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Mathematical Challenges to Secondary School Students in a Guided Reinvention Teaching-Learning Strategy towards the Concept of Energy Conservation

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Abstract

Guiding sixteen-year-old students to rediscover the concept of energy conservation may be done in three distinct learning steps. First, we have chosen for the students to reinvent what we call partial laws of energy conservation (e.g. $\sum m \cdot g \cdot h = k_1$). Secondly, the students are asked to combine these partial laws into more and more general laws of energy conservation (e.g. forming $\sum m \cdot g \cdot h + \sum m \cdot c \cdot T = k_3$). Because a new term may always be added this process of combining laws can be continued for a long time. The result may still be only a partial law of energy conservation. A third learning step is needed in which students are to extrapolate the process of combining partial laws. If the student becomes convinced that it is indeed always possible to add an extra term to the equation when necessary, the student must now be convinced as well that the law is applicable to any situation and has thereby reinvented the general law of energy conservation. In the first two learning steps we have uncovered mathematically challenging steps which remain obscure in more traditional teachings of the concept of energy conservation. The first mathematical challenge lies in retrieving a physical law from measured data, specifically quadratic ones. The second challenge lies in combining the special cases of energy conservation into a more widely applicable case of energy conservation. In this paper we will exemplify these mathematical problems and give some possible solutions for teachers to guide students in passing these obstacles. At least partially these steps clarify the abstractness of the concept of energy and contribute to explain problems students are having in understanding this concept.

Introduction

In the existing situation in the Netherlands students' ideas on energy in secondary education are diagnosed to be inflexible in formal examination tasks (Borsboom et al., 2008): amongst others students tend to leave some of the relevant forms of energy out of the equation when solving problems. Another flexibility problem has earlier been observed with students attending university chemistry courses on thermodynamics (Kaper, 1997): the students tended to stick to their secondary school conception of energy instead of adopting a new thermodynamic view of the concept of energy. In current education the law of energy conservation is taught as an indisputable fact detached from its scientific origin which may lead to the usefulness of the law not being immediately apparent to students (Borsboom et al., 2008; De Vos et al., 2002; Kaper, 1997). Freudenthal (1991) says knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others. Freudenthal therefore recommends a guided reinvention approach.

We think that a process of reinvention is needed to make students realize which forms of energy are relevant to which situations and to convince the students of the general validity of the law of energy conservation (cf. Feynman et al., 1963; Joule et al., 1884). In this process the law will, as we prefer, become debatable and will no longer be an indisputable fact. Knowing the way in which the law is constructed and critically thinking throughout its reinvention may cause the students to apply their conception of energy conservation more properly and make it more susceptible for later necessary adjustments: their conception may become more versatile.

Finding an adequate balance between freedom for the students to learn and guidance from above by the teacher (cf. Lijnse & Klaassen, 2004; Freudenthal, 1991) is part of our research. We try to find this balance by posing assignments and then giving the students as much freedom as possible within the constraints set by that assignment.

As opposed to the general law of energy conservation, for the ideal gas law such a reinvention approach is used more often. The students reinvent the various partial gas laws (Boyle's law, Gay-Lussac's law, etc.) and combine those into one, more general law: the ideal gas law (e.g. Van Baalen et al., 2008). Guided reinvention seems possible for reinventing the general law of energy conservation as well (Logman et al., 2010, 2011). Students however may find several steps in this reinvention process mathematically challenging. Therefore we focus on which parts of mathematics a teacher should address during such a reinvention:

To which extent does a teacher need to guide students during mathematically challenging steps while students are to reinvent the law of energy conservation?

Method

We assume that for most students it's not possible to reinvent the general law of energy conservation in one go. As a first learning step, we focus on reinventing what we call partial laws of energy conservation each with its own applicability domain (e.g. $\sum m \cdot h = k_1$ [1]).

In a second learning step the students will need to learn how to combine these partial laws into more and more general yet still partial laws of energy conservation (e.g. combining $\sum m \cdot h = k_1$ with $\sum m \cdot c \cdot T = k_2$ [2] to form $\sum m \cdot h + 426 \cdot \sum m \cdot c \cdot T = k_3$ [3]). Because it remains possible that new variables will show up, this combination process does not lead to a point where one can be sure that the law is complete: the result will still remain a – possibly – partial law of energy conservation.

Therefore the students need to take a third learning step in which the process steps for extraction and combination of partial laws are extrapolated and checked whether they are always possible. In this extrapolation no new mathematical steps need to be taken. If the student concludes that these process steps are indeed always possible, in the student's mind the law must now have become applicable to any situation and can therefore truly be called general (as opposed to partial): the general law of energy conservation is reinvented (see Figure 1).

1 If one extracts laws from experiments involving only gravitational energy one will not add the gravitational acceleration g into this equation because it has no use. k_1 is only a constant when there is little friction and all other forms of energy are constant. It may vary over different experiments.

2 The "*c*" in this equation describing the mixing of various hot and cold substances denotes the specific heat of a substance but not in SI units. Historically c was chosen to be 1 (kcal/kg·K) for water. k_2 is only a constant when the experiment is well insulated and all other forms of energy are constant. This coefficient is different for different experiments.

3 The specific heat c is here chosen to be 1 (kcal/kg·K) for water. The factor 426 (m/K) stems from Joule's experiment establishing the mechanical equivalent of heat. Multiplying both terms in the equation by g may change the factor *426* into *4180*: the specific heat of water in SI-units. The coefficient k_3 is only a constant when the experiment has only friction in places where the temperature is measured. Again all other forms of energy need to be constant and the coefficient k_3 may vary over different experiments.

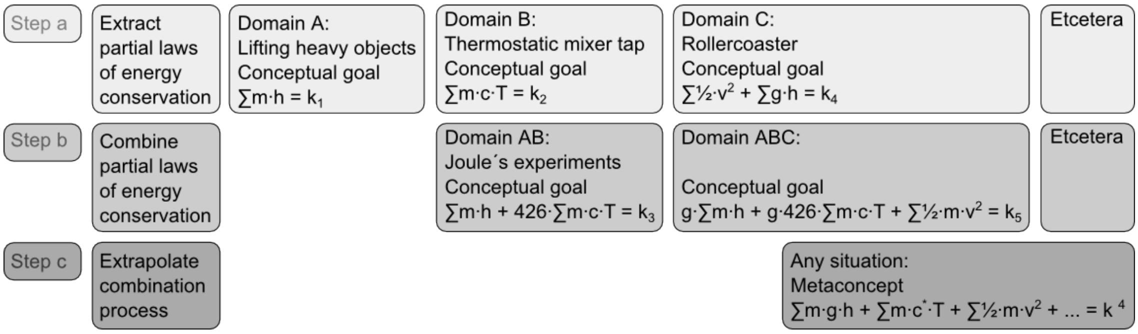

Figure 1. Intended learning trajectory towards the general law of energy conservation[1]

At this point the students should be able to apply this law to many different situations. Even in situations in which the students do not know all the terms involved we expect them to look for and reinvent the missing term to make the law complete for that specific situation (Feynman et al., 1963). If the student is able to do so he has acquired a versatile concept of energy conservation in the sense of its applicability.

During our first try-outs we focused on steps involving mathematics and observed which steps were the most problematic to the students. In our subsequent try-outs we tried to solve these problems to see whether we were able to make them less problematic. In the analysis of those steps we compared the intended reinvention of the general law of energy conservation to the somewhat similar combination of partial gas laws to form the ideal gas law to make our results more generally applicable.

In our final try-out the students received a description and data from a fictitious experiment which connects the electric potential energy of a capacitor to an already known form of energy (thermal energy) as a test. In this test the students were asked to extract a relationship between voltage and temperature and combine that law with the earlier laws. They were also asked how far this combination process could be continued and tell whether they believed a new combination could always be made when necessary. In order to answer this question we expected the students not only to perform the combination itself but pay attention to the process of combining as well.

We used three cycles to develop our educational design. The first try-out was performed by the researcher at his own school in a mixed group of 17 sixteen-year-olds. In the second try-out besides the researcher’s school five other schools were involved. In these five schools 4 teachers taught a total of 5 groups of sixteen-year-olds and 2 teachers each taught 1 group of seventeen-year-olds. In the third and last try-out two other schools besides the researcher’s school were involved concerning 2 teachers each teaching 1 group of seventeen-year-olds and 1 teacher teaching a group of sixteen-year-olds. The number of students in a group ranged from 8 to 30. The educational materials were used to replace the traditional quantitative introduction to energy. The students worked in groups of two or three students. All schools are located in the vicinity of Amsterdam.

Results and analysis

In the first try-out we encountered two major problematic mathematical steps in our approach. The first involved the extraction of a physical law from the measured data and rewriting that physical law into an easy to use notation, and the second the combination of such physical laws into one more general law.

In our learning trajectory students are supposed to extract linear and quadratic relationships. Extracting a linear law proved to be possible for the students even though the laws were not always in the notation we preferred. An example of a physical law that students came up with for mixing hot and cold water is shown in Figure 2.

1 The equation at the bottom right is meant to describe the general law of energy conservation including any terms as yet unknown to the students.

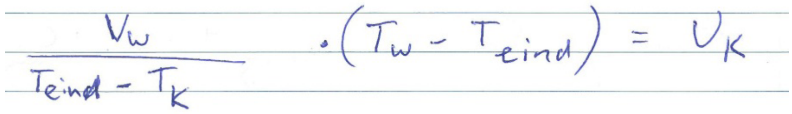

Figure 2. *A linear physical law extracted by students*

The extraction of a quadratic relationship from the data concerning a rollercoaster proved to be more problematic. In the first try-out only one out of eight groups managed to come up with a relationship. They did so by plotting the square of one of the measured variables against the other resulting in a linear graph. They were only able to do so after being guided by the teacher. The resulting linear graph made it easy to come up with a physical law by calculating the slope of the graph (see Figure 3).

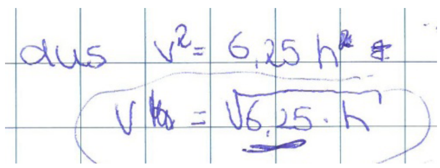

Figure 3. *A quadratic physical law extracted by students*

In subsequent try-outs we decided to refer to this method as a general method used by engineers and scientists to come up with physical laws from data. In our final try-out the students received a description and data from a fictitious experiment which connects the electric potential energy of a capacitor to thermal energy as a test. In this test the students had to extract a quadratic relationship between voltage and temperature. In that test situation 28 of the 65 groups divided over four teachers did not derive a relationship at all. We assume they weren't capable of doing so. The other groups, 37, mentioned a relationship that fitted the data. Seven of these groups however, did not show how they derived the right quadratic relationship so we are unable to say whether they were actually capable of performing the extraction themselves.

Table 1. *Student's capabilities in extracting a quadratic relationship*

Student's capability	Extract quadratic law
Capable	30/65 (46%)
Undecided	7/65 (11%)
Incapable	28/65 (43%)

Some guidance not stemming directly from the assignment was necessary to have a better number of students extract a quadratic relationship between two variables. The students were taught a general method of extracting a physical law but still had to reinvent the particular relationship themselves. In spite of learning about the general method about half of the students remained incapable of extracting a quadratic relationship.

From the examples we showed in figures 2 and 3 it is clear that in almost all cases the notation of the reinvented laws do not show many similarities and therefore do not invite students to think about a possible combination of these laws. This extends the first problematic mathematical learning step a little further. To guide the students to a more similar notation for all reinvented laws (e.g. our intended notation in Figure 1) a discussion on which notation would be the best was held guided by the teacher. We expected that criteria given by the students for the best notation would lead to more similar notations of the various laws. The best notation meant that the law would be easy to use and would be as widely applicable as possible in future similar problems. To start the discussion the students were asked to rewrite the reinvented laws in as many mathematically correct ways as possible. After an inventory of all possible notations the students were asked which notation would be the best:

Teacher: I must say there are more [notations] on the board than I myself expected. I think really all possible notations are there! I can't think of any more. But now I have 9 possibilities.

Student1: Yes.

Teacher: Which one are we gonna choose? Who has an idea?

[…]

Teacher: Here I notice something strange about this notation as opposed to some other notations...

Student2: On both sides we have 2 variables.

Teacher: On both sides 2 variables. I also notice something else about that formula that... I have groups of notations in which [identifiers] 1 [and] 2 are on one side and [identifiers] 2 [and] 1 are on the other side. And I see groups, like these, in which 1 1 are on one side and 2 2 are on the other. Which would physicists, if one had to choose, which would you choose? Would you choose all 1's on one side and all 2's on the other or would you mix them?

Student3: Each on one side.

Easy to use in the eyes of the students meant not making mistakes with the formula. Using symbols made the formula easier to understand than writing the variables in full. Some students said subtractions gave them more problems than additions because in subtractions the order of the variables is essential. The teacher realized this had to do with commutative operations and asked whether they preferred multiplications over divisions as well and the students confirmed this. It may be useful for teachers to notice this and use commutative notations of physical laws in their classes.

The discussion on making the law for a rollercoaster ($2 \cdot g \cdot h + v^2 = k_4$) as widely applicable as possible showed several aspects. Discussing whether the zero points for the variables are free to choose, whether the units are free to choose, and whether the law can be expanded to more objects or more materials than two were held. The discussion on zero points is illustrated in the quote below. It results in adding delta's to the notation when necessary.

Teacher: Read the question. Next think how many variables are in it.

Student1: Two variables. Three.

Teacher: Two variables. Yeah, actually three.

Student2: Three.

Student3: Yes. One needs to be aware.

Teacher: Yes, three. And does one need to take care with those? Well, let's check all three. The h, does one have to take care for the height? The zero height, where one chooses zero to be?

Student45&6: Yes.

Student7: No.

Student8: The zero height is zero.

Student9: No, that doesn't matter, does it?

Teacher: If I was to put the rollercoaster on a hill, would it do the same thing, or not?

Student1: Yes, in principle yes.

Student45&6: Yes.

Student10: The zero is the bottom of the rollercoaster.

Teacher: The zero needs to be at the bottom of the rollercoaster, yes. That is well. So one has to be aware of that. But if you don't want that, what do you need to use for that h?

Student7: h begin.

Teacher: No, not h begin

Student5: The height you're moving upwards.

Student10: The same.

Student7: Delta h.

Teacher: Delta h!

Student10: Delta h.

Teacher: Yes? And is that delta h plus or minus in our experiment? That is always important as well. Using delta's one has to be aware whether it is plus or minus. In our experiment we shot [the carts] from below upwards.

Student3: Plus.

Student4: That is plus.

Together with the students' preference for additions and multiplications the discussions guided them to our intended notations of the various partial laws of energy conservation.

In subsequent try-outs we incorporated such a discussion into the learning material. This classroom discussion was necessary to have the students come up with the intended notation. The students were guided by being asked to make the extracted physical law easy to use and as widely applicable as possible. However, during the discussion the teacher had to guide the students to come up with the right answers.

The second major problematic mathematical step involved combining the various partial laws of energy conservation into one more generally applicable law. In both the first and the second try-out we asked what a combination of the first two partial laws would look like. Many groups came up with a direct addition of the two original laws (see Figure 4):

$$[m_1 h_1 + m_2 h_2 + c_1 m_1 T_1 + c_2 m_2 T_2]_b =$$
$$[m_1 h_1 + m_2 h_2 + c_1 m_1 T_1 + c_2 m_2 T_2]_e$$

Figure 4. *A group's addition of partial laws as a suggestion for their combination*

Some other groups suggested some form of multiplication of the original two partial laws (see Figure 5):

$$(m_1 \cdot h_b \cdot c_1 \cdot T_b) + (m_2 \cdot h_b \cdot c_2 \cdot T_b) = (m_1 h_e \cdot c_1 \cdot T_e) + (m_1 h_e c_2 T_e)$$

Figure 5. A group's multiplication of partial laws as a suggestion for their combination

The students created possibilities based on mathematical rules for combining two equations, like in: $a = b$ together with $c = d$ implying $a + c = b + d$. The students were asked for suggestions before the law $m \cdot \Delta h = -426 \cdot m \cdot c \cdot \Delta T$ from Joule's experiment had been reinvented. At that point most teachers had written the partial laws like $\sum m \cdot h = k_1$ and $\sum m \cdot c \cdot T = k_2$ and students suggested adding, multiplying, subtracting, or dividing both the left and the right hand sides of these equations to combine the two laws. The first two options were the most suggested combinations of the partial laws.

This transfer of mathematical knowledge to physics shows that the students were unaware that physical laws have preconditions under which they are valid. Whereas in mathematics constants are usually absolutely constant, in physics many constants are only constant under specific preconditions. Only one teacher (in the second try-out) consistently used preconditions in his discussions of the partial laws of energy conservation. For him it was easier than for his colleagues to show that most suggested combinations could not be valid for they contradicted the original partial laws under their own specific preconditions. Addition results in contradicting the law for Joule's experiment, multiplication results in trivial equalities when either h or T equals zero. Simply adding, multiplying or performing any other mathematical operation on the two 'constant' terms in the two physical laws is not allowed because of the preconditions to the physical laws that in general remain silent. Rewriting the law governing Joule's experiment into a similar notation as the other partial laws ($m \cdot h + 426 \cdot m \cdot c \cdot T = k_3$) did however strongly suggest the right solution to combining the two original partial laws:

Teacher: We have found a law for lifting a heavy object and one for the mixer tap

Student1: Yes.

Teacher: We are trying to combine those.

Student1: Yes.

Teacher: Well, what do you expect to be that combination?

Student2: Isn't that just what's on the board?

Student1: Yes.

Teacher: Well, check it! With the work we've already done that should go much faster.

Student2: We checked it here.

Teacher: Yeah, that's right. Keep that.

In subsequent try-outs we decided to expand our earlier mentioned discussion to include under which circumstances the reinvented law would be valid or not.

Reverse engineering the solution from the answer is easy but students will not understand how physical laws are to be combined appropriately right away, as a question of a college student on an internet forum illustrates (see Figure 6).

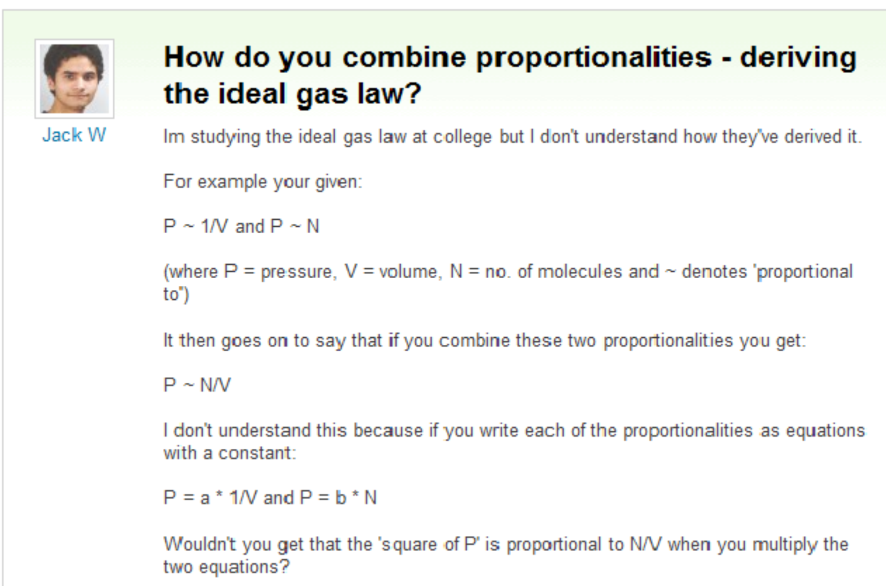

Figure 6. *A college student's question on an internet forum.*

(Source: http://answers.yahoo.com/question/index?qid=20080912061753AAjBNoY, last access 24-3-2012)

To generalize our findings on combining physical laws beyond the general law of energy conservation we analyze above example of combinations to form the ideal gas law. The student above shows the same transfer of mathematical rules to combining physical laws as we have shown earlier in our results without any consideration of preconditions. But in the case of the ideal gas law the situation is subtly different.

Multiplying Boyle's law ($P \cdot V = k_5$) with Avogadro's law ($P/n = k_6$) results in $P \cdot V \cdot P/n = k_7$ as suggested by the student. Addition whether or not including adding a new constant like in the case of combining partial laws of energy conservation would result in $P \cdot V + (x \cdot)P/n = k_7$. All combinations that result from these operations are wrong. Not many teachers will address this problem in class.

The constant k_5 in Boyle's law is only a constant when n (and T) is assumed to be constant. The constant k_6 in Avogadro's law is only a constant when V (and T) is assumed to be constant. This means k_5 can still be a function of n and similarly k_6 can still be a function of V resulting in $P \cdot V = k_5(n)$, and $P/n = k_6(V)$.

Proving which combination would be the right one calls for another level of mathematics (Kaper et al., 2012) and does not seem to be an option with sixteen-year-olds. Knowing the right combination ($P \cdot V/n = k_7$) however it is easy to reverse engineer the proper function $k_5(n)$ to being $k_7 \cdot n$ ($P \cdot V = k_7 \cdot n$ $k_5(n) = k_7 \cdot n$). The new constant k_7 now can no longer depend on n but can still depend on T, because we have been silently assuming T was constant in all experiments. Another option would be to assess suggested combinations by checking them with data from experiments.

Returning to our case of combining partial laws of energy conservation in the law $\sum m \cdot h = k_1$ the constant k_1 can still be a function of any variable kept constant during the experiment from which we derived the law (essentially coming down to any variable but h). Similarly in $\sum m \cdot c \cdot T = k_2$ the constant k_2 can depend on any other variable but T. In the right combination ($\sum m \cdot h + 426 \cdot \sum m \cdot c \cdot T = k_3$) the constant k_3 does not depend on either h or T anymore. Apparently the functions need to be $k_1(T) = -426 \cdot \sum m \cdot c \cdot T + k_3$ in which k_3 can no longer depend on T (nor h). Again proving the right combination from the partial laws would call for another level of mathematics (Kaper et al., 2012).

For education we can now choose from two possible approaches: reverse engineering from the right solution or assessing possible combinations with measured data. The first approach does not agree with our choice for guided reinvention because the solution needs to be known beforehand and is therefore not reinvented.

That is why in our subsequent try-outs we decided to use the second approach. First from a demonstration of Joule's experiment the law that governs it was reinvented in a classroom discussion guided by the teacher. Next the students were asked to rewrite that law into a similar notation as before ($m \cdot h + 426 \cdot m \cdot c \cdot T = k_3$). This notation suggested to most students the right combination, ($\sum m \cdot h + 426 \cdot \sum m \cdot c \cdot T = k_3'$) which was subsequently checked with the data and the reinvented laws governing the earlier experiments to see whether the new combination did indeed incorporate all the earlier laws as well as the law describing Joule's experiment.

[The formula ∑m·h + 426·∑m·c·T = constant is written on the board]

Teacher: What was the formula again that we discovered during lifting? In the end?

Student1: m times h, wasn't it?

Student2: Yeah, m times h.

Teacher: This, right? [Writes ∑m·h = constant on the board]

Student3: Yes.

Student2: Yes.

Teacher: Is this one very different [from the new combined formula]?

Student2: No.

Student3: You just...

Teacher: What did we find for the mixer tap?

Student3: Well... I don't know.

Student4: c times m1 times T or something like that.

Student2: That was eh...

Teacher: I hear someone mention it. [Writes ∑c·m·T = constant on the board] Is this one very different?

Student4: Well, no...

Teacher: Well, it is a little.

Student5: Eh, something needs to be added to it.

Student1: Yes, it only needs 426 in it.

Teacher: Is that allowed? Was I allowed to write 426 on both sides?

Student4: Yes.

Students1,2&6: Yes.

Teacher: Yes, that's allowed, isn't it? It would be nonsense again, like he said. [...] I can multiply by 426 but why would one do so? Well...

Student2: Than it becomes more complicated.

Teacher: Than it becomes more complicated, but now I can suddenly describe Joule's experiment when I include the 426, only when I include that 426 in it, does it describe Joule's experiment as well. If I don't add it, right, than I get that the temperature would rise a lot. If I drop something one meter the temperature would rise a degree Celsius as well. That is not the case. That's why the 426 is in it.

We hope that after combining the first two partial laws the students can get the hang of this approach and apply it to further expansions of the law themselves.

In our final try-out the students received a test containing a description and data from a fictitious experiment which connects the electric potential energy of a capacitor to an already known form of energy (thermal energy). Subsequently the 65 groups were asked to extract a law from that data and combine this new law with the earlier combined laws. Out of the 30 groups that managed to extract the quadratic relationship between voltage and temperature 8 groups weren't able to combine this law with the earlier combined laws at all. Five groups combined the extracted law in a wrong way whereas 17 out of those 30 groups did manage to come up with the right combination. One of the groups that earlier on in the learning process did not show the derivation of the extracted law now managed to combine it in a proper way. So in total 18 out of the 65 groups managed to combine a new physical law into the equation.

Table 2. Student's capabilities in combining partial laws of energy conservation

Student's capability	Combine laws (all students)
Capable	18/65 (28%)
Undecided	2/65 (3%)
Incapable	45/65 (69%)

We should not expect those students that were incapable of extracting the quadratic law to take the next learning step. Only looking at those students that succeeded in extracting the quadratic law shows that even for those students the combination step was still quite a big step to take.

Table 3. *Student's capabilities in combining partial laws of energy conservation limited to those students that were capable of extracting the quadratic law*

Student's capability	Combine laws (students capable of extracting quadratic law only)
Capable	17/30 (57%)
Undecided	0/30 (0%)
Incapable	13/30 (43%)

Conclusion

Reinventing the general law of energy conservation is not possible without conquering some big mathematical obstacles, the two most important ones being extracting a quadratic relationship from experiments and combining partial laws of energy conservation into more general laws of energy conservation.

Finding an adequate balance between freedom for the students to learn and guidance from above by the teacher (cf. Lijnse & Klaassen, 2004; Freudenthal, 1991) is part of our research. We try to find this balance by posing assignments and then giving the students as much freedom as possible within the constraints set by the assignment.

However, within the constraints set by the assignments many students did manage to take the first learning step of extracting a physical law from measured data without extra guidance as long as the relationship was linear. Using an extra form of guidance, being a general method of extracting a physical law by linearization, about half of our groups of sixteen-year-olds were capable of overcoming the first problematic mathematical step of extracting a quadratic relationship between variables themselves. The method consists of plotting various functions of the two measured variables until a linear graph is found.

Establishing the domain and preconditions of a physical law is not a natural thing for our students to do. We guided them through this process by a classroom discussion on which notation would be the easiest and most widely applicable. This discussion guided by the teacher concerned the following subjects:

- easy to use notations,
- zero points for involved variables,
- units to be used in the law,
- expansion to more than two objects or substances, and
- preconditions on the law.
- It was interesting to notice that students preferred commutative notations of physical laws whereas that is not always the case in physical laws as taught.

The preconditions on the law are especially important when combining partial laws to establish which possible combinations can be discarded and which others can be assessed experimentally. Guidance to combine partial laws can be done either by reverse engineering the partial law from the more general law or by rewriting physical laws stemming from crucial connecting experiments (like Joule's experiment) and subsequently testing the resulting possible combination by experimental data.

In traditional teaching the reverse proof is often used to 'explain' physics (e.g. Van Baalen et al., 2008). One has to be aware however that this approach may leave students wondering how such combinations are performed as shown in the example of the college student wondering about the combinations of Boyle's and Avogadro's law.

In our approach using an assessment of suggested combinations the process becomes more apparent even though a mathematical proof for the eventual solution is avoided. By showing the students crucial connecting experiments and rewriting the resulting physical law similarly to the partial laws in question (e.g. $m \cdot h + 426 \cdot m \cdot c \cdot T = k_3$) we were able to suggest an appropriate combination of partial laws ($\sum m \cdot h + 426 \cdot \sum m \cdot c \cdot T = k_3$). This combination could then be assessed using the data from all connected experiments. After guiding the students through two of such combinations about a quarter of our groups of students showed they were capable of combining a new partial law into a more general law themselves.

Clearly we underestimated the mathematics behind a reinvention of the law of energy conservation. For simple linear equations guided reinvention combined with a context-based approach should be possible for sixteen-year-olds. Perhaps the extraction of quadratic relationships and the combining of partial laws should be postponed to later classes. Another option would be to guide the students through more combinations before they are asked to attempt one themselves or pay more attention to the involved mathematics prior to our intended learning trajectory.

It is clear that school physics differs from school mathematics in the process of combining physical laws due to the preconditions on such laws. This difference is an addition to earlier described differences between mathematics and physics by Ellermeijer (2003) and Heck (2001). Students are much more used to the mathematical approach than the physical approach to combining equations and transfer mathematical rules to physics unaware of the preconditions under which physical laws are valid. To help students understand these differences better, preconditions deserve explicit attention in class. Besides that it appears that keeping every other variable constant in experiments has become too silent an assumption and in that way conceals those preconditions from the students.

The two mathematical steps hidden in traditional teaching show some of the abstractness behind the energy concept and may explain part of the difficulties students have in seeing the usefulness of the concept and in applying it to various situations.

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