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Cernohorsky, J.

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Chapter 4
Flux–limited Neutrino Diffusion In Static Stellar Backgrounds

J. Cernohorsky

Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, the Netherlands, and
Center for High Energy Astrophysics, P.O. Box 41882, 1009 DB Amsterdam, the Netherlands

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Abstract—The numerical implementation of multi–group Levermore–Pomraning Flux–limited Neutrino Diffusion Theory (FNDT) is presented. The behaviour of this transport scheme is investigated in five static stellar models. In the calculations the feedback of the neutrino flow on the stellar matter is neglected. The evolution of the neutrino energy distribution function is followed in time, starting from an initial local thermodynamic equilibrium (LTE) distribution throughout the star, until a stationary non–LTE solution is reached. Spectral and frequency–integrated sources, luminosities and distributions are presented. The influence of electron degeneracy on the neutrino transport is highlighted.

Energy deposition in regions of the stellar models relevant to the delayed–explosion mechanism, at matter densities broadly between $10^{12}$–$10^{9} \text{g cm}^{-3}$, is rule rather than exception. Absorption of high–energy neutrinos ($\omega > 20$) MeV depletes the high–energy end of the spectrum at densities ranging down to $10^{8} \text{g cm}^{-3}$. In order to simulate spectra seen by an observer at infinity, it is necessary to extend the transport calculation to this density. Emergent neutrino energy distributions are typically non–thermal. Thermal fits can be made only on the high–energy tail of the spectrum, which is most readily detectable and sampled on earth. However, using the ensuing fitting parameters in the evaluation of bulk luminosities may overestimate these by factors of several.
1 Introduction

Neutrino transport in a practical stellar application is a difficult problem. Extended optically thick, thin and semi-transparent regions must be smoothly connected, and the transport scheme must be able to operate in all three. Steep gradients in density, temperature and chemical potentials, as well as the Fermi–statistics of the neutrinos further complicate the problem, not in the least its numerical implementation. In two previous papers \cite{1,2} the Levermore–Pomraning (LP) Flux–limited Diffusion Theory \cite{3} (FDT) was extended and reformulated for neutrino transport. This Flux–limited Neutrino Diffusion Theory (FNDT) was developed for use in dense stellar environments, notably in the supernova and neutron star formation context.

The FNDT approach to the neutrino transport problem is a practical compromise between physically rigorous \cite{4} but computationally unaffordable approaches on the one hand, and highly convenient but rather ad hoc ones on the other. The method involves fewest assumptions in a multigroup formulation \cite{1} and is inherently flux–limited. It retains information about the neutrino energy distribution, which is in principle observable with earth–based detectors.

In this paper the implementation of the FNDT transport scheme in a computer code is presented. The final aim is to embed the transport code in a dynamical matter evolution code and simulate the neutrino–radiation driven evolution of the hot proto–neutron star. Such a calculation was done previously by Burrows and Lattimer in their pioneering paper Ref.\cite{5}, but employed a rather crude neutrino transport scheme. Recently, Suzuki and Sato \cite{6} simulated neutron star formation using a transport scheme very similar to the one presented here. The Kelvin–Helmholtz cooling phase and associated neutrino emission take place on a relatively long time scale and last some 10–20 sec real time. Because of this, both the matter evolution and the transport code must use an implicit algorithm, as one cannot afford the numerical time step to be limited by the Courant condition.

The transport code must be able to operate in different matter environments and on a finite computer budget. It must be automatically vectorizable, fast, accurate, and robust. In order to test its performance, the FNDT code has been run in five different stellar models. These have been chosen in such a way as to represent a variety of probable and interesting physical circumstances. The differences between them are set up systematically so that the influence of relatively minor variations in their equation of state (EOS) and the electron fraction $Y_e$ can be investigated. In particular the differences in electron degeneracy and their influence on the transport are highlighted.

The spirit of this work is that in order to achieve a complicated final goal, one should go towards it in small, systematic steps, making the model systematically more complicated and realistic, and study the effect of each refinement.

The aim of this paper is to test the transport scheme and code in a set of calibrated and simple stellar environments, which still contain the essential characteristics of more realistic settings. In order to do simple things first and to systematically study the effect and behaviour of the transport, the code is used to simulate $\nu_e$ and $\bar{\nu}_e$ transport in spherically symmetric static stellar matter backgrounds. Moreover, if the relaxation time scale on which the neutrino flows reach a steady state is found to be small compared to the dynamical evolution time scale of the star itself, the matter will not change much during a transport step.

The stellar models are constructed in hydrostatic equilibrium, so that they can be used as starting points for dynamical evolution calculations. In order to isolate the behaviour
of the transport from possible complications associated with the EOS of the matter, the latter is kept as simple as possible. In particular, the baryonic component consists of free nucleons only. At very high matter densities and in regions of the star through which the bounce shock has passed this ought to be a good approximation. At densities below some $10^{-3}g\text{ cm}^{-3}$ this becomes increasingly less realistic.

The neutrinos interact with the matter through the $\beta$–processes (2.8,2.9) and by elastic neutrino–nucleon scattering (2.7). The calculations presented in this paper do not involve neutrino–electron scattering, pair and plasmon processes. The reason is twofold. First, the cross sections of these reactions are two orders of magnitude smaller than those of the $\beta$–reactions. Second, these processes exchange energy with the matter without removing the neutrino from the stream. Therefore they couple neutrino energy–bins. This complicates the problem conceptually and technically, since the formulation of FNDT as given in Refs.[1, 2] deals strictly with monochromatic transport. A self–consistent extension involving bin–coupling sources has not been made to date. On the technical side, bin–coupling processes change the structure of the matrix (4.8), inverted in the code, from tri–diagonal to tri–block–diagonal. Instead of having single elements along the main diagonal plus and minus one column, it would have there blocks of size $NB \times NB$ with $NB$ the number of energy bins. The time needed for the calculation with uncoupled bins is linear in $NB$. With coupled bins this is at least $(NB)^2$. Moreover, the relaxation and convergence speeds would be limited by the slowest bin. The solution of a $NB$ times larger coupled system of non–linear equations may introduce added complications and instabilities. The incorporation of General Relativity (GR) introduces the same conceptual and technical problems as discussed above and is therefore also delegated to a later time. In my view it is not sensible to put great effort into the inclusion of $O(10-20\%)$ effects, if their inclusion in a more or less ad hoc fashion may introduce systematic errors of the same magnitude.

The organisation of the paper is as follows. In Sec. 2 the problem is stated. In Sec. 3 background stellar models are presented and their properties are discussed. In Sec. 4 the numerical implementation of the transport scheme is presented, and the numerical ‘experiments’ are outlined. Section 5 contains the results of the calculations. These are further discussed in Sec. 6. For details of F(N)DT the reader is referred to Refs.[1]–[3]

# 2 Transport model

The technical problem at hand is solving the energy balance equation, which is the first angular moment of the neutrino transport equation.[1, 2] The energy balance equation given below is identical for $\nu_e$ and $\bar{\nu}_e$ although, of course, the opacities and distributions are different for both species. For a spherically symmetric system it reads\(^1\) in standard notation

$$\frac{\partial \epsilon}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \epsilon f \right) = \kappa_e^b (b - e), \tag{2.1}$$

with $b(\omega, r) = (1/4\pi) \int d\Omega f^0_\nu(\omega, r)$ and $e(\omega, r, t) = (1/4\pi) \int d\Omega f_\nu(\Omega, \omega, r, t)$ the LTE and the non–LTE neutrino energy distributions respectively, defined as the normalised first angular moments of the (non–)LTE neutrino distribution function. The spectral energy densities $B(\omega, r)$ and $E(\omega, r, t)$ follow from $e$ and $b$ by multiplication with the factor $\omega^2/2\pi^2$, with $\omega$ the particle energy. The equilibrium energy density is thus given by

$$B = (\omega^2/2\pi^2) f^0_\nu, \tag{2.2}$$

\(^1\)We adopt natural units, setting $\hbar = c = k_B = 1.$
and involves the Fermi-Dirac distribution function

\[ f_\nu^0(\omega, r) = \frac{1}{e^{\beta(r)(\omega-\mu_\nu(r))} + 1} = b(\omega, r) \]

at local matter temperature \( T(r) = \beta^{-1}(r) \), with

\[ \mu_\nu(r) = \mu_e(r) + \mu_p(r) - \mu_n(r) \]  \hspace{1cm} (2.3)

the local equilibrium neutrino chemical potential.

The first Eddington factor \( f(\omega, r) \), defined in terms of the normalised second angular moment of the distribution function \( f_\nu \) and equal to the ratio of the spectral flux \( F(\omega) \) and spectral energy density \( E(\omega) \), \( f \equiv |F/E| \) is expressed as \(^1, 2, 3\)

\[ f = \coth R - 1/R, \] \hspace{1cm} (2.4)

in terms of the dimensionless quantity\(^2\)

\[ R \equiv \frac{-\partial_e e}{b\kappa_e^0 + e(\kappa_e - \bar{\kappa}_e/3)}. \] \hspace{1cm} (2.5)

In the absorption opacity \( \kappa_e^a(\omega, r) \) stimulated emission and absorption are included through the blocking factor

\[ \theta \equiv \frac{1 - f_\nu^0(\omega)}{1 - f_\nu^0(\omega)} , \] \hspace{1cm} (2.6)

with \( f_\nu^0 \) the LTE electron distribution function.

Elastic scattering on nucleons

\[ \nu + n, p \leftrightarrow \nu + n, p \] \hspace{1cm} (2.7)

is accounted for by \( \kappa_e(\omega, r) \) and \( \bar{\kappa}_e(\omega, r) \), the isotropic and anisotropic contributions respectively.\(^1, 2\) The expressions for the rates and cross sections used in this paper were taken from Bruenn.\(^7\) Elastic scattering on nucleons contributes to the total opacity and the momentum transfer, but because the recoil of the nucleons is neglected, it does not contribute to the energy transfer. The scattering opacities are the same for neutrinos and anti-neutrinos.

Energy and lepton number are exchanged with the matter through emission and absorption by the beta-reaction

\[ \nu_e + n \leftrightarrow p + e^- , \] \hspace{1cm} (2.8)

for neutrinos and

\[ \bar{\nu}_e + p \leftrightarrow n + e^+ , \] \hspace{1cm} (2.9)

for anti-neutrinos.

The matter background is static. All the opacities as well as the equilibrium energy distribution function \( b(\omega, r) \) are ‘frozen’ in time during the transport calculation. The feedback of the neutrino flows on the matter exchanging energy and lepton number with the background is neglected. Starting from some initial neutrino energy distribution, for which the LTE distribution \( b(\omega, r) \) is taken here, the distribution \( e(\omega, r, t) \) will evolve into a stationary state. The time scale on which \( e \) and \( f \) reach this steady state defines the relaxation time scale of the neutrino flow.
3 Matter background models

In order to simulate neutrino transport and test the code one needs to have some quasi realistic dense stellar matter background. Of course one could resort immediately to the use of a realistic stellar model as it comes out of a hydrodynamic collapse calculation some time after bounce and the start of the shock propagation through the stellar core. This approach has been followed e.g. by Janka and Hillebrandt in Ref.[4]. Apart from the obvious virtues of such a realistic approach, there are some disadvantages as well. Such a stellar model is the product of a complicated calculation that involved a plethora of input physics, approximations and the treatment and solution of many numerical problems. Typically it involved a complicated, tabulated EOS and a lot of nuclear physics. Obviously this is not an easily reproducible background. Also it may contain many features that are poorly understood, but which still may influence the behaviour of the neutrino transport performed on it. It inhibits systematic experimentation and the complexity of the background may obscure some characteristics of the transport.

In principle, the transport is indifferent to the question whether the matter background is in hydrostatic equilibrium, and what its EOS is. Given reasonably smooth profiles of mass density $\rho(\tau)$, temperature $T(\tau)$, and electron chemical potential $\mu_e(\tau)$ as functions of position $\tau$ in a spherical model the transport code is capable of calculating the $\nu_e$ and $\bar{\nu}_e$ non-LTE energy distributions $e(\omega, \tau, t)$ as they evolve from a given initial state for a given time $dt$, or into the stationary solution.

The approach to the construction of the matter backgrounds used in this paper is a compromise between realism and simplicity. The EOS (3.1) that was used to calculate temperatures and chemical potentials is kept very simple, see below. In particular the models do not include any nuclei. Their baryonic composition is made up of free ideal nucleons, except for the strong-interaction term above nuclear density $n_n$. However, the models are configurations which are in hydrostatic equilibrium so that in a later stage one can use them as initial models for a dynamical evolution calculation including this transport code.

The construction of a matter background model is realised in two stages. In the first stage a 'star' with specified total mass $M$ and central density $\rho_c$ is constructed in hydrostatic equilibrium. The hydrostatic equilibrium and mass-continuity equations are solved, using a cold polytropic EOS, $P_{\text{pol}}(\rho) = K\rho^n$, with a given polytropic index $n$. This yields profiles of the density $\rho(m_r)$, radial position $r(m_r)$ and the total polytropic pressure $P_{\text{pol}}(m_r)$ as functions of the enclosed mass $m_r$ which is transformed using the coordinate transformation (4.1) into the parameter $\xi(m_r)$. In the second stage the temperature $T(\xi)$ and the electron chemical potential $\mu_e(\xi)$ profiles are self-consistently constructed by specifying a hot EOS and equating the cold polytropic pressure to it, $P_{\text{pol}}(\rho(\xi)) = P_{\text{hot}}(\rho(\xi), T(\xi), \mu_e(\xi))$. In order to solve for both $T$ and $\mu_e$ an electron fraction at each position is specified simultaneously, giving a second equation $Y_e(\xi) = Y_e(\rho, T, \mu_e)$ (3.3). With the solution of these two equations and the density profile $\rho(\xi)$ already known from the first stage, a temperature $T(\xi)$ and electron chemical potential profile $\mu_e(\xi)$ are constructed. The four variables $\rho(\xi), T(\xi), \mu_e(\xi)$ and $\xi(\xi)$ as functions of enclosed mass $\xi(m_r)$ define the matter background, which is by construction in hydrostatic equilibrium. Here $\xi_e = \mu_e(\xi)/T(\xi)$ is the electron degeneracy parameter.

The second stage of the approach sketched above can in principle be applied to any EOS. The simple EOS used in this paper for constructing $T$ and $\xi_e$ as functions of position
Figure 1: Stellar mass–densities as functions of position $\xi(m_r)$.

Figure 2: Radial positions in km as functions of the enclosed mass parameter $\xi(m_r)$. 
is the following

\[ P_{\text{hot}} = \frac{K_0 n_e}{9 \Gamma} \left( \frac{n_B}{n_e} \right)^\Gamma + n_B T + \frac{\pi^2}{45} T^4 + \frac{2}{\pi^2} T^4 S_4(\xi_e). \] (3.1)

The terms on the rhs of Eq. (3.1) represent the $T = 0$ contribution to the baryonic pressure above nuclear density $n_n$, (with $K_0$ the nuclear incompressibility and $n_B$ the baryonic particle density), an ideal free gas of nucleons, a free photon gas, and an electron–positron mixture, respectively. The matter is in LTE with itself because it can equilibrate not only through the weak interaction but also electromagnetically. The polynomial $S_4(\xi) = (1/4!)[z^2(z^2 + 2\pi^2) + (7\pi^4/15)]$ is the exact Sommerfeld expansion of the sum of the Fermi–integrals in the expressions for the relativistic $e^-$ and $e^+$ pressure contributions.

The contribution of a $\nu_e\bar{\nu}_e$ mixture in LTE with the matter, given by

\[ P_{\nu e}^{\text{eq}} = \left( \frac{1}{\pi^2} \right) T^4 S_4(\xi), \] (3.2)

can in principle be added to (3.1), but is not included for the models in this paper. Its inclusion in LTE is only justified in the very dense opaque interior, and moreover, unrealistically raises the electron chemical potentials, because effectively it introduces just another species of ‘electrons’, be it with half the electron helicity factor. The electron fraction, defined as $Y_e = (n_{e^-} - n_{e^+})/n_B$ is given by

\[ Y_e = \left( \frac{2}{\pi^2} \right) \frac{T^3}{n_B} S_3(\xi_e), \] (3.3)

as a function of the matter variables with $S_3(\xi) = (1/3!)(z^3 + \pi^2 z)$.  

Figure 3: Electron fractions $Y_e = (n_{e^-} - n_{e^+})/n_B$.  

The drawback of the use of simplified stellar models such as these is the uncertainty as to whether any ‘special effects’ are also reproduced in more realistic models. On the other hand, the occurrence of such effects can be more easily investigated because the models allow more experimentation. The major drawback of this particular two–step approach to the construction of a ‘hot’ proto–neutron star is that the initial polytropic structure with
central densities in the relevant $10^{14}-10^{15}\,\text{g/cm}^3$ range never yields a star much bigger than some 30 kilometers. In the hot EOS-fit this compactness leads to artificially high electron degeneracy, see Fig. 5, even for modest electron fractions $Y_e = 0.2-0.3$.

Figure 4: Matter temperatures as functions of position.

Figure 5: Electron degeneracy parameters $\xi_e = \mu_e/T$. 
In this paper five different background models are used, called M1, M1.1, M2, M3 and M4. The first three models, M1, M1.1 and M2 are the simplest, and are related by their polytropic structure. They are single polytropes with polytropic index \( \gamma = 4/3 \), total mass \( M_* = 1.5 M_\odot \), and central density \( \rho_c = 7.10^{14} \, g \, cm^{-3} \). They are put in hydrostatic equilibrium by solving the Lane–Emden equation\(^8\) using a stepsize-controlled fourth order Runge–Kutta routine.\(^9\)

Models M1 and M1.1 have \( K_0 = 0 \), so that the \( T = 0 \) contribution to the baryonic pressure, the first term in (3.1), vanishes. Models M1 and M1.1 differ in the imposed electron fraction which in M1 is chosen as \( Y_e = 0.33 \) and in M1.1 as \( Y_e = 0.1 \) throughout the star, see Fig. 3. The difference in composition translates into differences in electron degeneracy and temperature, see Figs. 4 and 5. Model M1 is lepton rich, relatively cold and contains very degenerate electrons throughout the star. Hence the neutrinos in LTE in M1 are also very degenerate. Because of their soft cores M1 and M1.1 have very simple smooth and monotonic \( T(\xi) \) and \( \xi(\xi) \) profiles.

Model M2 is the same as M1 and M1.1 in polytropic structure. The nuclear incompressibility in the hot EOS is set to \( K_0 = 110 \, MeV \), with nuclear density at \( n_* = 0.12 \, fm^{-3} \) and \( \Gamma = 2 \). The electron fraction is set to \( Y_e = 0.28 \) throughout. This model has an incompressible core. The first component of (3.1) dominates the pressure above nuclear density so that there the temperature is low and rises, whereas \( \xi(\xi) \) is high and drops dramatically. The model is quite lepton rich, and contains extremely degenerate electrons in the dense inner core above nuclear density. Because of the hard core the ensuing \( T(\xi) \) and \( \xi(\xi) \) profiles are no longer monotonic but exhibit some simple structure.

Model M3 is more elaborate and closer to the physical environment in a hot protoneutron star shortly after collapse, bounce and shock-launch. It is a tri-polytrop in which three cold polytropic EOS

\[ P(\rho) = K_i \rho^{\Gamma_i} + D_i, \quad i = 1, 2, 3 \]  

(3.4)

are joined together at certain predefined matter densities \( \rho_{12}, \rho_{23} \) such that the pressure is continuous and differentiable at the interfaces. The \( D_i \)'s are constants. The constant \( D_3 \) at the surface of the star is chosen such as to keep the pressure profile continuous at the surface. The surface lies at \( \rho_{\text{edge}} = 10^7 g \, cm^{-3} \). The total mass of the star is \( M_* = 1.5 M_\odot \), and the central density \( \rho_c = 4.10^{14} g \, cm^{-3} \). The rest of the model parameters are \((\rho = g \, cm^{-3}) \Gamma_1 = 2 \) for \( 10^{14} < \rho < \rho_c, \Gamma_2 = 1.25 \) at \( 10^{10} < \rho < 10^{14} \) and \( \Gamma_3 = 4/3 \) for \( \rho < 10^{10} \). In this way the collapsed dense inner core, the shocked outer core and an outer unshocked region can be roughly modeled in the first stage. The hydrostatic equilibrium and mass-continuity equations are solved directly as a coupled set with a fourth order Runge–Kutta routine. Because of the composite-polytropic nature of the model, the ensuing profiles are no longer E-solutions of the Lane–Emden equation.\(^8\) The routine is iterated until the \( K_1 \) is found which corresponds to the predefined total mass of the star.

The parameters of the hot EOS (3.1) for M3 are \( K_0 = 120 \, MeV \) at nuclear density \( n_* = 0.14 \, fm^{-3} \), and \( \Gamma = 2.0 \). These values correspond to the prescription given in Ref. [10] for the electron fraction \( Y_e = 0.15 \), which is set equal to this value throughout the star. It is a star with a hard core, a lower central density, and it is moderately lepton rich. The temperature profile changes its slope at the edge of the inner core and the gradient of \( \xi(\xi) \) flips sign there, see Figs. 4, 5. Maximal electron degeneracy occurs at the beginning of the 'shocked' outer core, instead of in the centre of the star as in M2. The degeneracy in the outer regions of the star is lower in M3 than in the previous models.
Model M4 is M3 evolved for one msec of real time using a dynamical stellar evolution code, coupled to the neutrino transport code described in this paper. It is an aberration as far as easy reproducibility is concerned. Nevertheless, it is included because it is no longer a polytrope, it is twice as big as the other models, and it is evolved from M3, so that comparison of the two allows a glimpse into the future of M3. Also the electron fraction is no longer constant but has a position-dependent profile, see Fig. 3.

After turning on the neutrino transport, the star instantaneously expanded somewhat due to the sudden additional contribution to the pressure. During the msec evolution the region below $10^{12}\, g\, cm^{-3}$ expanded rapidly as it was heated up by energy deposited by the neutrino flow. After one msec the star is nearly twice as big. Most of the deleptonisation on this short time scale has occurred in the semi-transparent region between $10^{13}$ and $10^{11}\, g\, cm^{-3}$. The reaction driving this neutrino emission is electron capture on free protons. In the atmosphere below $10^{11}\, g\, cm^{-3}$ neutrinos were partly captured on neutrons, causing the rise in the electron fraction, see Fig. 3.

Given the matter variables, the opacities and equilibrium neutrino energy distributions which enter the equations (2.1,2.4,2.5) solved by the transport code, are calculated. As an illustration below in Fig. 6 the total opacity defined as\[^2\]

$$\kappa_{\text{tot}} = \kappa_{\text{e}} + (\kappa_s - \kappa_e / 3)$$

is plotted for M3. The opacities are all directly proportional to the number of scattering

![Figure 6: Total opacities for six energy bins in M3. The solid lines correspond to the $\nu_e$ and the dashed lines to the $\bar{\nu}_e$ opacities.](image)

or absorbing and emitting target particles and hence the matter density. Their energy dependence is in leading order proportional to $\omega^2$. The total $\nu_e$ and $\bar{\nu}_e$ opacities are shown for the following energy bins, going from top to bottom in the figure, 115.6, 77.0, 50.0, 30.4, 16.3 and 3.65 MeV respectively. The 'swirl' in the $\nu_e$ opacities is due to the blocking factor $\theta(\omega)$, Eq.(2.6). 29).

As an illustration the LTE $\nu_e$ and $\bar{\nu}_e$ energy distributions $b(\omega, \xi)$ are shown in Figs. 7 and 8 respectively as spectra at fixed positions for M3. A few positions are marked with their matter densities (c.f. Figs. 26 and
Figure 7: Equilibrium $\nu_e$ energy distribution $b(\omega, \xi)$ at fixed positions as a function of $\omega$, M3. Five curves are marked with their corresponding matter densities, $2.4e14 = 2 \times 10^{14} g \text{ cm}^{-3}$, using standard FORTRAN notation.

Figure 8: Equilibrium $\bar{\nu}_e$ energy distribution $b(\omega, \xi)$ at fixed positions as a function of $\omega$, M3.
In the dense inner core above $10^{13} g \ cm^{-3}$ the neutrinos are very degenerate. The fact that the peak in the degeneracy lies off-centre is reflected in Fig. 7 where the $\nu_e$ are slightly less degenerate in the centre of the star marked with $\rho_e$ than in the contours further out. The fact that $M_3$ is quite degenerate causes the $\bar{\nu}_e$ to be much less abundant in LTE than $\nu_e$, see Fig. 8. Where $\xi$ has its peak the $\bar{\nu}_e$-abundance is lowest. Going out from the centre, the LTE $\nu_e$ spectrum first decreases and rises again after the $\xi$-peak.

In Table 3 the model parameters are summarised.

<table>
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<th>Model</th>
<th>$\rho_e$</th>
<th>$\gamma$</th>
<th>$K_0$</th>
<th>$\Gamma$</th>
<th>$Y_e$</th>
<th>$n_e (fm^{-3})$</th>
<th>$\rho_{12}, \rho_{23}$</th>
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<tr>
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<td>$7.10^{14}$</td>
<td>$4/3$</td>
<td>0</td>
<td>-</td>
<td>0.33</td>
<td>single</td>
<td>-</td>
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<td>M1.1</td>
<td>$7.10^{14}$</td>
<td>$4/3$</td>
<td>0</td>
<td>-</td>
<td>0.1</td>
<td>single</td>
<td>-</td>
</tr>
<tr>
<td>M2</td>
<td>$7.10^{14}$</td>
<td>$4/3$</td>
<td>110</td>
<td>2</td>
<td>0.28</td>
<td>single</td>
<td>0.12</td>
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<tr>
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<td>$4.10^{14}$</td>
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<td>2</td>
<td>0.15</td>
<td>2,1,25,4/3</td>
<td>0.14</td>
</tr>
<tr>
<td>M4</td>
<td>$3.55 \times 10^{14}$</td>
<td>multi</td>
<td>120</td>
<td>2</td>
<td></td>
<td>M3 evolved 1 msec</td>
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</table>

Table 1: Summary of matter background model parameters.

### 4 Transport scheme

The transport code solves Eq. (2.1), while simultaneously imposing Eqs. (2.4, 2.5). The energy balance equation (2.1) is a partial differential equation of first order in time and second order in space. The independent variable is $\varepsilon(\omega, r, t)$. The equation can either be solved as a pure differential equation on a simple grid or, after being integrated over a shell volume, as a conservative integro-differential equation on a shifted grid consisting of sites and shells. Both approaches have been implemented. The latter turned out to be superior and is described and used here. Its convergence speed and numerical stability compared to the solution of the equation as a second order differential equation in space is about twice as large. Its robustness in dealing with extremely degenerate conditions is considerably larger.

The integro-differential equation is put on a discrete grid, and turned into a difference equation. The background model has been constructed on a Lagrangean grid, with the grid–points labeled by the enclosed mass $m_r$ within a radial distance $r$ from the centre. The grid–sites are numbered $i = 1, ..., NG$ with site $i = 1$ corresponding to the centre of the star. The number of sites $NG$ is $NG = 84$ for models M1, M1.1 and M2, and $NG = 83$ for M3 and M4. In constructing the matter background, all quantities were defined on the grid–sites. The underlying grid parameter is the enclosed mass $m_r$ transformed to the coordinate

$$\xi(m_r) = \log(1 - \left(\frac{m_r}{M_*(1 + \eta)}\right)^{1/\alpha}),$$

with $M_*$ the star–mass, and $\eta < < 1$ a small parameter. This coordinate transformation, based on one given in Ref.[11], achieves a sufficient resolution of the stellar ‘atmosphere’ at low densities, labeling shells that contain little mass. The parameter $\eta$ determines this resolution and is best chosen between $e^{-25} < \eta < e^{-15}$. The power $1/\alpha$ with $\alpha = 3$ in the logarithm ensures that for small $m_r$, near the centre, $r(\xi) \propto \xi$.

In the transport code version discussed here, the grid is ‘backward’ centered, which means that the $i^{th}$ shell lies between site $i - 1$ and $i$. In the conservative formulation, vectors, such as $F_i, R_i$, and the metric quantities $m_r$ and $r_i$ are defined on the grid–sites.
Scalars, like $e_i, b_i, \kappa_i$ and tensors of the second grade are defined in between the sites, in the shells.

Equation (2.1) is integrated over the volume of each shell, and with Gauss’ Law turned into:

$$\int dV_i \partial_t e + \oint dS_i \cdot fe - \int dV_i \kappa_i^2 (b - e) = 0.$$  \hspace{1cm} (4.2)

As was argued in the introduction, the transport code must be implicit. The implicit conservative formulation leads to the following difference equation

$$\frac{e_i^{t+1} - e_i^t}{\Delta t} dV_i - (\kappa_i^2)_i (b_i - e_i^{t+1}) dV_i + 4\pi [r_i^2 f_i^{t+1} < e^{t+1} >_i - r_{i-1}^2 f_{i-1}^{t+1} < e^{t+1} >_{i-1}] = 0,$$  \hspace{1cm} (4.3)

with $dV_i = (4\pi/3)(r_i^3 - r_{i-1}^3)$ the volume of shell $i$. The bracket $< e >_i$ puts a quantity defined in the shell on the site and is defined as

$$< e >_i = \frac{e_i + e_{i+1}}{2},$$

the average of two scalars in adjacent shells $i$ and $i+1$.

The first Eddington-factor $f(R(e, \partial, e))$ is defined on the grid–sites and calculated separately using Eqs. (2.4) and (2.5). Written out as a difference equation the latter gives

$$R_i^{t+1} = \frac{-\Delta e^{t+1}/\Delta r_i}{< \kappa^2 b >_i + < e^{t+1}(\kappa - \bar{\kappa}/3) >_i}.$$  \hspace{1cm} (4.4)

The discrete spatial derivative is defined as

$$\frac{\Delta e^{t+1}}{\Delta r_i} = 4\pi r_i^2 < \rho >_i \left( \frac{d \xi}{dm} \right)_i \left( \frac{e_i^{t+1} - e_i^t}{\xi_{i+1} - \xi_i} \right),$$  \hspace{1cm} (4.5)

with $[\xi_i] = (\xi_i + \xi_{i-1})/2$ in the shell–middles. It is important to use this representation of the derivative with the difference taken in terms of the underlying Lagrangean grid–parameter $\xi$ instead of taking simply

$$\frac{\Delta e^{t+1}}{\Delta r_i} = \frac{e_i^{t+1} - e_i^t}{\bar{r}_{i+1} - \bar{r}_i}.$$  \hspace{1cm} (4.5)

Especially at the edge of the star where the density gradient is the largest, the $\xi$–grid remains well resolved whereas the $r$ tends to become rather singular. When the transport code is embedded in a dynamical evolution calculation on the Lagrangean grid, $r$ is a dynamical variable and $\xi$ is not.

Because of the complicated $f(e, \partial, e)$–dependence Eq.(2.1) is highly nonlinear. The implicit equation as a function of its unknowns at each position $i$ is of the form

$$H_i(e_i^t, e_i^{t+1}, e_{i+1}^{t+1}) = 0.$$  \hspace{1cm} (4.6)

Assuming that $e_i^t$ is known at time $t$, with $e_i^{t+1} = e_i^t + w_i^t$, the unknown in (4.6) is the time increment $w_i^t$. At each time step the corrections $d w_i^t(a)$ to the time increments $w_i^t(k-1)$ are solved iteratively by a Newton–Raphson algorithm from (4.6) written as a linearised matrix–equation

$$M_{ij}(w_i^{t(k-1)}) \cdot (d w_i^t(a))_j = -H_i(w_i^{t(k-1)}),$$  \hspace{1cm} (4.7)
with \( j = i - 1, i, i + 1 \). After \( (k) \) iterations \( w^{(k)} = w^{(k-1)} + d\omega^{(k)} \). In all iteration sequences, except for the first one, \( w^{(0)} = 0, d\omega^{(0)} = 0 \). The iteration is continued until the corrections to the time increments in each shell are small, i.e. \((1/NG)\sum |d\omega_i/e_i| \leq \epsilon_1\), with \( \epsilon_1 = 10^{-3} \) and Eq. (4.6) is well satisfied, requiring \((1/NG)\sum |H_i/M_{ij}| \leq \epsilon_2\), with \( \epsilon_2 = 10^{-6} \). Because each shell is coupled to its nearest neighbours the matrix \( M_{ij} \) is tridiagonal and contains

\[
M_{ij} = \frac{\partial H_i}{\partial e_j^{\pm1}},
\]

where \( j = i - 1, i, i + 1 \). The matrix-inversion required for the solution of \( d\omega^{(k)} \) from (4.7) is done by a standard algorithm from Ref. [9]. When the iteration is sufficiently converged, \( e_i^k \) is updated, time incremented and a new ‘inner iteration loop’ started.

The time evolution is stopped when the solution is time-independent requiring

\[
(1/NG)(\sum |u_k/e_k|) \leq \epsilon_3 \delta t,
\]

with \( \epsilon_3 = 10^{-3} \). This defines the stationary state.

Because Eq. (2.1) is a partial differential equation of first order in time and second order in space, the specification of the problem requires two boundary conditions and an initial condition. The first physical inner boundary condition (IBC) that follows from the spherical symmetry of the system and which performs best on the backward centered grid is \( \partial_v f(r = 0) = 0 \). It implies that \( f(r = 0) = R(r = 0) = 0 \) in the centre. In addition the conservative discretization scheme effectively imposes \( \partial_v f(r = 0) = 0 \), which fixes the second physical boundary condition.

On the outer boundary the discrete formulation makes an additional prescription necessary on how to define the spatial derivative there. The system is not very sensitive to this discrete outer boundary condition (OBC), and several have been tested. The one used imposes the equation itself (4.3) on the surface site, with the condition that in the spatial derivative (4.5) on \( NG \) the difference is taken as \((e_i^{NG} - e_i^{NG-1})/(\bar{\xi}_{NG} - \bar{\xi}_{NG-1})\). This OBC has the virtue that no explicit assumption on the transparency of the last shell is required, and that it becomes trivial in the continuum limit.

The initial condition is chosen to be \( e_i(t = 0) = b_i \). In the dense interior of the star at \( \rho > 10^{13} g cm^{-3} \) the neutrinos are expected to be in LTE, so that there the source term \( \kappa^2(\beta - \epsilon) = 0 \). This implies \( \beta = \epsilon \). In principle different initial conditions can be considered, and e.g. \( e_i(t = 0) = b_i/2 \) has been tried. Of course the evolution of \( e(t) \) towards the stationary state is different in that case, however the steady state itself is not.

At each time step the algorithm calculates roots of \( NG \) coupled nonlinear equations in a \( NG \)-dimensional solution space with Newton’s method. For smooth convergence it is vital to make an inspired ‘first guess’ at the solution vector \( w^k \) in the first iteration cycle at \( t = 0 \). This first guess must bring the system into that particular, but unknown, part of phase-space which constitutes the attractive region of the stationary solution. It can be found either by knowing the solution of the problem beforehand, or by trial and error. For an illuminating discussion of this problem see Ref.[9]. For intermediate and high-energy neutrinos \( \omega > 9 MeV \), the working first guess is taking \( w^k = 0 \) until the point where \( b_i \leq 10^{-50} \). Starting from there the initial energy density is geometrically diluted from the previous shell i.e. \( e_i^0 = (rr_{f-1}/rr_{f})e_i^{p-1} \). For low-energy neutrinos \( \omega < 9 MeV \), geometrical dilution begins at the point where the total mean free path \( \kappa^2_{tot} \) becomes larger than the typical shell–size, conveniently defined for this purpose as \( r_{NG}/NG \).

The transport was performed on energy–bins situated at the roots \( z_j \ (j = 1, .., 15) \) of a 15th–order Gauss–Laguerre polynomial. For the calculation of frequency–integrated bulk quantities a Gauss–Laguerre quadrature algorithm was used in order to obtain good integration accuracy with not too many energy–bins. The roots \( z_j \) are a set of numbers
which must be assigned an energy scale $\omega_j$ MeV. The actual energy of bin $j$ is given by $\omega_j = \omega_* x_j$ MeV. For nearly all calculations presented in this paper $\omega_* = 3$ MeV was used. For the $\bar{\nu}_e$, energy bins below the reaction threshold (1.805 MeV) of the beta-reaction (2.9) are left out.

Starting from the initial condition, the flow–evolution was calculated until the stationary state was reached. At predefined times snapshots of the evolution were taken. The snapshot–times are $t_1 = 10^{-6}$, $t_2 = 10^{-5}$, $t_3 = 10^{-4}$, $t_4 = 5.10^{-4}$ sec, and $t_{\text{stat}}(\omega)$ which turned out to be of the order of a few tenths of a millisecond, depending on the energy–bin. The code must be run in double precision. The time step is limited by the requirement that the $e$–profile is not allowed to change more than 2.5% per time step. The code vectorizes automatically and completely except for the routine that inverts the tri–diagonal matrix, which uses a recursive algorithm.

The code has run on a number of machines. The time–performance of the evolution of 15 bins from initial condition to stationary state on a 84 site spatial grid is listed below. The values are normalised on the NP1, and expressed in CPU–seconds of each machine. The time performance and robustness of the code is sufficient for a full proto–neutron star evolution that uses it for the neutrino transport to be completed in a few hours CPU time on a supercomputer like the SX2.

### Table 2: Code performance on different machines normalised on the NP1.

<table>
<thead>
<tr>
<th>Machine</th>
<th>nominal speed</th>
<th>vector</th>
<th>run time</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENCORE NP1</td>
<td>40 Mflop</td>
<td>no</td>
<td>10 CPU (sec) ≡ 1</td>
</tr>
<tr>
<td>NEC SX2</td>
<td>1.3 Gflop</td>
<td>yes</td>
<td>1/130</td>
</tr>
<tr>
<td>IBM 3090–600J/6VF</td>
<td>138 Mflop</td>
<td>yes</td>
<td>1/20</td>
</tr>
</tbody>
</table>

5 Transport results

In the following the results of the five model calculations are presented and discussed. The backbone of the discussion is provided by figures. It may at times be difficult to infer detailed quantitative information from the figures alone. Where quantitative results are discussed, they have been taken from the numbers and not by eye from these figures.

A top–down approach is followed. First, frequency integrated, or bulk–quantities are discussed. All bulk quantities presented are calculated with the steady–state solutions at time $t = t_{\text{fin}}$. Second, the bulk quantities are resolved into their constituent spectral components, which are the quantities that are directly calculated by the code. Third, the time–dependent non–equilibrium energy distribution $e(\omega, r, t)$, the first Eddington–factor $f(\omega, r, t)$ along with some other quantities featuring in FNDT are presented.

5.1 Bulk quantities

The bulk quantities presented in the following are all computed in the stationary state. The integrations were carried out using a 15–point Gauss–Laguerre quadrature algorithm. The integration accuracy was estimated by doing the integrations with the left and right adjusted and centered trapezium rule and comparing with the Gauss–Laguerre results. It depends on the quantity considered, but is always better than 10%. In most cases it is better than 1% and the integration errors are invisible to the naked eye.
5.1.1 Densities

The bulk non-equilibrium $\nu_e$ energy and number (particle) densities are defined as (c.f.(2.2) and above)

$$E_{\text{bulk}}(r, t_{\text{fin}}) = \int_0^\infty d\omega E(\omega, r, t_{\text{fin}})$$  \hspace{1cm} (5.1)

$$n_{\text{bulk}}(r, t_{\text{fin}}) = \int_0^\infty d\omega E(\omega, r, t_{\text{fin}})/\omega$$  \hspace{1cm} (5.2)

and are shown as functions of position in Figs. 9 and 10 together with their LTE counterparts. The bulk $\bar{\nu}_e$ energy and particle densities are presented in Figs. 11 and 12, and are defined analogically. The particle densities are plotted in units per nucleon. In all models three regions in the star can be distinguished. The first is the region deep inside the star where the neutrinos are in LTE with the matter. Hence $E_{\text{bulk}} = E_{\text{eq}}^{\nu_e}$ and $n_{\text{bulk}} = n_{\text{eq}}^{\nu_e}$. The LTE regions for $\nu_e$ lie at matter densities $\rho > 10^{12} \text{g cm}^{-3}$, $-\xi < 6$ for the most degenerate models M1 and M2, and at $\rho > 5.10^{12} \text{g cm}^{-3}$, $(-\xi < 5)$ for the rest.

![Figure 9: Bulk $\nu_e$ LTE and non-LTE energy densities. The curves with horizontal branches represent the non-LTE values.](image)

In the atmospheres, the non-LTE bulk densities are much higher than their equilibrium counterparts, $E_{\text{bulk}} > E_{\text{eq}}^{\nu_e}$ and $n_{\text{bulk}} > n_{\text{eq}}^{\nu_e}$. The bulk non-LTE energy densities level off at values of order $10^{29} \text{erg cm}^{-3}$ in the atmospheres and correspond to the horizontal branches in Figs. 9 and 11. For the most degenerate models M1 and M2 the $\nu_e$ energy densities in the atmospheres are highest, leveling off at $2.10^{29} \text{erg cm}^{-3}$ for M1 and around $10^{38} \text{erg cm}^{-3}$ for M4, the least degenerate model. Because the particle densities are depicted in units per nucleon, the non-LTE branches are the rising ones in Figs. 10 and 12. The total lepton fractions in the atmospheres are clearly dominated by neutrinos by several orders of magnitude. In all models the atmospheres lie at $\rho < 10^{11} \text{g cm}^{-3}$, $(-\xi \geq 9)$. 
Because in the atmospheres the non–LTE bulk densities are much higher than the LTE densities, and the spectral emission rates are proportional to \( b(\omega) \) whereas the absorption rates are proportional to \( e(\omega) \), the atmospheres will act as net neutrino absorbers.

Between the LTE–cores and the atmospheres, i.e. between \( \sim 10^{12} > \rho > 10^{11} \text{g cm}^{-3} \), \( (5 \leq -\xi \geq 10) \) there are net emitting regions where \( E_{\text{bulk}} < E_{\text{eq}}^{\nu_e} \) and \( n_{\text{bulk}} < n_{\text{eq}}^{\nu_e} \), this is most clearly seen in M4. Here the deleptonisation will start and proceed at the highest rate.

For \( \bar{\nu}_e \) the picture is qualitatively the same, see Figs. 11 and 12, although the locations of the three regions and the corresponding values of the densities are not. The \( \bar{\nu}_e \)-densities are much more sensitive to the degeneracy of the models than the \( \nu_e \)-densities. The \( \bar{\nu}_e \) LTE regions lie deeper inside at \( \rho > 10^{12} \text{g cm}^{-3} \), \( (-\xi < 4) \), and reflect the degeneracy profiles of the inner core (cf. Fig. 5) much more than the \( \nu_e \)-densities. The \( \bar{\nu}_e \) atmospheres also begin at higher densities \( \rho \approx 10^{12} \text{g cm}^{-3} \), \( (-\xi \approx 8) \) in the \( \bar{\nu}_e \)-poorest models M1 and M2. In M3 the atmosphere starts out relatively far outside at \( -\xi \approx 11 \) corresponding to \( \rho \approx 10^{10} \text{g cm}^{-3} \). Both the \( \bar{\nu}_e \) energy and the number densities are some 2–3 orders of magnitude lower than the \( \nu_e \)-densities. With the exception of M4, the models that are most abundant in \( \nu_e \) are \( \bar{\nu}_e \)-poorest (M1 and M2) and vice-versa (M1.1 and M3). This can be understood in terms of the degeneracies of the models, keeping in mind that \( \mu_\nu = -\mu_{\bar{\nu}} \).

The \( \bar{\nu}_e \) densities in M2 in the dense inner LTE core deviate slightly from the norm. Above roughly nuclear density \( \rho > 2.7 \times 10^{14} \text{g cm}^{-3} \), \( E_{\text{bulk}} > E_{\text{bul}}^{\nu_e} \) and \( n_{\text{bulk}} > n_{\text{eq}}^{\nu_e} \), which makes the inner core a net \( \bar{\nu}_e \)-absorber. This is caused by an inward flux driven by the positive \( T \) gradient and the steep \( \xi \) drop in the inner core, cf. Figs. 4, 5 and 34.

The average energy per particle, defined as the ratio of the bulk energy and particle densities

\[
\bar{\omega} = \frac{E_{\text{bulk}}}{n_{\text{bulk}}}
\]
Figure 11: Bulk $\bar{\nu}_e$ LTE and non-LTE (horizontal branches) energy densities.

Figure 12: Bulk $\bar{\nu}_e$ LTE and non-LTE (rising branches) particle densities per nucleon, $Y_p = n_{bulk}/n_B$. 
Transport results

Figure 13: Bulk average $\nu_e$ particle energies.

Figure 14: Bulk average $\bar{\nu}_e$ particle energies.
is shown in Figs. 13 and 14 for \( \nu_e \) and \( \bar{\nu}_e \) respectively.

The inner cores down to \( 10^{12} \, g \, cm^{-3} \) are dominated by high particle energies, in the 100–200 MeV range, corresponding to the electron chemical potentials, cf. Figs. 4, 5. Neutrinos in models M1 and M2 have the highest \( \bar{\omega} \) and correspondingly the \( \bar{\nu}_e \) have the lowest average energy. In all models at low matter densities \( \bar{\omega} \) levels out at energies between 10 MeV (M1.1) and 16 MeV (M1). Although the differences in electron degeneracy in the models lead to large differences in average particle energies deep inside the stars, the \( \bar{\omega} \)'s of the escaping neutrinos in the atmospheres are not very sensitive to this.

5.1.2 Sources

The bulk energy, lepton number and momentum sources, defined as

\[
SQ(\xi) = \frac{1}{n_B(\xi)} \int d\omega \kappa^2_\omega(\omega, \xi)[B(\omega, \xi) - E(\omega, \xi)],
\]

\[
SI(\xi) = \frac{1}{n_B(\xi)} \int d\omega \frac{\kappa^2_\omega(\omega, \xi)}{\omega}[B(\omega, \xi) - E(\omega, \xi)],
\]

\[
SA(\xi) = -(1/n_B(\xi)) \int d\omega \kappa_{tot}(\omega, \xi) F(\omega, \xi)
\]

respectively, in units per baryon, are shown in Figs 15–20. The spectral flux is given by \( F(\omega, \xi) = f(\omega, \xi)E(\omega, \xi) \). The momentum source \( SA(\xi) \) features in the momentum balance (hydrostatic equilibrium) equation. These sources account for the exchange of

Figure 15: Neutrino energy source term \( SQ(\xi) \). Different models have different scales, as depicted in the figure. In this way e.g. the peak in M1 lies at 1.2 \( 10^{-12} \, erg \, cm^{-1} \) per baryon.

energy, lepton number and momentum between neutrinos and matter. In the equations that govern the matter evolution they enter as sources into the energy conservation, lepton
Figure 16: Neutrino lepton number source term $S_I(\xi)$.

Figure 17: Anti-neutrino energy source term $S_Q(\xi)$.
number conservation and hydrostatic equilibrium equations, respectively. They establish the feedback of the neutrino-flow on the stellar background which drives the proto-neutron star evolution.

Where \( SQ \) (\( SI \)) is positive there is a net transfer of energy (lepton number) from the matter into the neutrino fluid. Where it is negative, energy (lepton number) is deposited into the matter. The energy and lepton number sources \( SQ \) and \( SI \) are pure non–LTE quantities, as their integrands are functions of the deviation from LTE \( (b(\omega) - e(\omega)) \).

The division of the stellar models into three regions that was sketched above in the discussion of the energy and number densities is reflected in the sources. In the inner LTE–cores at \( \rho > 10^{13} - 5.10^{12} \text{g cm}^{-3} \), \( (-\xi \leq 3) \) the neutrinos are in LTE with the matter, hence the energy and lepton number sources are zero.

Between the LTE–cores and the atmospheres at \( 3 \leq -\xi \leq 9 \) corresponding to \( 10^{13} - 5.10^{12} > \rho > 3. -5.10^{11} \text{g cm}^{-3} \), there are emitting regions where energy and lepton number are transported out of the matter. Electron capture on free protons (2.8) is responsible for the \( \nu_e \)-emission. In a dynamical situation the electron fraction \( Y_e \) would decrease rapidly here. This is illustrated in Fig. 3, in the comparison of M3 and M4. The \( \nu_e \)-emission is driven by \( e^+ \) capture on neutrons (2.9) which creates free protons that in turn become available for electron capture. A situation is feasible in which there is only energy but no net lepton number emission in this region. The emitting region in M4 is anomalous in comparison to the other models. It is a net lepton number emitter, but a net energy absorber. High–energy neutrinos coming from the interior, see the small peak around \(-\xi \approx 4 \) in Fig. 15 which corresponds to the \( Y_e \) gradient there, (cf. Fig. 3) are absorbed and reemitted again at lower energies, while the emission of low–energy neutrinos goes on unabated. As a net effect the total energy carried away from this region is smaller than the energy deposited by incoming high–energy \( \nu_e \).

The atmospheres at \(-\xi \geq 8 \) absorb energy and lepton number. Dynamically, the \( \nu_e \)-
absorption on neutrons (2.8) will raise the electron and proton fractions in the atmospheres, cf. M3 and M4 in Fig. 3. In turn, the capture of $\bar{\nu}_e$ on protons (2.9) will decrease $Y_e$. A dynamical equilibrium in which both processes proceed at equal rates will try to establish itself, creating an nearly stable $Y_e$. In that case almost no net lepton number deposition will occur. However, the energy deposition will be enhanced because the opposing reactions will keep the abundances of the absorbing particles at optimal levels, keeping both absorption channels open. This effect is illustrated to some degree by M4 when compared with its 'parent' M3. In M4 the $\bar{\nu}_e$ emission and absorption, Figs. 17-18, is much higher than in the other models. The $\bar{\nu}_e$-flow is being turned on by the local changes in the electron fraction, Fig. 3, while these choke off the $\nu_e$-exchange.

In the most degenerate models M1 and M2, which also had the highest $\nu_e$-densities, the $\nu_e$-LTE-core is biggest. Also the $\nu_e$ emission and absorption rates are highest. For the $\nu_e$ the situation is also consistent with the discussion of the densities. The $\bar{\nu}_e$-LTE-cores are much smaller in M1 and M2, and their $\bar{\nu}_e$ emission and absorption rates are respectively about 7 and 5 orders of magnitude smaller than in the rest of the models. Although in M1.1 the relatively high $\bar{\nu}_e$ exchange is due to the low electron degeneracy, the magnitude of its $\bar{\nu}_e$-LTE-core is in the same category with M1 and M2. Their common polytropic structure and high central density is responsible for this.

![Figure 19: Neutrino momentum source term $SA(\xi)$.](image)

The bulk momentum sources $SA(r, t_{fin})$ (5.5) are shown in Figs. 19 and 20. Both the LTE and the total sources are shown, always in the same line-type. The curves depicting the total sources level out at nonzero values in the atmospheres, whereas the LTE branches go rapidly to zero.

The momentum source $SA$ (5.5) is a function of the spectral flux $F(\omega) = f(\omega) E(\omega)$ and is therefore nonzero even in LTE as long as there are gradients in the matter. Where
Figure 20: Anti-neutrino momentum source term $SA(\xi)$.

Figure 21: Integrand of $\nu_e - SA$ as function of $\omega$ at fixed positions in the star (M3).
it is negative, momentum is transferred to the matter, acting as a positive contribution to the total pressure, see Figs. 19 and 20. Because of the occurrence of inward \( \bar{\nu}_e \) fluxes in M2, M3 and M4 the effective contribution of their momentum sources to the pressure can be negative in the inner core.

The momentum sources are largest in the inner cores. There they are LTE quantities, dominated by high-energy neutrinos, see Fig. 21. Deviations from LTE become apparent in the emitting region \( 10^{11} < \rho < 10^{12} \, g \, cm^{-3} \), decreasing the total momentum transfer. The LTE contribution, with the equilibrium flux \( F^{eq}(\omega) = f^{eq}(b(\omega))B(\omega) \) in (5.5), rapidly vanishes at densities below \( 10^{11} \, g \, cm^{-3} \). In the atmospheres \( SA \) is dominated by its non-LTE contribution.

### 5.1.3 Luminosities

In the preceding two sections it was established that the stars can be distinguished into three contiguous regions characterized by LTE and no energy and lepton number transfer, emission and hence positive \( SQ \) and \( SI \) (5.3,5.4), and absorption with negative sources in the atmosphere, respectively. In this section it will be shown how this stratification translates into bulk energy and particle luminosities, both inside the stars and as seen by an observer at infinity. The bulk luminosities offer the simplest handle for establishing where and how much energy and lepton number is being deposited into the matter by the neutrino stream passing through it. The bulk energy and particle luminosities are shown in Figs. 22-25, and are defined respectively as

\[
LE(\xi, t_{fin}) = 4\pi r^2(\xi) \int d\omega f(\omega, \xi, t_{fin})E(\omega, \xi, t_{fin}), \quad (5.6)
\]

\[
NL(\xi, t_{fin}) = 4\pi r^2(\xi) \int d\omega f(\omega, \xi, t_{fin})E(\omega, \xi, t_{fin})/\omega. \quad (5.7)
\]

In all models and for both \( \nu_e \) and \( \bar{\nu}_e \) the luminosities behave globally the same. In the LTE-cores and for most of the emitting region they rise. They reach a peak at the transition from the emitting into the absorptive region around \( \rho \approx 5.10^{11} \, g \, cm^{-3} \), (2.5 \( 10^{11} \) for M1) and decrease between \( 5.10^{11} > \rho > 10^{10} \, g \, cm^{-3} \) where the energy and number absorption rates are highest. They level out on constant values further out in the atmospheres at \( \rho \approx 10^{10} - 10^9, -\xi \approx 12 \), and assume their output values observable at infinity. Although the luminosities level off, the absorption of energy and lepton number in units per nucleon never actually stops. However, the nucleon density is so low in the atmospheres that the exchange no longer makes a visible dent in the luminosities. The spectra still change slightly at densities lower than this, modifying the ‘colour’-composition of the luminosities, but for the bulk luminosities the free-streaming region sets in at \( \rho \approx 10^{10} \).

In the \( \bar{\nu}_e \) luminosities also inward (negative) fluxes occur at the edge of the dense inner core in models M2, M3 and M4. These can be attributed to the temperature inversion in M2, and to the \( \xi_e \)-bump (and therefore \( \xi_{\nu_e} \)-trough with corresponding gradients) in M3 and M4.

All models are very luminous in \( \nu_e \), putting out in energy between \( 3.8 \, 10^{52} \) (M4) and \( 4.41 \, 10^{52} \, \text{erg sec}^{-1} \) (M1) and with a particle output between \( 2.1 \, 10^{67} \) (M4) and \( 1.6 \, 10^{58} \, \nu_e \, \text{sec}^{-1} \) (M1). The \( \bar{\nu}_e \) luminosities are some 2–4 orders of magnitude lower than the \( \nu_e \) luminosities in M1, M3 and M4, and 9 and 7 orders respectively in M1 and M2.
Figure 22: Bulk $\nu_e$ energy luminosities. Curves corresponding to different models have been scaled with a different scale factor.

Figure 23: Bulk $\nu_e$ number luminosities.
The fact that the $\nu_e$ luminosities are so high and the $\bar{\nu}_e$ luminosities so much lower, indicates that all the models must be considered to be 'early'. They are more representative for the neutrino flash associated with rapid electron capture than for the long-time cooling phase. In the cooling phase the $\bar{\nu}_e$-flow is expected to turn on, and develop a luminosity comparable to the that of $\nu_e$. Moreover, if the luminosities were to remain as high in the course of time during the cooling phase the latter could never last 10–20 sec. Consistently with this classification the luminosities are most evenly divided in the more evolved model M4, $\nu_e$ luminosities being just a factor 20 higher than the $\bar{\nu}_e$ luminosities.

The $\nu_e$ number luminosities peak slightly further out, at matter densities of about a factor two lower than the energy luminosities. The $\nu_e$ energy luminosities peak in all models, and drop much more after reaching their peak values than the number luminosities. Therefore the $\nu_e$-flows deposit more energy than lepton number into the semi-transparent region. On average high-energy neutrinos are being absorbed through the inverse $\beta$-reaction (2.8) (the absorption rate depends on $\epsilon(\omega, r, t)$). Subsequently they are (partly) reemitted with the LTE distribution (2.2). This energy 'downgrading' of the flow is further illustrated in the next section where the bulk quantities are resolved into their constituent spectral components. The downgrading is also the reason that the number luminosities do not peak in all models, but rise continuously in M1 and M4. The energy deposition at matter densities broadly between $10^{12} > \rho > 10^9 g cm^{-3}$ is quite substantial, ranging from $6.8 \times 10^{52}$ (M2) to $9.1 \times 10^{51} erg sec^{-1}$ (M4). This corresponds to 17% and 19% of their peak luminosities respectively, and per second amounts to several times a total supernova explosion energy in kinetic energy and light. The maximal lepton number deposition occurs in M3 at a rate of $7.1 \times 10^{56} sec^{-1}$. However, it must be stressed at this point that these are 'flash'-values, which in a dynamical situation will decrease rapidly, c.f. M3 and M4.

The energy luminosity peak in model M4 lies deeper inside the star than in the other models, at $\rho \approx 7.1 \times 10^{12} g cm^{-3}$ ($\xi \approx 4$), corresponding to the first peak in its emitting region, see Fig. 15. The luminosity peak corresponds to the maximal slope of the drop in electron-fraction $Y_e$, see Fig. 3. The primary emission peak is driven by the steep gradient in the electron degeneracy, but lies at high matter density. Neutrinos that are emitted there with the high average particle energy corresponding to the local LTE chemical potential, are promptly reabsorbed where the electron degeneracy is lower. The neutrino energy re-emission at lower $\omega$ in the rest of the emitting region cannot keep up with the energy absorption because the chemical potentials and temperatures are much lower there than at the primary emission peak. The number luminosity does not drop at all in the emitting region, which indicates that the absorbed high-energy $\nu_e$ are indeed promptly reemitted, making the number luminosity rise continuously. This extreme energy flow downgrading is consistent with the anomalous behaviour of the sources $SQ$ and $ST$ in M4, as discussed in the previous section.

The $\bar{\nu}_e$ luminosities shown in Figs. 24 and 25, peak slightly deeper inside the stars at $\rho \approx 10^{12} g cm^{-3}$, quite consistently with the behaviour of their sources and densities discussed in the previous sections. Most energy in $\bar{\nu}_e$ is deposited in M1.1, $5.2 \times 10^{49} erg/sec$, which is 8% of the peak luminosity. The $\bar{\nu}_e$ luminosities in M3 rise monotonically. Energy flow downgrading and energy deposition in the matter occurs for $\bar{\nu}_e$ in the same way as for $\nu_e$, at slightly higher matter densities.

It must be stressed that the models considered here may not be typical of a proto-neutron star or a supernova after shock launch and during shock stall. First, they constitute merely one 'snapshot' in the evolution of the matter background. Second, the
Figure 24: Bulk $\nu_e$ energy luminosities. Curves corresponding to different models have been scaled with a different scale factor.

Figure 25: Bulk $\nu_e$ number luminosities.
fact that the baryonic component of the matter does not include nuclei at densities below 10^{10} g cm^{-3} is unrealistic and enhances the energy deposition in the matter at low densities. Also, the constant low Y_e in the outer atmospheres, and hence the high neutron abundance aids the deposition. Moreover, it is impossible to predict here how much energy would be deposited in the course of time, as this is a question involving the matter dynamics. This will be the subject of a subsequent paper.\cite{19} Nevertheless, in realistic collapse models the bounce shock stalls roughly in the density region 10^{10} < \rho < 5 \times 10^{11} g cm^{-3} where we found that the neutrino flows deposit most of their energy. Where the shock has ploughed through it has dissociated nuclei into \alpha-particles and nucleons, so that perhaps at least in some part of the main absorptive region the free nucleon gas composition may be a reasonable approximation. Hence, the neutrino flow energy downgrading and the associated energy deposition in the matter in this region may be an indication of the delayed-explosion mechanism in its infancy.

5.2 Spectral quantities

The following presentation of spectral quantities and the resolution of bulk quantities at fixed positions in their frequency-dependent constituents is not exhaustive but concentrates on M3. The individual curves in the figures below depict spectral quantities as functions of \omega at different positions in the star. Each curve corresponds to one position.

![Figure 26: Energy \nu_e distribution spectrum e(\omega, \xi = constant) of M3. The curves are at fixed positions in the star.](image)

In Fig. 26 the \nu_e spectrum e(\omega) in M3 is shown at fixed positions in the star, some of which are labeled with their matter density. At densities above 10^{14} g cm^{-3} the \nu_e are highly degenerate and in LTE. First the low-energy gap appears between 10^{14} > \rho >
$10^{13} \text{ g cm}^{-3}$, (cf. Fig. 7, showing the LTE spectrum $b(\omega)$). Going further out the peak in the $\nu_e$ spectrum becomes lower and shifts towards higher frequencies because the low frequency gap widens. This trend is reversed and the peak starts shifting to lower energies at the beginning of the emitting region, corresponding to the peak in bulk emissivity and luminosity cf. Figs. 15, 16 and 22. In the emitting region the matter becomes increasingly transparent to the lower frequencies, while high-energy neutrinos are being absorbed and partly re-emitted again with lower energies. This is further illustrated in Fig. 27, depicting the spectral composition of the source $SQ$ (5.3). Between $5.10^{12} > \rho > 10^{11} \text{ g cm}^{-3}$ the

![Figure 27: Integrand of $\nu_e$-$SQ$ as function of $\omega$ at fixed positions in the star (M3).](image)

matter is absorptive at high energies and an emitter at low energies. This flow downgrading accounts for the rise in the luminosity at lower frequencies around 10 MeV in this region, see Fig. 28 showing the spectral composition of the luminosity at different positions. In the atmosphere the spectrum shifts slightly more to lower frequencies until at densities between $10^{10} > \rho > 10^{9} \text{ g cm}^{-3}$ the stable output spectrum is reached in the truly free-streaming region.

The $\bar{\nu}_e$-spectrum is presented in Fig. 29, and behaves very differently. In the very dense inner core it is relatively high and in LTE, cf. Fig. 8, but decreases as $\xi_\nu$ increases, cf. Fig 5. After the peak in $\xi_\nu$ is passed the $\bar{\nu}_e$ abundance rises, peaks at $\rho \approx 5.10^{12} \text{ g cm}^{-3}$ and afterwards decreases in the emitting and absorptive regions shifting its peak to lower frequencies, due to flow downgrading.

In this atmosphere only the (very) high end of the spectrum ($\omega > 30 \text{ MeV}$) is still depleted. See for this Fig. 30 where spectral luminosities are shown as functions of position for a number of energy bins. The luminosities of the high–energy bins are being depleted until relatively far out in the star, and finally level out completely between $10 < -\xi < 12$, corresponding to densities between $10^{10} > \rho > 10^{9} \text{ g cm}^{-3}$. For the bulk luminosity this
Figure 28: Spectral $\nu_e$ energy–luminosity as function of $\omega$ at fixed positions (M3).

Figure 29: Energy $\bar{\nu}_e$ distribution spectrum $e(\omega, \xi = \text{constant})$ of M3.
Figure 30: Spectral $\nu_e$ luminosities as functions of position (M3) for 6 energy bins

is not significant. However, for the detection of neutrino signals on earth this depletion at low density is not without importance. Cross sections rise quadratically with the particle energy so that high-energy neutrinos are more readily detectable.

In Fig. 31 the first Eddington factor $f(\omega, \xi)$ (2.4) is presented as a function of $\omega$. The energy dependence of the opacities causes low–energy ($< 10$ MeV) neutrinos to decouple first from LTE, at matter densities of about $10^{13}$ g cm$^{-3}$. High–energy neutrinos remain diffusive and in LTE for $p > 10^{11}$ g cm$^{-3}$.

In Fig. 32 $f$ is shown as a function of position for a selection of energy bins. The definition of the neutrinosphere as the position where $f$ reaches some predefined value, e.g. $f = 1/2$, such as proposed in Ref.[12] is meaningful only spectrally. For different bins the spatial spread in this position encompasses the whole emitting region between $23 < r < 33$ km ($5 < -\xi < 9$) as bins of increasing $\omega$ decouple further out at lower density. The thickness of the neutrinosphere amounts to 30% of the total radius of the star.

Also included in Fig. 32 is the energy averaged Eddington factor $< f > = F_{\text{bulk}}(\xi)/E_{\text{bulk}}(\xi)$. The spatial spread of $< f >$ shown in Fig. 33 for all models is up to a factor two comparable to the spread for different energy–bins per model, cf. Fig. 32. Note in Fig. 34 the small region of negative $< f >$ for $\bar{\nu}_e$ in M2. This corresponds to the inward flux that causes the non-LTE $\bar{\nu}_e$ energy and particle densities to be above their LTE values in the dense inner core above nuclear density.

This spectral analysis completes the picture sketched in the discussion of the bulk quantities. The semi–transparent region is characterized by energy downgrading of the neutrino flow. The energy is deposited in the matter and will heat it up. The downgrading and deposition mechanism is caused by the $\omega$–dependence of the opacities and redistributes
Figure 31: First $\nu_e$ Eddington factor $f(\omega)$ as a function of $\omega$ at fixed positions (M3).

Figure 32: First $\nu_e$ Eddington factor $f(\xi, \omega)$ as a function of position for a selection of 6 energy bins. The solid curve marked with the 'swiss cross' depicts the bulk Eddington factor. (M3)
Figure 33: Bulk $\nu_e$ first Eddington factors $F/E$ as function of position for all models. They are depicted from left to right in the order as given on the figure. The solid line corresponds to M4.

Figure 34: Bulk $\nu_e$ first Eddington factors $F/E$ as function of position for all models. They are depicted from left to right in the order as given on the figure. The solid line corresponds to M4.
the composition of the spectrum. Because its working depends on the local non–LTE spectra it is difficult to see how it could be reliably reproduced by a non–multigroup transport approach, without putting it in by hand.

Between $10^{10} > \rho > 10^9 \, g \, cm^{-3}$ the spectrum reaches its value observable at infinity. In order to get a complete picture of its high energy tail and to observe the energy deposition in the matter it is necessary to follow the transport down to these densities.

5.2.1 Thermal fit

Assume that the non–equilibrium energy distribution $e(\omega)$ is thermal of the Fermi–Dirac form, and involves two parameters that could be associated with a distinct neutrino temperature $T_\nu$ and a non–LTE neutrino degeneracy parameter $\xi_\nu$, such as was proposed in Ref.[12]. In that case the function $\phi(\omega) = \log(e^{-1}(\omega) - 1)$ should be a linear function of $\omega$.

In Fig. 35 this function is shown as calculated for $\nu_e$ on M3, the curves being at fixed positions in the star. Obviously the spectrum as a whole cannot be well represented this way, except for perhaps in the very dense interior above $10^{14} g \, cm^{-3}$. This conclusion was also drawn by Janka and Hillebrandt, [13] and Myra and Burrows, [14]. Although the high–energy tail at energies $\omega > 20$ MeV can be reproduced quite well by

$$e_{fit}(\omega) = \frac{1}{e^{a + b \omega} + 1},$$

the spectrum as a whole resists such a fit. However, it can be parametrized in two parts. In addition to the parametrization of the high–energy tail as given above, the low–energy part of the spectrum for $\omega < 20$ MeV, can be fitted quite well with three parameters and the form

$$e^h_{fit}(\omega) = \frac{1}{e^{a^h + b^h \omega + c^h \omega^2} + 1}.$$  

In Fig. 36 the complete fit in two parts combined, for $\omega < 20$ and $\omega > 20$MeV is shown. Such a combined 5 parameter fit is the best and involving the least fitting parameters.
found, that represents the total spectrum with statistically meaningful accuracy, but it is not very simple and does not add much to one's physical insight.

It is tempting to interpret the parameters $a$ and $b$ in the high-energy tail fit as a non-equilibrium neutrino degeneracy parameter $a = -\xi_\nu$ and an inverse neutrino temperature $b = T_\nu^{-1}$. We yield to this temptation in Figs. 37 and 38, where $T_\nu$ and $\xi_\nu$ are plotted with their LTE matter counterparts.

In the dense inner core and emitting region where the high-energy neutrinos are diffusive, this identification is obviously quite good. Outside their diffusive region at densities below $10^{11}\ g\ cm^{-3}$ the high-energy neutrinos to which the fit was made finally decouple from the matter and their temperature and chemical potential start to deviate from LTE significantly.
Figure 38: $\nu_e$ non-equilibrium degeneracy parameter, obtained from high-energy tail fit (M3).

Figure 39: Spectral $\nu_e$ energy luminosity (dashed curve), together with the one fitted with the high-energy tail parameters (dotted curve) (M3) as seen at infinity.
The neutrino signal from SN 1987A largely sampled the high-energy tail of the spectrum. Certainly this was true for the IMB detections, with the IMB detector threshold at 20 MeV. Most, if not all analyses of the neutrino signal assumed, imposed or concluded that the observed spectrum was thermal of the form (5.8), at zero or nonzero chemical potential, and identified the fitting parameters as above. It was then assumed that the whole spectrum, the low-energy part included, can be represented with these parameters.

The fitted spectrum was used in the calculation of the total energy and particle luminosities. These were then in turn used in arguments e.g. on the neutrino magnetic moment, the existence of neutrino oscillations, and axion emission.

Let us pretend for the moment that the spectra calculated here are ‘real’, and identify the fitting parameters of the thermal fit to the high-energy tail with \( T_\nu \) and \( \xi_\nu \). Assume that the whole spectrum is thermal and represented by these parameters. Calculate the bulk luminosity using this fitted quasi-thermal spectrum. This approach results in Fig. 39 where the luminosity spectrum as it leaves the star is given together with its fit, and in Fig. 40 in which the real and fitted bulk energy luminosities throughout the star are plotted. Although the high-energy tail of the luminosity is in very good agreement with the thermal fit, the low-energy part and peak are not. This can be understood in terms of the low-energy gap. In fact the fitted luminosity overestimates the real one by a factor of nearly three.

### 5.2.2 Time dependence

Until now the time-dependence of the neutrino flow has not been discussed and all results were presented in the steady state. For the situation at hand the stationary state is of higher physical interest than the time-evolution of \( e(\omega, \xi, t) \). The latter strongly reflects the initial condition, which was to some degree arbitrary. A different initial condition, taking, e.g., \( e(\omega, \xi, t = 0) = b(\omega, \xi, t = 0)/2 \) results in a different evolutionary track. In contrast, the stationary solution is unique, because in the steady state (2.1) reduces to a boundary value problem. In fact, the time evolution in a static background constitutes

---

**Figure 40:** Bulk \( \nu_e \) energy luminosity (dashed), and the one fitted with the high-energy tail parameters (dotted).
Figure 41: Time-evolution of the bulk $\nu_e$ energy luminosity in M3. Snapshot-times: $t_1 = 10^{-6}, t_2 = 10^{-5}, t_3 = 10^{-4}, t_4 = 5.10^{-4}$ sec.

Figure 42: Time-evolution of the bulk $\bar{\nu}_e$ energy luminosity in M3. Snapshot-times: $t_1 = 10^{-6}, t_2 = 10^{-5}, t_3 = 10^{-4}, t_4 = 5.10^{-4}$ sec.
Figure 43: Time evolution of the 3.65 MeV $\nu_e$ bin energy distribution in M3, as function of position.

Figure 44: Time evolution of the 3.65 MeV $\nu_e$ bin Eddington factor $f$ in M3, as function of position.
Transport results

no more than the relaxation of a transient, and is of little dynamical importance. Nevertheless, for a full evolution calculation in which the matter is also dynamical, being able to solve (2.1) with the time dependence included is important. In that case the initial condition for each transport step is no longer arbitrary but is the state of the system at the previous time. In Figs. 41 and 42 the time evolution of the bulk $\nu_e$ and $\bar{\nu}_e$ energy luminosities in M3 is plotted.

![Graph](image1)

Figure 45: Time evolution of the 30.36 MeV $\nu_e$ bin energy distribution in M3, as function of position.

![Graph](image2)

Figure 46: Time evolution of the 30.36 MeV $\nu_e$ bin Eddington factor $f$ in M3, as function of position.

The evolution takes place on a $10^{-5}$–$10^{-4}$ sec time scale. This relaxation time scale is the same for all models. Individual (lower) energy bins evolve on a somewhat longer time scale, but after 0.1 msec bulk quantities no longer change very much. Individual bins reach the stationary state after at most 0.5–1 msec, see Figs. 43–46, when we apply the
rigorous local requirement for stationarity mentioned in Sec. 4.

The relaxation time scale of the neutrino flow is much shorter than the dynamical evolution time scale of the matter in the proto-neutron star, which is of the order of at least milliseconds. If the proto-neutron star evolution can be calculated with discrete time steps of $O(10^{-4}-10^{-3})$ sec, the neutrino flows will be evolving essentially from one stationary state into the next.

5.2.3 Knudsen plateau

The first Eddington factor $f(\omega)$ (2.4) of the high-energy bins $\omega > 30$ MeV, develops a 'shoulder' in its position dependent profile at the transition from the semi-transparent into the free-streaming region. This effect was previously reported in Ref. [15]. In the $f-\xi$ plot (Fig. 32) the steep rise in $f$ levels off somewhat, a plateau develops, which is followed again by a steeper rising region towards $f = 1$. This behaviour only occurs in the high-energy bins. The value of $f$ in this plateau depends relatively weakly on the energy of the bin, and lies at $f \approx 0.78$. This value does depend on the matter background model, in particular on the value of $Y_e$ at the position of the plateau.

With these observations in mind an analysis of this behaviour can be made in an attempt to understand it. All assumptions made in the following analysis were afterwards checked and found to be correct. The $e(\xi)$ profile of a high-energy bin that exhibits a $f(\xi)$-plateau in the steady state is a solution of the time-independent energy balance equation, (2.1) with the $\partial_t e$ term dropped. Splitting out the gradient term and dividing the equation by $e$ leads to

$$\frac{1}{r^2} \partial_r(r^2 f) + \frac{f \partial_e e}{e} = -\kappa_a^*(1 - b/e). \quad (5.10)$$

Because the gradient of $f$ is small in the plateau, and it occurs in the outer regions of the star, the first term on the lhs can be neglected with respect to the others. The plateau develops in the atmosphere only for high-energy bins, for which in this region $b(\omega)/e(\omega) \ll 1$, so that the second term on the rhs can be dropped. Because $f$ is not far from its asymptotic value $f = 1$, it can in good approximation be expressed as

$$f(R) \approx 1 - 1/R. \quad (5.11)$$

Substituting this, inserting the definition of $R$ (2.5) and dropping another term of $O(b/e)$ leads finally to the prediction

$$\frac{\partial_e e}{e} = \kappa_{tot}, \quad (5.12)$$

with $\kappa_{tot}$ given in Eq. (3.5). Using (2.5) and $f(R) \approx 1 - 1/R$ this can be translated into

$$f = \kappa_a^*/\kappa_{tot}. \quad (5.13)$$

If the expressions for the opacities and the value of $Y_e = 0.15$ are substituted in (5.12), and under the assumption that there is no phase space inhibition in this region for high-energy bins, i.e. the blocking factor (2.6) $\theta = 1$ in $\kappa_a^*$, see Fig. 47, the observed value of $f$ in the plateau is obtained. The plateau will only develop if the following circumstances are simultaneously met. First, $b(\omega) \ll e(\omega)$, the medium must be a pure absorber at that particle energy. Second, in the region where it occurs, phase space inhibitions must be absent, i.e. $\theta = 1$ see Fig. 47. These two requirements are only met by high-energy bins. Third, $Y_e$ must be constant in this region, because otherwise with (5.12) it is clear that $f(\xi)$ could not form a plateau.
The prediction (5.11) can be interpreted on a somewhat more fundamental level by noting that the ratio of the lhs and the rhs of Eq. (5.11) is the ratio of the microscopic and macroscopic length scales, given by the total mean free path $l = \kappa_{\text{tot}}^{-1}$, and energy scale height $L = \left|(-\partial e/e)^{-1}\right|$ respectively, and can be interpreted as the local Knudsen number ($Kn$) of the system. Apparently, the $e(\xi)$ profile develops in such a way that in the stationary state $Kn$ is constant and of order unity in this shoulder plateau region. That this is indeed the case is clear from Fig. 48, where it is shown for a number of energy bins in M3.

Figure 47: The blocking factor $\theta$, which is included in $\kappa_\alpha^* (\omega)$. It is plotted for four energy bins. Note that only for the two highest energy bins $\theta = 1$ in the region where the Knudsen–plateau develops.

Figure 48: The Knudsen number as a function of position for four selected energy bins. Note the plateau forming at $Kn \approx 1$ between $9 < -\xi < 11$ in the high–energy bins.

The reason for this behaviour at the transition from the hydrodynamic regime, $Kn =$
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\( l/L \ll 1 \), to the streaming regime, \( Kn \gg 1 \), is unclear. However, the resulting \( c(\xi) \) profile is a perfectly good stationary solution of Eq.(2.1), and therefore unique. This behaviour is exhibited independently of the numerical details of the solution method, and was reproduced in all models by all versions of the transport code, irrespective of whether these were formulated as a conservative scheme or as an explicit differential equation,\(^{15}\) and with several different OBC's implemented. It is therefore not likely that the effect is a purely numerical artefact, nor that it is caused by the OBC. Although unexpected and remarkable, the 'shoulder' effect is little more than of academic interest. It is probably a peculiarity of FNDT as it has not been reported in other transport schemes. However, this may be due in part to the fact that in the realistic models usually used \( Y_e \) is not constant, so that a shoulder would not appear.

5.2.4 Anisotropy

The angular dependence of the distribution function is averaged over, and this information is lost as a consequence of the use of an angular moment method such as this. Nevertheless it is instructive to investigate the angular dependence \( \psi \) of the intensity \( I(\omega) = (\omega/2\pi)^3 f(\omega, r, t) \) which was defined\(^{31}\) as

\[
\psi(\omega, \Omega, r, t) = I(\omega, \Omega, r, t)/E(\omega, r, t),
\]

and features prominently in the derivation of F(N)DT, see Refs.[1]–[3]. Within the formalism of F(N)DT it is expressed as

\[
\psi(\Omega, \omega, r, t) = \frac{1}{4\pi} \frac{1}{1 + f \cdot R - R \cdot \Omega},
\]

The angular dependence of \( \psi \) lies in the combination \( R \cdot \Omega = Ru \) with \( u = \cos(\theta) \). From this it immediately follows that \( f \cdot R = fR \). Of course, calculating \( \psi \) with values \( f \) and \( R \) calculated in the transport code does not yield \( \psi \) as a solution of the Boltzmann equation.

Taking the values of \( f \) and \( R \) in the stationary state of the 16.65 MeV bin of M3, and using these to calculate \( \psi \) as a function of \( u \) at different positions in the star, we obtain the plot shown in Fig. 49. In FNDT it is natural for \( R (2.5) \) to become large when the mean free paths of the neutrinos become large compared to the macroscopic scale height. In Ref.[1] it was shown that in this limit

\[
\psi \approx \frac{1}{4\pi} \frac{1}{R(1 + \delta(R) - u)},
\]

with the regulator \( \delta = 2e^{-2R} \) keeping the expression finite in the singular point \( u = 1 \). Because \( R \to \infty \) in this limit, any contribution to an angular integral effectively comes from this pole. The distribution function will be forward peaked. This prediction is clearly vindicated in Fig. 49. In the diffusive inner core at \( \rho > 2.10^{12} g cm^{-3} \) \( \psi(u) \) is constant and the distribution is isotropic. Going to lower densities it becomes increasingly more forward peaked. In the free-streaming region near the edge of the star the angular distribution is a sharp delta-like peak. The extended semi-transparent region acts as an effective collimator, and the neutrinos transported with FNDT leave the star as radially directed bundles or 'rays'. To an observer equipped with neutrino-sensitive eyes the star would appear as a single dot, not as a disc. Forward peaking behaviour is expected to happen on very general geometrical considerations, although Monte Carlo simulations\(^{14}\) seem to suggest that it occurs in a less extreme fashion than is sketched here.
Figure 49: The angular part of the $\nu_e$ distribution in M3 at different positions. The distribution is isotropic ($\psi = 1/4\pi$) at densities above $2.10^{12} g cm^{-3}$, and becomes increasingly forward peaked in the semi-transparent and free-streaming regions.

The forward peaking as presented here does not prove anything because $\psi$ is not a solution of the Boltzmann equation. Still it suggests that the anisotropy of the neutrino flow may occur easily and may be quite extreme. This would strongly suppress the efficiency of $\nu\bar{\nu}$-annihilation as a heating source\cite{17} in the delayed explosion scenario, as was previously argued by Cooperstein et al. in Ref. [18].

6 Conclusions

The semi-transparent region in all models lies broadly between $5.10^{12}$ and $10^{10} g cm^{-3}$ and contains an emitting and an absorptive part. As a whole it is characterized by the down-grading of the energy in the neutrino flow which shifts the peak in the neutrino spectrum towards lower particle energies. In the emitting region the neutrinos gradually decouple from the matter background, which first becomes increasingly transparent to low-energy neutrinos, while it still absorbs high-energy neutrinos. These are largely re-emitted with lower energies. Because of this re-emission, more energy than lepton number is deposited in the layers where the emitting region turns into a net absorptive one. Therefore, the bulk lepton number luminosities peak at lower matter densities than the energy luminosities, or do not peak at all. The flow downgrading decreases the average particle energy. The emerging average particle energy as detectable by an observer at infinity is not very sensitive to the degeneracy of the deep interior of the star.

The energy that is lost from the neutrino flow is deposited in the matter. The total $\nu_e$ energy deposition in the layers around $10^{11} g cm^{-3}$ ranges from $6.8 \times 10^{52}$ (M2) to $9.10^{51} erg sec^{-1}$ (M4). Per second this would amount to several times the total explosion energy of a supernova. The $\bar{\nu}_e$ flow deposits less than one percent of this amount in the models considered, because they are much less luminous in $\bar{\nu}_e$ than in $\nu_e$.

In the results presented here, the energy deposition is exaggerated at $\rho < 10^{10} g cm^{-3}$ because the baryonic component of the matter in these models is a free nucleon gas and does not include any nuclei, which have a much lower $\nu_e$ capture cross section than nucle-
ons. Inclusion of neutrino–electron scattering, pair and plasmon processes and GR may modify the deposition picture further. Nevertheless, in view of the relative magnitude of the cross sections involved, it seems reasonable to assume that the $\beta$–reactions are responsible for the bulk of the behaviour.

It is impossible to predict at this point how the luminosity and energy deposition will develop in time. This question involves the dynamical evolution of the matter, and will be the subject of a subsequent paper. The time–integrated energy deposition over the whole cooling phase will certainly be much lower than a linear extrapolation of the deposition rates found here would suggest. However, the initial ‘flash’ energy deposition rate bears some promise. It occurs in the right place, and if it remains relatively high, it would be certainly more than sufficient to drive a delayed explosion.

High electron, and hence neutrino, degeneracy ($M_1$ and $M_2$) causes the decoupling of the neutrino flow from the matter to occur at lower matter densities than in less degenerate settings ($M_1.1, M_3, M_4$), for obvious reasons. High electron degeneracy produces higher $\nu_e$, and very much lower $\bar{\nu}_e$ luminosities.

Spectra still change at densities as low as $10^{10} - 10^{9} g\, cm^{-3}$, mainly in the high–energy tail at $\omega > 40$ MeV. Because it is the high–energy neutrinos that are most easily detectable, it is necessary to extend neutrino transport to these densities in order to calculate the spectrum at infinity. This has been previously pointed out by Janka and Hillebrandt.

The outgoing stationary spectra are ‘pinched’, showing a deficit in both the low and high–energy ends of the spectrum. This feature has been reported by many authors, in particular Janka and Hillebrandt from Monte–Carlo (MC) simulations, Myra and Burrows, and Suzuki.

The emergent neutrino spectra are without exception non–thermal, mainly due to the low–energy gap. It is impossible to represent the entire spectrum with a thermal Fermi–Dirac distribution even if it involves a nonzero chemical potential. Such a fit can be made only on the high–energy tail of the spectrum. In practice, neutrino detections on earth heavily sample this part of the spectrum. The 19 events from SN 1987A were no exception. When neutrino temperatures and chemical potentials derived from such a thermal fit to the high–energy tail are used to represent the spectrum, the calculated bulk energy luminosity leaving M3 is overestimated by a factor of three.

Because of the $\omega$–dependence of the opacities, neutrinos of different energies decouple from the matter background at very different positions. The concept of a neutrinosphere as a geometrical entity for the neutrino flow as a whole is therefore meaningless. It is difficult to envision a realistic bulk transport scheme. First, such a scheme always involves an assumption on the functional form of the non–equilibrium spectrum, like in Ref.[12]. Because the spectrum is intrinsically non–thermal, it is problematic to construct a realistic Ansatz. In addition, some prescription is needed to calculate a bulk first Eddington factor $< f >$. No obvious simple candidate for such a prescription which would not require a spectral calculation, springs to mind.

Apart from these conceptual objections, it is unlikely that a bulk transport scheme could in a natural way reproduce the downgrading of the neutrino flow. The downgrading requires knowledge of the non–LTE spectrum $e(\omega)$ at each position in the star.

It is no wonder that bulk and leakage schemes which simply declare the neutrinos to be free–streaming below $10^{12} - 10^{11} g\, cm^{-3}$ fail to produce delayed explosions. They throw the effect a priori away in the region where it occurs. Fancier bulk transport schemes, even flux–limited ones, will probably fail as well, or succeed by construction where FNDT displays the effect naturally. Stopping the calculation at $10^{11}g\, cm^{-3}$ misses most of the
energy deposition and therefore misrepresents the output spectra.

The neutrino flow evolves with a relaxation time scale of $10^{-5} - 10^{-4}$ sec. After maximally 0.5–1 msec the stationary state is reached. This time scale is much shorter than the dynamical matter evolution time scale. Treating the matter as a static background during one transport-step is therefore justified. If in a fully dynamical calculation the evolution can be followed with time steps of $O(10^{-4} - 10^{-3}$ sec), the neutrino flows will evolve from one stationary state into the next.

The results produced by FNDT are qualitatively consistent with other transport approaches,\cite{14,6,13,14} including physically more rigorous ones, like the MC simulations of Janka and Hillebrandt.\cite{14,13} A quantitative comparison is complicated by the differences in matter backgrounds. However, the FNDT transport scheme is orders of magnitude faster, and is therefore suitable for use in dynamical calculations.

The transport code is sufficiently robust to simulate the neutrino transport in a hot neutron star in a wide range of environments. The code transports $\nu_e$ with the same ease and using the same time step and accuracy control as $\nu_x$. It can deal with steep gradients, inward fluxes, inverse temperature and chemical potential gradients and high electron and neutrino degeneracy. The accuracy in $e$ and $f$ is better than one percent. This has been estimated by using different grids, different differencing schemes and different time step controls on the same models and measuring the ensuing differences in the stationary profiles.

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References


