The universe on edge: Limits of the effective field theory approach in the very early universe
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CHAPTER 1

EFFECTIVE THEORIES FOR GRAVITY IN COSMOLOGY

1.1 THE REASONABLE EFFECTIVENESS OF PHYSICS IN THE COSMOLOGICAL SCIENCES

Why does physics work? Why is it a predictive science whose mathematical methods we can use to theoretically produce an observable and experimentally testable result?

The world around us is complex. Most processes involve a multitude of particles which are connected to each other with a multitude of different interactions governed by physics we might not even know yet. Hence, at first sight, it seems as if there was no reason to hope to be able to calculate anything useful, let alone correct.

The reason why this is still possible is that physics has scales. Different physics is relevant at different scales. A scale is a range of, say, energy, momentum, inverse length or time, which are all equivalent for that matter, within which the dynamics can be described to any desired level of accuracy by using a particular theory. There is a well-known procedure how to determine this theory as a simplification of physics at higher scales, or how to devise a theory which works although the physics at higher scales is completely obscure. Such a theory is known as a (low energy) effective theory. It describes the essential phenomena for a certain part of the parameter space.

The idea of reducing complexity on lower energy scales originates in statistical physics. Systems, which generically consist of an order of the Avogadro number \( N_A = 6.022 \times 10^{23} \) particles and would be far too complicated to describe on the microscopic level of their mutual interactions, can be understood much better on a low energy scale in terms of only a few macroscopic
variables like temperature, pressure, entropy, chemical potential or magnetization. In this case, the low energy approximation simplifies the description of the system. We use these thermodynamic variables instead of the fundamental ones like momenta and positions of the individual particles. What makes it somewhat natural to perform this limit is the fact that thermodynamical systems very often have an inherent upper bound on the energy scale. When describing a gas or a fluid, the low energy physics is governed by the inter-atomic interactions and atomic physics itself can be safely discarded. Therefore, the size of an atom can be taken as the cut-off, the inverse of the maximum energy scale. Of course, this cut-off represents a limit of the validity of the approximation. The closer the energy scale of interest lies to the cut-off, the more the effects of atomic physics will play a rôle and can no longer be ignored. In other words, at the cut-off scale, new physics enters the description.

Very interesting phenomena that can be studied in the low energy effective theory are phase transitions. At a generic point, thermal fluctuations of a continuous field are correlated only over a few atomic distances. At a critical point, however, the correlation length diverges and becomes of the order of magnitude of the material sample. A critical point is the point in parameter space where the order parameter, which emerges due to a first order phase transition, becomes continuous and the difference between the two adjacent phases vanishes. In other words, at the critical point, a single thermodynamic state bifurcates into two distinct phases. This leads to long-range thermal fluctuations, which characterize a second order phase transition. It turns out that, remarkably, very different systems can share essential properties around a critical point, e.g. in the vicinity of the critical temperature. Assuming just a few general symmetries, Landau theory predicts, that a wide range of system show universal behavior of characteristic quantities like the order parameter, the correlation length and other thermodynamic quantities, which universally depend on the temperature exponentially like \((T - T_C)^\alpha\). This universality is also remarkable, because it shows that most of the relevant physics at low energies does not depend on the high energy behavior. However, the power with which the thermodynamic quantities depend on the temperature, the critical exponent \(\alpha\), can differ between different systems. Systems with the same critical exponent form a universality class. It should be noted that at the critical point, those quantities do not scale at all, i.e. they are scale independent.

When turning from a classical to a quantum theory, we can replace the statistical fluctuations with quantum fluctuations and describe the statistical system near the critical point with a quantum field theory. The mass of the quantum field is taken to be well below the cut-off scale and must vanish at the critical point. In this case, the field theory at the critical point has no scales and is a conformal field theory. The universal behavior of the multitude of phase transitions in nature is caused by the low energy physics being independent of the physics at the cut-off. The quantum field theory is renormalizable and predictive, after determining a finite number of parameters from experiment. Renormalizability and universality are the two sides of one coin. The parameters which cannot be determined theoretically are linked to the infamous ultra-violet divergences of quantum field theories. They signal that an infra-red quantity depends on the UV physics. If physics above the cutoff is unknown, as is generically
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the case in a quantum field theory, the value of such quantities, like the charge and mass of an elementary particle, cannot be derived but only measured. On the other hand, if a specific interaction in the UV involving heavy fields has been integrated out, the interaction can enter the effective theory as non-local.

It is a remarkable feature that we can write down a theory which accurately describes the physics at accessible scales, without the need to resort to knowledge about fundamental physics. Though it is possible to simplify physics by distilling the relevant physics from a UV complete theory, it is impossible to infer the fundamental degrees of freedom from the low energy physics. Yet, we know that new physics must enter at several scales. The aim of this thesis is to contribute to the quest for such new physics.

Since the standard model of particle physics, as a very successful application of these ideas to high-energy particle physics, unifies three of the fundamental forces of nature, the electromagnetic, strong and weak interactions, into one framework of quantum gauge field theory, new physics is expected to come mainly from gravity. It turns out that gravity cannot be described by the same framework which we used above because it is non-renormalizable. The scale at which new degrees of freedom of gravity enter the description is conjectured to be the Planck-scale, which is at $1.22 \times 10^{19}$ GeV. In terms of a length scale, this would be $1.612 \times 10^{-35}$ m, which is about $10^{20}$ times smaller than a proton. At this size, it is not even clear if it makes sense to talk about space-time as a geometric concept, let alone define a quantum field theory. This scale seems to be nearly inaccessible experimentally, too. As a comparison, the Large Hadron Collider (LHC), which seems to be on its way to measure the last parameters needed to complete the standard model, reaches a top energy of 14 TeV, still an order of $10^{16}$ short of probing this scale.

We have one system at hand, which at some time in the past most certainly has been governed by physics at the highest possible energy range: namely the universe. In our current understanding the universe is now expanding and has been doing so since its beginning, the big bang. This means that its density is becoming smaller and smaller, whereas if we return in time towards the big bang, its density grows and with it the temperature and the energy. The furthest back in time we can experimentally look is observing the cosmic microwave background radiation (CMB), which was formed approximately 360,000 years after the big bang. This was the time when, due to recombination of hydrogen and helium ions with free electrons, the universe became transparent for photons. At this point, the universe had a temperature of about 3000 K or 0.2 eV, much less than the Planck scale. However, the gravitational physics governing the behavior of the plasma descends from Planck scale physics, which is thus imprinted on the CMB. It has been measured to very high accuracy over the past years by the COBE and WMAP missions and by the PLANCK satellite, whose results are expected to come forward in the coming year at the time of writing. This means, that there is a good chance to learn something about new physics at the Planck scale from such high-precision measurements of the CMB.

1 However, it might be possible to explore the Planck-scale via the effects its physics has on the physics at accessible scales. The Alpha-experiment might be able to discover a breaking of the Lorentz- or CPT-symmetry in comparing the spectra of hydrogen and anti-hydrogen which might come from Planck scale effects [7].
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This thesis is not so much concerned with the observational wealth of the early universe but with the theoretical puzzles which it imposes on us. For only if we understand the early universe well enough such that our model of it is without any conundrums can we hope to interpret the data in a useful way. There are quite a few problems in the very early universe, which we can not understand very well with the technology of effective field theory as explained above. Instead, we have to resort to techniques coming from candidates for a quantum theory of gravity to tackle them. String theory has for a long time been such a candidate, which changes the rules of the effective field theory game quite a bit. While in this framework it is possible to solve the renormalizability issues of gravity, it has not yet been predictive to the point where it could have been supported by experimental evidence.

I have picked two topics which shed light on the problems which arise when combining novel with established techniques in the early universe. The first problem is connected to its very beginning, the big bang itself. In the standard cosmological model it is viewed as a singularity, because as the contraction of the universe is extrapolated into the past, its density diverges and so does the curvature. It is important to notice, though, that this singularity is an artefact of using an effective theory, namely general relativity, outside of its range of validity. It is a theory which is in particular only valid at large distances and has in fact not been tested below about 55 \( \mu \text{m} \). At the Planck scale, new physics is expected to resolve the initial singularity. For instance, in string theory, gravitons are oscillations of the string, which itself shrinks to a point at low energies and the effective theory treats it as a conventional particle. I am presenting an attempt to understand this using the Anti-de Sitter/conformal field theory (AdS/CFT) correspondence (chapter 2). With the help of this duality, new physics originating from string theory becomes tractable in the dual conformal field theory. There, a space-time at strong coupling, i.e. at high curvature, is described as a field theory at weak coupling. Briefly after the big bang at about \( 10^{-36} \) to \( 10^{-33} \) s, current cosmological models assume an exponentially accelerated expansion of the universe. This so-called inflation is driven by the potential energy of some new degree of freedom. Although the universe is already below the Planck energy at this time, this potential must descend from a fundamental theory which includes gravity. Thus, inflation is a prototypical example of the effective field theory approach: All the physics above some cut-off, which is assumed to be close to the Planck scale, gets integrated out to obtain an effective theory, which has to contain at least one scalar degree of freedom that can serve as the inflaton. The shape of its potential, which is subject to conditions ensuring that inflation works, descends from the UV physics. While this is currently a matter of taste, there are generic operators which any UV safe theory of gravity will deliver. And from these, general conclusions for the physics of inflation can be drawn. Supergravity is the low energy effective theory of string theory. String theory has numerous scalar fields, which are assumed not to take part in the inflationary dynamics. I present an analysis in supergravity, in which I examine to what extent the new inflationary degrees of freedom can be separated from all the other degrees of freedom that are usually silently assumed not to participate in the dynamics of the inflationary period (chapter 3). It turns out that strong restrictions apply to building such models, even stronger than usually assumed.
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1.2 Renormalization and low energy effective theories

I will begin this exposition by linking the problems of quantum field theories with the ideas of an effective field theory. A quantum field theory is generically defined perturbatively in a small coupling constant $\lambda$ around some free point. The perturbative series is conveniently represented in terms of Feynman diagrams. To calculate the amplitude of a scattering process or interaction with specified in- and out-states we draw, evaluate and sum all Feynman diagrams with these states as external legs. The perturbative order in the coupling at which each diagram contributes depends on the number of loops and calculating all diagrams containing a particular number of loops means approximating the theory to that level of accuracy.

A particularly simple case is the so-called $\phi^4$-theory of one scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4.$$  \hspace{1cm} (1.1)

Like any quantum field theory, Feynman diagrams containing loops are UV-divergent. This comes from the fact that particles can go around a closed loop with any momentum, which needs to be integrated over. This can be avoided by introducing a UV cut-off, like the spacing of a lattice. If there is no natural cut-off, those divergences can be cancelled by counter-terms to obtain a finite answer. They are introduced by defining the parameters in the theory to get

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + \frac{1}{2} \delta \frac{\delta}{\partial_{\mu} \phi)^2 - \frac{1}{2} \delta m^2 \phi^2 - \frac{\delta \lambda}{4} \phi^4.$$  \hspace{1cm} (1.2)

For the one loop order, the structure of the four-field interactions is depicted in figure 1.1. These radiative corrections will change the value of the parameters like masses and couplings depending on the scale, at which they are measured by a specific experiment. They are called the physical couplings, whereas the parameters $\lambda, m$ in the Lagrangian are the bare couplings, which have no real meaning. Therefore, those parameters and with them the theory needs to be defined at a specific scale. This needs to be done for every parameter and every divergence. A useful set of renormalization conditions is, for instance, to define

- $m^2$ to be the pole of the renormalized propagator and
- $\lambda$ to be the 4-point amplitude of the scattering amplitude at zero momentum.
Now, summing the Feynman diagrams to any desired order while maintaining the renormalization conditions will produce a finite result which is independent of the regulator. In that way, the counterterms transform the UV divergences into scale dependence.

This procedure can be performed if the theory is \textit{(super-)renormalizable}. This is the case if all the coupling constants of a theory have non-negative mass dimension. If there is an interaction which has a coupling constant with negative mass dimension, it is \textit{non-renormalizable}.

When imposing a cutoff, we basically discard all the dynamics of the higher momentum contributions. Consider the generating function (or for that matter the action, the Hamiltonian or any other object which describes the full theory)

\[
Z[J] = \int D\phi e^{i \int (L + J \phi)} = \left( \prod_k \int d\phi(k) \right) e^{i \int (L + J \phi)} .
\] (1.3)

The effect of a UV cutoff \( \Lambda \) is to set \( \phi(k) = 0 \) for \( |k| > \Lambda \). The difference between this and the full theory is precisely the integral over the Fourier components with momenta higher than the cutoff. These modes are being \textit{integrated out}. This can be done in several steps, lowering the cutoff more and more. If the steps are taken to be infinitesimally small, integrating out high momentum modes leads to a continuous transformation of the parameters of the theory. Thus, going from higher to lower energies introduces a flow of the coupling constants. Since the coupling constants define the theory, one can perceive this flow as a trajectory in the space of possible theories. This idea has become known as the \textit{renormalization group (RG)} \[9\].

Of course, around a point where all coupling constants vanish \( m^2 = \lambda = \cdots = 0 \), the theory does not change any more under a scale transformation. This point is called a free \textit{fixed point}, where the theory is scale invariant or a \textit{conformal theory}. There are also fixed points which are not free, like the Wilson-Fisher fixed point in \( \phi^4 \) theory.

The picture of integrating out higher momentum modes also sheds some new light on the systematics of renormalizability with different kinds of couplings in a theory. Those couplings can be seen as local operators which perturb the fixed-point Lagrangian. Operators whose coefficients grow while going down the energy scale are called \textit{relevant} operators, because in the statistical picture they determine the low energy physics, whereas operators whose coefficients diminish are called \textit{irrelevant} operators. An operator is relevant if its mass dimension

\[
d_i = N \frac{d - 2}{2} + M ,
\] (1.4)

with \( N \) the number of scalar fields, \( M \) the number of derivatives in the operator and \( d \) the number of space-time dimensions smaller than the space-time dimension, \( d_i < d \), and irrelevant if it is larger, \( d_i > d \). If the mass dimension is the same as the space-time dimension, \( d = d_i \), it is called \textit{marginal}, which means that its relevance is determined by quantum corrections. Operators which are exactly marginal to all orders of perturbation theory do not perturb the theory away from a conformal point. An example of such a case is \( \mathcal{N} = 4 \) Super Yang-Mills theory. Note that the RG transformation is lossy and works only one way to lower energies.
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It now becomes clear why a low energy theory is always fairly simple irrespective of how complicated it was in the UV. The cutoff $\Lambda$, which was originally introduced as an artificial regulator, now plays the rôle of a physical scale at which a theory contains some rich physics. As momenta $k$ become smaller with respect to this cutoff, the bigger part of this physics scales away as $(k/\Lambda)^{d-1}$. At every order in $1/\Lambda$, non-renormalizable operators are introduced into the effective theory. However, since it is only valid for small energies $k \ll \Lambda$ only renormalizable operators play a rôle. Furthermore, theories within one universality class are distinguished only by irrelevant operators.

Now, a theory could start at a UV fixed point and be perturbed by some relevant deformation which, as we go down in energy, would make the theory flow away to a new IR fixed point. Again, it is not possible to go the other way round, because it is not a priori clear which of the infinitely many possible irrelevant operators to add, unless some UV symmetry restricts them. A theory is called UV complete or UV safe, if there are not any further irrelevant operators entering at some scale. The theory then captures all physics. If this is connected with a UV fixed point, the theory can either be asymptotically free like QCD, which means that the coupling becomes arbitrarily small at high energies, or asymptotically safe if the UV fixed point is not free. It is also possible that the coupling becomes infinite at a finite energy as in QED. In that case, it is clear that the theory is not UV complete but if it was, such a pole would signal that the perturbative approximation breaks down.

The way to find fixed points and determine the precise trajectory of the RG-flow in the space of possible Lagrangians is to exploit the properties of the UV divergences of the theory. Having removed such divergences by introducing counterterms and adjusting the amplitudes to match the renormalization conditions, the result is dependent on the renormalization scale, the momentum scale at which the conditions are applied. This dependence encodes the information of the renormalization group flow. Around a critical point with $m^2 = 0$, the renormalization conditions we have been using earlier would lead to singular counterterms. We avoid this by imposing the renormalization conditions at arbitrary space-like momenta $p^2 = -M^2$, namely

- the 2-point function vanishes at $p^2 = -M^2$,
- the derivative of the 2-point function vanishes at $p^2 = -M^2$ and
- the 4-point function is $-i\lambda$ at $s = t = u = -M^2$.

Thus, the Green's functions are fixed at a certain point and UV-divergences are removed. The theory is defined at some scale $M$.

From these conditions we can now work out the flow equation of all the couplings, the so-called Callan-Symanzik equation [10][11]. There is no preferred scale to define the theory and we could have just as well used $M' \neq M$ as our renormalization scale. This change of scale would only affect the renormalized Green's functions, whereas the bare theory would not see it at all. The connected $n$-point function in renormalized perturbation theory is

$$G^{(n)}(x_1, \ldots, x_n) = \langle \psi_0 | T\phi(x_1) \ldots \phi(x_n) | \psi_0 \rangle_{\text{connected}}.$$  

(1.5)
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Under an infinitesimal shift $M \rightarrow M + \delta M$, the couplings and the fields have to scale as

$$\lambda \rightarrow \lambda + \delta \lambda, \quad \phi \rightarrow (1 + \delta \eta) \phi$$

(1.6)

(1.7)

to keep the bare Green’s function invariant. The rescaling of the fields will then introduce a
shift in the renormalized Green’s function

$$G^{(n)} \rightarrow (1 + n \delta \eta) G^{(n)}.$$  

(1.8)

This can be written as a differential

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M} \delta M + \frac{\partial G^{(n)}}{\partial \lambda} \delta \lambda = n \delta \eta G^{(n)}.$$  

(1.9)

Introducing the dimensionless parameters

$$\beta = \frac{M}{\delta M} \delta \lambda, \quad \gamma = - \frac{M}{\delta M} \delta \eta,$$

(1.10)

we can write this relation as

$$\left( M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial \lambda} + n \gamma \right) G^{(n)}(x_1, \ldots, x_n, M, \lambda) = 0.$$  

(1.11)

The $\beta$-function and the anomalous scaling $\gamma$ are universal for all $n$ and independent of the coordinates. Both depend on the coupling $\lambda$. The $\beta$-function describes the running of the coupling and the anomalous dimension $\gamma$ the shift in the scaling dimension. Both relate the shift of the couplings, which compensates for the shift in the renormalization scale. A vanishing $\beta$-function means that a theory is conformal, while a negative $\beta$-function means that it is asymptotically free.

A complete theory should include all fundamental forces of nature. The standard model, albeit very successful, does not contain gravity because gravity is non-renormalizable (cf. e.g. [12]). The Einstein-Hilbert action

$$S = \frac{1}{16\pi G_N} \int d^d x \sqrt{g} R[g]$$  

(1.12)

is derived from general coordinate covariance as a symmetry principle. The mass dimension of Newton’s constant is

$$[G_E] = 2 - d.$$  

(1.13)

With the redefinition $2\kappa^2 = \frac{1}{16\pi G_N}$ using linearized gravity $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, this yields

$$S = \frac{1}{2} \int d^d x \left[ (\partial h)^2 + \kappa (\partial h)^2 h + \ldots \right],$$  

(1.14)

which looks like a perturbation around a fixed point. Now we see that the gravitational interaction is irrelevant for $d > 2$. Therefore, gravity is non-renormalizable, which means that it can only be treated with the use of an effective theory. We will return to the question whether there might be a UV fixed point in the next section. Since the cutoff scale for gravity is the unimaginably large Planck scale $M_{pl}$, this almost never produces any problems. Cosmology, however, probes this energy scale and the effective field theory description will break down at the beginning of the cosmological evolution. This breakdown, when carefully approached, can teach us lessons about the new physics, which we want to discover at the Planck scale.
1.3 **The Weinberg-Witten No-Go Theorem and Holography as a Way Out**

I have argued that in general, an effective field theory containing gravity is not UV complete, because it is non-renormalizable but that the early universe is sensitive to physics at the UV. We observe in gauge theories, though, that a gauge symmetry does not necessarily have to be present at the fundamental level in the UV. We rather see in the case of spin 1 particles, that a gauge symmetry can be an emergent phenomenon, which we only perceive in the IR at large distances. A natural idea is then that gravity could be an “emergent phenomenon”, in which the graviton is a composite particle at low energies, which only appears elementary. However, it has been shown in the Weinberg-Witten theorem [13] that this is not possible. I am going to briefly outline below, why gravitons cannot emerge from a quantum field theory and how we can circumvent this no-go theorem.

We know that the graviton must be

- a massless particle, because gravity is the longest range force that we know in nature,
- a spin 2 particle, because it is sourced by the stress-energy tensor, which is second rank. Besides, it can been shown that any massless spin-2 field must interact with the stress-energy tensor just as the gravitational field and would be indistinguishable from gravity.

The Weinberg-Witten theorem states, that the assumption of a 3+1-dimensional local QFT with a conserved and Poincaré-covariant stress-energy tensor does not admit a massless (composite) particle with helicity $|h| > 1$ and thus excludes gravitons.

The reason for this restriction lies in the nature of the stress-energy tensor. If a particle or a state with spin two or higher interacts with the stress-energy operator in its rest-frame its helicity would have to change from positive to negative, which is a total change of 4. To allow for a Lorentz-invariant spin 2 state, the stress-energy tensor would have to be a spin-4 state, but it is bounded to be maximally spin 2.

In the following, I sketch the proof of the theorem. We look at a process which measures the energy of the particle, i.e. the interaction of its state with the stress-energy operator. With $p^\mu$ the value of the null-component of the stress energy tensor $p^\mu = \int d^3x T^{\mu 0}$ and $E$ the eigenvalue for the energy operator $p^0$, we have

$$E\delta^3(p' - p) = \langle p', h|p^0|p, h \rangle = \int d^3 x \langle p', h|T^{00}(0, \vec{x})|p, h \rangle$$

$$= \int d^3 x \, e^{i(\vec{p}' - \vec{p})\cdot \vec{x}} \langle p', h|T^{00}(0, 0)|p, h \rangle$$

$$= (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \langle p', h|T^{00}(0, 0)|p, h \rangle$$

for single-particle states with momenta $p, p'$, respectively and helicity $h$, and hence we have

$$\langle p, h|T^{00}(0, 0)|p, h \rangle = \frac{E}{(2\pi)^3}$$

(1.18)
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where \( E \neq 0 \). If the momentum transfer between two states \( |p^\mu\rangle \) and \( |p'^\mu\rangle \) is assumed to be space-like, i.e. such that \( (p - p') \) is not null, we can always transform to a reference frame, where \( p' + p \) is along the time direction such that the momentum of the ingoing particle is \( \left( \frac{q}{2}, 0, 0, -\frac{q}{2} \right) \) and \( p' - p \) along the space direction of motion, \( z \) say, such that the momentum of the outgoing particle is \( \left( \frac{q}{2}, 0, 0, \frac{q}{2} \right) \). Under a rotation of an angle \( \theta \) about the direction of motion, the single-particle states transform as \( |p, \pm \hbar\rangle \rightarrow e^{\pm i\theta \hbar} \) and \( |p', \pm \hbar\rangle \rightarrow e^{\mp i\theta \hbar} \) and thus, the left-hand side of (1.18), \( \langle p, \hbar|T^{00}(0, 0)|p, \hbar\rangle \) transforms as \( e^{2i\theta \hbar} \). To preserve rotational invariance, the matrix elements must transform to

\[
R_{\mu \rho}^\prime(\theta) R_{\nu \sigma}^\prime(\theta) \langle p', \pm \hbar|T^{\mu \nu}\rangle |p, \pm \hbar\rangle,
\]

(1.19)

where \( R(\theta) \) is the rotation matrix, which has Fourier components \( e^{\pm i\theta} \), only. Hence, to preserve rotational invariance, the matrix elements of \( T^{\mu \nu} \) must vanish unless \( |\hbar| = 0, \frac{1}{2}, 1 \). For these values, the rotation coincides with the and Lorentz-invariance can be preserved. In particular, there is no spin-2 state. Therefore it is proven that gravitons cannot be described in and gravity cannot emerge from a local quantum field theory.

Another indication for this fact comes from the observation that in a local quantum field theory, the entropy scales with the volume of a system. In a gravitational theory, the bound is stronger and the entropy can only grow with the area of the boundary of the system. This is based on the argument that the entropy of a black hole is the area of its horizon measured in Planck units \[14–16\]

\[ S = \frac{A_{\text{horizon}}}{4l_{\text{Planck}}^2} , \]

(1.20)

This as well hints to the fact that gravity cannot be obtained from local degrees of freedom.

It should be noted that a gauge symmetry never comes from a global symmetry. The gauge symmetry only arises in the IR. In gravity, space-time points themselves are not gauge invariant and are changed under a gauge transformation. Therefore, we can conclude that in any theory, in which gravity emerges, the geometry of space-time must emerge with it, just as everything has to emerge, that lives on this space-time like gauge and matter fields.

This last observation points to a possible solution of the problem: Instead of having gravity emerge from a field theory defined on the same space-time, gravity can emerge from a quantum field theory in a dimension less. The symmetries of the field theory in such a setup give rise to the isometries of the emerging space-time. In the case of a conformal field theory, which has a scaling symmetry, the emergent space-time is an anti-de Sitter space with a metric

\[
ds^2 = R_{\text{AdS}} \frac{dz^2 + dx_{\mu}dx^{\mu}}{z^2} , \]

(1.21)

where \( R_{\text{AdS}} \) is the AdS radius. The scaling symmetry under \( x \rightarrow \lambda x \) of the CFT leaves the metric invariant, if also \( z \rightarrow \lambda z \). Since gravitational interactions decay with the distance between two particles, the Hilbert space of the theory should allow for a Fock space structure. Gauge theories with \( SU(N) \) gauge group have such a structure in their large-\( N \) limit. It has been speculated for a long time that such theories are related to string theory as a theory of gravity (\[17\], see \[18\] for a review).
The large-$N$ limit uses the observation, that a gauge field in the adjoint representation of SU($N$) can be recast as a direct product of a fundamental and an anti-fundamental field with separate indices. Those are depicted as separate lines adjacent to each other in a Feynman diagram. Then we observe that the Feynman diagrams of the theory fall into two separate classes (cf. figure 1.2):

1. **planar diagrams** which can be drawn on a flat surface such that none of the lines cross and
2. **non-planar diagrams** which can only be drawn on a surface with a higher genus to avoid crossing of lines.

In every closed loop, the gauge indices are not fixed and are summed over all $N$ possibilities. Thus, such a loop contributes a factor $N$ to the value of the diagram. Carefully counting the different combinations of powers of the gauge coupling $g$ and of $N$ for each closed index loop shows that we can split off an effective coupling of $\lambda_{t \text{ Hooft}} = g^2 N$, the so-called 't Hooft coupling. If this coupling is held fixed in a limit

$$N \to \infty , \quad g^2 N = \text{constant} , \quad (1.22)$$

we see that

1. planar diagrams survive the large-$N$ limit, whereas
2. non-planar diagrams are sub-leading in $\frac{1}{N^{\gamma}}$ and die in this limit, where $\gamma > 0$ is the genus of the surface on which they are drawn.

The field theory is now expanded in a double series in $g^2 N$ and in $\frac{1}{N}$. The latter corresponds effectively to an expansion in the genus of the corresponding surface.
This is very resemblant of string theory, where particles are replaced by strings with an inner degree of freedom and the Feynman diagrams of QFT are replaced by world-sheet diagrams of connecting strings (see fig. 1.3). Now, if we consider a closed chain made up of particles which each carry two of the $N$ different colors such that two adjacent particles always share a color, the probability of two such chains with the same color combination passing by each other is $\frac{1}{N}$. Only such strings would interact and the string coupling would be

$$g_s \sim \frac{1}{N} .$$  \hspace{1cm} (1.23)  

In the strict large-$N$ limit of an infinite chain, this coupling vanishes and the corresponding string theory is free. This means that all the loop string scattering diagrams are suppressed and all the higher genus surfaces in fig. 1.3 do not contribute.

If we apply this observation to the interpretation of a surface with a specific genus to represent the diagrams at a specific order in a $\frac{1}{N}$ expansion of a field theory, this means that the contribution of all the non-planar diagrams vanishes. The field theory is expanded only in the finite, weak ’t Hooft coupling and its Hilbert space automatically has the Fock space structure we have required earlier. The strings, conversely, are effectively treated as point particles, whose interactions are suppressed by $\frac{1}{N}$. Of course, up to this point we do not know at all, what this string theory would be and we will have to look for a concrete realization of this idea as a quantum field theory, from which a known string theory emerges.

The best known example is the duality between $\mathcal{N} = 4$ Super-Yang-Mills theory in 4 dimensions and type IIB string theory on $\text{AdS}_5 \times S^5$ [20–22]. Here, we examine the field theory around its conformal fixed point because $\mathcal{N} = 4$ SYM is a conformal theory in four dimensions. The relation between the geometric parameters and the field theory quantities is

$$\lambda_{\text{t Hooft}} = \left( \frac{R_{\text{AdS}}}{l_{\text{string}}} \right)^4$$  \hspace{1cm} (1.24)  

$$g_{\text{YM}}^2 = g_s .$$  \hspace{1cm} (1.25)  

We see that this correspondence is, indeed a duality, because in a region where the AdS curvature radius $R_{\text{AdS}}$ is small as compared to the string length $l_{\text{string}}$, which means that gravity is strongly coupled, the ’t Hooft coupling $\lambda_{\text{t Hooft}}$ is small and vice versa. Note that taking the ’t Hooft limit is essential to that observation, because a large Yang-Mills coupling $g_{\text{YM}}$ would lead to a large string coupling $g_s$. In turn, if the ’t Hooft coupling is fixed, the string coupling, indeed, scales inversely with $N$. 

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This so-called AdS/CFT correspondence allows to describe the dynamics in a strongly coupled gravitational bulk, where quantum gravity effects are important, by a perturbative quantum field theory and vice versa, if the fields in the bulk are related to the operators in the boundary field theory. To perturb the theory around the conformal point, we can add single trace operators to the field theory, i.e. operators of the form

\[ O = g^2 \text{tr } \phi^4 = \phi_j^i \phi_k^j \phi_l^k \phi_s^l , \]  

where upper and lower indices are fundamental and anti-fundamental, respectively. The prescription is now, that the source of such an operator is the boundary condition for a field in the bulk.

A very intriguing feature of the AdS/CFT correspondence is that it embeds the ideas of the renormalization group (see section 1.2) in a geometric way [23–26]. It turns out that the Hamilton-Jacobi equations of supergravity take the form of the Callan-Symanzik equations of the field theory

\[ \frac{1}{\sqrt{g}} \left( g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} + \beta^I (\phi) \frac{\delta}{\delta \phi^I} \right) \Gamma[\phi, g] = 4\text{-derivative terms} , \]  

where \( \Gamma \) is the non-local part of the action. The 4-derivative terms on the right hand side of this equation stem from cross-terms of the potential, functional derivatives thereof and the non-local effective action \( \Gamma \), as well as curvature squared terms and products of the curvature with space-time derivatives of the scalar fields. These terms drop out upon variation of the action and do not play a role. With the metric \( g_{\mu \nu} = a^2 \eta_{\mu \nu} \) this yields the Callan-Symanzik equation upon replacing the functional derivatives with ordinary ones by virtue of

\[ \int g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} = a \frac{\partial}{\partial a} , \quad \int \frac{\delta}{\delta \phi^I} = \frac{\partial}{\partial \phi^I} . \]  

This is depicted in figure 1.4. If the field theory is not taken to be at the boundary but at a finite distance \( z = z_{\text{cutoff}}, \) this theory would correspond to a renormalized version of the boundary field theory, which has some multi-trace operators added [28]. Note that the stress-energy tensor, which measures the energy, should never be renormalized.
Figure 1.4: The AdS/CFT correspondence provides a geometric picture of Wilson RG flow. The theory at the boundary is the UV version. Defining the theory at a finite distance from it corresponds to integrating out UV degrees of freedom just like in a series of block spin transformations labeled by a parameter $r$ on the left. Figure from [27].