# The probability of a random straight line in two and three dimensions

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# The probability of a random straight line in two and three dimensions

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*Abstract:* Using properties of shift- and rotation-invariance probability density distributions are derived for random straight lines in normal representation. It is found that in two-dimensional space the distribution of normal coordinates $(r, \varphi)$ is uniform: $p(r, \varphi) = c$, where $c$ is a normalisation constant. In three dimensions the distribution is given by: $p(r, \varphi, \theta, \zeta) = cr \sin\theta$.

*Key words:* Straight lines, parameter representation, normal representation, length estimators, randomness.

## 1. Introduction

The statistically sound design of image measurement procedures requires an explicit formulation for the probability distribution describing the relative occurrence of realisations of the originals. In the design process the following type of integrals are evaluated:

$$\int_{t \in D} F(g - \hat{g}) \cdot p(O(\boldsymbol{t}))\, \mathrm{d}\boldsymbol{t},$$

where $F$ is the error function expressing the difference between the property $g$ of the original instance $O$ and $\hat{g}$ the property estimated from an observed instance of $O$. For MSE, $F$ is given by $F(g - \hat{g}) = (g - \hat{g})^2$. The originals $O$ are described by a set of parameters $\boldsymbol{t}$, with $p(O(\boldsymbol{t}))$ the probability distribution of $O$. In the absence of further *a priori* knowledge, random distributions for $p(O(\boldsymbol{t}))$ are commonly assumed.

In this communication we will address the problem of finding an expression for $p(O(\boldsymbol{t}))$ in the case of random straight lines, i.e. in the general case when no *a priori* knowledge is available about a preferred direction or location of the lines. For $\boldsymbol{t}$ we will choose the standard normal representation of straight lines. Appropriate parameterization allows to efficiently study sets of lines. The results form a basis for the optimal estimation of length (Dorst & Smeulders, 1987), distance (Borgefors, 1984) and orientation in digital images, and the development of line detectors (Duda & Hart, 1972). These results have a further impact on image processing in that pseudo-Euclidean distance transformations (Borgefors, 1986; Vossepoel, 1988; Beckers & Smeulders, 1989) can be based on them, which may lead to accurate and efficient morphological and topological analysis (Dorst, 1986) and path-planning (Verbeek et al., 1986). In pattern recognition, applications may be found in cluster analysis and nearest neighbour classification in 2- and 3-D problems.

To the extent of our knowledge in the literature no formal derivations of random probability density distributions are given for straight lines. Duda and Hart (1973) give an intuitive treatment on random straight lines, using principles of shift and rotation invariance. Let a line in normal representation be given by the parameters $(r, \varphi)$, where $r$ is the length of the support vector of the line and $\varphi$ the orientation. They propose randomness of straight lines to be described by a uniform distribution in parameter representation:

$$p(r, \varphi) = c, \tag{1}$$

where $c$ is a normalisation constant. An expression for the probability distribution of randomness in three dimensions can, however, not easily be seen. Therefore we followed a more general approach to derive the probability distributions in both 2 and 3 dimensions, based on invariance principles described in the following section.

## 2. Principles

Assume that a line with a certain reference point $R$ is tossed into two- or three-dimensional space $\mathbb{R}^n$ ($n \in \{2,3\}$), then randomness for straight lines can be based on the following principles.

**Principle 1.** Shift invariance is equivalent with $R$ having no preference for position.

Regardless of dimension, from this principle it follows that $R$ is uniformly distributed in a Cartesian coordinate frame in $\mathbb{R}^n$.

**Principle 2.** Rotation invariance implies a line having no preference for orientation with respect to $R$.

The second principle implies an orientationally uniform distribution. The meaning of this will be explained later in detail.

For straight line segments of finite length in addition a principle for the length of a segment must be introduced.

**Principle 3.** There is no preference for a specific segment length.

As a direct consequence, the segment length has a uniform distribution.

And, finally:

**Principle 4.** Position, orientation and length are statistically independent.

This last principle initially allows the consideration of the separate probability density distributions for position, orientation and length and multiply them later in order to obtain the joint probability distribution of random straight lines.

## 3. Random lines in $\mathbb{R}^2$

To calculate the distribution of random lines the following approach is taken. As a direct consequence of the principles of Section 2, we obtain an expression for randomness in a Cartesian coordinate frame. From that expression, we find the distribution in normal representation by a coordinate transformation.

Locate at the position of an observer a Cartesian coordinate frame in $\mathbb{R}^2$-space. Figure 1a shows the situation: an infinitely long straight line $l(x, y, \alpha)$ has a reference point $R(x, y)$ and orientation $\alpha$. For random lines, the first principle states that $R(x, y)$ is uniformly distributed on $\mathbb{R}^2$:

$$p(x, y) = c_1. \tag{2a}$$

The second principle implies that the orientation of a line has a uniform distribution:

$$p(\alpha) = c_2. \tag{2b}$$

From the independence of position an orientation, as stated in principle 4, it follows that:

$$p(x, y, \alpha) = p(x, y) \cdot p(\alpha) = c_3. \tag{2c}$$

In the normal representation the line is specified by $l(r, \varphi, u)$, for which the situation is given in Figure 1b. Here, $u$ is the position of the reference point on the line, relative to the normal point. Rewriting $p(x, y, \alpha)$ as a function of the normal coordinates $(r, \varphi, u)$, we get:

$$p(r, \varphi, u) = |J| \cdot p(x, y, \alpha), \tag{3a}$$

$$J = \frac{\partial(x, y, \alpha)}{\partial(r, \varphi, u)}, \tag{3b}$$

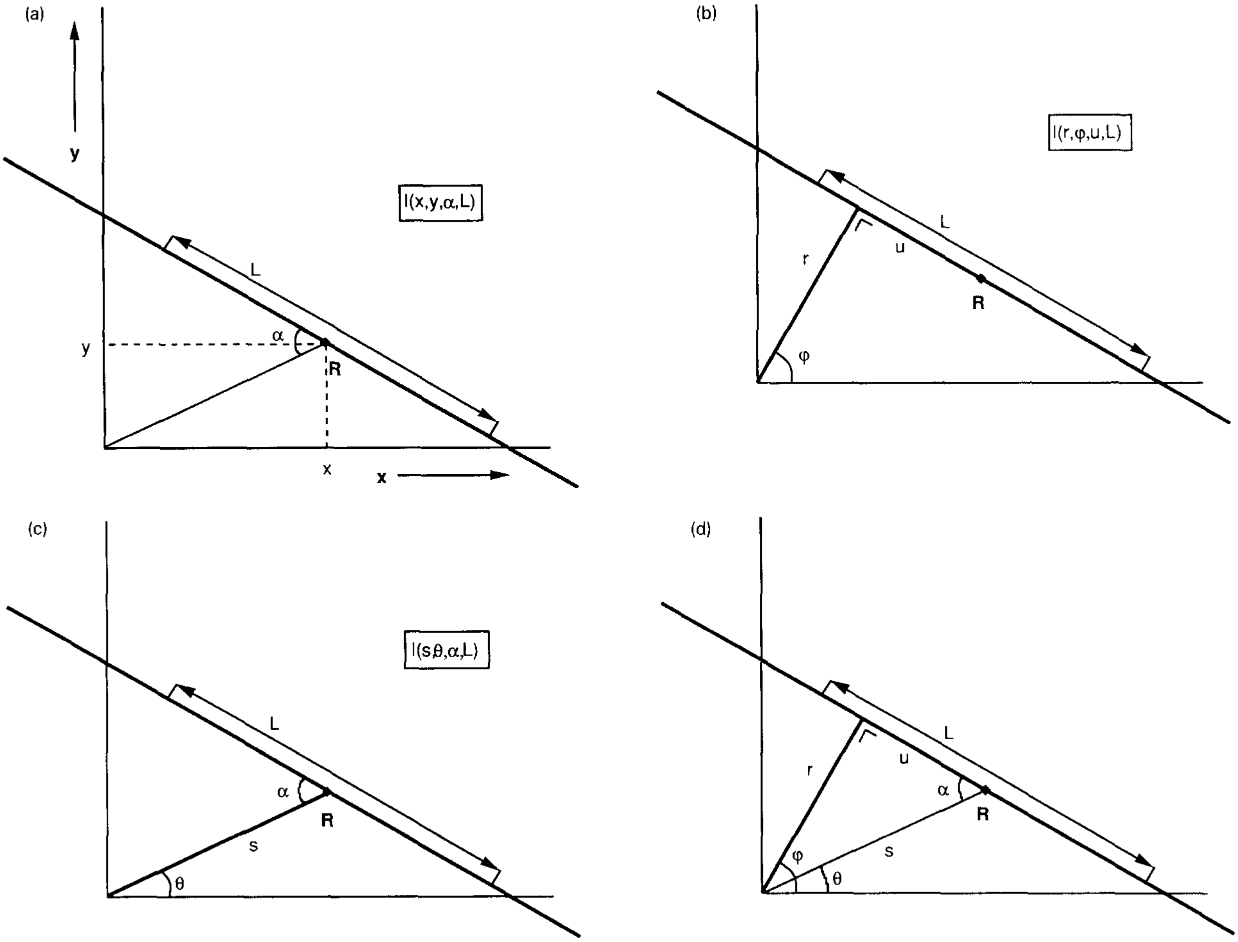


Figure 1. (a) A straight line segment in cartesian representation. (b) A straight line segment in normal representation. (c) A straight line segment in polar representation. (d) The relation between the polar and normal representations.

where Jacobian $J$ accounts for the coordinate transformation. To express the probability distribution in normal coordinates, equation (3b) can best be evaluated in two steps:

$$J = J_1 \cdot J_2. \tag{4a}$$

The first step is a transformation of $(x, y, \alpha)$ into polar coordinates $(s, \theta, \alpha)$ (see Figure 1c), with Jacobian $J_1$:

$$J_1 = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \alpha} \\ \frac{\partial \alpha}{\partial s} & \frac{\partial \alpha}{\partial \theta} & \frac{\partial \alpha}{\partial \alpha} \end{vmatrix}. \tag{4b}$$

And the second step is a transformation from polar coordinates into normal ones with Jacobian $J_2$:

$$J_2 = \begin{vmatrix} \frac{\partial s}{\partial r} & \frac{\partial s}{\partial \varphi} & \frac{\partial s}{\partial u} \\ \frac{\partial \theta}{\partial r} & \frac{\partial \theta}{\partial \varphi} & \frac{\partial \theta}{\partial u} \\ \frac{\partial \alpha}{\partial r} & \frac{\partial \alpha}{\partial \varphi} & \frac{\partial \alpha}{\partial u} \end{vmatrix}. \tag{4c}$$

We will first evaluate $J_1$. From Figures 1a and 1c it follows that: $x = s\cos\theta$ and $y = s\sin\theta$. Calculating $J_1$ from (4b) then gives:

$$J_1 = \begin{vmatrix} \cos\theta & -s\sin\theta & 0 \\ \sin\theta & s\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = s. \tag{5}$$

For the computation of $J_2$ the following expressions are obtained from Figure 1d:

$$s = \sqrt{r^2 + u^2},$$

$$\theta = \varphi + \arctan(r/u) - \pi/2, \tag{6}$$

$$\alpha = \arctan(r/u).$$

Evaluating (4c), using (6) gives:

$$J_2 = \begin{vmatrix} r/s & 0 & u/s \\ u/s^2 & 1 & -r/s^2 \\ u/s^2 & 0 & -r/s^2 \end{vmatrix} = -1/s. \tag{7}$$

Combining the results (5) and (7) gives:

$$|J| = 1. \tag{8}$$

With eqs. (2c) and (3a), we have:

$$p(r, \varphi, u) = c. \tag{9}$$

For an arbitrarily positionned reference point $R(x, y)$ we finally arrive at the expression of the distribution of infinitely long random straight lines by integrating $p(r, \varphi, u)$ over $u$:

$$p(r, \varphi) = c. \tag{10}$$

This equation is consistent with eq. (1) in (Duda & Hart, 1973).

In the case of straight lines of finite length, the description of a line is extended with a variable $L$ describing the length of a segment. According to principle 3, $L$ has a uniform distribution. Since $L$ is independent of all other variables, the transfor-

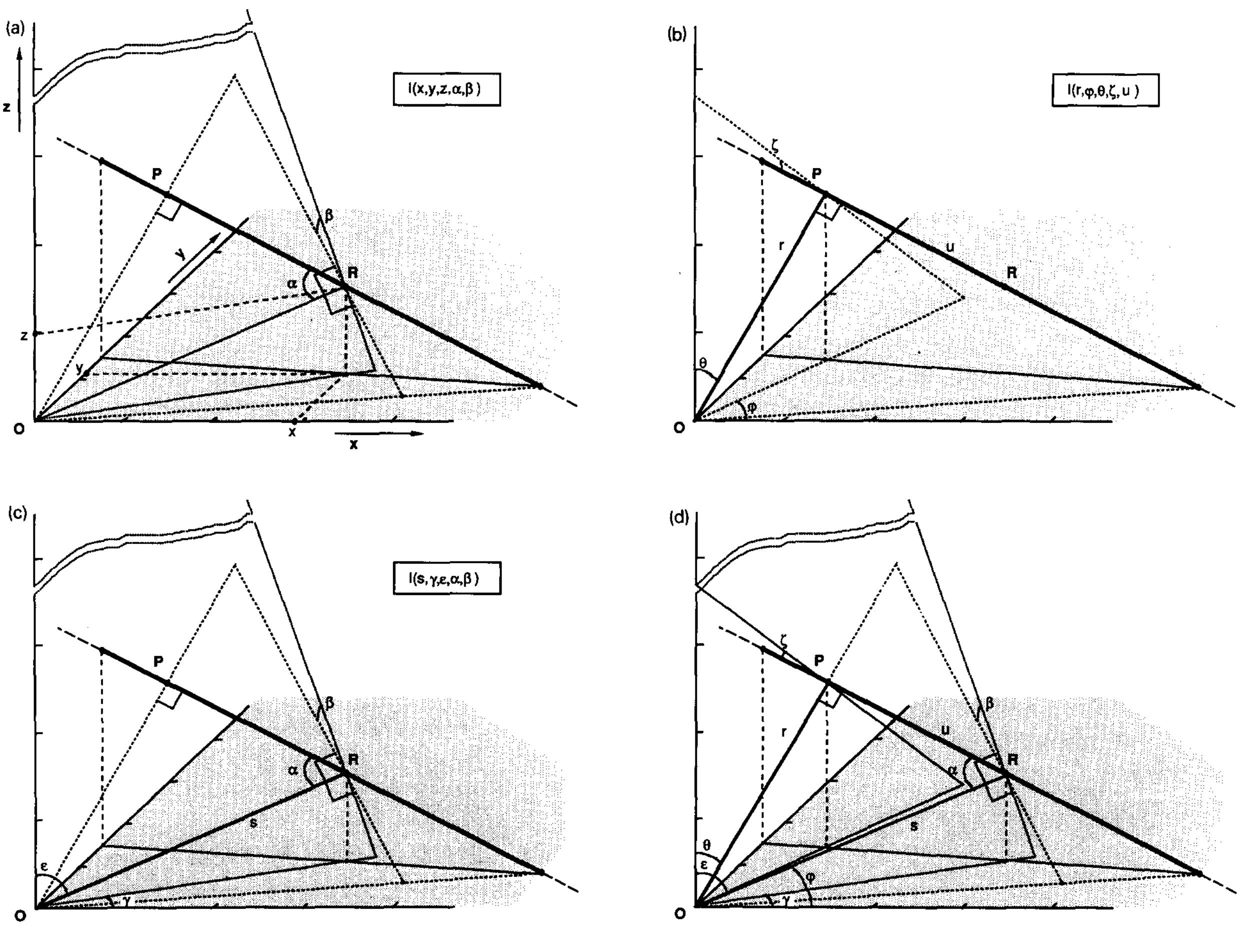


Figure 2. (a) Representation of a straight line in three dimensions in cartesian coordinates. For line segments an additional coordinate for the length has to be introduced as in Figure 1. Note that $\alpha$ and $\beta$ describe the line in spherical coordinates in a frame with origin $R$. (b) Representation of a straight line with reference point $R$ in normal coordinates. (c) Representation of a straight line in spherical coordinates. (d) The relation between the representations in (b) and (c).

mations do not affect $L$. Random straight line segments are therefore described by a uniform distribution:

$$p(r, \varphi, L) = c. \tag{11}$$

## 4. Random lines in $\mathbb{R}^3$

For the three-dimensional problem of expressing the probability distribution of a random straight line, the same approach was followed as in $\mathbb{R}^2$. First, using the principles of Section 2 it is possible to specify the distribution of random lines in a Cartesian reference frame. Then again a transformation is performed to the normal representation. After calculating the Jacobian an expression is found for the distribution of random straight lines in normal form.

As in the previous section a Cartesian coordinate frame is situated at the position of an observer, now in $\mathbb{R}^3$-space. As shown in Figure 2a, in the observation field a straight line is situated with reference point $R(x, y, z)$. The orientation $(\alpha, \beta)$ is given in spherical coordinates in a tilted reference frame with centre $R$.

From principle 1 it was concluded that random lines have a uniformly distributed reference position $R(x, y, z)$:

$$p(x, y, z) = c_1. \tag{12a}$$

The second principle implies a homogeneous distribution for the orientation of the line in $\mathbb{R}^3$-space. So, from Figure 3 it follows that for a given orientation $(\alpha, \beta)$ the number of lines in that direction is $p(\alpha, \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta$. For a unit sphere, the size of the solid angle for that orientation is $\sin\alpha\,\mathrm{d}\alpha\,\mathrm{d}\beta$. Since the distribution of lines in $\mathbb{R}^3$ is homogeneous, the number of lines in any direction should be proportional to the size of the solid angle:

$$p(\alpha, \beta) = c_2 \sin\alpha. \tag{12b}$$

Position and orientation of a line are independent, as was stated in principle 4, so random lines have the distribution:

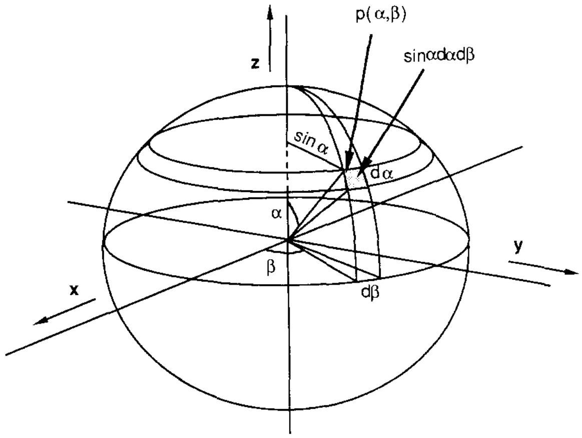


Figure 3. The number of lines in direction $(\alpha, \beta)$ is proportional to the size of the solid angle for that direction.

$$p(x, y, z, \alpha, \beta) = p(x, y, z) \cdot p(\alpha, \beta) = c_3 \sin\alpha. \tag{12c}$$

The normal representation of the line, shown in Figure 2b, is given by $(r, \varphi, \theta, \zeta, u)$. In this representation the parameters $(r, \varphi, \theta)$ describe the normal vector of the line, where $r$ is the length of the vector and $\varphi$ and $\theta$ determine the orientation. Given $r$, $\varphi$ and $\theta$, the orientation of the line within the plane orthogonal to $(r, \varphi, \theta)$ is still free to choose as well as the position of the reference point $R$. So, define the orientation of the line by the angle $\zeta$ in the normal plane and define the position of $R$ relative to $P$ by parameter $u$.

Similar to the two-dimensional case, the distribution $p(x, y, z, \alpha, \beta)$ is transformed into normal coordinates:

$$p(r, \varphi, \theta, \zeta, u) = |J| \cdot p(x, y, z, \alpha, \beta), \tag{13a}$$

with

$$J = \frac{\partial(x, y, z, \alpha, \beta)}{\partial(r, \varphi, \theta, \zeta, u)} \tag{13b}$$

where the transformation is incorporated in the Jacobian $J$. Here we also perform the coordinate transformation in two steps:

$$J = J_1 \cdot J_2. \tag{14a}$$

In the first step the Cartesian coordinates $(x, y, z)$ are transformed into spherical coordinates $(s, \gamma, \varepsilon)$ (see Figure 2c), with Jacobian $J_1$:

$$J_1 = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial \gamma} & \frac{\partial x}{\partial \varepsilon} & \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial \gamma} & \frac{\partial y}{\partial \varepsilon} & \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial \gamma} & \frac{\partial z}{\partial \varepsilon} & \frac{\partial z}{\partial \alpha} & \frac{\partial z}{\partial \beta} \\ \frac{\partial \alpha}{\partial s} & \frac{\partial \alpha}{\partial \gamma} & \frac{\partial \alpha}{\partial \varepsilon} & \frac{\partial \alpha}{\partial \alpha} & \frac{\partial \alpha}{\partial \beta} \\ \frac{\partial \beta}{\partial s} & \frac{\partial \beta}{\partial \gamma} & \frac{\partial \beta}{\partial \varepsilon} & \frac{\partial \beta}{\partial \alpha} & \frac{\partial \beta}{\partial \beta} \end{vmatrix}. \tag{14b}$$

In the second step spherical coordinates are transformed into normal coordinates, with Jacobian $J_2$:

$$J_2 = \begin{vmatrix} \frac{\partial s}{\partial r} & \frac{\partial s}{\partial \varphi} & \frac{\partial s}{\partial \theta} & \frac{\partial s}{\partial \zeta} & \frac{\partial s}{\partial u} \\ \frac{\partial \gamma}{\partial r} & \frac{\partial \gamma}{\partial \varphi} & \frac{\partial \gamma}{\partial \theta} & \frac{\partial \gamma}{\partial \zeta} & \frac{\partial \gamma}{\partial u} \\ \frac{\partial \varepsilon}{\partial r} & \frac{\partial \varepsilon}{\partial \varphi} & \frac{\partial \varepsilon}{\partial \theta} & \frac{\partial \varepsilon}{\partial \zeta} & \frac{\partial \varepsilon}{\partial u} \\ \frac{\partial \alpha}{\partial r} & \frac{\partial \alpha}{\partial \varphi} & \frac{\partial \alpha}{\partial \theta} & \frac{\partial \alpha}{\partial \zeta} & \frac{\partial \alpha}{\partial u} \\ \frac{\partial \beta}{\partial r} & \frac{\partial \beta}{\partial \varphi} & \frac{\partial \beta}{\partial \theta} & \frac{\partial \beta}{\partial \zeta} & \frac{\partial \beta}{\partial u} \end{vmatrix}. \tag{14c}$$

We first evaluate $J_1$. From Figures 2a and 2c, it can easily be seen that the coordinates of reference point $R(x,y,z)$ are given by: $x = s\sin\varepsilon\cos\gamma$, $y = s\sin\varepsilon\sin\gamma$ and $z = s\cos\varepsilon$. According to eq. (14b), $J_1$ is as given in equation (15) below.

The second step, the transformation to normal coordinates, appears to be a complex one. Abbreviate

$$r^2\sin^2\theta + u^2 - u^2\cos^2\zeta\sin^2\theta + 2ru\cos\zeta\sin\theta\cos\theta$$

as $T$. From Figure 2d, it can be derived that the coordinate transformation from $(s,\gamma,\varepsilon,\alpha,\beta)$ to $(r,\varphi,\theta,\zeta,u)$ is determined by the following expressions:

$$\begin{aligned} s &= \sqrt{r^2+u^2}, \\ \gamma &= \varphi - \arcsin\left(\frac{u\sin\zeta}{T}\right), \\ \varepsilon &= \arccos\left(\frac{r\cos\theta - u\cos\zeta\sin\theta}{\sqrt{r^2+u^2}}\right), \\ \alpha &= \arctan(r/u), \\ \beta &= \arccos\left(\frac{u\cos\theta + r\cos\zeta\sin\theta}{T}\right). \end{aligned} \tag{16}$$

After evaluation of eq. (14c), Jacobian $J_2$ is given as in equation (17), see next page. Some elements of the determinant are marked by a '·', as their value is irrelevant for the value of the determinant.

Substituting the definitions of $s$ and $T$ and inserting eqs. (15) and (17) in (14a) gives:

$$|J| = \sqrt{r^2+u^2}\sin\theta. \tag{18}$$

From eqs. (12c) and (16), it follows that

$$p(x,y,z,\alpha,\beta) = cr/\sqrt{r^2+u^2}.$$

Evaluating eq. (13a), using (18) yields:

$$p(r,\varphi,\theta,\zeta,u) = cr\sin\theta. \tag{19}$$

After integration over the relative position $u$ of the reference point, we finally arrive at the desired result:

$$p(r,\varphi,\theta,\zeta) = cr\sin\theta. \tag{20}$$

$$J_1 = \begin{vmatrix} \sin\varepsilon\cos\gamma & -s\sin\varepsilon\sin\gamma & s\cos\varepsilon\cos\gamma & 0 & 0 \\ \sin\varepsilon\sin\gamma & s\sin\varepsilon\cos\gamma & s\cos\varepsilon\sin\gamma & 0 & 0 \\ \cos\varepsilon & 0 & -s\sin\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -s^2\sin\varepsilon \tag{15}$$

$$J_2 = \begin{vmatrix} r/s & 0 & 0 & 0 & u/s \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 0 & \dfrac{(r\sin\theta + u\cos\zeta\cos\theta)}{\sqrt{T}} & \dfrac{-u\sin\zeta\sin\theta}{\sqrt{T}} & \cdot \\ u/s^2 & 0 & 0 & 0 & -r/s^2 \\ \cdot & 0 & \dfrac{us\sin\zeta}{T} & \dfrac{s\sin\theta\,(r\sin\theta + u\cos\zeta\cos\theta)}{T} & \cdot \end{vmatrix} = \sin\theta/\sqrt{T} \tag{17}$$

This formula gives an expression in normal form for the probability density distribution of a line of infinite length, tossed randomly on a grid in $\mathbb{R}^3$.

For straight lines of finite length, in complete analogy of the two-dimensional case in Section 3, it can be derived that

$$p(r, \varphi, \theta, \zeta, L) = cr\sin\theta. \tag{21}$$

## 5. Discussion

The result of eq. (20) is not immediately clear. It can be explained as follows. The factor of $\sin\theta$ originates from orientation invariancy, as was likewise arrived at in eq. (12c). The linearity of $r$ in eq. (20) is less obvious. An explanation can be found when these results are compared with the uniform result in the two-dimensional case. Consider two concentric circles as in Figure 4a, one with radius $r$ and one with radius $R$ ($R > r$). If $\mathrm{d}p'$ is part of the distribution of lines passing through a single point $P$ on the outer circle for a specific orientation, then $P$ contributes $2\,\mathrm{d}p'$ to the distribution $p(r)$ on the inner circle. For the complete outer circle the contribution is:

$$\mathrm{d}p(r) = 2\pi R \cdot 2\,\mathrm{d}p'.$$

Then, the problem of finding an expression for $p(r)$ is reduced to a one-dimensional one. It is obvious that integration over all outer circles yields a uniform result. In a similar way the three-dimensional case can be analyzed, where we have two concentric spheres, with radii $r$ and $R$ ($R > r$), see Figure 4b. Here again $\mathrm{d}p'$ is part of the distribution of lines passing through a point $P$ on the outer sphere for a given direction. The contribution of $P$ to the distribution $p(r)$ is

$$2\pi r \cdot \mathrm{d}p'\sqrt{(R^2 - r^2)}/R.$$

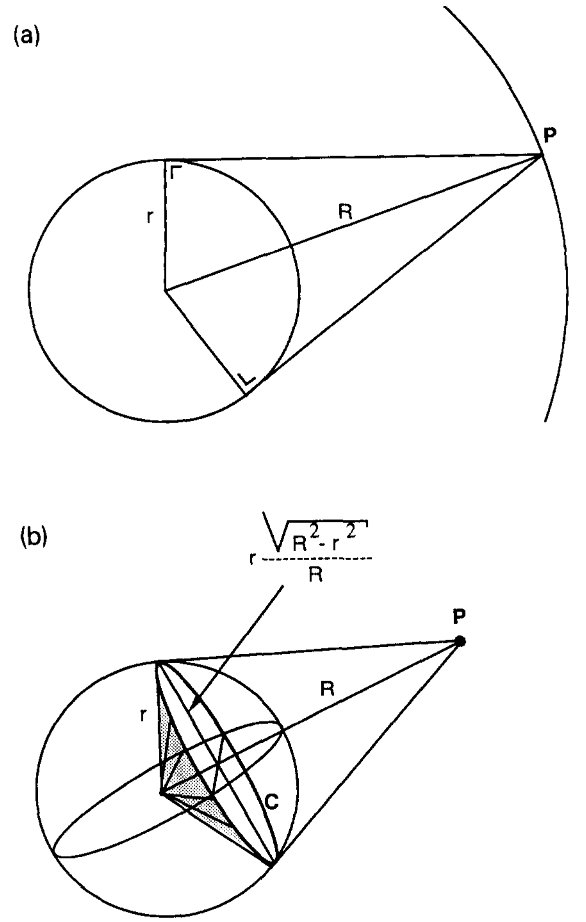


Figure 4. (a) In two dimensions the point $P$ contributes at two points to the distribution $p(r)$. (b) In three dimensions the contribution of point $P$ to the distribution $p(r)$ is proportional to the circumference of circle $C$.

To incorporate all points on the outer sphere a factor of $4\pi R^2$ has to be added to make the problem one-dimensional:

$$\mathrm{d}p(r) = 8\pi^2 Rr\sqrt{(R^2 - r^2)} \cdot \mathrm{d}p'.$$

Compared to the two-dimensional case a linear dependence on $r$ arises approximately, because here a circle $C$ contributes to $p(r)$, instead of just 2 points in the 2-D case.

## 6. Conclusion

Principles of invariance for position, rotation and length were introduced to formally define randomness of lines. It was shown that infinitely long straight random lines in two dimensions yield a uniform distribution in parameter space: $p(r, \varphi) = c$, consistent with the intuitive formula by Duda and Hart (1973). Random straight line segments of finite length also are described by a distribution uniform in the normal parameters: $p(r, \varphi, L) = c$.

In three dimensions random infinitely long straight lines have a distribution:

$$p(r, \varphi, \theta, \zeta) = cr \sin\theta.$$

For line segments the distribution is:

$$p(r, \varphi, \theta, \zeta, L) = cr \sin\theta.$$

These are significant results in that they demonstrate for 3-D, that the distribution deviates from a first intuition equating randomness with a uniform distribution in the normal representation. The sinusoidal dependence on $\theta$ makes the result orientation invariant, while the linear dependence on $r$ originates from the fact that $p(r)$ is calculated on a sphere with radius $r$.

The expressions for the distributions of random lines and line segments are applicable in the optimization of length (Dorst & Smeulders, 1987) and distance estimators (Borgefors, 1984) in the absence of a priori knowledge. These estimators themselves have various applications in the morphological and topological analysis of objects in images and possibly for the design of line detectors.

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