k^\mu)(\nabla_\nu k_\nu) = \frac{1}{4}\theta^2 \]

\[ = e^{-2\alpha}(-\frac{1}{4}(\partial_\lambda a + \partial_\lambda a)^2 - \frac{1}{2t} \frac{\partial_\lambda a + \partial_\lambda a}{(\partial_\lambda a + \partial_\lambda a)} + (1+4\chi^2) \]

\[ \times ((\partial_\lambda \psi)^2 + (\partial_\lambda \psi)^2) + (\partial_\lambda \chi)^2 + (\partial_\lambda \chi)^2 - \frac{1}{t}(1+4\chi^2) (\partial_\lambda \psi + \partial_\lambda \psi) \]

\[ + \frac{2\chi}{t}(\partial_\lambda \chi + \partial_\lambda \chi) + \frac{1}{2t^2} (1+2\chi^2) + 2(1+4\chi^2) \partial_\lambda \psi \partial_\lambda \psi + 2\partial_\lambda \chi \partial_\lambda \chi \]

\[ - 4\chi(\partial_\lambda \psi \partial_\lambda \chi + \partial_\lambda \psi \partial_\lambda \chi) + 4\chi(\partial_\lambda \chi \partial_\lambda \psi + \partial_\lambda \psi \partial_\lambda \chi) \],  

(31)
and the Weyl tensor components $\psi_0 = C_{\mu\nu\rho\sigma} k^\mu m^\nu k^\rho m^\sigma$, $\psi_2 = C_{\mu\nu\rho\sigma} m^\mu n^\nu k^\rho m^\sigma$ and $\psi_4 = C_{\mu\nu\rho\sigma} m^\mu n^\nu m^\rho n^\sigma$. They can be used in the determination of the Petrov type and the mass parameter of blackholes eventually formed. From the propagation equations and back-reaction equations of §3.3 we can derive in the case of the metric of equation (26) the following system for $\psi, \alpha, \bar{\psi}, \hat{h}_{22}$ and $\hat{h}_{23}$

$$\begin{align*}
\dot{\alpha}^2 \psi - \dot{\bar{\alpha}}^2 \bar{\psi} &= 8 \chi (\alpha \chi \alpha \psi - \alpha \chi \bar{\alpha} \bar{\psi}) + 8 \chi^2 ((\alpha \chi)^2 - (\alpha \chi)^2) + 2 ((\alpha \chi)^2 - (\alpha \chi)^2) \\
\frac{-1}{(1+8 \chi^2)} \dot{\alpha}^2 \psi + \frac{4 \chi}{(1+8 \chi^2)} \dot{\bar{\alpha}}^2 \bar{\psi} &= \frac{-8 \pi G}{1+8 \pi G \phi^2} (\bar{\alpha} \bar{\alpha} \bar{\alpha} \bar{\alpha} \\
-2 \psi \bar{\psi} (\alpha \chi \bar{\alpha} \chi - \alpha \chi \bar{\alpha} \chi) - \bar{\psi} \bar{\psi} e^{2 \alpha \phi \phi} \frac{d \psi}{d \phi},
\end{align*}$$

(32)

$$\begin{align*}
\dot{\alpha}^2 \alpha - \dot{\bar{\alpha}}^2 \bar{\alpha} &= (1+4 \chi^2) ((\alpha \chi)^2 - (\alpha \chi)^2) + 2 ((\alpha \chi)^2 - (\alpha \chi)^2) + 4 \chi (\alpha \chi \alpha \chi - \alpha \chi \bar{\alpha} \chi) \\
\frac{-1}{(1+4 \chi^2)} \dot{\alpha}^2 \alpha + \frac{2 \chi}{(1+4 \chi^2)} \dot{\bar{\alpha}}^2 \bar{\alpha} &= \frac{-4 \pi G}{1+8 \pi G \phi^2} (1+4 \phi - 4 \phi \phi \phi) ((\alpha \chi)^2) \\
- (\alpha \chi \phi) - 2 e^{2 \alpha \phi \phi} (\psi + 2 \phi \phi \phi) \frac{d \psi}{d \phi},
\end{align*}$$

(33)

$$\begin{align*}
\dot{\alpha} \bar{h}_{22} + \dot{\bar{\alpha}} \alpha \bar{h}_{22} + \bar{e} \bar{h}_{22} &= \frac{(1+8 \chi^2)}{8 \pi G \phi} - 4 (1+4 \chi^2) ((\alpha \chi \bar{\alpha} \chi) + 8 \chi (\alpha \chi \bar{\alpha} \chi) + \alpha \chi \bar{\alpha} \chi) \\
+ \frac{8 \pi G \phi}{1+8 \pi G \phi} (\alpha \chi \bar{\alpha} \chi \bar{\alpha} \chi) \bar{h}_{22} - \frac{4 \chi}{(1+4 \chi^2) (\alpha \chi \bar{\alpha} \chi \bar{\alpha} \chi) + 4 \chi (\alpha \chi \bar{\alpha} \chi) + \alpha \chi \bar{\alpha} \chi} &= 0,
\end{align*}$$

(34)

$$\begin{align*}
\dot{\alpha} \bar{h}_{23} + \dot{\bar{\alpha}} \alpha \bar{h}_{23} + \bar{e} \bar{h}_{23} &= \frac{1}{8 \pi G \phi} (\alpha \chi \bar{\alpha} \chi \bar{\alpha} \chi) \bar{h}_{23} - \frac{16 \chi}{8 \pi G \phi} (\alpha \chi \bar{\alpha} \chi)^2 (\alpha \chi \bar{\alpha} \chi) + \alpha \chi \bar{\alpha} \chi + \alpha \chi \bar{\alpha} \chi \\
+ \frac{8 \pi G \phi}{1+8 \pi G \phi} (\alpha \chi \bar{\alpha} \chi \bar{\alpha} \chi) \bar{h}_{23} - \frac{16 \chi}{8 \pi G \phi} (\alpha \chi \bar{\alpha} \chi)^2 (\alpha \chi \bar{\alpha} \chi) + \alpha \chi \bar{\alpha} \chi + \alpha \chi \bar{\alpha} \chi) \\
&= 0.
\end{align*}$$

(35)
and
\[ a^2 + 2a \frac{e^2 M}{M} = 0. \]  \( \text{(36)} \)

We used the gauge condition of §3.3, such that \( h_{\mu
u} = 0 \).

Further
\[ h_{33} = -t^2 e^{-4\psi} (1+4\chi^2) h_{22} + 4\chi e^{-2\psi} h_{23}. \]  \( \text{(37)} \)

For the constraint equations we obtain
\begin{align*}
\frac{a^2}{\hat{t}} &\equiv -\frac{32\pi GF \psi \frac{e^2}{e^2} + 24 (16\pi GF \psi \frac{e^2}{e^2} + 2) (1+8\pi GF \psi \frac{e^2}{e^2} - 2)}{1+8\pi GF \psi \frac{e^2}{e^2} + 16\pi GF \psi \frac{e^2}{e^2}} \\
&= \frac{-32\pi GF \psi \frac{e^2}{e^2} + 24 (1+8\pi GF \psi \frac{e^2}{e^2} + 2) (1+8\pi GF \psi \frac{e^2}{e^2} - 2)}{1+8\pi GF \psi \frac{e^2}{e^2} + 16\pi GF \psi \frac{e^2}{e^2}} \\
&\left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} + 2 (1+4\pi GF \psi \frac{e^2}{e^2} + 2) \right\} \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\}
\end{align*}

\[ \text{(38)} \]

\begin{align*}
-\frac{8\pi G}{1+8\pi GF \psi \frac{e^2}{e^2}} &\left\{ (1+2\pi GF \psi \frac{e^2}{e^2} + 2) \right\} \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} + 2 (1+4\pi GF \psi \frac{e^2}{e^2} + 2) \right\} \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\} - 2 \left\{ (\frac{\dot{\psi}}{\hat{t}})^2 \right\}
\end{align*}

\[ \text{(38)} \]
\[- \frac{8\pi G}{1+8\pi Gf \phi^2} \left\{ (1+2\phi) \partial_t^2 \phi \partial_\phi^2 \phi + 2\phi \partial_t \phi \partial_\phi \phi \right\} + \frac{e^{2a}}{t} \phi \delta \xi, \quad (39)\]

where

\[E = \frac{1}{2}(1+4\chi^2) e^{-4\psi} \frac{\hbar^2}{\hbar^2 - \frac{2\chi}{t} e^{-2\psi} \hbar^2 \hbar^2} , \quad (40)\]

We make the observation that the equations for $\alpha$ and $\psi$ are coupled. This is in contrast with the vacuum situation. In the vacuum case, the fact that the equations are uncoupled leads to the generation of new exact solutions referred to as gravitational solitons (Belinskii and Zakharov, 1980). In vacuum we would have (diagonal case)

\[\partial_t^2 \psi - \partial_t^2 \psi - \frac{1}{t} \partial_t \psi = 0. \quad (41)\]

Equation (41) can be rewritten as

\[\partial_t U - \partial_x V = 0, \quad (42)\]

where

\[U \equiv t \partial_x y^{-1} , \quad V \equiv t \partial_x y^{-1} , \quad (43)\]

and with $\gamma_{ab}$ the two-metric of equation (22). Now one can consider the eigenvalue problem

\[D_1 \phi \equiv (\partial_x - \frac{2\lambda}{\lambda^2 - \frac{2\chi}{t} e^{-2\psi}}) \phi = - \frac{tV + \lambda U}{\lambda^2 - \frac{2\chi}{t} e^{-2\psi}} \phi , \quad (44)\]

\[D_2 \phi \equiv (\partial_t - \frac{2\lambda t}{\lambda^2 - \frac{2\chi}{t} e^{-2\psi}}) \phi = - \frac{tU + \lambda V}{\lambda^2 - \frac{2\chi}{t} e^{-2\psi}} \phi , \quad (44)\]
where $\theta(r,t;\lambda)$ is a two dimensional matrix that satisfies
\begin{equation}
\theta(r,t;0) = \gamma(r,t) \tag{45}
\end{equation}
and where $\lambda$ is a complex spectral parameter. Given a particular solution $\gamma$ of equation (42), equation (44) and (45) must be integrated to find the corresponding $\bar{\theta}(r,t;\lambda)$. A solution $\theta$ can be generated by algebraic operations if one assumes that $\theta$ is the product of a two dimensional matrix with $n$ simple poles in the complex $\lambda$-plane, and $\bar{\theta}$. Equation (45) shows that an $n$-soliton solution for $\gamma(r,t)$ then can be found.

One has reduced the equations to a more general overdetermined system of matrix equations related to an eigenvalue-eigenfunction problem for some linear differential operators. It is easy to verify that
\begin{equation}
[D_1,D_2] = 0. \tag{46}
\end{equation}

It then turns out that the compatibility conditions for the equations (44) and (45) coincide with the equation (42). The procedure continues by specifying whether the pole trajectories
\begin{equation}
\mu_k = u_k - r + \sqrt{(u_k - r)^2 - t^2} \quad (k=1,\ldots,n) \tag{47}
\end{equation}
are real or complex. Here the $u_k$ are solutions of the equations
\begin{equation}
\not{\partial}_x u_k = -\frac{2u_k^2}{t^2 - u_k^2}, \quad \not{\partial}_t u_k = -\frac{2tu_k}{t^2 - u_k^2} \tag{48}
\end{equation}
and the $u_k$ are arbitrary constants. $n$ represents the number of solitons. $u_k$ defines the location and width of the soliton. The two dimensional metric $\gamma_{ab}$ corresponding to the new spacetime is obtained from that of the old one $\bar{\gamma}_{ab}$ by
\begin{equation}
\gamma_{ab} = \frac{1}{t^n} \sum_{k=1}^{\infty} u_k \left\{ \text{\gamma}_{ab} - \frac{(r^{-1})^{k,l} \mu_k c_{m_k} a_{m_k} d_{m_k} \text{\gamma}_{ab}}{\mu_k' l} \right\} \tag{49}
\end{equation}
where
\[ \Gamma_{kl} \equiv \frac{\gamma_{cb} \rho_{cb}^{c-b}}{\nu_{k}^{c-b}} \quad \text{and} \quad m_{ka} \equiv m_{k}^{c-b} \left| \tilde{g}^{-1} (r, t; \nu_{k}' \right|_{ca} (50) \]

and \( m_{a}^{b} \) free parameters (\( b=x, y; a=1, \ldots n \)). Solutions of this type are highly irregular near the initial singularity but tend to the "seed" spatially homogeneous solutions asymptotically. The curvature has \( n \) maxima which correspond to solitons propagating on expanding background. After \( \gamma \) has been found, we can calculate the metric coefficient \( a \) by integrating equation (38) and (39) in the vacuum case. The final result is (Belinskii and Zakharov, 1980)

\[ a = \bar{a} t^{-\frac{1}{2} n^2} \quad \text{det} \Gamma_{k}^{l} \quad \text{and} \quad \text{det} \Gamma_{k}^{l} \quad \text{(51)} \]

where \( \bar{a} \) represents the seed solution. In cosmological context, one is interested in whether the solution in the far region will be different from the seed metric or not and if the solution will be stable against the high-frequency perturbations. From equation (32) and (33) we conclude that the scalar field interacts with both the transverse and the longitudinal part of the gravitational field. So it will not be easy to obtain (for example from the vacuum solution) solutions in compact form. If we write equation (32) for the diagonal case as

\[ \dot{u} \dot{U} - \dot{v} \dot{V} = 16 \pi G e^{2a} v \quad \text{(52)} \]

where the 2x2 matrices \( \dot{U} \) and \( \dot{V} \) are

\[ \dot{U} = a^2 U + \dot{a} t a^2 I \quad \dot{V} = a^2 v + \dot{a} r a^2 I \quad \text{(53)} \]
and $\sigma^2 = 1 + 8 \ast G \phi^2$, $I$ the identity matrix and $V$ the potential of $\psi$, one could try to proceed in the same way as equation (44) and (45). In the righthand side of equation (52) one then has to substitute the seed solution $\tilde{a}$.

We can make some remarks. First, it turns out, from equation (35) of §3.3, that $\sigma^2 h_{\mu \nu} \dot{h}_{\mu \nu}$ is a conserved quantity, because $\frac{\partial}{\partial t} (\sigma^2 k^\rho h_{\rho \mu} \dot{h}_{\mu \nu}) = 0$. So if $\dot{\phi}_C$, $h_{\mu \nu}$ will grow unlimited and quantum effects will probably become important. Secondly, in the case of a Kasner metric as seed metric,

$$ds^2 = t^{\Delta^2 - 1} (dt^2 - dr^2) + t^{1+\Delta} dx^2 + t^{1-\Delta} dy^2,$$  \hspace{1cm} (54)

where $\Delta$ is a real parameter, one can investigate the behaviour of the $h_{\mu \nu}$. $\Delta = 0$ corresponds to the axisymmetric Kasner solution, while $\Delta = 1$ corresponds to Minkowski spacetime. The $r$-axis is expanding for $\Delta > 1$ and contracting for $\Delta < 1$. Then from equations (38) and (39) we obtain ($\dot{\phi} = \text{constant}$; the contribution of $\dot{\phi}$ decreases the $t$-exponent)

$$\Delta = 0: \quad \dot{h}_{22} \sim t^{5/8}, \quad \dot{h}_{23} \sim t^{5/8}.$$  \hspace{1cm} (55)

$$\Delta = 1: \quad \dot{h}_{22} \sim t^{3/2}, \quad \dot{h}_{23} \sim t^{1/2}.$$  \hspace{1cm} (56)

So there is a growing mode in the $\Delta = 1$ case and the evolution of the background will be influenced.

3.5.3 Numerical solutions

In the non-diagonal case, we can apply a numerical method in order to say something about the behaviour of the background metric and the perturbations. Let us consider the initial values

$$\ddot{\phi}(0,r) = \text{const.}, \quad \dot{\phi}(0,r) = 0, \quad \phi(0,r) = -r^2,$$  \hspace{1cm} (57)

$$\chi(0,r) = 0.1, \quad \dot{\chi}(0,r) = 0, \quad \chi(0,r) = re^{-r+1}, \quad \dot{h}_{22}(0,r) = re^{-r}, \quad \dot{h}_{23}(0,r) = e^{-r}.$$  \hspace{1cm} (58)
Then, from the constraint equations (38) and (39) we obtain

\[ a_t(0,r) = t(2(1+4\chi^2)((a_t\psi)^2 + \frac{1}{r^4}) - \frac{2}{r}(1+4\chi^2)a_t\psi + \frac{2\chi^2}{r^2}(3-2\chi^2) \]  \tag{59} 

\[ + \frac{16\pi G\eta^{-2}dY}{1+8\pi G\eta^{-2}} + E \}, \]

\[ a_t(0,r) = \frac{1}{2r(2+4\chi^2(1-4\chi^2))} \left( \frac{2r(1+4\chi^2)}{t} \right) \pm \frac{1}{4}E \], \tag{60} 

where

\[ E = e^{-2r}\left\{ \frac{1}{2}(1+4\chi^2)e^{-4r^2+2} + \frac{1}{2t^2} - \frac{2\chi r e^{-2r^2}}{t} \right\} \]. \tag{61} 

For \( r=0.1-0.7 \) we plotted \( \exp(\psi) \) (figure 4), \( \hat{\eta}_{22} \) (figure 5), \( \hat{\eta}_{23} \) (figure 6) and the expansion parameter \( \theta \) (figure 7) for several times. The smooth \( h_{\mu \nu} \) tends to break up (increased inhomogeneity) and moves to the outside, while the stability of \( \psi \) is disturbed. An interesting question is how the behaviour will be at later times. Maybe a more advanced computer program (we used the ACM algorithm of IMSL, see §3.3) could produce results. Another interesting case occurs when the initial \( \varphi \)-distribution is not constant. For two different initial \( \psi(0,r) \) we plotted \( h_{22} \) and the shear for \( r=0.5-1.8 \) (figures 8 and 9).

We conclude that the specific form of the initial \( \psi(0,r) \) determines the overall behaviour of the high-frequency perturbations (the occurrence of the irregularities is due to the discontinuities in \( \varphi(0,r) \)). In both cases \( \psi(t,r) \) remains stable. So the solitary behaviour could be transported to the \( h_{\mu \nu} \).
Figure 4. Behaviour of the metric component $\psi$.

$\psi_{\text{initial}} = -r^2$ and $\tilde{\psi}_{\text{initial}} = \text{constant}$. The stability is disturbed.
Figure 5. Behaviour of the (2-2)-component of the gravitational perturbation as function of time. The initial function was $re^{i-r}$. 
Figure 6. Behaviour of the (2-3)-component of the gravitational perturbation as function of time. The initial function was $e^{-r}$. 
Figure 7. Behaviour of the expansion $\theta = \nu \cdot k^\mu$. The formation of a trapped surface occurs.
Figure 8. Plot of the shear $\sigma$, the expansion $\theta$ and the gravitational perturbation for initial $\psi=1/r$. The dashed line represents the initial $\bar{\phi}$. 
Figure 9. As figure 4, but with initial $\zeta = -r^2$. There is a remarkable difference with respect to figure 4 concerning the behaviour of $\hat{h}_{22}$. The dashed line represents the initial $\zeta$. 
3.5.4 Conclusions

The Einstein-Rosen cosmological spacetimes provide a theoretical basis for analysis of chaotic universes. These models include a large class of spatially homogeneous universes and their inhomogeneous extensions obtained by an unidirectional homogeneity breaking. One expects many features found in these models to be present in more general chaotic universes, inhomogeneous in all three directions. Together with the scalar coupling the model could be a preliminary step towards the quantum description of the early universe (Berger, 1982).

In the search for vacuum solutions with cosmological interest, it is natural to start from Bianchi metrics as seed metrics. The persistence of the solitons found suggests that these solutions may be singled out as modelling the most likely form of irregularity in the early universe. By calculating soliton solutions with Bianchi backgrounds one finds inhomogeneous solutions in which the Kasner frame will change during the time evolution and the so-called sequence of "Kasner epochs" is broken. So the Hamilton description will always break down. The corner-run solutions of the oscillatory approach to singularities can also be better understood in the soliton approach. The extension to the Einstein-Rosen metric will lead to solutions different from the seed background metric, a signal that the solitonic perturbation is physically singular. However, this singular behaviour will probably emerge due to the incomplete description of the usual one-soliton approach. The singular behaviour would probably disappear when higher order approximations are used. In the high-frequency approximation, the solitonic behaviour is visualized in the high-frequency perturbations (in the non-vacuum situation). It would be a subject for further research to investigate the occurrence of shockwaves owing to the existence of a jump discontinuity across the light-cone (acting as wave front).
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