Learning in overlapping generations models

Tuinstra, J.

Learning in Overlapping Generations Models

Jan Tuinstra*

Preliminary version

Abstract

We consider a standard two generations version of the overlapping generations model with different learning algorithms. Agents predict inflation rates on the basis of some mis-specified (linear) perceived law of motion, which is estimated by running a regression on past prices or inflation rates. Beliefs might converge, although inflation rates keep on fluctuating. These fluctuations are, in some sense, consistent with the (mis-specified) limit belief.

1 Introduction

Recently the shortcomings of the perfect foresight or rational expectations approach have led an increasing number of people to study models of learning (for nice reviews on bounded rationality see Sargent (1993,1998)). The overlapping generations model with two generations has played a prominent role in this recent literature. The reasons for this seem to be twofold. First, the overlapping generations model provides a tractable dynamic general equilibrium model in which all agents are utility maximizers (or, given their perceptions, believe they are) and markets clear in every period. Therefore it has become a modelling tool often utilized by proponents of the rational expectations approach, especially when it comes to problems in the field of monetary economics. Secondly, under rational expectations or perfect foresight the overlapping generations model features indeterminacy: there is a continuum of perfect foresight orbits, of which all but one have the unsatisfactory characteristic that they converge to the autarkic steady state in which money has no value.[1] Indeed, some of the first

*Department of Quantitative Economics and CeNDEF, Faculty of Economics and Econometrics, University of Amsterdam, NL-1018 WB, The Netherlands. Phone: (+31)20-525-4117, fax: (+31)20-525-4349, e-mail: tuinstra@fee.uva.nl

[1] This is provided that we are in the *Samuelson* case. For the *classical* case almost all perfect foresight paths tend to the monetary steady state. This classical monetary steady state, however, cannot be supported by money and therefore is seen as less important in macroeconomic analysis. For an analysis of indeterminacy in more general overlapping generations model see Kehoe and Levine (1985).

models of learning in overlapping generations models were developed to provide a selection device in picking the "most likely" of the different perfect foresight paths.

In this paper we will study a number of learning models in a standard version of the overlapping generations model. Our main focus is not on learning as a selection device but on the dynamics of different learning models itself. We belief the analysis of learning dynamics is an important subject since it may provide us with a better description of human behaviour than the rational expectations hypothesis does and it may account for some of the dynamical phenomena observed in reality. The actual dynamics of our model are given by a complicated nonlinear map. It is assumed that the agents do not know this complicated map but that they believe that the dynamics evolves according to some simple linear relationship. They estimate this relationship by using a least squares algorithm. The estimated model then provides agents' expectations about some variable, for example inflation rates. Since the actual realizations of this variable depend upon this expectation, the learning feeds back into the actual dynamics. These realizations of the variable again lead to an update of the estimated relationship. In this way a closed model of actual dynamics and learning dynamics arises. An important aspect of this model, in which it differs from most of the existing literature on learning in economic models, is the structural misperception of agents. The model they have of the economy is mis-specified. We believe that this is the most sensible approach to learning in economic models. The perceptions of an agent have to be much simpler than the unknown economic model itself which, among other things, depends upon the perception of this agent and the perceptions of all other agents in the economy and therefore will be very complicated. Agents will in general use simple "rules of thumb" to make predictions.

We investigate some different types of perceived laws of motion in this paper. We find that different types of beliefs may lead to different stability properties of the actual dynamics and in fact various dynamical phenomena, such as the existence of periodic cycles, quasi periodic orbits and strange attractors, might occur. Furthermore it is important whether the perceived law of motion is put in terms of price levels or in terms of inflation rates, in particular when there is a positive net inflation rate in equilibrium. Clearly the number of possible perceived laws of motion is unbounded. We hope to provide more insight into this "wilderness of bounded rationality" by studying which of these perceived laws of motion are, in some sense, reasonable.

Our work is related to that of Hommes and Sorger (1998). They introduce the notion of consistent expectations equilibria. In their model agents use simple linear expectation rules in a nonlinear environment. A consistent expectations equilibrium then emerges when there are no structural errors (that is, sample averages and sample autocorrelations are equal for the perceived and the actual models) in the forecasts agents make given their expectations function and the actual dynamics implied by this expectations function. In such an equilibrium agents cannot improve upon their forecasts in a linear statistical sense. Other related work can be found in the literature on learning in temporal equilibrium dynamics (Böhm and Wenzelburger (1999), Chatterji (1995), Duffy (1994) and Lettau and Van Zandt (1995)) and in the literature on least squares learning in macroeconomic models (Bray (1982), Bray and Savin (1986) and Marcet and Sargent (1989a,b)). A number of contributions has focused on the possibility of nonconvergence of least squares learning or other learning algorithms in temporal equilibrium dynamics (Benassy and Blad (1989), Bullard (1994), Grandmont and Laroque (1991) and Schönhofer (1996)). Our approach falls within this strand of the literature. Although a popular way to model learning, the least squares learning approach is not the only learning model. For example, there has been a growing interest in the genetic algorithm as a learning model. Applications of the genetic algorithm in overlapping generation models can be found in Arifovic (1995,1996) and Bullard and Duffy (1998,1999).

This paper is organized as follows. Section 2 introduces the overlapping generations model we study in this paper and briefly discusses stability properties of the perfect foresight paths. This model is taken from Bullard (1994) (and is also studied by Schönhofer (1996)). In section 3 several forms of expectation formation are discussed. These are static in the sense that they are invariant over time and are not updated as new information becomes available. In section 4 we study the least squares learning algorithm as put forth by Bullard (1994) and give a critical assessment of his results. We also present a learning algorithm on inflation rates, closely related to his, with significantly different stability properties. In section 5 and 6 we present learning algorithm on perceived laws of motions that only involve inflation rates and not price levels. We show, mainly by numerical investigation, that complicated dynamical phenomena might occur, although the learning dynamics converges. Section 7 summarizes.

2 The overlapping generations model

In this section we follow Bullard (1994) and consider a standard two period overlapping generations model where in each period a generation is born that lives for two periods. Of course this is a very simple model and it can only serve as a first approximation to more complicated models with more generations, different agents and more commodities per period. We believe, however, that this simplified model is sufficiently rich to illustrate the main features of learning in more general models. The generation born in period t solves the following problem

$$\max_{c_0,c_1} U(c_0,c_1) \quad \text{subject to} \quad p_t c_0 + p^e_{t+1} c_1 \leq p_t w_0 + p^e_{t+1} w_1,$$

where $U : I\!R^2_+ \to I\!R$ is a strictly monotone, strictly quasi concave utility function, c_0 and c_1 are consumption in the first and second period of the agent's life, w_0 and w_1 are endowments in the first and second period of the agent's life, p_t is the price in period t and p^e_{t+1} is the expected price in period $t+1$. This optimization results in a savings function $S\left(\frac{p^e_{t+1}}{p_t}\right) = w_0 - c_0\left(\frac{p^e_{t+1}}{p_t}\right)$ for the young consumer. We assume that the only means of saving is money and, at least for the price paths we will study, the savings function will always be nonnegative.[2] The demand for money in period t then is

$$\frac{M_t}{p_t} = S\left(\frac{p^e_{t+1}}{p_t}\right).$$

The government has a series of exogenous expenditures, possibly negative, which it finances by seignorage. This results in the following rule for the growth of the money stock

$$M_t = \theta M_{t-1}.$$

Equilibrium on the money market then is given by

$$S\left(\frac{p^e_{t+1}}{p_t}\right) p_t = \theta S\left(\frac{p^e_t}{p_{t-1}}\right) p_{t-1}, \tag{1}$$

or in terms of gross inflation rates $\pi_t \equiv \frac{p_{t+1}}{p_t}$,

$$\pi_{t-1} S\left(\pi^e_t\right) = \theta S\left(\pi^e_{t-1}\right). \tag{2}$$

From this last expression it can be seen that this system can have two different types of steady state equilibria: the monetary steady state $\pi^* = \theta$ and, if it exists, the autarkic steady state $\pi^* = \pi^a$, where π^a is such that $S(\pi^a) = 0$. If endowments in both periods are positive the autarkic steady state exists and if preferences are smooth and strictly convex, the autarkic steady state is unique. We will occasionally make the following assumptions on the savings function $S(.)$

Assumptions

1) *S is twice differentiable and* $S'(.) < 0$,

2) $S(\theta) > 0$.

Assumption 1 states that savings are a decreasing function of the inflation rate. This implies that c_0 and c_1 are gross substitutes. Assumption 2 states that we are in the Samuelson case where in the monetary steady state savings are positive and people transfer income from the present to the future. Assumptions 1 and 2 together imply that $\pi^a > \theta$. For convenience we introduce the following variable

$$a(\pi) = -\pi \frac{S'(\pi)}{S(\pi)}.$$

$a(\pi)$ is the (negative of the) *inflation elasticity of savings* and plays an important role in the dynamical behaviour of the overlapping generations model. Under assumptions 1 and 2 we have $a(\theta) > 0$.

[2] We are therefore focusing on the Samuelson case, where agents save when young and dissave when old.

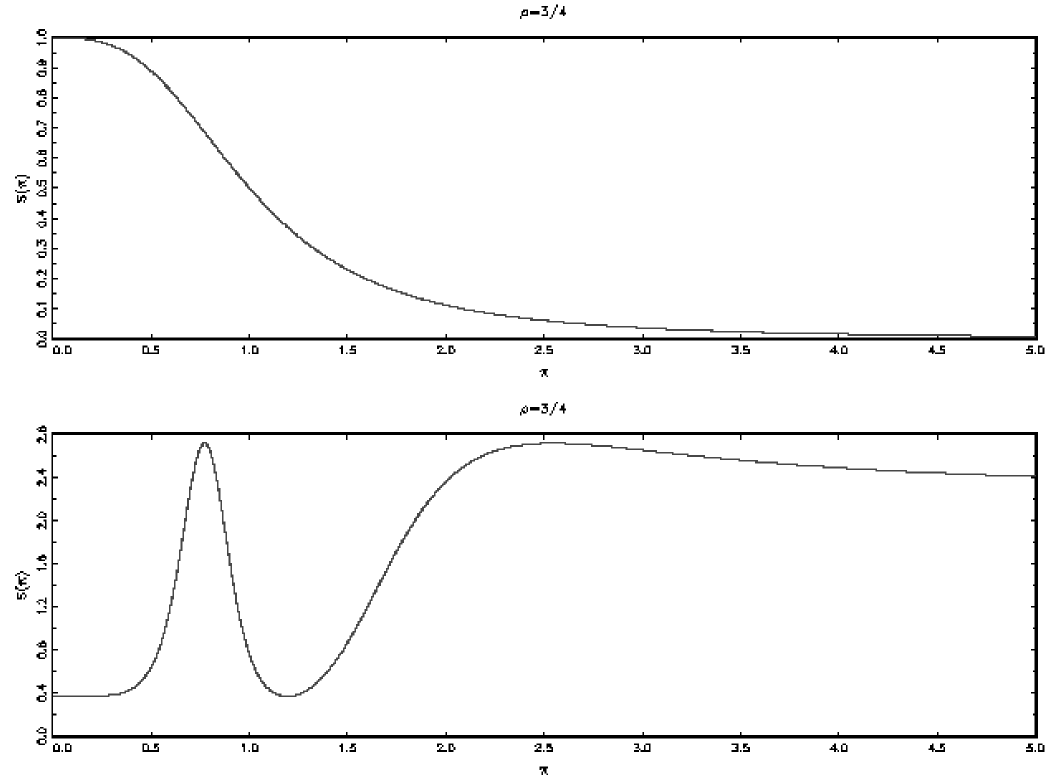

Figure 1: Savings functions. Upper diagram: CES savings function, lower diagram: complicated non-monotonic savings function

In order to do some numerical investigations we have to specify the savings function. Following Bullard (1994) we consider the following two typical examples. The first one derives from the well-known CES utility function $U(c_0, c_1) = (c_0^\rho + c_1^\rho)^{\frac{1}{\rho}}$ with endowments $w_0 = 1$ and $w_1 = 0$ and $\frac{1}{2} < \rho < 1$. The savings function then becomes

$$S(\pi) = \frac{1}{1 + \pi^{\frac{\rho}{1-\rho}}}.$$

Notice that it satisfies assumptions 1 and 2. Furthermore, since $w_1 = 0$ the autarkic steady state does not exist in this case. The inflation elasticity of savings for this savings function is

$$a(\pi) = \frac{\rho}{1-\rho} \frac{1}{1 + \pi^{\frac{\rho}{\rho-1}}}.$$

We will also consider a more complicated example of a savings function to illustrate some complicated phenomena that might occur in our model. The savings function is an aggregate excess demand function and therefore by the Sonnenschein-Debreu-Mantel results (see Debreu (1974), Mantel (1974,1976) and Sonnenschein (1973)) it can take on almost any form. We therefore also consider the following non-monotonic transformation of the CES savings function

$$S(\pi) = \exp\left[-\cos\left(\frac{10}{1+\pi^{\frac{\rho}{\rho-1}}}\right)\right].$$

Figure 1 shows these two savings functions for $\rho = \frac{3}{4}$.

Before we turn to the study of different modes of learning and expectation formation in this overlapping generations model, we briefly consider the case of perfect foresight. Under perfect foresight agents know exactly which inflation rates will obtain in the future. Therefore $\pi_{t+1}^{e} = \pi_{t+1}$. The overlapping generations model (2) then reduces to

$$S(\pi_t) = \theta \frac{S(\pi_{t-1})}{\pi_{t-1}}. \tag{3}$$

Under the assumption that savings decrease as the inflation rate rises, the temporal equilibrium map, implicitly defined by (3) is upward sloping. Therefore no complicated perfect foresight dynamics, such as cycles and chaos can occur (if the savings function is non-monotonic, cycles and even chaotic perfect foresight paths may occur (see Grandmont (1985)). Figure 2 shows this temporal equilibrium map. From this figure the following well known result immediately follows.

Proposition 1 *Under assumptions* 1 *and* 2 *the overlapping generations model with perfect foresight (3) has a locally stable autarkic steady state* π^a *and an unstable monetary steady state* θ*.*

Proof. Linearizing (3) around a steady state π^* gives

$$S'(\pi^*)\, d\pi_t = \theta \frac{\pi^* S'(\pi^*) - S(\pi^*)}{(\pi^*)^2} d\pi_{t-1}.$$

For $\pi^* = \theta$ we have

$$d\pi_t = \left[1 - \frac{S(\theta)}{\theta S'(\theta)}\right] d\pi_{t-1} = \left[1 + \frac{1}{a(\theta)}\right] d\pi_{t-1}.$$

Since $a(\theta) > 0$ the monetary steady state is unstable. For $\pi^* = \pi^a$ we have

$$d\pi_t = \frac{\theta}{\pi^a} d\pi_{t-1},$$

and since $\pi^a > \theta$, the autarkic steady state is locally stable. ■

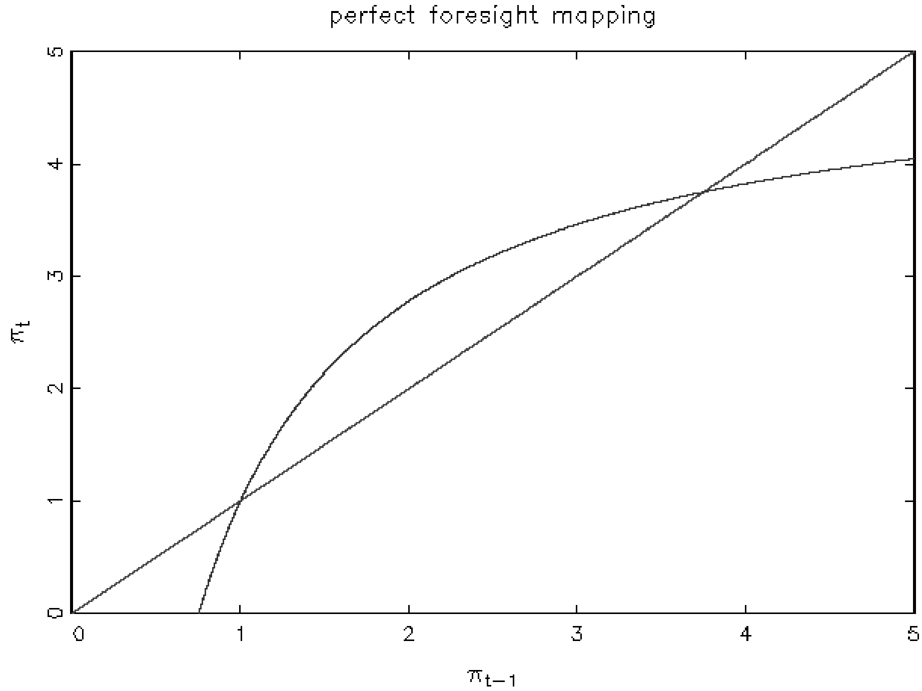

Figure 2: The perfect foresight mapping. The lower equilibrium corresponds to the monetary steady state θ and the higer equilibrium corresponds to the autarkic steady state π^a.

All orbits starting from the right of the monetary steady state converge to the autarkic steady state[3] and all orbits starting from the left of the monetary steady state become infeasible in finite time, that is the inflation rates keep falling until for a certain π_t there is no inflation rate π_{t+1} such that the money market clears.

In the macroeconomic literature the dynamical behaviour of the overlapping generations model under perfect foresight has been perceived as disturbing for the following reason. The perfect foresight equilibrium paths are *indeterminate*, that is, there is an infinity of perfect foresight equilibrium paths, none of which is locally unique (for a discussion see Farmer (1993)). Only one of these perfect foresight paths gives the monetary steady state θ, all other perfect foresight paths converge to a steady state where money has no value. A number of contributions have focused on the multiplicity of perfect foresight paths and argued that by introducing some kind of adaptive expectations or learning the "most likely" perfect foresight path can be singled out. Lucas (1986) shows in a standard version of the overlapping generations model that if agents predict the (inverse of the) price level by using a sample average of previ-

[3] For our example with CES utility functions no autarkic steady state exists and all orbits starting from the right of the monetary steady state then diverge to infinity.

ous observations of the (inverse of the) price level, there will be convergence to the monetary steady state. Marcet and Sargent (1989b) study the dynamical behaviour of an overlapping generations model of hyperinflation similar to ours, when agents use a least squares regression on past prices to predict future prices. They show, for their model, that there can only be convergence to the monetary steady state. They however also point out that the least squares learning algorithm may lead to instability.

It is important to note that all these results refer to the Samuelson case, where income is transferred from the young to the old. All stability results are reversed when we consider the classical case, where young agents consume more than their endowment and therefore "dissave" (that is, saving of the young is negative). Old agents then pay back the debt they incurred when they were young.

3 Static expectations

In the previous section we have discussed the dynamics of the overlapping generations model under perfect foresight. Clearly, the assumption of perfect foresight is a demanding one, since it requires that agents exactly know the market equilibrium equations and are able to use these to compute the market clearing prices for the next periods. From now on we will consider less divine agents, who use observations on past prices or inflation rates to make predictions about future inflation rates. In this section we will consider simple forms of backward looking expectations. First we have to be clear about the information agents posses at the moment they have to make predictions. Agents born in period $t+1$ know all prices up to time t, and therefore the last inflation rate they have observed is $\pi_{t-1} = \frac{p_t}{p_{t-1}}$. The savings decision of these newborn only depends upon the ratio of price levels between periods $t+1$ and $t+2$. The expectation π^e_{t+1} of this price ratio is one of the determinants of the price level p_{t+1} in period $t+1$. We make the assumption that agents have already made their savings decision at the moment they enter the money market and that the prices they observe at this market therefore cannot effect this decision anymore. Hence, the expectation π^e_{t+1} and the savings decision is based upon information on inflation rates up till time $t-1$. An alternative would be to let the expectation formation of agents for π^e_{t+1} also depend upon the value of π_t. Lettau and Van Zandt (1995) show that these differences in information structure lead to significant differences in the dynamical behaviour of the overlapping generations model.[4]

We will first consider *naive* expectations. Under naive expectations agents believe that the inflation rate in the next period will be the same as in this period. Due to the information structure discussed above we then obtain the following expectations

[4] Our approach is more common in the literature. One of the reasons for this is that it would complicate the analysis considerably, if we would assume that the expectation would depend upon current inflation rates. This would imply that we have to solve for an equilibrium in each period and a closed form expression for the inflation dynamics would then not be available.

scheme: $\pi^e_{t+1} = \pi_{t-1}$. The overlapping generations model (2) now becomes

$$\pi_{t+1} = \theta \frac{S(\pi_{t-1})}{S(\pi_t)}. \tag{4}$$

A local stability analysis of the monetary steady state then reveals the following.

Proposition 2 *Assume that $a'(\theta) \neq 0$. Under assumptions* 1 *and* 2 *the overlapping generations model with naive expectations (4) generically undergoes a Hopf bifurcation at the monetary steady state when $a(\theta) = 1$. For $a(\theta) < 1$ the monetary steady state is stable and for $a(\theta) > 1$ the monetary steady state is unstable. If assumptions* 1 *and* 2 *are not satisfied and $a(\theta) < 0$, then generically a period-doubling bifurcation occurs at the monetary steady state for that value of θ for which $a(\theta) = -\frac{1}{2}$. Then if $-\frac{1}{2} < a(\theta) < 1$ the monetary steady state is locally stable and if $a(\theta) < -\frac{1}{2}$ the monetary steady state is unstable.*

Proof. The overlapping generations model can be written as a system of two first order difference equations

$$\begin{aligned} \pi_{t+1} &= \theta \frac{S(\gamma_t)}{S(\pi_t)}, \\ \gamma_{t+1} &= \pi_t. \end{aligned}$$

Linearizing this system around its steady state $(\pi^*, \gamma^*) = (\theta, \theta)$ gives

$$\begin{pmatrix} d\pi_{t+1} \\ d\gamma_{t+1} \end{pmatrix} = \begin{pmatrix} a(\theta) & -a(\theta) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d\pi_t \\ d\gamma_t \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix are

$$\mu_{1,2} = \frac{1}{2} a(\theta) \pm \frac{1}{2} \sqrt{(a(\theta) - 4) a(\theta)}.$$

These eigenvalues are complex for $a(\theta) \in (0, 4)$. The eigenvalues cross the unit circle when

$$\mu_1 \mu_2 = a(\theta) = 1$$

Therefore a Hopf bifurcation occurs when $a(\theta) = 1$. If $a(\theta) < 0$ the eigenvalues are real. It can be easily checked that the positive root is always smaller than $+1$. The negative root equals -1 when $a(\theta) = -\frac{1}{2}$. ■

The Hopf bifurcation referred to in the proposition may be *subcritical* or *supercritical.* For a subcritical Hopf bifurcation a repelling invariant closed curve around the monetary steady state exists for $a(\theta)$ close to but smaller than 1. This closed curve coalesces with the monetary steady state at $a(\theta) = 1$. It implies that, as $a(\theta)$ approaches 1, the basin of attraction of the monetary steady state shrinks and

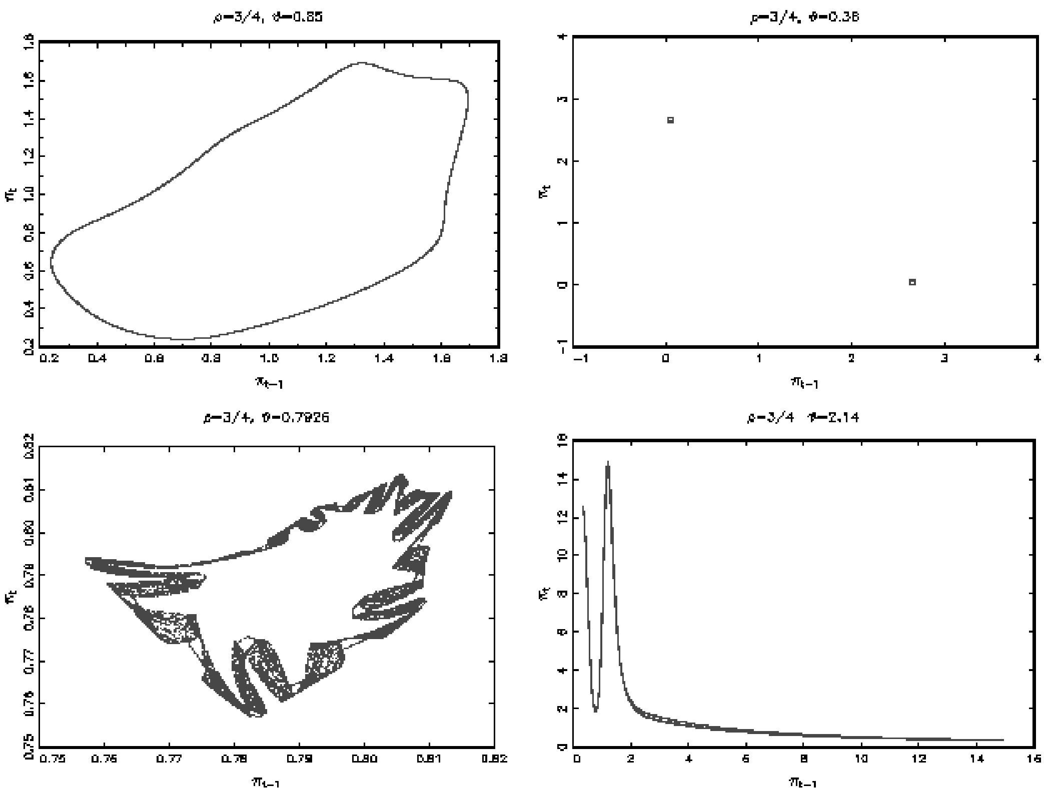

Figure 3: Attractors for the OG model with naive expectations. a) Invariant circle created through Hopf bifurcation for OG with CES savings function, $\rho = \frac{3}{4}$ and $\theta = 0.85$. b) Period two cycle created through flip bifurcation for the complicated savings function, $\rho = \frac{3}{4}$ and $\theta = 0.36$. c) and d) Strange attractors for OG with complicated savings function, $\rho = \frac{3}{4}$ and $\theta = 0.7926$ and $\theta = 2.14$, respectively.

finally disappears. For a supercritical Hopf bifurcation an attracting closed curve exists for $a(\theta)$ close to and larger than 1. On the invariant closed curve created in the Hopf bifurcation the dynamics are quasi periodic. Simulations suggest that for our overlapping generations model with CES utility functions the Hopf bifurcation is supercritical. The bifurcation value of θ can be explicitly calculated as

$$\theta^* = \left(\frac{2\rho - 1}{1-\rho}\right)^{\frac{\rho-1}{\rho}}, \qquad \frac{1}{2} < \rho < 1. \tag{5}$$

Figure 3.a) shows an example of the attracting closed curve created through this supercritical Hopf bifurcation, for the model with CES utility functions and $\rho = \frac{3}{4}$, $\theta = 0.85 > \sqrt[3]{\frac{1}{2}} = \theta^*$.[5] The other pictures in figure 3 are attractors for the overlapping

[5] This corresponds to a situation of deflation where the government also supplies the commodity

generations model with the more complicated savings function. As can be seen from figure 1.b) the slope of this savings function becomes positive at a certain time and therefore $a(\theta)$ becomes negative. For $\rho = \frac{3}{4}$ and $\theta \approx 0.3535$, $a(\theta) = -\frac{1}{2}$ and a period doubling bifurcation occurs resulting in a period two orbit. This attracting period two orbit is shown in figure 3b), for $\theta = 0.36$. For this complicated savings function, also some strange attractors might emerge. As can be seen from figure 1.b) as θ increases $a(\theta)$ increases again and becomes positive. In fact, for $\theta \approx 0.7890$, the inflation elasticity of savings equals 1 and a Hopf bifurcation occurs. This Hopf bifurcation results in an invariant circle. This invariant circle undergoes a bifurcation route to a strange attractor as θ increases even more. This strange attractor is shown for $\theta = 0.7926$ in figure 3.c). It coexists with the period two orbit created in the primary bifurcation. The strange attractor soon becomes unstable however, and then almost all orbits converge to the period two cycle again. As θ increases the monetary steady state becomes stable again and loses stability through a period doubling bifurcation again (compare figure 1.b)). Figure 3.d) shows another strange attractor, for $\theta = 2.14$. This attractor is created through a cascade of period doubling bifurcations. As θ increases even more the period two orbit becomes the unique attractor once again. It disappears through a final period doubling (or period halving) bifurcation after which the monetary steady state is stable again. That this will happen can also be seen from figure 1.b).

Under naive expectations agents believe that the inflation rate in the current period will be the same as in the inflation rate in the previous period. Clearly for the attractors in figure 3 the agents make forecast errors. If there is no structure in these forecast errors and they look like white noise, the agents might attribute them to some stochastic noise in the economy, although in reality they might be generated by a deterministic nonlinear system. For the attractors in figure 3, however, there appears to be some structure in the forecast errors.[6]

Another form of static expectations are given by so-called *adaptive expectations*. These are of the following form

$$\pi_{t+1}^e = \alpha\pi_{t-1} + (1-\alpha)\,\pi_t^e, \qquad 0 < \alpha \leq 1. \tag{6}$$

This can also be written as

$$\pi_{t+1}^e = \pi_t^e + \alpha\left(\pi_{t-1} - \pi_t^e\right),$$

which shows the adaptive character of this type of expectations formation: expectations are updated as new information becomes available. The parameter α gives the

and extracts a certain amount of money from the economy. For other specifications of the CES utility functions (in particular for $\rho < \frac{2}{3}$) periodic behaviour only occurs if $\theta > 1$ and there is positive inflation. The qualitative results remain the same however.

[6]Notice that due to the information structure in the overlapping generations model, it could be argued that there is no forecasting error under naive expectations if the dynamical system converges to a period two orbit. In fact, if agents would expect a period two cycle they would make the same forecasts as when they have naive expectations, but they would always be correct.

importance that the agents attach to new observations. As is well-known we can, by repeated substitution, write the adaptive expectations scheme as

$$\pi_{t+1}^{e} = \sum_{j=1}^{t} \alpha (1-\alpha)^{j-1} \pi_{t-j} + (1-\alpha)^{t} \pi_{1}^{e}.$$

Therefore expectations depend upon all previous realizations albeit with decreasing weights. It is also clear that the influence of the initial expectation, π_1^e, decreases as time goes by. The adaptive expectations scheme is a typical example of a so-called *constant gains* learning algorithm, each new observation is as important as previous observations were (the gain of each new observation is the same as the gain from previous observations). This implies that observations in the past becoming increasingly less important as the above formula shows.

We can again write the overlapping generations model (2) with adaptive expectations (6) as a system of two first order difference equations in the following way

$$\begin{aligned} \pi_{t+1}^{e} &= (1-\alpha)\pi_t^e + \alpha\theta \frac{S(\gamma_t)}{S(\pi_t^e)} \\ \gamma_{t+1} &= \pi_t^e \end{aligned} \tag{7}$$

Linearizing (7) around the monetary steady state gives the following Jacobian matrix

$$\mathbf{J}_{adap} = \begin{pmatrix} (1-\alpha) + \alpha a(\theta) & -\alpha a(\theta) \\ 1 & 0 \end{pmatrix}. \tag{8}$$

The eigenvalues of this matrix are complex if $1+\alpha-2\sqrt{\alpha} < \alpha a(\theta) < 1+\alpha+2\sqrt{\alpha}$. A Hopf bifurcation then generically occurs when the determinant of the Jacobian matrix equals 1, that is, when $\alpha a(\theta) = 1$. In all other case we have real eigenvalues and a period doubling bifurcation occurs when $a(\theta) = \frac{1}{2} - \frac{1}{\alpha}$. The local dynamics around the steady state therefore are qualitatively equivalent with those of (4), except that the longer memory, represented by $\alpha > 0$, stabilizes the dynamics.

The dynamical phenomena that were encountered in the study of the overlapping generations model with naive expectations, such as periodic orbits, aperiodic orbits and strange attractors can also be found in model (7). Since this is not our main concern at this point we will not pursue this issue any further here.

4 A regression on price levels

In the previous section we saw that when agents employ static expectations in order to make their forecasts for future inflation rates may result in a stable equilibrium, periodic behaviour, quasi periodic behaviour and even strange and erratic fluctuations. In this section we will describe and discuss a learning model for the overlapping generations model from the previous sections, which was developed by Bullard (1994).

He assumes that agents believe that prices evolve according to the following perceived law of motion

$$p_t = \beta p_{t-1}. \tag{9}$$

Notice that this implies that the inflation rate is constant and equal to β. However, β is unknown to the agents and it is estimated by a regression of the price level on lagged values of the price level. That is, β is estimated by an ordinary least squares regression, using information up to time $t-1$, which gives

$$\beta_t = \frac{\sum_{s=1}^{t-1} p_{s-1} p_s}{\sum_{s=1}^{t-1} p_{s-1}^2}, \tag{10}$$

as the estimate of agents born in period t. Their forecast of the inflation rate then becomes $\pi_t^e = \beta_t$. Given this, the price dynamics of the overlapping generations model becomes

$$p_t = \theta \frac{S\left(\beta_{t-1}\right)}{S\left(\beta_t\right)} p_{t-1}. \tag{11}$$

Clearly (10) and (11) together form an *expectations feedback system.* Realized prices influence perceptions agents have about economic reality and these perceptions feed back into the actual dynamics and determine which prices will be realized. We can make a distinction between two types of dynamics: the learning dynamics (10) which determine how agents beliefs about the economy are adapted as they receive information about the price levels and the dynamics of price levels or inflation rates (11) that determine how the actual price levels evolve given the beliefs of the agents.

We can write the complete system (10-11) as a recursive dynamical system by introducing the variable $g_t = p_{t-1}^2 \left[\sum_{s=1}^{t} p_{s-1}^2\right]^{-1}$. Using (11) the least squares estimate at time $t+1$ can be written as

$$\begin{aligned}
\beta_{t+1} &= \frac{\sum_{s=1}^{t-1} p_{s-1} p_s + p_{t-1} p_t}{\sum_{s=1}^{t} p_{s-1}^2} = \frac{\beta_t \left(\sum_{s=1}^{t} p_{s-1}^2 - p_{t-1}^2\right) + p_{t-1} p_t}{\sum_{s=1}^{t} p_{s-1}^2} \\
&= \beta_t + \frac{p_{t-1}^2 \left(\frac{p_t}{p_{t-1}} - \beta_t\right)}{\sum_{s=1}^{t} p_{s-1}^2} = \beta_t + g_t \left[\theta \frac{S\left(\beta_{t-1}\right)}{S\left(\beta_t\right)} - \beta_t\right].
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
g_{t+1} &= \frac{p_t^2}{\sum_{s=1}^{t} p_{s-1}^2 + p_t^2} = \frac{p_t^2}{p_{t-1}^2 g_t^{-1} + p_t^2} \\
&= \left[\left(\frac{p_{t-1}}{p_t}\right)^2 g_t^{-1} + 1\right]^{-1} = \left[\left(\theta \frac{S\left(\beta_{t-1}\right)}{S\left(\beta_t\right)}\right)^{-2} g_t^{-1} + 1\right]^{-1}.
\end{aligned}$$

The whole model can then be written as a three-dimensional system of first order difference equations as follows

$$
\begin{aligned}
\beta_{t+1} &= \beta_t + g_t \left[\theta \frac{S(\gamma_t)}{S(\beta_t)} - \beta_t \right], \\
\gamma_{t+1} &= \beta_t, \\
g_{t+1} &= \left[g_t^{-1} \left(\theta \frac{S(\gamma_t)}{S(\beta_t)} \right)^{-2} + 1 \right]^{-1}.
\end{aligned} \tag{12}
$$

Notice that the dynamics of the beliefs (β_t) coincides with the dynamics of the inflation rates, implying that if the least squares estimator β_t converges then so do inflation rates. The system (12) has steady state $(\beta^*, \gamma^*, g^*) = \left(\theta, \theta, 1 - \theta^{-2}\right)$.

Proposition 3 *(**Bullard (1994)**) Assume $\theta > 1$ and assumptions* 1 *and* 2 *are satisfied. Then the monetary steady state generically undergoes a Hopf bifurcation at that value θ^* of θ, for which*

$$\left(1 - \theta^{-2}\right) a(\theta) = 1,$$

for $\theta < \theta^*$, the monetary steady state is stable and for $\theta > \theta^*$ it is unstable.

Proof. The Jacobian matrix of (12) evaluated at the equilibrium $\left(\theta, \theta, 1 - \theta^{-2}\right)$ equals

$$
\begin{pmatrix}
\theta^{-2} + \left(1 - \theta^{-2}\right) a(\theta) & -\left(1 - \theta^{-2}\right) a(\theta) & 0 \\
1 & 0 & 0 \\
\frac{\partial g}{\partial \beta} & \frac{\partial g}{\partial \gamma} & \theta^{-2}
\end{pmatrix}. \tag{13}
$$

One of the eigenvalues is equal to θ^{-2} and hence stable for $\theta > 1$. The eigenvalues of the upper 2×2 matrix are complex and lie on the unit circle when $\left(1 - \theta^{-2}\right) a(\theta) = 1$. ■

For our CES example with $\rho = \frac{3}{4}$ we have $\theta^* = \frac{1}{2} + \frac{1}{2}\sqrt{3} \approx 1.3660$. Two attractors for the CES savings function are shown in the upper pictures in figure 4. The first picture shows the attracting invariant closed curve created in the Hopf bifurcation, the second picture shows a strange attractor that is created from this invariant closed curve. The lower pictures in figure 4 show strange attractors that might emerge from system (12) when we use the complicated savings function. For this complicated savings function (or in fact for any savings function with $a(\theta) < 0$) it can be shown that (12) undergoes a period doubling bifurcation at the monetary steady state at that value of θ for which $\frac{\theta^2 - 1}{\theta^2 + 1} a(\theta) = -\frac{1}{2}$.

Previous work on least squares learning in economic models mainly deals with establishing convergence of the beliefs to some finite limit. We now want to develop an intuition why this does not happen in the current model. Ordinary least squares

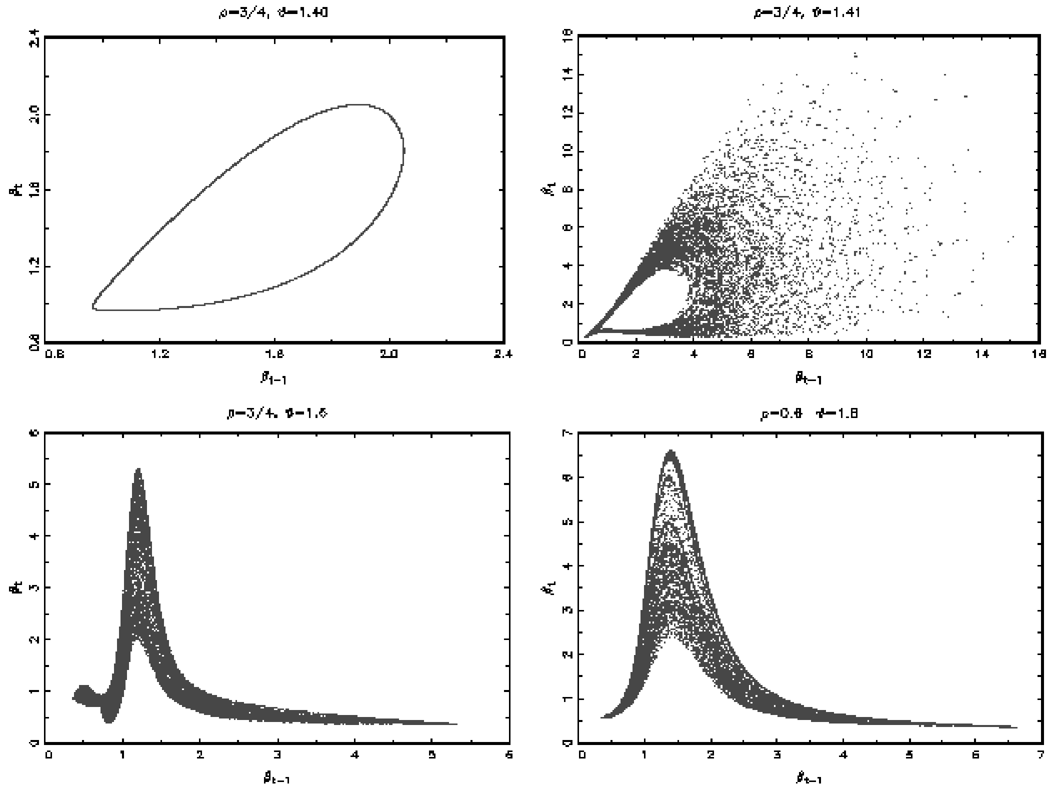

Figure 4: Attractors for the OG model with least squares learning on price levels. a) CES savings function, $\rho = \frac{3}{4}$ and $\theta = 1.40$. b) CES savings function, $\rho = \frac{3}{4}$ and $\theta = 1.41$. c) Complicated savings function, $\rho = \frac{3}{4}$ and $\theta = 1.5$. d) Complicated savings function, $\rho = \frac{3}{5}$ and $\theta = 1.8$.

algorithms are so-called *decreasing gains* algorithms. Different observations receive the same weights in the regression which implies that, as time goes by, the impact of new observations (that is the gain of new observations) becomes smaller. In (12) this gain is represented by the variable $g_t = p_{t-1}^2 / \sum_{s=1}^{t} p_{s-1}^2$. Now if price levels are bounded then, maybe except for some very special time paths, g_t will converge to 0. This would result in convergence of the inflation rate. In the model at hand, however, price levels are not bounded and in fact, in equilibrium they grow at a constant rate $\theta > 1$. This implies that the weight g_t does not converge to 0 but to a positive number $1 - \frac{1}{\theta^2}$, provided the monetary steady state is stable. Therefore, even for very large t new observations may lead to a significant change in the beliefs of the agents. In fact, if we put in the equilibrium value for g_t, (12) reduces to

$$\begin{aligned} \beta_{t+1} &= \frac{1}{\theta^2}\beta_t + \left(1 - \frac{1}{\theta^2}\right)\theta \frac{S(\gamma_t)}{S(\beta_t)}, \\ \gamma_{t+1} &= \beta_t, \end{aligned} \tag{14}$$

which is just the overlapping generations model under adaptive expectations, where the weight α equals $1-\frac{1}{\theta^2}$. The stability condition for (12) is exactly the same as the stability condition for the adaptive case with $\alpha = 1-\frac{1}{\theta^2}$. This follows from the fact that the upper 2×2 matrix of (13) (which is the relevant part) is equal to the Jacobian of the overlapping generations model with adaptive expectations (8). Therefore the local stability properties of (12) are equivalent with those of (7). This explains why it can be possible that the learning and inflation dynamics do not converge.[7] Notice that the belief parameters in this case keep fluctuating.

Due to the nonstationarity of the time series of the price level an alternative approach, and a more sensible one from an econometrician's viewpoint, would be to forecast inflation rates by using the average of previous inflation rates. The agents in the previous model believe that the inflation rate is constant, and they estimate this constant by the ordinary least squares estimate of a regression of the price level on itself. The following question now arises: why don't agents estimate the inflation rate by computing the average of past inflation rates? This estimate would then be

$$\beta_t = \frac{1}{t-1}\sum_{s=0}^{t-2}\pi_s,$$

which is also the least squares estimate one obtains when regressing the inflation rate on a constant. β_t can be written recursively as

$$\beta_t = \frac{1}{t-1}\left[\sum_{s=0}^{t-3}\pi_s + \pi_{t-2}\right] = \frac{t-2}{t-1}\beta_{t-1} + \frac{1}{t-1}\pi_{t-2}.$$

The dynamical system becomes

$$\beta_{t+1} = \frac{t-1}{t}\beta_t + \frac{1}{t}\theta\frac{S\left(\beta_{t-1}\right)}{S\left(\beta_t\right)}. \tag{15}$$

Now compare this with system (14). It is immediately clear that the stability properties of this system are different. Taking averages corresponds to a decreasing gains algorithm as can be seen by rewriting (15) as

$$\beta_{t+1} = \beta_t + \frac{1}{t}\left(\theta\frac{S\left(\beta_{t-1}\right)}{S\left(\beta_t\right)} - \beta_t\right) = \beta_t + \frac{1}{t}\left(\pi_{t-1} - \beta_t\right).$$

It is clear from this formulation that, as time goes to infinity, the contribution of new observations will decrease to 0. Therefore it seems reasonable to assume that

[7] The above results only hold if there is a positive amount of money creation, $\theta > 1$.If $\theta < 1$, the learning dynamics breaks down. The equilibrium value for g then becomes negative, whereas g is a ratio of strictly positive quantities, so clearly this equilibrium cannot be stable. Simulations show that in this case g approaches 0 very fast and this implies that beliefs are updated at a very low level and will not reach their equilibrium level. Inflation rates, on the other hand, do converge.

inflation rates and beliefs will converge to the monetary steady state. Simulations support this conjecture, though convergence might be rather slow (which is also due to the decreasing gains character of (15)). It is however well known that time averages need not converge. Situations might arise where beliefs and inflation rates keep on changing, albeit at an increasingly slower rate. This might also happen in this case. For example, when $S(\pi)$ gets arbitrarily close to 0 for some values of π, there might be no convergence at all, although it looks like it due to the fact that the magnitude of the adjustment becomes negligible. In the next section we will briefly study an example with the same property. Other examples of nonconvergence of time averages can be found in Shapley (1964) and Gaunersdorfer and Hofbauer (1995).

To summarize, it seems that the way in which the perceived law of motion is put is an important determinant of the stability of the inflation dynamics. In order to make forecasts about future inflation rates it seems to be wiser to consider old inflation rates than to consider the nonstationary series of old prices.

5 A regression on inflation rates – part I

Above we saw that regressing prices on past prices might induce fluctuating inflation rates and fluctuating beliefs if the money growth rate is high enough. An intuition was provided for this result and it was argued that the same belief could also be learned in a different way that would almost always lead to convergence of beliefs. In this section we consider a different kind of learning dynamics. We let agents regress inflation rates on past inflation rates, that is, agents belief that inflation rates develop according to the following relationship

$$\pi_t = \beta \pi_{t-1}. \tag{16}$$

Given the estimate of β_{t+1} of the generation born in period $t+1$, their forecast of the inflation rate becomes $\pi^e_{t+1} = \beta^2_{t+1}\pi_{t-1}$. The estimate β_{t+1} is obtained by a least squares regression on inflation rates, which is

$$\beta_{t+1} = \frac{\sum_{s=1}^{t-1} \pi_{s-1}\pi_s}{\sum_{s=1}^{t-1} \pi^2_{s-1}}.$$

Again it is important to be clear about the timing of the process. Agents born in the beginning of period $t+1$ do not know p_{t+1} yet, so the last inflation rate they have observed is $\pi_{t-1} = \frac{p_t}{p_{t-1}}$. They therefore make their estimates on the basis of all inflation rates up to π_{t-1}.

The estimate β_{t+1} can be written recursively again. First define $R_{t+1} = \left[\sum_{s=1}^{t} \pi^2_{s-1}\right]^{-1}$. We then have

$$R_{t+1} = \left[\sum_{s=1}^{t-1} \pi^2_{s-1} + \pi^2_{t-1}\right]^{-1} = \left[R_t^{-1} + \pi^2_{t-1}\right]^{-1} \tag{17}$$

using this we can write

$$\begin{aligned}
\beta_{t+1} &= R_t \sum_{s=1}^{t-1} \pi_{s-1}\pi_s = R_t \left[\sum_{s=1}^{t-2} \pi_{s-1}\pi_s + \pi_{t-2}\pi_{t-1}\right] \\
&= R_t \left[\beta_t R_{t-1}^{-1} + \pi_{t-2}\pi_{t-1}\right] = R_t \left[\beta_t \left(R_t^{-1} - \pi_{t-2}^2\right) + \pi_{t-2}\pi_{t-1}\right] \\
&= \beta_t + R_t \pi_{t-2} \left[\pi_{t-1} - \beta_t \pi_{t-2}\right]
\end{aligned}$$

Therefore the learning dynamics can be written recursively as

$$\begin{aligned}
\beta_{t+1} &= \beta_t + R_t \pi_{t-2} \left[\pi_{t-1} - \beta_t \pi_{t-2}\right] \\
R_{t+1} &= \left[R_t^{-1} + \pi_{t-1}^2\right]^{-1}
\end{aligned} \tag{18}$$

Giving these learning dynamics the actual inflation rates evolve according to

$$\pi_t = \theta \frac{S\left(\beta_t^2 \pi_{t-2}\right)}{S\left(\beta_{t+1}^2 \pi_{t-1}\right)}. \tag{19}$$

Given the values of β_t, R_t, π_{t-1} and π_{t-2} new values β_{t+1}, R_{t+1} and π_t are determined according to the following four dimensional system of first order difference equations.

$$\begin{aligned}
\pi_t &= \theta \frac{S\left(\beta_t^2 \gamma_{t-1}\right)}{S\left(\left(\beta_t + R_t \gamma_{t-1}\left[\pi_{t-1} - \beta_t \gamma_{t-1}\right]\right)^2 \pi_{t-1}\right)} \\
\gamma_t &= \pi_{t-1} \\
\beta_{t+1} &= \beta_t + R_t \gamma_{t-1}\left[\pi_{t-1} - \beta_t \gamma_{t-1}\right] \\
R_{t+1} &= \left[R_t^{-1} + \pi_{t-1}^2\right]^{-1}
\end{aligned} \tag{20}$$

Notice that the difference of timing of the learning variables β and R and the inflation variables π and γ stems from the fact that expectations for people born in period $t+1$ depend upon information about inflation rates up to time $t-1$. Of course we could formulate the model in terms of the learning variables $\beta_t^* \equiv \beta_{t+1}$ and $R_t^* \equiv R_{t+1}$, but this would obscure the interpretation of β_{t+1} as the belief of the generation born in period $t+1$.

In contrast to the learning model studied in the previous section in the present model convergence of beliefs does not imply convergence of the inflation rate. It could be the case that learning parameters converge but that the corresponding inflation dynamics (19) is unstable and generates erratic inflation dynamics. Consider the possible equilibria of (20). Clearly, at an equilibrium we must have $\pi = \gamma = \theta$ and $\beta = 1$. At these values of π, γ and β the value of R does not matter, but from (17) it follows that R_t must then approach 0. Notice however that at $R = 0$ the system

(20) is not defined. To get some insights into the dynamics of (20) we can look at the Jacobian evaluated at $(\pi,\gamma,\beta,R)=(\theta,\theta,1,R)$. This Jacobian is

$$J_R=\begin{pmatrix} \left[1+2R\theta^2\right]a(\theta) & -\left[1+2R\theta^2\right]a(\theta) & \theta\left[1-\theta R\right]a(\theta) & 0 \\ 1 & 0 & 0 & 0 \\ R\theta & -R\theta & 1-\theta^2 R & 0 \\ -2\theta\left[R^{-1}+\theta^2\right]^{-2} & 0 & 0 & \left[1+R\theta^2\right]^{-2} \end{pmatrix}$$

First of all notice that this matrix has an eigenvalue equal to $\left[1+R\theta^2\right]^{-2}$ which, for all positive R, lies strictly between 0 and 1. Now taking the limit of this matrix as R approaches 0 gives

$$J_0=\lim_{R\to 0}J_R=\begin{pmatrix} a(\theta) & -a(\theta) & \theta a(\theta) & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

The Jacobian matrix J_0 has two eigenvalues equal to $+1$. These eigenvalues correspond to the learning dynamics and it is easy to see why they equal $+1$. At a point with $\pi=\gamma=\theta$ and $\beta=1$, we saw that there is no value of R disrupting this equilibrium. Furthermore, if the weight R would be equal to 0, and abstracting from the nonexistence problem of R_{t+1}, there would be no updating of β, and therefore any β would be neutrally stable. It is important to observe that both eigenvalues are strictly between 0 and 1 for R small but positive.[8]

The other two eigenvalues of (21) correspond to the inflation dynamics of (20), and they are

$$\mu_{1,2}=\frac{1}{2}a(\theta)\pm\frac{1}{2}\sqrt{\left(a(\theta)^2-4a(\theta)\right)}.$$

Notice that these are exactly the same eigenvalues as for the case of naive expectations (they correspond to the upper 2×2 matrix of (21)). This is intuitively clear since, if the learning dynamics converges to $\beta=1$, agents in fact have naive expectations. Therefore, the local stability conditions of the more sophisticated learning process (20) are the same as that of the simplest learning process (4) we studied sofar, namely $-\frac{1}{2}<a(\theta)<1$.

[8]It might be instructive to consider a model where the weight is updated as follows

$$R_{t+1}=\left[(R_t+e)^{-1}+\pi_t^2\right]^{-1},$$

where $e>0$ is small. In this case the equilibrium weight would be $R_e=\frac{1}{2}\sqrt{e^2+4\frac{e}{\theta^2}}-\frac{1}{2}e>0$ and no problems would arise at the equilibrium $(\theta,\theta,1,R_e)$. We then get results similar as the ones in the paper. Of course, in this case the estimate β_t will not coincide with the least squares estimates of β.

We are interested in the dynamical behaviour of (20), in particular when the monetary steady state under naive expectations is unstable. We can make a clear distinction between learning dynamics and inflation dynamics here. Due to the decreasing gains character of the least squares learning algorithm the estimate β will change very slowly as time goes by. There can however be large changes in inflation rates from period to period. Therefore it might be worthwhile to study the *short-run dynamics* of the inflation rates for a given, fixed value of β, e.g. β^0. The short-run dynamics then are given by

$$\pi_{t+1} = \theta \frac{S\left(\left(\beta^0\right)^2 \pi_{t-1}\right)}{S\left(\left(\beta^0\right)^2 \pi_t\right)}. \tag{22}$$

From section 3 we know that these short-run dynamics can exhibit all kinds of dynamical phenomena. In particular, the inflation rates might converge to the monetary steady state, to a periodic orbit, to an invariant closed curve or even to a strange attractor. Now suppose that the short-run dynamics (22) indeed converge to some attractor $S_{\beta^0} \subset I\!\!R^2_+$. What would then happen with the long run dynamics (20)? Let $\{\pi_t\}_{t=1}^{\infty}$ be an orbit generated by (22) and lying on S_{β^0}. Then a least squares regression can be performed on these inflation rates and this will reveal an estimate for β, e.g. β^1. In general $\beta^1 \neq \beta^0$. Therefore we expect the learning dynamics to move from β^0 to β^1. But, given β^1, the short run dynamics will in general not converge to the attractor S_{β^0}. Therefore the estimate of β will change again. In this way we can in fact construct an implicit map of beliefs

$$\beta^{k+1} = T\left(\beta^k\right),$$

where the mapping $T(.)$ gives the ordinary least squares estimate β for the relationship $\pi_t = \beta\pi_{t-1}$, given that the inflation rates are generated by the short-run dynamics (22) with fixed β^k. This artificial beliefs map can help us understand the dynamics of the long run dynamics (20). Our main concern are the equilibria of $T(.)$, since at such an equilibrium β^* agents believe that $\pi_t = \beta^*\pi_{t-1}$ and running a regression on inflation rates indeed does not give them reason to change their beliefs. So, although agents beliefs are mis-specified, new observations on inflation rates are consistent with these beliefs in the sense that these new observations do not modify these beliefs. This equilibrium belief is the best one in the class of beliefs that is considered by the agents. This leads us to give, loosely, the following general definition of a *beliefs-equilibrium.*

Definition 4 *Let G be a class of perceived laws of motion. An element g of G has the following general form*

$$\pi^e_{t+1} = g\left(\pi_{t-1}, \pi_{t-2}, \ldots, \pi_{t-l}\right) = g\left(\overrightarrow{\pi}_{t-1}\right),$$

where the functional specification of g and the number of lags l is prescribed by the class G. Consider the dynamical system

$$\pi_t = \theta \frac{S\left(g^2\left(\overrightarrow{\pi}_{t-2}\right)\right)}{S\left(g^2\left(\overrightarrow{\pi}_{t-1}\right)\right)}, \tag{23}$$

where $g \in G$. Then we call $g^* \in G$ a Beliefs-Equilibrium *(BE) if it is the perceived law of motion in G that fits the time series of inflation rates generated by (23) with g^* better*[9] *than all other elements of G.*

In the present section the class G corresponds to all perceived laws of motion of the form $\pi_t = \beta\pi_{t-1}$.[10] One example of a beliefs-equilibrium is the situation, discussed above, where the overlapping generations model with naive expectations has a stable monetary steady state. In that case $\beta^* = 1$ and indeed one has (for t high enough) $\pi_t = \beta^*\pi_{t-1} = \pi_{t-1}$. More interesting is a BE where inflation rates do not converge to the monetary steady state. An example is shown in figure 5.a). Here we have considered the overlapping generations model with learning on inflation rates (20) with a CES savings function with $\rho = \frac{3}{4}$ and $\theta = 1$. From section 3 we know that for these values the monetary steady state is unstable under naive expectations. For our model (20) the beliefs then converge to $\beta^* \approx 0.97346$ and the inflation dynamics converge to an attracting invariant closed curve. It can easily be checked numerically that here β^* is indeed an equilibrium of $T(.)$. Figure 5.a) shows the attractor of the inflation rates (the closed curve) and the beliefs of the agents (the straight line through the origin with slope β^*). One might argue that, when the agents observe this picture they might realize that their belief is mis-specified and try to learn something different. Figure 5.b) shows the results for the same model, only with a small noise on the parameter of the money growth. In fact, we assume $\theta_t = \theta + \varepsilon_t$ where ε_t are IID disturbances that have a normal distribution with mean 0 and standard deviation 0.02. If agents see this diagram it might indeed be sensible to run a regression on past inflation rates.

Figure 6 shows the time series for π_t and β_t, corresponding to figure 5.a). It is clear that the latter converges and the former keeps on fluctuating (notice that the time scales are different for these two time series).

There can also be nonexistence of an equilibrium of these learning dynamics. Examples of this nonexistence can be found for the overlapping generations model with the complicated savings function. The corresponding pictures are the lower ones in figure 5. Here no equilibrium β^* of the map $T(.)$ exists. To explain what happens consider the time series for the case with $\rho = \frac{3}{4}$ and $\theta = 1$, depicted in figure 7.

[9] "Better" clearly is a rather subjective notion. In this chapter, where we only consider linear perceived laws of motion, one belief is "better" than another belief if its unweighted sum of squared forecast errors is smaller. This, of course, is consistent with using the (recursive) least squares algorithm as a learning model.

[10] We already encountered another general class G where we considered perceived laws of motion of the type $\pi^e_{t+1} = \beta$. Clearly for this class the unique BE is given by $\pi^e_{t+1} = \theta$.

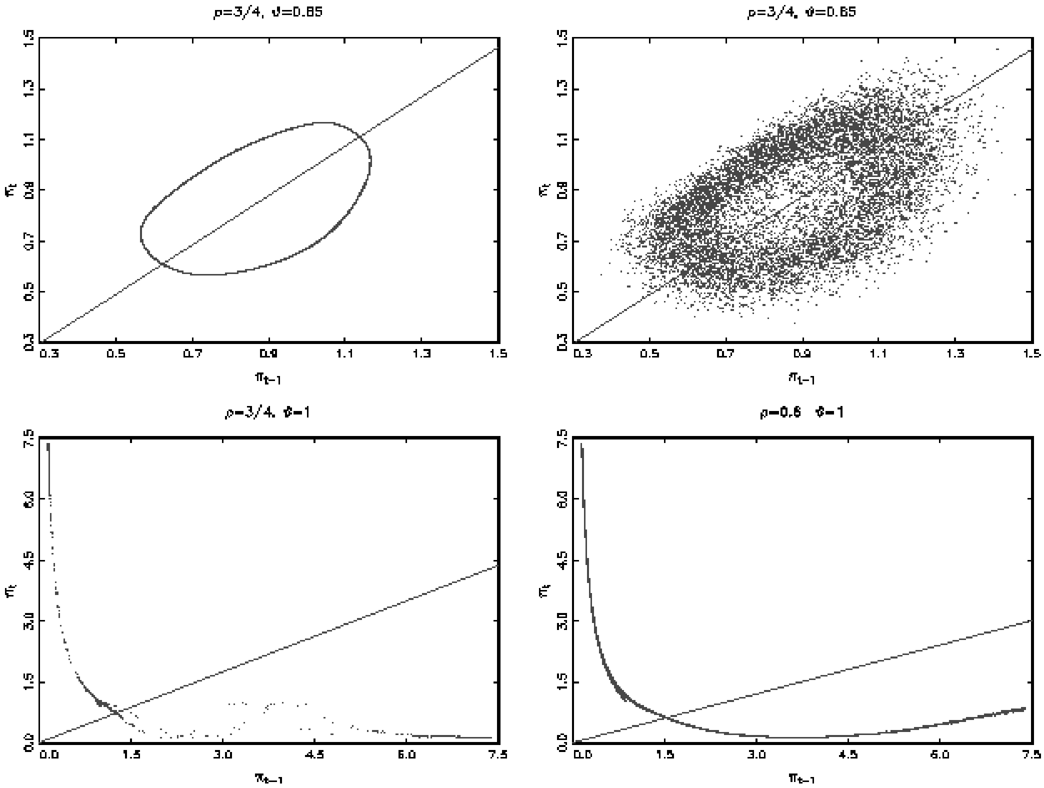

Figure 5: Attractors of (20) and examples of beliefs-equilibra. a) BE for CES savings function with $\rho = \frac{3}{4}$ and $\theta = 0.85$. b) BE from a) with small noise. c) and d) Nonexistence of BE for complicated savings function with $\rho = \frac{3}{4}$ and $\rho = \frac{3}{5}$, respectively and $\theta = 1$.

Suppose that initial beliefs are such that β is rather small and the corresponding short-run dynamics (22) are stable. In that case inflation rates converge to $\theta = 1$ and the estimate of β will increase. At a certain time this β will be so high that the short-run dynamics become unstable and a period two orbit emerges. This happens at $\beta \approx 0.5930$. Then β decreases again[11] and this stabilizes the short-run dynamics (22). This repeats over and over. Notice that the inflation rate is stable for a long time which implies that agents will start believing more and more that it will be the

[11] In fact, we can explicitly compute the value of β corresponding to a period two orbit (π_1, π_2). Since we know that for a period two orbit of the overlapping generations model one must have $\pi_1 \pi_2 = \theta^2$, the least squares regression estimate β converges to

$$\beta = \frac{\sum_t \pi_t \pi_{t-1}}{\sum_t \pi_{t-1}^2} = \frac{2\theta^2 \pi_1^2}{\theta^4 + \pi_1^4},$$

which is always smaller than 1 for $\pi_1 \neq \theta$.

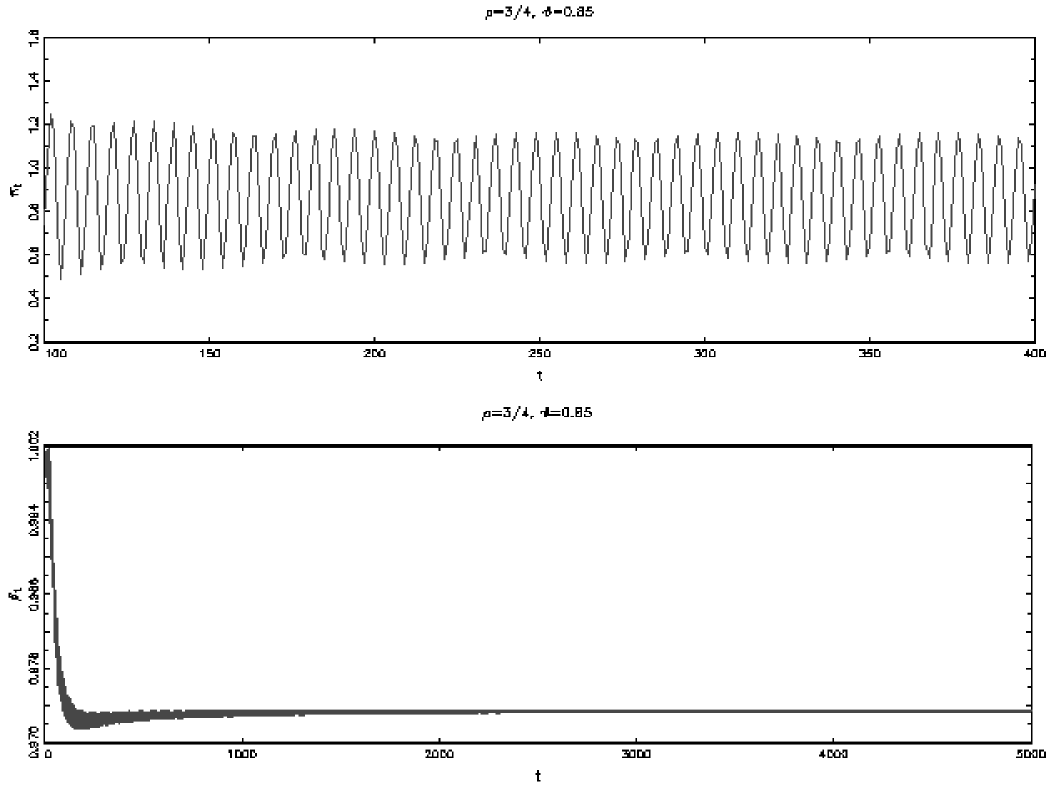

Figure 6: Time series of π_t and β_t for (20) with CES savings function and $\rho = \frac{3}{4}$ and $\theta = 0.85$.

same as in the previous period (which seems to be a plausible expectation) and this leads the inflation to become unstable. As discussed in the previous section we here have a situation where the least squares algorithm does not converge, though it moves slower and slower as time goes by.

6 A regression on inflation rates – part II

In the previous section we saw that regressing inflation rates on past inflation rates may lead to a situation where the belief parameters, which are "slow" variables, converge, whereas inflation rates keep on fluctuating. It can be argued that the class of perceived laws of motion we considered there is too simple. Unless $\beta = 1$, agents believe that the inflation rate will converge to zero or diverge to infinity, which clearly is not the case. In particular, sample averages are inconsistent with the perceived law of motion. In this section we want to consider a more general class of perceived laws of motion, namely

$$\pi_t = \alpha + \beta \left(\pi_{t-1} - \alpha \right). \tag{24}$$

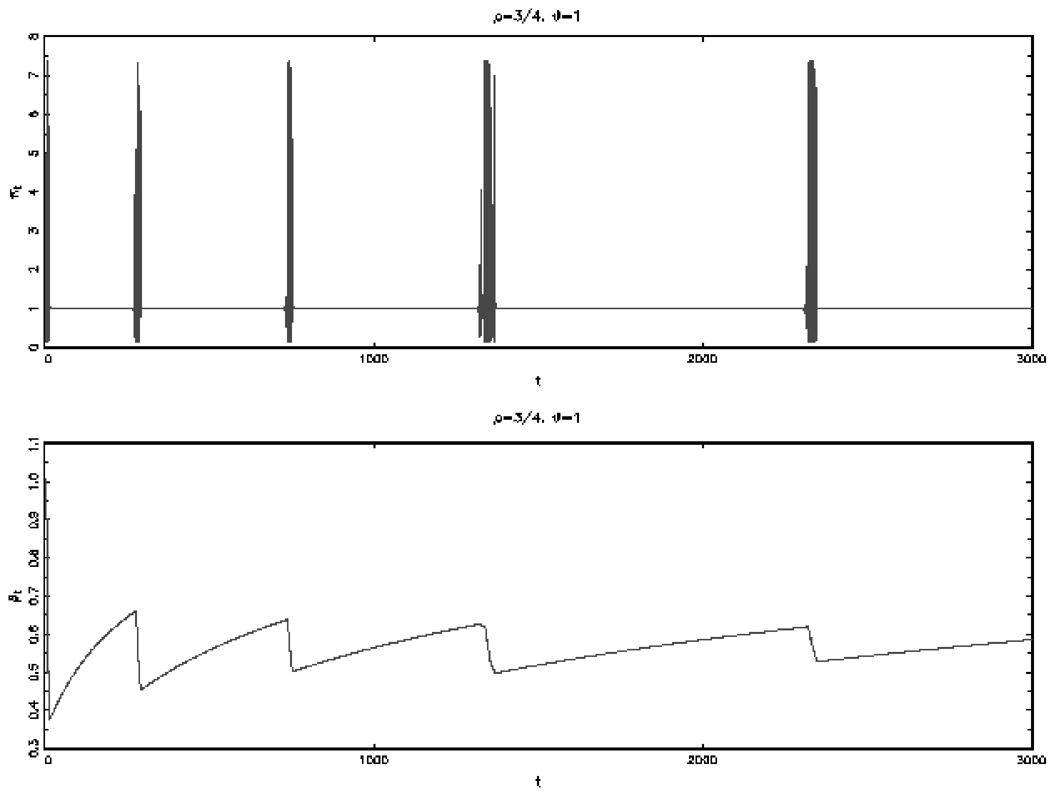

Figure 7: Time series of π_t and β_t for (20) with complicated savings function and $\rho = \frac{3}{4}$ and $\theta = 1$.

Notice that $\alpha = 0$ corresponds to the perceived law of motion from the previous section and $\beta = 1$ corresponds to naive expectations. For general α and β, an agent beliefs that the long run equilibrium equals α and he beliefs the inflation rate will converge to this long run equilibrium at a rate β (provided that $|\beta| < 1$). If $\beta < 0$ he expects some overshooting, if $\beta > 0$ he expects monotonic convergence. The two periods ahead prediction with (24) becomes

$$\pi^e_{t+1} = \alpha + \beta^2 (\pi_{t-1} - \alpha)$$

and given these expectations the overlapping generations model (2) becomes

$$\pi_{t+1} = \theta \frac{S\left(\alpha + \beta^2 (\pi_{t-1} - \alpha)\right)}{S\left(\alpha + \beta^2 (\pi_t - \alpha)\right)}. \tag{25}$$

In the previous section we saw that if expectations would be correct in equilibrium, which seems to be a minimal consistency condition to be imposed on the expectations scheme, one needs $\beta = 1$. For expectations schemes of the form (24) to be consistent in equilibrium we need

$$\pi_t^e = \alpha + \beta(\theta - \alpha) = \theta.$$

In this case the equilibrium belief parameters α and β are unidentified. That is, there are a lot of values of α and β that satisfy the above consistency condition. In particular, all pairs with $\alpha = \theta$ and all pairs with $\beta = 1$ satisfy it. This multiplicity of beliefs equilibria becomes important when we introduce learning. In particular, we know that the inflation dynamics depend upon the parameters α and β so there are always consistent expectations schemes for which the monetary steady state is locally stable.[12].

The expectations scheme (24) also admits the possibility of perfect prediction when the inflation rates move in a period two cycle. Let (π_1, π_2) be such a two-cycle of the dynamics. Then correct prediction requires α and β to satisfy

$$\pi_2 = \alpha + \beta(\pi_1 - \alpha) \text{ and } \pi_1 = \alpha + \beta(\pi_2 - \alpha),$$

or equivalently, the two periods ahead prediction, given that at this time the inflation rate is π_1, must be

$$\pi_1 = \alpha + \beta^2(\pi_1 - \alpha)$$

given that $\pi_1 \neq \pi_2 \neq \alpha$ this requires that $\beta^* = -1$. α then has to equal $\frac{1}{2}(\pi_1 + \pi_2)$. Of course this situation only occurs when the inflation dynamics given these values of α^* and β^* indeed has a period two-cycle. That is

$$\pi_{t+1} = \theta \frac{S\left(\alpha^* + (\beta^*)^2(\pi_{t-1} - \alpha^*)\right)}{S\left(\alpha^* + (\beta^*)^2(\pi_t - \alpha^*)\right)} = \theta \frac{S(\pi_{t-1})}{S(\pi_t)}$$

must have a period two cycle.

Thus expectations can be correct along the steady state and along a period two cycle. These expectations correspond to the so-called *consistent expectations equilibria* (CEE) introduced by Hommes and Sorger (1998). In their framework an expectations scheme is a CEE if agents make no structural forecast errors in a linear statistical sense. Besides steady state and two-cycle CEE they also have chaotic CEE, where the dynamics of the state variable moves over a chaotic attractor but there is no

[12] It can easily be calculated that, for a monotone decreasing savings function, a Hopf bifurcation occurs at that value of θ^* for which we have

$$\frac{\beta^2\theta^*}{\alpha + \beta^2(\theta^* - \alpha)} a\left(\alpha + \beta^2(\theta^* - \alpha)\right) = 1,$$

which at an equilibrium where agent have consistent expectations becomes

$$\beta^2 a(\theta^*) = 1.$$

Clearly, there are then always values of β for which the inflation dynamics are stable.

structure in the forecast errors. These CEE are related to the beliefs-equilibria we defined in the previous section. We now turn to the analysis of the learning model where agents have expectations according to (24) and the belief parameters α and β are learned trough a least squares regression on inflation rates. We assume agents estimate the following perceived law of motion

$$\pi_t = \beta_0 + \beta_1 \pi_{t-1},$$

which is equivalent to the perceived law of motion (24) up to the transformation $\beta_0 = \alpha(1-\beta)$ and $\beta_1 = \beta$. Define

$$X_{t+1} = \begin{pmatrix} 1 & \pi_0 \\ 1 & \pi_1 \\ \vdots & \vdots \\ 1 & \pi_{t-2} \end{pmatrix}, \; Y_{t+1} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{t-1} \end{pmatrix} \text{ and } x_{t+1} = \begin{pmatrix} 1 & \pi_{t-1} \end{pmatrix}.$$

X_{t+1} is the set of regressors for the generation that is born in period $t+1$. Recall, that the last observation on inflation rates of these agents is π_{t-1}. The estimator b for $(\beta_0, \beta_1)'$ at time $t+1$ is

$$b_{t+1} = \left(X'_{t+1} X_{t+1}\right)^{-1} X'_{t+1} Y_{t+1} \equiv R_t X'_{t+1} Y_{t+1}$$

As before we can write this estimator recursively,[13] and the learning dynamics then are given by

$$\begin{aligned} b_{t+1} &= b_t + R_t x'_{t+1} \left[\pi_{t-1} - x_{t+1} b_t\right], \\ R_{t+1} &= \left(R_t^{-1} + x'_{t+2} x_{t+2}\right)^{-1}, \end{aligned}$$

and the inflation dynamics are given by

$$\pi_t = \theta \frac{S\left(\alpha_t + \beta_t^2 \left(\pi_{t-2} - \alpha_t\right)\right)}{S\left(\alpha_{t+1} + \beta_{t+1}^2 \left(\pi_{t-1} - \alpha_{t+1}\right)\right)}.$$

In terms of β_0 and β_1 the whole system becomes

$$\begin{aligned} b_{t+1} &= b_t + R_t x'_{t+1} \left[\pi_{t-1} - x_{t+1} b_t\right] \\ R_{t+1} &= \left(R_t^{-1} + x'_{t+2} x_{t+2}\right)^{-1} \\ \pi_t &= \theta \frac{S\left(\beta_{0t}\left(1+\beta_{1t}\right) + \beta_{1t}^2 \pi_{t-2}\right)}{S\left(\beta_{0,t+1}\left(1+\beta_{1,t+1}\right) + \beta_{1,t+1}^2 \pi_{t-1}\right)} \end{aligned} \tag{26}$$

[13] We have $R_t = \left(X'_t X_t + x'_{t+1} x_{t+1}\right)^{-1} = \left(R_{t-1}^{-1} + x'_{t+1} x_{t+1}\right)^{-1}$ and

$$\begin{aligned} b_{t+1} &= R_t X'_{t+1} Y_{t+1} = R_t \left[X'_t Y_t + x'_{t+1} \pi_{t-1}\right] = R_t \left[R_{t-1}^{-1} b_t + x'_{t+1} \pi_{t-1}\right] \\ &= R_t \left[\left(R_t^{-1} - x'_{t+1} x_{t+1}\right) b_t + x'_{t+1} \pi_{t-1}\right] = b_t + R_t x'_{t+1} \left[\pi_{t-1} - x_{t+1} b_t\right]. \end{aligned}$$

Notice that these dynamics constitute an 8-dimensional system, since b_t is a 2×1 vector, R_t is a 2×2 matrix and the inflation rate in time t depends upon π_{t-1} and π_{t-2}.

In the model of the previous section we saw that there was only one equilibrium of the dynamics (corresponding to $\beta = 1$) which implies that the stability of this equilibrium is equivalent with the stability properties of the overlapping generations equilibrium under naive expectations. In the model studied in this section the situation is a little more complicated. Since there is a continuum of beliefs equilibria consistent with the monetary steady state and the stability of this monetary steady state depends upon the belief parameters α and β, there are always a number of stable equilibria and a number of unstable equilibria. In particular, there exists a number $\overline{\beta} > 0$, such that all equilibria with $|\beta^*| < \overline{\beta}$ are stable under the learning dynamics discussed above and all other equilibria are not. The initial conditions on inflation rates and the initial beliefs therefore seem to be quite important for the dynamics. It might be worth mentioning here that the least squares learning algorithms we have been studying can have an interpretation as a Bayesian learning scheme. In particular, the above algorithm also specifies the case where agents have a normally distributed prior on α and β and after each realization update this prior according to Bayes' rule (see Bray and Savin (1986) and Zelner (1971)). The matrix R_0 then corresponds to the precision of the initial beliefs. If R_0 is large, then agents are not to sure about their initial beliefs (their prior distribution has a relatively high variance) and if R_0 is small they have a lot of confidence in their initial beliefs. Therefore, if for example according to their prior the agents know for sure that $\beta_1 = 0$, (that is, they attach probability 1 to this event) the updating scheme discussed above is exactly equivalent to an updating scheme where agents forecast inflation rates by taking the average of previous inflation rates, which converges almost always as we saw before.

Simulations of the overlapping generations model with the CES savings function suggest that, for all parameter values, there will be convergence to the monetary steady state and to belief parameters α and β for which the steady state is stable and which correctly predict this steady state. Which belief parameters that will be depends upon the initial conditions. For the complicated savings function the situation is a little different. Figure 8 shows some typical examples of the learning and inflation dynamics for this case.

Initial beliefs appear to be very important. The first three pictures show the possible dynamics of the overlapping generations model with $\rho = \frac{3}{4}$ and $\theta = 1$. For all three pictures, the initial belief on α equals 5, they only differ in the initial belief over β. The initial inflation values are the same for each simulation and are chosen near the period two orbit (which is the attractor for the overlapping generations model under naive expectations). In the first picture $\beta_0 = -\frac{1}{4}$ and as can be seen the inflation dynamics converges. The learning dynamics also converges to a situation with $\alpha = \theta = 1$. As discussed before the final value of β is indeterminate and very much depends upon the way in which the inflation rate converges to the monetary

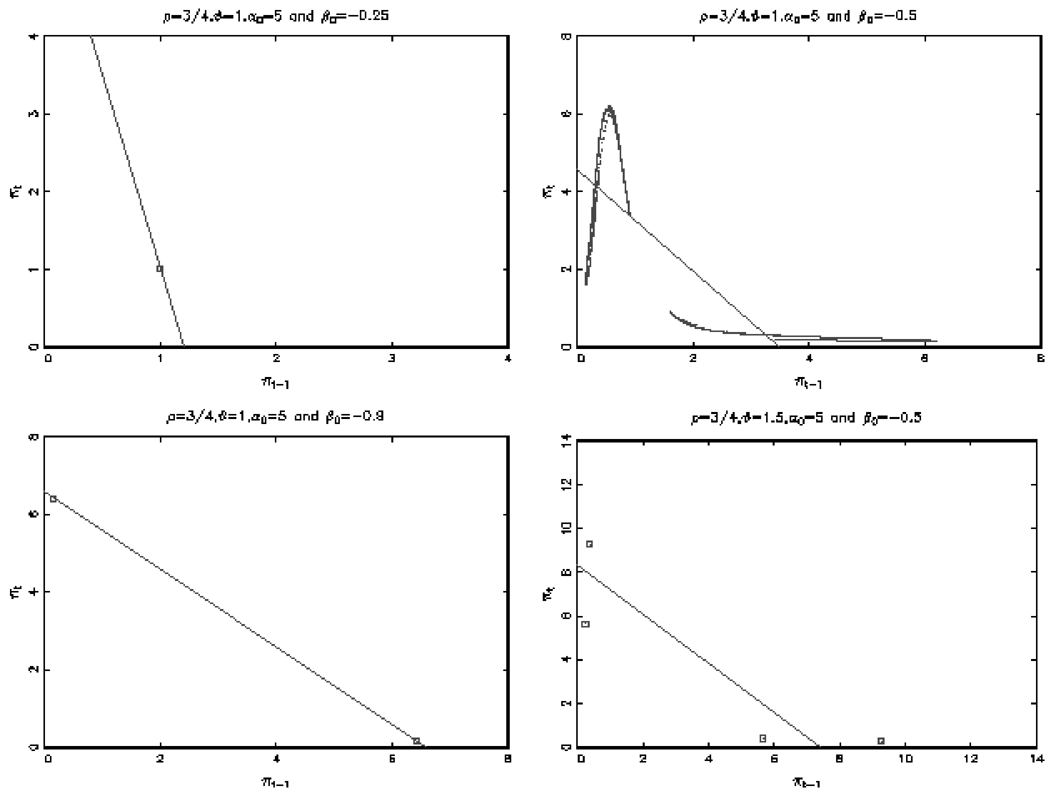

Figure 8: Attractors of (26) and examples of beliefs-equilibria with complicated savings function. a) BE corresponding to steady state, $\rho = \frac{3}{4}$, $\theta = 1$, $\alpha_0 = 5$ and $\beta_0 = -\frac{1}{4}$. b) Nonexistence of BE, $\rho = \frac{3}{4}$, $\theta = 1$, $\alpha_0 = 5$ and $\beta_0 = -\frac{1}{2}$. c) BE corresponding to period two cycle, $\rho = \frac{3}{4}$, $\theta = 1$, $\alpha_0 = 5$ and $\beta_0 = -0.9$. d) BE corresponding to four cycle, $\rho = \frac{3}{4}$, $\theta = 1.5$, $\alpha_0 = 5$ and $\beta_0 = -\frac{1}{2}$.

steady state θ. The second picture shows what happens when the initial value of β is a bit lower, $\beta_0 = -\frac{1}{2}$. In this case the inflation dynamics seems to converge to a strange attractor. However, this is a case of nonexistence, similar to the one discussed in the previous section. Starting with $\alpha = 5$ and $\beta = -\frac{1}{2}$ the inflation dynamics converges to a two-cycle. This makes β converge to -1, however as β becomes approximately -0.76 the period two cycle becomes unstable and a strange attractor emerges. Across this strange attractor we have a regression coefficient β smaller than -0.76 which makes the system return to a stable period two cycle again and the whole story repeats. A brief look at the time series would then reveal the following. The economy moves in an approximate two-cycle for a long period. As agents start believing more and more in this two-cycle it is disturbed and inflation rates start fluctuating more and more causing the beliefs to move in the reverse direction which leads to the two-cycle again. The third picture shows the inflation and learning dynamics when the initial belief on β equals -0.9. The inflation rates converge to a period two orbit (π_1, π_2) and the learning dynamics converges to $(\alpha, \beta) = \left(\frac{\pi_1+\pi_2}{2}, -1\right)$. Agents really have to belief in a period two cycle in order for it to indeed emerge. The fourth picture shows the dynamics for $\theta = 1\frac{1}{2}$. We then find that for initial beliefs $\alpha = 5$ and $\beta = -\frac{1}{2}$ the inflation rates converge to a period four orbit.

7 Summary

In this paper we have considered a standard overlapping generations model and discussed several ways in which agents might learn about the dynamics of inflation. We have seen that, dependent upon the specific learning procedure at hand and the specification of the overlapping generations model, the monetary steady state might be stable or unstable and periodic, quasi periodic and strange behaviour might occur. An important aspect of our models is that agents perceptions are mis-specified. It is clear that for any agent it is impossible to understand the whole economy including the specific psychology and behaviour of all agents in it. Therefore, an agent always has some extremely simplified model of the economy in his mind when he tries to make forecasts of some economic variable. Therefore the use of mis-specified models of the economy is dominant in economic life.

Some important lessons may be learned from the analysis conducted above. First of all it is clear that convergence to a steady state depends very much upon the specific type of learning models people use. Secondly, erratic dynamics can not be dismissed by a higher degree of sophistication of the learning models. Thirdly, though the learning dynamics might converge to a limit belief the corresponding inflation dynamics might be erratic. These fluctuating inflation rates are then in some sense consistent with the limit belief of the agents: in their class of models of the economy, the limit belief explains the inflation rates the best. Of course models from other classes might perform better, however the class of learning models has to be taken exogenously.

A next step in the analysis would be to consider more general classes of perceived laws of motion and investigating which of these laws of motion and corresponding beliefs-equilibria are the most relevant for the model under study. In this way one might be able to restrict the wilderness of bounded rationality by ruling out laws of motion that make structural forecast errors.

Of course the analysis in this paper has only been a starting point in a more definite characterization of the dynamic properties of learning procedures. However we believe that it carries in it all the phenomena that one would find in more realistic dynamic economic models where agents can use more sophisticated learning algorithms.

References

[1] Arifovic, J. (1995): Genetic algorithms and inflationary economies. *Journal of Monetary Economics,* **36,** 219-243.

[2] Arifovic, J. (1996): The behavior of the exchange rate in the genetic algorithm and experimental economics. *Journal of Political Economy*, **104,** 510-541.

[3] Benassy, J.-P. and M.C. Blad (1989): On learning and rational expectations in an overlapping generations model. *Journal of Economic Dynamics and Control*, **13,** 379-400.

[4] Böhm, V. and J. Wenzelburger (1999): Expectations, forecasting and perfect foresight – a dynamical systems approach. *Macroeconomic Dynamics,* forthcoming.

[5] Bray, M.M. (1982): Learning, estimation, and the stability of rational expectations. *Journal of Economic Theory*, **26,** 318-339.

[6] Bray, M.M. and N.E. Savin (1986): Rational expectations equilibria, learning, and model specification. *Econometrica*, **54**, 1129-1160.

[7] Bullard, J. (1994): Learning Equilibria. *Journal of Economic Theory,* **64,** 468-485.

[8] Bullard, J. and J. Duffy (1998): On learning and the stability of cycles. *Macroeconomic Dynamics,* **2,** 22-48.

[9] Bullard, J. and J. Duffy (1999): Using genetic algorithms to model the evolution of heterogenous beliefs. *Computational Economics,* **13,** 41-60.

[10] Chatterji, S. (1995): Temporary equilibrium dynamics with Bayesian learning. *Journal of Economic Theory,* **67,** 590-598.

[11] Debreu, G.. (1974): Excess Demand Functions. *Journal of Mathematical Economics,* **1,** 15-21.

[12] Duffy, J. (1994): On learning and the nonuniqueness of equilibrium in an overlapping generations model with fiat money. *Journal of Economic Theory,* **64,** 541-553.

[13] Farmer, R.E. (1993): *The Macroeconomics of Self-Fulfilling Prophecies,* Cambridge: The MIT Press.

[14] Gaunersdorfer, A. and J. Hofbauer (1995): Fictitious play, Shapley polygons and the replicator equation. *Games and Economic Behavior,* **11,** 279–303.

[15] Grandmont, J.M. (1985): On endogenous competitive business cycles. *Econometrica,* **53**, 995-1045.

[16] Grandmont, J.-M. and G. Laroque (1991): Economic dynamics with learning: some instability examples. In: W.A. Barnett et al (eds.) *Equilibrium Theory and Applications,* Cambridge: Cambridge University Press.

[17] Hommes, C.H. and G. Sorger (1999): Consistent expectations equilibria. *Macroeconomic Dynamics,* **2,** 287-321.

[18] Kehoe, T.J. and D.k. Levine (1985): Comparative statics and perfect foresight in infinite horizon economies. *Econometrica*, **53**, 433-453.

[19] Lettau, M. and T. Van Zandt (1995): "Robustness of adaptive expectations as an equilibrium selection device", working paper Tilburg University.

[20] Lucas, R.J. (1986): "Adaptive behaviour and economic theory", *Journal of Business,* **59,** 401-426.

[21] Mantel, R. (1974): On the characterization of aggregate excess demand. *Journal of Economic Theory,* **7,** 348-353.

[22] Mantel, R. (1974): Homothetic preferences and community excess demand functions. *Journal of Economic Theory,* **12,** 197-201.

[23] Marcet, A. and T.J. Sargent (1989a): Convergence of least squares learning mechanisms in self-referential linear stochastic models. *Journal of Economic Theory*, **48,** 337-368.

[24] Marcet, A. and T.J. Sargent (1989b): Least squares learning and the dynamics of hyperinflation. In: W.A. Barnett et al. (eds.) *Economic Complexity: Chaos, Sunspots, Bubbles and Nonlinearity.* Cambridge: Cambridge University Press.

[25] Sargent, T.J. (1993): *Bounded Rationality in Macroeconomics,* Oxford: Clarendon Press.

[26] Sargent, T.J. (1998): *The Conquest of American Inflation,* Princeton: Princeton University Press.

[27] Schönhofer, M. (1996): Least squares learning leads to chaos. Working paper University of Bielefeld.

[28] Shapley, L. (1964): Some topics in two-person games. *Annals of Mathematical Studies,* **4,** 1-28.

[29] Sonnenschein, H. (1973): Do Walras' identity and continuity characterize the class of community excess demand functions? *Journal of Economic Theory,* **6,** 345-354.