The universe on edge: Limits of the effective field theory approach in the very early universe
Oberreuter, J.M.

Citation for published version (APA):
Oberreuter, J. M. (2013). The universe on edge: Limits of the effective field theory approach in the very early universe
CHAPTER 2

THE RESOLUTION OF COSMIC SINGULARITIES WITH THE HELP OF THE GAUGE/GRAVITY DUALITY

Our whole universe was in a hot dense state,
Then nearly fourteen billion years ago
expansion started. Wait...
The Earth began to cool,
The autotrophs began to drool,
Neanderthals developed tools,
We built a wall (we built the pyramids),
Math, science, history, unraveling the mysteries,
That all started with the big bang!

Barenaked ladies,
The Big Bang theory theme song

In this chapter, I am going to investigate the effects of quantum gravity on the big bang singularity. That there is a singularity at the beginning of our universe poses a severe problem to our understanding of the cosmos and of gravity. A singularity means that an important quantity of the theory, in this case the Ricci scalar, becomes infinite. Not only does the theory at hand lose its predictability, it is also impossible to impose initial conditions. In general, a singularity signals the breakdown of the approximation used for the specific problem under investigation, in this case general relativity \cite{29,30} as the low energy theory of UV complete gravity.

If a theory of gravity contains a singularity, there are two possible scenarios. Either, they are resolved in the full quantum theory. If string theory is such a theory, this means including
higher curvature and string coupling corrections. Or the theory was ill-defined to start with, meaning that even the UV complete theory has a singularity. No consistent theory should have singularities. However, there might be different types of singularities some of which a quantum theory is not required to resolve.

Usually, unphysical solutions to the equations of motion of a theory can be ruled out by analyzing their stability. In general relativity, the potential energy is only bounded if an energy condition is applied. However, this still does not rule out singularities, such as for instance black holes or a big crunch, to form. Such singularities occur in a well-defined classical theory and therefore need to be resolved by its quantum version, whereas, if a solution is perturbatively unstable, we might have to discard it as a whole. Still, in gravity, there is no general relation between the stability of a theory and the occurrence of singularities.

The gravitational potential considered in the following is perturbatively stable. Therefore it appears as if the situation is not fundamentally flawed and we think that quantum effects will play an important rôle. Such quantum effects can be perturbative, but do not necessarily show up at the lowest order in perturbation theory, or they can even be non-perturbative.

### 2.1 A Singularity at Strongly Coupled Gravity

I will first present the reason for having a singularity at the beginning of the universe. Already in 1929, Edwin Hubble and Milton L. Humason realized that the universe was expanding when discovering the proportionality of the red-shift of the spectra of distant galaxies to their distance [31, 32]. They thus confirmed the conjecture, which was put forward a couple of years earlier by Georges Lemaître [33, 34], that the universe was expanding and started from a "unique atom, the atomic weight of which is the total mass of the universe". He was building his model on the solutions of Albert Einstein [35] and Willem de Sitter [36–38] to the theory of general relativity. Their solutions suffered from being unstable and only allowed for a universe expanding at a declining rate or contracting increasingly fast.

This problem was turned into what later should become the standard model of cosmology by Alexander Friedmann, Georges Lemaître, Howard Robertson and Arthur Walker [33, 39–42]. It is built on the cosmological principle, that no observer is at the center of the universe, which looks the same viewed from any point, i.e. it is isotropic and homogeneous. This leads to a maximally symmetric metric, in which a scale factor \(a(t)\) accounts for the expansion or the collapse of the universe

\[
ds^2 = dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\]  

(2.1)

The parameter \(k\) encodes the spatial curvature of the universe as

\[
k = \begin{cases} 
-1 & \text{for negatively curved} \\
0 & \text{for flat} \\
1 & \text{for positively curved}
\end{cases}
\]  

(2.2)
spatial hyper-surfaces. With this ansatz, the Einstein field equations
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}, \tag{2.3} \]
yield the Friedmann equations. The first one describes the evolution of the scale factor, the so-called Hubble parameter
\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G_N}{3} \rho(t) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{2.4} \]
where \( c \) is the speed of light. Besides, \( \Lambda \) denotes the cosmological constant related to the vacuum energy. The energy density of matter is indicated with \( \rho \).

The Hubble parameter is currently measured to be \( 73.8 \pm 2.4 \text{ km/s Mpc} \) using the Wide Field Camera 3 on the Hubble space telescope \[43\] or \( 67.0 \pm 3.2 \text{ km/s Mpc} \) using the 6dF Galaxy Survey \[44\]. Hubble himself obtained a value of around \( 500 \text{ km/s Mpc} \). The Hubble time, which is \( H^{-1} \), is the approximate age of the universe. The cosmic expansion drives anything which is further than the Hubble radius \( \frac{c}{H} \) apart from a given observer away from it faster than the speed of light. Hence, the Hubble radius is the size of the observable universe. However, if the Hubble parameter \( H \) is not constant, the observable region changes, which is an important feature for cosmological model building.

The second Friedman equation or Raychaudhuri equation describes the acceleration of the scale factor and is derived from the spatial components of the Einstein equations
\[ \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3} \left( \rho + \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3}. \tag{2.5} \]
On top, conservation of energy yields the continuity equation
\[ \dot{\rho} = -3H \left( \frac{p}{c^2} + \rho \right), \tag{2.6} \]
which describes the dilution of energy during the expansion. To solve the above set of cosmological equations, we need to specify the relation between the pressure \( p \) and the energy density \( \rho \), which is the so-called equation of state. The different contributions to the energy in the universe are well-described by a linear equation of state \( p = c^2 w\rho \), which can also describe a cosmological constant as dark energy and curvature with effective pressure and energy density. Different values of \( w \) describe different such components, namely

- \( w = 0 \): cold matter
- \( w = \frac{1}{3} \): radiation
- \( w = -\frac{1}{3} \): e.g. (negative) curvature
- \( w = -1 \): cosmological constant/dark energy.

We can then describe a universe filled with different kinds of energy by summing up the different contributions. Thus, we rewrite the first Friedman equation in terms of the critical density of a flat universe
\[ \rho_c = \frac{3H^2}{8\pi G_N}, \tag{2.7} \]
Chapter 2. The resolution of cosmic singularities

such that all the contributions need to sum up to

\[ 1 = \sum_{\text{matter, radiation, } k, \Lambda} \frac{\rho_i}{\rho_c}. \]  

The evolution of the energy density can be determined from integrating \((2.6)\)

\[ \frac{d\rho}{\rho} = -3(1 + w) \frac{da}{a} \Rightarrow \rho \propto a^{-3(1+w)} \text{ for } w \neq -1. \]  

We see that the density will generally be diluted as the scale factor grows at later times. Note that \(\rho = \text{constant for } w = -1\), which means that the cosmological constant is, indeed, not diluted by the expansion as the name suggests.

In turn, when evolving the density backwards in time as the scale factor shrinks, we see that the density is bigger for earlier times, which leads to a divergence of the values in the stress-energy tensor. Correspondingly, from \((2.3)\), also the Riemann scalar \(R\) has to diverge and we arrive at a (space-like) curvature singularity. Not only at the singularity itself, but also in its neighborhood is general relativity unpredictable. This can be seen in the linearized approach \((1.14)\), where the coupling constant \(\kappa\) would have to become strong, if \(h_{\mu\nu}\) is still to be considered a small perturbation. Therefore, gravity is strongly coupled around the singularity and the perturbative approximation breaks down.

The red-shifting of galactic spectra is not the only evidence for an expanding universe. The FLRW model got much more convincing support from the accidental discovery of the Cosmic Microwave Background (CMB) by Arno Penzias and Robert Woodrow Wilson in 1965 \[45, 46\]. This radiation was just around the same time discussed to be a left-over from the big bang by Robert H. Dicke, Jim Peebles and David Wilkinson \[47\] but had a history of being conjectured by George Gamow \[48, 49\], Gamow, “Hans Bethe” and Ralph Alpher \[50\] and the latter with Robert Herman \[51\]. In a theory of an expanding universe, this radiation was created about 380,000 years after the big bang. Before that time, the temperature of the universe would be too high for neutral hydrogen to exist and free electrons scatter photons very efficiently such that the early plasma was opaque. As the temperature dropped with the expansion of the universe, neutral hydrogen formed and the universe became transparent with a mean free path of photons larger than the Hubble radius. These photons were since redshifted by a factor \(\frac{T_{\text{recombination}}}{T_{\text{now}}} \sim 1100\) to a temperature of about 2.725 K \[52\] and now form the CMB. This is one of the features of big bang cosmology, which is the most difficult to attain by alternative models.

So far, I have laid out good and generally accepted arguments supporting the idea that the beginning of the universe is a strange singular state, the big bang, which marks the beginning of what we describe as the evolution of our universe in the paradigm of general relativity. This is a frustrating situation, since this very beginning is a point of utter interest. It is a very natural question, what happened “before” the big bang and which dynamics led to the onset of the expansion at a certain point. It is here, where we need to impose boundary conditions, if there is no dynamics before to ensure, e.g. that the universe starts out in a low, even zero
Chapter 2. The resolution of cosmic singularities

entropy state, to satisfy the second law of thermodynamics (Penrose in [53]). Of course, we would rather have this special state to be selected dynamically.

To understand what a low entropy state means, we first need to understand the micro-states of any theory at hand. In the case of the very early universe, where gravity is the dominating force, those would mainly be the micro-states of gravity. The notion of gravitational micro-states comes from the aforementioned observation that black holes have a finite entropy, given by the area of their horizon in Planck units as in equation (1.20). The micro-states of a system are accessible in the UV regime. In the case at hand this is a theory of quantum gravity. The observation that the effective description of gravity becomes strongly coupled and breaks down close to the singularity is another fact supporting the idea that a theory of quantum gravity should naturally solve the problems associated with a big bang singularity.

A number of different approaches to quantum gravity seem to produce the correct density of micro-states for black holes [54–56]. Under the assumption, that only the near-horizon geometry accounts for the degrees of freedom, the conditions, which need to be imposed on any such theory of quantum gravity to produce a black hole seem to enforce a conformal symmetry there. A situation, in which we understand the rôle which is played by conformal field theory and in particular the Cardy-Verlinde formula [57–59] in the counting of gravitational micro-states [60] is the AdS/CFT duality within string theory mentioned in section 1.3. It suggests holography as a way to define quantum gravity. Being a duality, it relates a strongly coupled theory on one side to a weakly coupled theory on the other. This appears very useful in our case at hand, since gravity, whose microscopic description we are after, is strongly coupled around the singularity. The corresponding quantum field theory is weakly coupled, which is very well understood perturbatively. Intuitively, general relativity is a theory valid at large scales, but close to the big bang the scales are small and a quantum field theory is a more suitable theory, there. The AdS/CFT correspondence is only well-understood for the very specific case of an $\text{AdS}_5 \times S^5$ dual to an $\mathcal{N} = 4$ Super-Yang-Mills theory. However, we expect that the holographic principle, derived from the scaling of black hole micro-states with the area of the horizon, holds universally. If the correspondence is taken to have a universal meaning as the “gauge-gravity-duality”, every well-defined gravity background should be described by some QFT. A breakdown of one side should be reflected in a breakdown of the other. Since quantum gravity needs to remove the big bang singularity from the gravitational theory, according to the duality it should be possible to describe it by some well understood (deformation of a) conformal field theory.

In the following I am first going to introduce some technical aspects of the AdS/CFT correspondence before applying it to a specific model of a space-like gravitational singularity as a model of the big bang.
Chapter 2. The resolution of cosmic singularities

Figure 2.1: When reducing to a subsystem, the fields become entangled over the boundary. This resembles the situation of a black hole, where the degrees of freedom inside the horizon are inaccessible to an outside observer.

2.2 The AdS/CFT correspondence

The idea that there is a duality between the microscopic theory of gravity and a quantum field theory on the boundary of the space under consideration is conceptually built on the holographic principle \[61,62\]. This in turn is a conjecture which is fed by the observation that the entropy of a black hole doesn’t scale with its volume as one would assume for a quantum field theory but with the area of its horizon. The horizon is a hyper-surface with one dimension less than the black hole. The entropy which can be contained in a given volume \(\Gamma\) is bounded by the entropy of a black hole occupying this volume. For theories of gravity in \(d\) dimensions, the entropy is hence bounded by its surface in terms of the Planck length

\[
S_{\text{Bekenstein}} \leq \frac{\text{Vol}(\partial \Gamma)}{4l_{\text{Planck}}^{d-2}}.
\]

(2.10)

Although surprising from the point of view of statistical mechanics, where we expect entropy as an extensive quantity to grow like the volume of the system, such a scaling is well-known for entanglement entropy. The entanglement entropy quantifies the amount of information observers lose about a system, if they cannot access a part of it any more. It arises, because quantum states are defined globally. When projecting it onto the subsystems, the eigenstates become entangled over the boundary, which separates the two subsystems, see fig. 2.1. We reduce the system to subsystem \(B\), say, by tracing the density matrix over the degrees of freedom of subsystem \(A\) and obtain the reduced density matrix \(\rho_B\). The entropy, associated with this loss of information is

\[
S_B = -\text{tr} (\rho_B \log \rho_B) \sim \partial A = \partial B,
\]

(2.11)

which scales with the boundary between the two systems \[63,64\]. This can be understood heuristically, because for any theory with short-ranged interactions, the biggest contribution to the entanglement comes from the states close to the boundary, the number of which is
proportional to it. This also means, that the entanglement entropy is symmetric

\[ S_A = S_B , \] (2.12)
i.e. that it doesn't matter if we trace out system \( A \) or \( B \).

This resembles the situation with a black hole insofar as an observer at the outside also looses information about the states inside the horizon. The vacuum state of all the fields including gravity inside the horizon region become entangled with the fields outside. Thus, entanglement entropy is seen to contribute at least a large part of the black hole entropy [65, 66]. Actually, equation (2.11) is divergent, unless we invoke a UV cutoff. If we interpret the Bekenstein-Hawking entropy as an entanglement entropy, we see that this cutoff is the Planck scale. This suggests, that the Planck scale is the minimal length scale up to which classical gravity can serve as a good approximation to the fundamental theory, which is in line with the expectations. The understanding of how this cutoff arises as a physical length scale is related to understanding the physics is characteristic at this scale and remains to be uncovered. The area law suggests that the horizon of a black hole is the place where the degrees of freedom of quantum gravity live and that that they are described for any system by a quantum field theory on its boundary.

The AdS/CFT correspondence in string theory is the only concrete realization that we know of the holographic principle so far. Here, string theory as a theory of quantum gravity on an Anti-de Sitter background, the bulk, is dual to and can be described by a quantum gauge field theory living on its asymptotic boundary, a flat Minkowski space of one dimension less. In the original setup, a string theory in ten dimensions is compactified on a five-sphere such that supergravity on an AdS\(_5\) background remains, which is dual to an \( \mathcal{N} = 4 \) super-Yang Mills theory on four-dimensional Minkowski space [20–22]. Since then, the the correspondence has been extended to other dimensions and spaces with less symmetry [67] and new realizations are much sought after. In particular, the generalization of the duality to de Sitter space presents us with conceptual problems and is still badly understood [68]. This is why I will present an application of the correspondence to cosmology in AdS space, although our universe resembles de Sitter space [69]. The lessons to be learned about the generic behavior of quantum gravity at the beginning of the universe and about the problems of applying the correspondence to gravity at strong coupling are still invaluable.

In the remainder of this section I am going to define Anti-de Sitter space and explain some of its peculiarities, before describing the large-\( N \) limit of quantum field theories. Then I am going to sketch how to relate the two and emphasize the importance of boundary conditions for this setup. There are numerous reviews, from which the following material can be extracted [18,70–74].

### 2.2.1 Anti-de Sitter Space

The Einstein field equations have three classes of maximally symmetric vacuum solutions, namely flat space, positively curved space and negatively curved space (cf. e.g. [75,77]). Anti-
de Sitter space (AdS) is the one with negative scalar curvature, which corresponds to having a negative cosmological constant. This is one of the reasons why it is easier to formulate quantum gravity on Anti-de Sitter than on de Sitter space, because a positive vacuum energy necessarily breaks supersymmetry, which complicates e.g. the extrapolation of black hole solutions to strong coupling.

A $d + 1$-dimensional Anti-de Sitter space, $\text{AdS}_{d+1}$, is a homogeneous space

$$\text{AdS}_{d+1} \cong \frac{\text{SO}(d, 2)}{\text{SO}(d, 1)}$$

and forms the Lorentzian analogue of a hyperbolic space. Therefore, $\text{AdS}_{d+1}$ can be embedded into a $d + 2$ dimensional space-time $\mathbb{R}^{2,d}$ as as a quadratic surface

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + \cdots + (X^d)^2 - (X^{d+1})^2 = -1. \quad (2.14)$$

This quadric is invariant under the $\text{SO}(d, 2)$ isometries of the embedding manifold and therefore maximally symmetric. The flat metric on $\mathbb{R}^{2,d}$

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + \cdots + (dX^d)^2 - (dX^{d+1})^2 \quad (2.15)$$

induces a Lorentzian metric on the hyperboloid with a scalar curvature

$$R = -d(d + 1). \quad (2.16)$$

The induced metric solves Einstein’s equations with a negative cosmological constant

$$\Lambda = -\frac{d(d - 1)}{2}. \quad (2.17)$$

For an arbitrary curvature radius, the ambient metric is rescaled to

$$ds^2 = -R_{\text{AdS}}^2 \left[(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + \cdots + (dX^d)^2 - (dX^{d+1})^2\right] \quad (2.18)$$

such that the scalar curvature will change to

$$R = -\frac{d(d + 1)}{R_{\text{AdS}}^2}. \quad (2.19)$$

In the following it will thus be sufficient to focus on the embedding as a unit hyperboloid.

We can use various coordinate systems on the hyperboloid (2.14), which differ in terms of the resulting metric and the amount of AdS space they cover. The so-called global coordinates cover all of $\text{AdS}_{d+1}$ and are defined by

$$X^0 = \cosh \mu \cos t,$$

$$X^i = \sinh \mu \omega^i, \quad i = 1, \ldots d,$$

$$X^{d+1} = \cosh \mu \sin t,$$
Chapter 2. The resolution of cosmic singularities

Figure 2.2: AdS space can be represented as the interior of a cylinder (left). Top and bottom of this cylinder are identified and closed time-like curves arise. For the universal covering, an infinite number of cylinders is glued on top of each other. The boundary of the cylinder represents the boundary of AdS, where the dual field theory lives. The Penrose diagram of Anti-de Sitter space reveals the causal structure. Since it is a conformal projection, angles and thus the light-cones remain unchanged. A curious fact is that light rays can reach the boundary in finite global time, although it is infinitely far away. There are several useful coordinate systems such as global coordinates (middle) or Poincaré coordinates (right). One set of Poincaré coordinates covers each of the diamond shaped regions, which are delimited in the figure on the left and which each are conformal to flat Minkowski space. Figure from [78]

where $\omega^i = \sin \theta_1 \ldots \sin \theta_{i-1} \cos \theta_i$ is a unit $d$-vector on the $d$-sphere and the range of the other coordinates is

$$0 \leq \mu \leq \infty ,$$

$$0 \leq t \leq 2\pi .$$  \hspace{1cm} (2.23)

The AdS$_{d+1}$ metric in global coordinates reads

$$ds^2 = -\cosh^2 \mu dt^2 + d\mu^2 + \sinh^2 \mu d\Omega_{d-1}^2 ,$$

where $d\Omega_{d-1}$ denotes the line element of a unit $(d-1)$-sphere. The topology of this space is $S^1 \times S^d$, since in the $d + 1$ dimensional embedding, the time coordinate is required to be periodic. This implies the existence of unphysical closed time-like curves. Therefore, we need to unwrap the time circle and consider $t \in \mathbb{R}$. Then AdS space is the universal cover of the Lorentzian space defined by (2.14). The causal structure of AdS space is best understood by drawing its Penrose diagram, which is a finite conformal projection of the full space (see fig. 2.2). To obtain it, we introduce the tortoise radial coordinate

$$\sinh \mu = \tan \rho , \quad 0 \leq \rho \leq \frac{\pi}{2} .$$

(2.26)

With this substitution, the metric reads

$$ds^2 = -\sec^2 \rho dt^2 + \sec^2 \rho d\rho^2 + \tan^2 \rho d\Omega_{d-1}^2$$

$$= \frac{1}{\cos^2 \rho} (-dt^2 + d\rho^2 + \sin^2 \rho d\Omega_{d-1}^2) .$$

(2.27)
Chapter 2. The resolution of cosmic singularities

In these coordinates, the boundary lies at $\rho = \pi/2$, which corresponds to $r = \tan \rho = \infty$, and has the topology of a $S^{d-1} \times R$. When suppressing all but one angular coordinates, the Penrose diagram is a full cylinder, which can be projected onto an infinite stretch, when suppressing also the last angular coordinate. Surprisingly, null geodesics can reach the boundary in finite global time and return to where they started.

For our purposes, the relevant parametrization are Poincaré coordinates, which slice the hyperboloid with hyperplanes given by

\begin{align*}
X^{0} - X^{d} &= \frac{1}{z}, \quad z > 0 , \\
X^{i} &= \frac{x^{i}}{z}, \quad i = 1, \ldots, d - 1 , \\
X^{d+1} &= \frac{x^{0}}{z} ,
\end{align*}

where we take $z = 1/r$. We obtain the metric

\begin{equation}
\text{d}s^2 = \frac{\text{d}z^2 + \eta_{ab} \text{d}x^{a} \text{d}x^{b}}{z^2} , \quad a, b = 0, \ldots, d - 1 ,
\end{equation}

with $\eta_{ab}$ the Minkowski metric. The position of the boundary is now at $z = 0$. These coordinates make the Poincaré symmetry manifest

\begin{equation}
x^{a} \to \Lambda^{a}_{\ b} x^{b} + b^{a} , \quad \Lambda \in \text{SO}(1, d - 1) .
\end{equation}

The full group of $\text{SO}(2, d)$ isometries is realized by the inversions

\begin{align*}
z &\to \frac{z^2}{z^2 + \eta_{ab} x^{a} x^{b}} , \\
x^{a} &\to \frac{x^{a}}{z^2 + \eta_{ab} x^{a} x^{b}}
\end{align*}

and the dilations

\begin{equation}
z \to cz , \quad x^{a} \to cx^{a} .
\end{equation}

Poincaré coordinates make the connection to a field theory on the boundary Minkowski space more tangible. Via a conformal rescaling of the metric $g_{\mu \nu} \to z^2 g_{\mu \nu}$, such that

\begin{equation}
\text{d}s^2 = \text{d}z^2 + \eta_{ab} \text{d}x^{a} \text{d}x^{b} ,
\end{equation}

it can now also be seen that $z = 0$ is the conformal boundary. Note that the Poincaré patch is geodesically incomplete, because $X^{0} - X^{d} > 0$, which means in global coordinates

\begin{equation}
cosh \mu \cos t - \sinh \mu \sin \theta_{1} \ldots \sin \theta_{d-1} \cos \theta_{d} > 0 .
\end{equation}

Both null- and time-like geodesics can reach $z = \infty$ at a finite value of their affine parameter and escape from the Poincaré patch.

We are now going to investigate some general properties of this conformal boundary and which general properties we can derive for a field theory living on it.
2.2.2 THE LARGE-N LIMIT OF BOUNDARY QUANTUM FIELD THEORIES

As predicted by the holographic principle, the quantum gravity in this AdS bulk should be described by a quantum field theory in one dimension less. I will now review a couple of general properties, that this dual theory must have on general grounds. Already in section 1.3 we have seen that the geometry of higher loop string scattering worldsheet diagrams resembles the topological properties of the $1/N$ expansion around the ’t Hooft limit. It turns out that the boundary theory must be

1. conformal,
2. a large-$N$ limit of a
3. gauge field theory,
4. which is dual in the sense that it relates strong to weak coupling.

I will reason on physical grounds why this is the expectation.

1. We have already established conformality of the boundary in the previous subsection 2.2.1. The isometries of the bulk act on the boundary as the group of conformal transformations in $d$ dimensions

$$x^a \rightarrow \Lambda^a_b x^b + b^a \quad \text{(Poincaré)}, \quad (2.39)$$
$$x^a \rightarrow c x^a \quad \text{(dilatations)}, \quad (2.40)$$
$$x^a \rightarrow \frac{x^a}{x^2} \quad \text{(inversions)}. \quad (2.41)$$

These transformations leave the boundary invariant and henceforth, the field theory living thereon must be conformal.

2. The theory must allow for a large-$N$ limit. This will be important to match the parameters of the gravitational theory to the ones of the gauge theory. At the core of the construction is the attempt to explain the black hole entropy as the entropy of the field theory in one dimension less. A Schwarzschild black hole in AdS space has the metric

$$ds^2 = R^2_{\text{AdS}} \left( -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \right) \quad (2.42)$$
$$f(r) = r^2 + 1 - \frac{2\tilde{G}_N m}{r^{d-2}}, \quad (2.43)$$

where

$$\tilde{G}_N = \frac{G_N^{d+1}}{R^{d-1}_{\text{AdS}}} \quad (2.44)$$

is the (dimensionless) effective gravitational coupling at the scale of the AdS radius. The Schwarzschild radius of a large black hole is located at $r_s \sim G_N m$ so that its entropy scales with the surface

$$S \sim \frac{r_s^{d-1}}{G_N} \sim \frac{1}{G_N \beta^{d-1}}, \quad (2.45)$$
where in the last part we have written the entropy in terms of the Hawking temperature \( \beta \propto \frac{1}{r_s} \). This is the entropy expected for a (conformal) field theory in \( d \) dimensions with massless scalar fields at a temperature \( \beta \)

\[
S \propto c \frac{V_{d-1}}{\beta^{d-1}},
\]

(2.46)

where \( c \) measures the effective number of fields. Comparing the gravitational entropy of a black hole with the entropy of the field theory, we fix this coefficient to be

\[
c \sim \frac{1}{G_N} = \frac{R_{\text{AdS}}^{d-1}}{G_N^{d+1}}.
\]

(2.47)

This relation means that the effective number of fields is inversely proportional to the effective gravitational coupling at the AdS scale. Therefore, if the duality is to be defined (also) for a weakly coupled bulk, the corresponding field theory must allow for a large number of fields. In the very classical limit with \( \tilde{G}_N \approx 0 \), the number of fields would be infinite.

3. The ’t Hooft limit for quantum gauge theories provides a well defined prescription for how to achieve this. In an SU(\( N \)) gauge theory with the fields in the adjoint representation, the ’t Hooft coupling

\[
\lambda = g^2 N = \text{constant}
\]

(2.48)

remains constant in the limit \( N \to \infty \) and perturbative field theory remains valid also for large \( N \), if \( \lambda \ll 1 \). The Hilbert space of such theories naturally has a Fock space structure, for which the energy of a multi-particle state is proportional to the sum of the energies of single-particle states up to small corrections. This is an important feature of weakly coupled theories, which the dual field theory needs to inherit.

A gauge invariant local operator can be built by taking the trace over fundamental fields, like \( \text{tr} F_{\mu\nu}(x) F^{\mu\nu}(x) \). A product of two of such operators would be a multi-trace operator. The scaling dimension of a multi-trace operator is the sum of the dimensions of its constituent single trace operators up to \( \frac{1}{N^2} \) corrections which are negligible in the large-\( N \) limit. Expanding the gauge theory in both \( \lambda \) and \( \frac{1}{N} \) organizes the Feynman diagrams according to their genus which corresponds to a fixed order in \( \frac{1}{N} \). The expansion of the field theory in the genus of a manifold resembles the world sheet of the loop expansion of a closed string with string coupling \( g_s \). Adding an extra genus would correspond to an extra loop order of a string theory with coupling

\[
g_s \sim \frac{1}{N}.
\]

(2.49)

The classical limit of this string theory as a quantum theory corresponds to taking \( g_s = 0 \) and therefore, having a classical limit for a holographic theory of quantum gravity suggest to use a large-\( N \) gauge theory.

4. Another condition for having classical gravity as a limit is that the graviton can be treated as a point-like particle. In a string theory, as suggested by the previous argument, the
graviton is the lowest oscillation mode of a string. Generically, a string theory will also have a tower of massive states with spin > 2, which we ignore in the classical limit. If the string is seen as an extended object, the graviton has the size of the string length $l_s$. The characteristic size of the AdS space needs to be large compared to the string length $R_{\text{AdS}} \gg l_s$. \hfill (2.50)

for the graviton to be point-like. In the dual gauge theory, higher spin operators have small scaling dimension at weak coupling. The bulk mass of the corresponding particles is then comparable to the inverse AdS radius $R_{\text{AdS}}^{-1}$ and are therefore rather light in the point-like approximation and render classical gravity invalid. Therefore, the coupling of the dual gauge theory must be strong enough to render the masses of the higher spin states large in the limit where gravity is classically weakly coupled.

Having argued these properties on general grounds, it seems most natural to realize the holographic principle in terms of a string theory dual to the large-$N$ limit of an SU($N$) gauge theory. In [20–22], such a realization has been constructed, which I will describe in the next subsection.

2.2.3 THE RELATION BETWEEN THE TYPE IIB ACTION AND SUPER-YANG-MILLS THEORY

The best understood instance of the gauge/gravity duality and the one I am going to use in this thesis is the correspondence between type IIB string theory on AdS$_5 \times S^5$ and $\mathcal{N} = 4$ superconformal Yang-Mills theory with an SU($N$) gauge group in 4 dimensions. We have already argued the general properties that this gauge theory needs to have in the previous subsection. We are now giving a concrete example and explain how it fulfills the above conditions.

Quantum field theories on four-dimensional Minkowski space are usually not conformal. We can adapt a theory which is known in nature, quantum chromodynamics, which is an asymptotically free gauge theory with an SU(3) gauge group to our purposes. First, we will generalize the gauge group to SU($N$) such that the theory has $N$ colors with the gauge field in the adjoint representation. Also, we make the theory maximally, i.e. $\mathcal{N} = 4$, supersymmetric. This is maximal supersymmetry with four fermions $\chi_\alpha$ and six scalars $\phi^I$ all in the adjoint representation [79,80]. The Lagrangian is uniquely determined by super- and gauge symmetry to be

$$L_{\text{SYM}} = -\frac{1}{4g_{\text{YM}}^2} \int d^4x \text{tr} \left[ F^2 + 2(D_\mu \phi^I)^2 + \chi \Gamma^a D_a \chi + \chi \gamma^i [\phi_i, \chi] - \sum_{I,J} [\phi^I, \phi^J] \right]$$ \hfill (2.51)

$$+ \frac{\theta}{8\pi^2} \int \text{tr} F \wedge F \ .$$ \hfill (2.52)

The two free parameters are the coupling constant $g_{\text{YM}}$ and the angle $\theta$. The theory is conformal due to the non-renormalizations coming from supersymmetry. The ‘t Hooft coupling is
defined as
\[ \lambda = g_{YM}^2 N . \] (2.53)

It can be seen as the effective coupling of the theory. All the fields in the adjoint representation are \( N \times N \) matrices with one fundamental and one anti-fundamental index. For two fields for which one of their indices are contracted, i.e. for which a color and an anti-color are entangled, there are \( N \) color degrees of freedom which can still be exchanged between them. This factor shows up as a closed index loop in the Feynman diagrams. The theory has a global SO(6) or SU(4) R-symmetry, which rotates the six scalars or the 4 fermions into each other. It does not commute with supersymmetry, since the fermions and bosons are in different representations of the R-symmetry group.

To find the corresponding theory of quantum gravity we now to match the parameters and symmetries of the field theory to a bulk theory. I have already argued in the previous section, that this is expected to be a string theory. Supersymmetric string theories are naturally living in ten dimensions. However, according to the holographic principle, we are looking for a five-dimensional theory, such that we have to compactify five dimensions.

A string theory, which contains only closed strings and reduces to a well defined supersymmetric theory of gravity at large distances is type IIB string theory with type IIB supergravity as its low energy effective theory [81]. Supersymmetry requires that this theory contains some massless fields besides the metric, in particular a five-form field strength \( F_5 \), which is completely anti-symmetric in all its indices and constrained to be self-dual \( F_5 = \ast F_5 \) and a dilaton \( \varphi \) and the axion \( \chi \). The action of this theory is
\[ S = \frac{1}{(2\pi)^7 \ell_{\text{Planck}}^8} \int d^{10} x \sqrt{g} (R + F_5^2) + \ldots , \] (2.54)
where the Planck length is related to the string length and coupling
\[ \ell_{\text{Planck}} = \frac{1}{g_s^{\frac{1}{2}} l_s} . \] (2.55)

The string coupling is related the vacuum expectation value of the dilaton
\[ g_s = \langle e^{\varphi} \rangle . \] (2.56)

The equations of motion admit solutions of the form \( \text{AdS}_5 \times S^5 \), which provides the desired five-dimensional AdS factor. These solutions have a five-form field along both directions with both electric and magnetic fields. Due to the Dirac quantization condition, the flux of \( F_5 \) over the sphere is quantized
\[ \int_{S^5} F_5 \propto N . \] (2.57)

The number of flux quanta \( N \) is the same as the number of colors in the gauge theory. The equations of motion give a relation between the rank of the gauge group and the radius of the \( \text{AdS}_5 \) and \( S^5 \) as
\[ R_{\text{AdS}}^4 = 4\pi N l_{\text{Planck}}^4 = 4\pi g_s N l_s^4 . \] (2.58)
Since we are interested in relating the five-dimensional AdS space to the holographic theory, we dimensionally reduce over the five-sphere. The dimensionally reduced action is

\[ S_{\text{IIB}} = \frac{2 R_{\text{AdS}}^5 \Vol(S^5)}{(2\pi)^7 l_{\text{Planck}}^8} \int d^5 x \sqrt{-g} \left( \frac{R^{(5)}}{2} + \frac{6}{R_{\text{AdS}}^2} \right) + \ldots \]  

(2.59)

\[ = \frac{N^2}{4\pi^2} \int d^5 x \sqrt{-g} \left( \frac{R^{(5)}}{2} + 6 \right) + \ldots . \]  

(2.60)

The boundary of this compactified space is four-dimensional Minkowski, indeed, since the conformal rescaling of the metric

\[ d\hat{s}^2 = dz^2 + \eta_{ab} dx^a dx^b \]  

(2.61)

shrinks the radius of the S^5 to zero at the boundary.

We now relate the dimensionless parameters of the field theory and the gravity theory. The axion is related to the angle in the field theory and the Yang-Mills coupling is related to the string coupling

\[ g_{YM}^2 = 4\pi g_s , \quad \theta = \langle \chi \rangle . \]  

(2.62)

Also the global symmetries of both theories match as required, namely

- SO(2, 4) is the conformal group and the isometry group of AdS_5.
- There are 32 real supercharges in the field theory and AdS_5 \times S^5 is a maximally symmetric solution of N = 2 type IIB supergravity.
- The SO(6) R-symmetry is the isometry group of the S^5.

Contemplating once more on the relation between the couplings in the two theories

\[ \frac{R_{\text{AdS}}}{l_s} = g_{YM}^2 N = g_s = \lambda_{\text{t Hooft}} , \]  

(2.63)

we see that this correspondence, indeed, satisfies the general principles explained in section 2.2.2. If we take N → ∞ at λ_{t Hooft} ≪ 1 fixed, the string coupling vanishes, g_s → 0. In the field theory's perturbative expansion only planar diagrams survive, whereas in the string theory higher genus contributions to the string scattering amplitude vanish. For a weakly coupled bulk theory, we need a large rank of the gauge group N ≫ 1. For Einstein gravity to be trustworthy, the effective coupling must also be large and we have two regimes

- \( \lambda_{\text{t Hooft}} \gg 1 \): classical gravity is valid, field theory is strongly coupled;
- \( \lambda_{\text{t Hooft}} \ll 1 \): classical gravity is invalid, field theory is weakly coupled;

Thus, for each regime, there is a predestinated theory, in which to do the calculations. Either, a well-established theory of gravity describes the non-perturbative regime of a quantum field theory or a well-understood perturbative quantum field theory sheds light on gravity beyond the classical approximation. The latter case in an application cosmology is of interest in this thesis.
2.2.4 THE IMPORTANCE OF BULK BOUNDARY CONDITIONS

What makes the AdS/CFT correspondence so powerful is that the equivalence holds at the full quantum level. This equivalence is encoded in the equivalence of the partition functions of the bulk and boundary theories

\[ Z_{\text{gravity}}[\phi_0(x)] = Z_{\text{field theory}}[\phi_0(x)] \] (2.64)

which is a central statement to the duality. From it, the AdS/CFT dictionary can be derived. It states that to each field in the bulk corresponds a primary operator in the field theory.

The boundary of the space-time is where the field theory lives. The boundary data of the bulk fields are sources of gauge invariant operators of the field theory. Therefore, the boundary conditions, which we impose on the bulk (scalar) fields, determine the solution in the bulk, which corresponds to picking specific operators in the field theory. Hence, for each field, we impose a boundary condition

\[ \phi(0, x^a) \sim \phi_0(x^a) \] (2.65)

Then the partition function of the bulk theory, subject to the boundary conditions, is identical to the generating functional of the field theory with the boundary values of the fields as sources

\[ Z_{\text{bulk}}[\phi_0] = \langle e^{\int d^4 x \phi_0(x) O(x)} \rangle_{\text{boundary}} \] (2.66)

This relation can be employed to concretely obtain correlators on either side from the other by functional differentiation. It becomes particularly predictive and useful in the 't Hooft limit, where \( g_S \to 0 \) and the bulk is essentially classical. The path integral determining the partition function \( Z_{\text{bulk}} \) is then largely dominated by the solution to the classical field equations.

To make the effect of boundary conditions concrete, let us consider the Euclidean action of a free bulk scalar field of mass \( m \)

\[ I = \int d^4 x d^4 z \sqrt{g} \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 \right] \] (2.67)

Connected correlation functions in the boundary theory can be computed by functional differentiation of the bulk partition function

\[ Z_{\text{bulk}}[\phi_0] = e^{W[\phi_0]} \] (2.68)

\[ \langle O(x_1) \cdots O(x_n) \rangle_{\text{connected}}[\phi_0] = \frac{\delta^n W}{\delta \phi_0(x_1) \cdots \delta \phi_0(x_n)} \bigg|_{\phi_0} \] (2.69)

where

\[ W_{\text{bulk}}[\phi_0] = I_{\text{on-shell}}[\phi_0] + \text{quantum corrections} \] (2.70)

where \( I_{\text{on-shell}}[\phi_0] \) is the Euclidean action of the classical solution satisfying the boundary conditions set by \( \phi_0 \). Imposing a finite boundary condition corresponds to adding a source term to the boundary action

\[ I_{\text{CFT}} \to I_{\text{CFT}} - \int d^4 x \phi_0(x) O(x) \] (2.71)
The linearized field equations in Poincaré coordinates show that the general solution behaves near the boundary as

$$\phi|_{z \to 0} \sim \left( \alpha(x) z^{4-\Delta} + \beta(x) z^\Delta \right) \left( 1 + \mathcal{O}(z^2) \right),$$

(2.72)

$$\Delta = 2 + \nu, \quad \nu = 4 + m^2.$$  

(2.73)

Here, $\alpha$ and $\beta$ are the mode functions defined on a Poincaré slice $z = \text{constant}$. We set the boundary conditions for the mode $\alpha$, which diverges near the boundary

$$\alpha(x) = \phi_0(x). \quad (2.74)$$

The solution to the free field equation respecting this condition is

$$\phi(z, x) = \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \int d^4y \left( \frac{z}{z^2 + (x - y)^2} \right)^\Delta \phi_0(y). \quad (2.75)$$

From that expression, the value of the decaying mode $\beta$ can be extracted

$$\beta(x) = \frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \int d^4y \frac{\phi_0(y)}{(x - y)^{2\Delta}}. \quad (2.76)$$

The on-shell action of the boundary theory is defined by an integral on the AdS boundary. It is divergent and needs to be renormalized by introducing suitable boundary counterterms, which do not affect the bulk equations of motion. The finite part of the action is

$$I_{\text{on-shell}} = -\frac{(\Delta - 1)(\Delta - 2)}{\pi^2} \int d^4x d^4y \frac{\phi_0(x)\phi_0(y)}{(x - y)^{2\Delta}}. \quad (2.77)$$

Let us now examine the one- and two-point functions of the boundary theory. The vacuum expectation value is calculated to be

$$\langle \mathcal{O}(x) \rangle = \frac{2(\Delta - 1)(\Delta - 2)^2}{\pi^2} \int d^4y \frac{\phi_0(y)}{(x - y)^{2\Delta}} = 2\nu \beta(x). \quad (2.78)$$

We see that the normalizable mode corresponds to the VEV of the dual field theory operator. This must vanish in the absence of sources not to break conformal invariance. The two-point function

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle_{\text{connected}} = \frac{2(\Delta - 1)(\Delta - 2)^2}{\pi^2} \frac{1}{(x - y)^{2\Delta}} \quad (2.79)$$

has the form expected for a two-point function of an operator with dimension $\Delta$ in a conformal field theory. Hence, the mass of the bulk (scalar) field determines the dimension of the operator.

We can even generalize the boundary conditions, which we impose by deforming the field theory action \[82,83\] and re-write (2.71) as

$$I_{\text{CFT}} = I_{\text{CFT}} + 2\nu W[\hat{\beta}(x)], \quad (2.80)$$

$$\hat{\beta}(x) = \frac{\mathcal{O}(x)}{2\nu}, \quad (2.81)$$
Chapter 2. The resolution of cosmic singularities

where
\[
W[\hat{\beta}(x)] = \int d^4 x \, \phi_0(x) \hat{\beta}(x) .
\] (2.82)

The bulk boundary condition are then re-written as
\[
\alpha(x) = -\frac{\delta W}{\delta \beta(x)} .
\] (2.83)

This is a way of imposing general, non-linear boundary conditions.

Such generalized bulk boundary conditions will generically correspond to multi-trace deformations of the field theory. They can also be imposed as field equations, if suitable boundary terms are added to the bulk action. The field equations of a scalar field in \(d + 1\) dimensional AdS space
\[
\Box_x \phi - \frac{d - 1}{z} \partial_z \phi + \partial_z^2 \phi = \frac{m^2}{z^2} \phi
\] (2.84)
lead to the general asymptotic behavior of the field
\[
\phi|_{z \to 0} \sim \left( \alpha(x) z^{d-\Delta} + \beta(x) z^{\Delta} \right) \left( 1 + \mathcal{O}(z^2) \right) ,
\] (2.85)
\[
\Delta = \frac{d}{2} + \nu , \quad \nu = \sqrt{\frac{d^2}{4} + m^2} ,
\] (2.86)
where we choose the larger root for \(\nu\). We see that \(\Delta\) is real even for fields with a negative mass squared, as long as the inequality \(m^2 \geq m^2_{\text{BF}} = -\frac{d^2}{4}\) is obeyed, which is called the Breitenlohner-Friedman bound [84]. Such fields do not destabilize AdS space. For a scalar field which saturates the bound \(m^2 = m^2_{\text{BF}}\), the two roots of \(\nu\) are the same and the asymptotic behavior of the field is
\[
\phi|_{z \to 0} \sim \left( -\alpha(x) z^{\frac{d}{2}} \log z + \beta(x) z^{\frac{d}{2}} \right) \left( 1 + \mathcal{O}(z^2) \right) .
\] (2.87)

The two variables \(\alpha\) and \(\beta\) are canonically conjugate and have the interpretation as the source and expectation value (2.78) of the scalar field.

For the example at hand, we can focus on this latter case where \(m^2 = m^2_{\text{BF}}\), which is a slight variation on the general case with \(m^2 > m^2_{\text{BF}}\). To impose the boundary condition \(\alpha(x) = \phi_0(x)\), we have to add a boundary term to the bulk action
\[
I_b[\phi_0] = \frac{\pi^3 R^8_{\text{AdS}}}{2\kappa^2} \int_{z=\epsilon} d^d x \, z^{-d} \left( \frac{\Delta}{2} \phi^2 - z^{d-\Delta} \phi_0 \phi \right) ,
\] (2.88)
and we know from (2.58) that \(R^8_{\text{AdS}} \propto N^2\) in the context of the AdS_5/CFT_4 correspondence [22]. This leads to the general solution
\[
\phi(z, x) = \frac{\Gamma(\Delta + 1)}{d\pi^{\frac{d}{2}}} \int d^d y \left( \frac{z}{z^2 + (x - y)^2} \right)^\Delta \phi_0(y) .
\] (2.89)

For the boundary term to vanish in general, the classical solution must satisfy
\[
\left. z^{-\Delta} (z \partial_z - \Delta) \phi \right|_{z=\epsilon} = -\phi_0 ,
\] (2.90)
Chapter 2. The resolution of cosmic singularities

which reduces to $\alpha = \phi_0$ when taking the cut-off $\epsilon \to 0$. In the large-$N$ limit, the equivalence of the partition functions can be re-stated as

$$\langle e^{-I_b[\phi_0]} \rangle_{\text{bulk}} = \langle e^{I[\phi_0(x)\mathcal{O}(x)]} \rangle_{\text{boundary}}.$$  \tag{2.91}

The complete action can be written such that only the boundary term is non-vanishing on-shell

$$I + I_b[\phi_0] = -\frac{N^2}{2} \int_{\epsilon}^\infty dz d^dz z^{-d+1} \phi \left( \square_z - \frac{d-1}{z} \partial_z + \partial_z^2 - \frac{m^2}{2^2} \right) \phi$$  \tag{2.92}

$$+ N^2 \int_{z=\epsilon} \mathcal{L} x z^{-d} \phi \left( -\frac{1}{2} z \partial_z \phi + \frac{\Delta}{2} \phi - z^{-d-\Delta} \phi_0 \right).$$  \tag{2.93}

Using the boundary condition (2.90), the on-shell action is

$$\langle I + I_b[\phi_0] \rangle_{\text{on-shell}} = -\int_{z=\epsilon} d^dz z^{-d} \phi \phi .$$  \tag{2.94}

We then use the general solution (2.89) and remove the regulator $\epsilon \to 0$

$$\langle I + I_b[\phi_0] \rangle_{\text{on-shell}} = -N^2 \frac{\Gamma(\Delta + 1)}{2 \gamma^2} \int d^d x d^d y \phi_0(x) \phi_0(y) \frac{\phi_0(x)}{(x-y)^{2\Delta}} .$$  \tag{2.95}

From that, we obtain the expectation value

$$\langle \mathcal{O}(x) \rangle = N^2 \frac{\Gamma(\Delta + 1)}{\gamma^2} \int d^d x \frac{\phi_0(x)}{(x-y)^{2\Delta}} = N^2 \beta(x) .$$  \tag{2.96}

For the field theory, we consider a deformation of the Euclidean action by a generic functional of $\mathcal{O}$

$$I_{\text{CFT}} \to I_{\text{CFT}} + N^2 W[\beta] , \quad \beta = \frac{\mathcal{O}(x)}{N^2} .$$  \tag{2.97}

For the partition functions, this deformation leads to

$$\langle e^{-N^2 W[\beta]} e^{I[\phi_0(x)\mathcal{O}(x)]]} \rangle_{\text{boundary}} =$$  \tag{2.98}

$$e^{-N^2 W} \left[ N^2 \frac{\delta}{\delta \phi_0(x)} \right] \langle e^{I[\phi_0(x)\mathcal{O}(x)]]} \rangle_{\text{boundary}} = e^{-N^2 W} \left[ N^2 \frac{\delta}{\delta \phi_0(x)} \right] \langle e^{-I_b} \rangle_{\text{bulk}}$$  \tag{2.99}

$$= \langle e^{-N^2 W[z^{-\Delta} \phi(\epsilon,x)]} - I_b \rangle_{\text{bulk}} .$$  \tag{2.100}

That means that the deformation of the field theory imposes a change in the bulk boundary term

$$I_b^W = I_b + N^2 W[e^{-\Delta} \phi(\epsilon,x)] ,$$  \tag{2.101}

which changes the on-shell constraint to

$$z^{-\Delta} \left( z \partial_z - \Delta \right) \phi \bigg|_{z=\epsilon} = \phi_0 + \frac{\delta W}{\delta \beta(x)} \left[ e^{-\Delta} \phi(\epsilon,x) \right] .$$  \tag{2.102}

Since the argument of the functional derivative of $W$ is divergent for $\epsilon \to 0$

$$e^{-\Delta} \phi(\epsilon,x) \sim z^{d-2\Delta} \alpha(x) + \beta(x) .$$  \tag{2.103}
That means, that the deformation of the field theory leads to ill-defined boundary conditions, which is generally the case, if \( W \) is itself independent of the cut-off.

In general, this means that we need to renormalize the deformed field theory. This is not a surprise, since with the deformation, we have broken conformal invariance of the theory, which now must be renormalized (see e.g. [85]). We assume that this can always be done consistently and that the finite part of (2.103) corresponds to the bare deformation of the dual field theory. Then, using the prescription of [82] imposing the boundary condition

\[
\alpha(x) = -\frac{\delta W}{\delta \beta(x)} \tag{2.104}
\]

corresponds to a deformation of the dual CFT’s Euclidean action by

\[
I_{\text{CFT}} \rightarrow I_{\text{CFT}} + N^2 W[\hat{\beta}] , \quad \langle \hat{\beta} \rangle = \beta . \tag{2.105}
\]

We will use this way of incorporating boundary conditions for a double trace deformation of the field theory in section 2.3.3. It will break both supersymmetry and conformal invariance. In the bulk, this means that the asymptotic AdS invariance is broken by back-reaction of the scalar field.

In the following, I am going to present a way to use such a deformation to construct an AdS space with a cosmological singularity and its dual field theory.

## 2.3 A singularity toy model

The AdS/CFT correspondence carries the exciting prospect of being able to address some long-standing issues in (Anti-de Sitter) quantum gravity in terms of well-defined and usually better understood quantum field theories. Of particular interest are “big bang” and “big crunch” singularities in the supergravity theory which in principle should have a holographically dual description in terms of a conformal field theory. In this section, I will describe a setup, in which an unstable bulk theory, which exhibits a singularity, is related to the deformation of a conformal field theory, using the prescription of generalized boundary conditions.

The idea to describe spacelike singularities in a dual theory reaches back to matrix models in two space-time dimensions [86, 87]. Light-like singularities have been investigated with matrix theory in higher space-time dimensions [88] and a non-commuting matrices have been suggested as a model of space-time near a singularity [89–92]. Insights can be gained from the study of singularities inside black holes [93–96] but the horizon concealing it protects the CFT from ever seeing the singularity. There are numerous other models of cosmological singularities in AdS/CFT [97–103].

The results of a first attempt to consider such an application of the AdS/CFT correspondence to four-dimensional space-times [104, 105] were somewhat inconclusive. In AdS_5/SYM_4 [106],
Chapter 2. The resolution of cosmic singularities

the cosmological setup is more precise and better understood. The generalized scalar field boundary conditions correspond to a double trace deformation of the field theory [82] and to leading order in $1/N$ the singular nature of the evolution in the bulk is reflected by an unbounded double trace potential in the field theory. In a more detailed string theoretical construction of the holographic set-up, D3-branes induce an instability, which leads to the crunch [85].

A negative and unbounded potential for the scalar fields leaves the field theory without a proper vacuum and is in obvious conflict with unitarity. In the large $N$ limit, loop corrections cannot improve on this situation since the double trace deformed theory is 1-loop exact and asymptotically free. One particular proposal to deal with this problem is to impose special self-adjoint boundary conditions for the field theory at infinity, which basically means to reduce the Hilbert space ad hoc such that it contains only symmetric states which have a wave packet coming in from infinity for every one vanishing there. When the field rolls up the potential, the singularity retracts from the boundary and this procedure results in a bouncing cosmology for the bulk [106, 107]. This is precisely whereon the aforementioned ekpyrotic model of cosmology is based, in which the initial conditions for the big bang are created by previous cosmological cycles. This idea has its problems, most notably having to do with particle creation and induced back-reaction, and relies on the assumption that the unboundedness of the potential remains an unavoidable consequence for all values of the parameters in the field theory. Strictly speaking, however, the unboundedness of the potential has only been checked in the limit $N \to \infty$. We have seen that this limit corresponds to effectively turning off quantum gravity in the bulk. A plausible alternative is that at finite $N$, a full (perturbative) analysis results in an effective potential with a new stable minimum in the far UV, which would drastically change the qualitative behavior of the field theory. In the bulk this picture would suggest a resolution of the big crunch singularity by higher order string corrections that become more and more important as the bulk scalar field flows down the unbounded supergravity potential.

Another important reason to suspect that the dynamics is more subtle than so far presented is that the double trace deformation breaks the conformal symmetry. I expect that this gives rise to the running of not just the double trace coupling, but also a scale-dependence of the gauge coupling at higher non-planar order. This implies a coupled set of (nonlinear) flow equations that should be studied carefully to determine the UV behavior. In particular this might result in the double trace deformation becoming marginally irrelevant, instead of asymptotically free, requiring the introduction of a UV cut-off in the theory. In the bulk gravitational description this should be related to the appearance of an additional dilatonic scalar degree of freedom, which might have important consequences on the solutions in the bulk and the validity of the supergravity limit near the crunch singularity. The appearance of a new stable minimum in the far UV stabilizing the dynamics was also discussed in recent work on AdS instanton solutions in the bulk and their dual interpretation in conformal field theory [108], although their proposal depends crucially on a positive conformal single trace contribution to the quartic potential that was added by hand. In our case we will only be interested in the effects of corrections suppressed by $1/N$ at higher loop order in the gauge theory deformed by a double
Chapter 2. The resolution of cosmic singularities

trace interaction. If they are sufficient to stabilize the potential, regularization by hand as in e.g. [109] is not necessary.

In the remainder of this section I will review of the specific AdS bulk and then relate it to the deformed field theory. I will comment on the the 1-loop, large $N$, effective potential in the presence of the double trace deformation.

2.3.1 COSMOLOGY IN ANTI-de SITTER SPACE

To setup a cosmology which has a field theory dual I look at five-dimensional supergravity in AdS space. I use precisely the same model as [106, 107]. Type IIB string theory is defined in ten dimensions, five of which need to be compactified on a five-sphere. The low energy effective theory is $\mathcal{N} = 8$ gauged supergravity [110–112], which is a consistent truncation of ten-dimensional type IIB supergravity dimensionally reduced on an $S^5$. In total, this compactification produces 42 scalars. For our purpose, we focus on the subset of the five scalars thereof, $\alpha_i, i = 1 \ldots 5$, which describe the different quadrupole distortions of $S^5$.

Their action is [113]

$$ S = \int \sqrt{-g} \left[ \frac{R}{2} - \sum_{i=1}^{5} \frac{1}{2} (\nabla \alpha_i)^2 - V(\alpha_i) \right] , $$

(2.106)

where units are such that the five dimensional Planck mass is unity. Supergravity is defined by its superpotential $W$, in terms of which the F-term potential for the scalar fields is given by

$$ V(\alpha_i) = \frac{1}{R_{\text{AdS}}^2} \sum_{i=1}^{5} \left( \frac{\partial W}{\partial \alpha_i} \right)^2 - \frac{4}{3 R_{\text{AdS}}^2} W^2 . $$

(2.107)

The easiest, most symmetric way to write the superpotential is in terms of new fields $\beta_i$, which are defined such that $\sum_i \beta_i = 0$. They are related to the five original scalars by

$$ \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 0 & 1/2\sqrt{3} \\ 1/2 & -1/2 & -1/2 & 0 & 1/2\sqrt{3} \\ -1/2 & -1/2 & 1/2 & 0 & 1/2\sqrt{3} \\ -1/2 & 1/2 & -1/2 & 0 & 1/2\sqrt{3} \\ 0 & 0 & 0 & 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 0 & 0 & -1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} . $$

(2.108)

In terms of these new fields, the superpotential is given by

$$ W = -\frac{1}{2\sqrt{2}} \sum_{i=1}^{6} e^{2\beta_i} . $$

(2.109)

If all the fields vanish, $\alpha_i = 0 \ \forall i$, the compact $S^5$ is unperturbed. There, the scalar potential has a local maximum, which is the maximally supersymmetric AdS state. Around this state, each scalar obeys a free wave-equation with a mass that saturates the Breitenlohner-Freedman
Chapter 2. The resolution of cosmic singularities

Figure 2.3: The truncated 5d supergravity potential for an SO(5) invariant scalar field, which is unbounded from below for large scalar field values.

bound \( m_B^2 = -\frac{4}{R_{AdS}^2} \). This means that the background considered here is perturbatively stable and it appears as if at least the gravity theory is not fundamentally flawed. Therefore, we expect that quantum effects should play an important rôle in resolving the cosmological singularity. One possibility to truncate the theory further to a single scalar is

\[
\beta_i = \frac{\varphi}{\sqrt{30}}, \quad i = 1, \ldots, 5, \quad \beta_6 = -\frac{5\varphi}{\sqrt{30}}. \tag{2.110}
\]

This theory is an SO(5) invariant scalar coupled to gravity \[^{[111]}\] with a potential

\[
\mathcal{V}(\varphi) = -\frac{1}{4R_{AdS}^2} \left( 15e^{2\gamma \varphi} + 10e^{-4\gamma \varphi} - e^{-10\gamma \varphi} \right), \tag{2.111}
\]

with \( \gamma = \sqrt{\frac{2}{15}} \), which is shown in figure \([2.3]\).

Note that the supergravity potential is unbounded from below for positive values of the scalar field, a feature that is replicated in the dual gauge theory description. In global coordinates the \( AdS_5 \) metric reads

\[
ds^2 = R_{AdS}^2 \left( -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_3 \right). \tag{2.112}
\]

As a reminder, scalar field perturbations behave as follows near the boundary as \( r \rightarrow \infty \)

\[
\phi(r) = \frac{\alpha \ln r}{r^2} + \frac{\beta}{r^2}, \tag{2.113}
\]

where the coefficients \( \alpha \) and \( \beta \) depend on the other coordinates \( (t, x) \) and are related to each other in some specific way, in terms of a specified boundary condition, for the dynamics of the theory to be well-defined. For cosmological, time-dependent, behavior of the background to occur one adopts boundary conditions of the form

\[
\alpha = -\frac{\partial W}{\partial \beta}, \tag{2.114}
\]
where the function $W(\beta)$ is a priori an arbitrary function of the remaining coordinates, which will appear as an additional potential term in the dual field theory. The case of interest here is when $\alpha = f\beta$, corresponding to a double trace deformation of the SYM theory. By allowing $\alpha \neq 0$ the scalar field falls off more slowly than in the standard (empty) AdS case, where $\alpha = 0$, and as a consequence the full AdS isometry group is partly broken.

Before moving on to briefly discuss the cosmological nature of the bulk solutions, let us remind the reader of the supergravity limit and the corresponding parameter map to the dual field theory. Taking supergravity as the low energy limit of string theory first of all requires that the string tension $\alpha'$ becomes large and strings shrink to point particles in comparison to the AdS radius $R_{\text{AdS}}$, i.e. $R_{\text{AdS}} \gg 1$. In the dual field theory this corresponds to the non-perturbative regime of large 't Hooft coupling, to be precise

$$\lambda \equiv g_{\text{YM}}^2 N = \left( \frac{R_{\text{AdS}}}{\sqrt{\alpha'}} \right)^4 \gg 1. \quad (2.115)$$

The other requirement of a valid supergravity limit is that of small string coupling $g_s \ll 1$, such that string loop effects can be neglected. The AdS/CFT dictionary dictates that $2\pi g_s = g_{\text{YM}}^2$ and as a consequence a fixed 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ necessarily implies a large $N$ planar limit. The effects of quantum string corrections are mapped to $g_s \propto 1/N$ non-planar corrections in the dual field theory, whereas the free planar field theory limit ($\lambda \to 0$) should describe the (free) string theory to all orders in $\alpha'$, for which a bulk description in terms of supergravity breaks down completely. We would like to stress that the strict $N \to \infty$ limit corresponds to a free, classical, AdS string theory, in which the effects of quantum gravity are effectively turned off. As a consequence one should be careful to extend results derived in the (classical) $N \to \infty$ limit to the large, but finite, $N$ case with gravity turned on. When confronted with singularities in the AdS bulk one would naively expect that an ever increasing strength of gravitational interactions should play an important, if not crucial, role in any mechanism to resolve the singularity and therefore a strict $N \to \infty$ limit could give rise to misleading results. As a corollary, non-planar contributions might result in drastically different conclusions regarding the effective potential and the corresponding behavior in the gravitational bulk. Looking at this from a pure bulk perspective this might be related to a non-perturbative inconsistency of the single $SO(5)$ invariant scalar field supergravity truncation. Since the double trace deformation is marginally relevant, breaking the super-conformal symmetries, higher order running of the gauge coupling should be expected and correspondingly the dilaton in the bulk should become dynamical, which is not described by the truncated supergravity Lagrangian and the corresponding instanton solutions.

### 2.3.2 Singular Cosmology as Instanton Solution

I now explain in more detail, how the authors of [106] construct a solution to (2.111), which satisfies the generalized boundary conditions $\alpha = f\beta$ and develops a space-like singularity.
Those data are constructed as a slice of an $O(5)$ invariant Euclidean instanton solution with metric
\[ ds^2 = R^2_{\text{AdS}} \left( \frac{d\rho^2}{b^2(\rho)} + \rho^2 d\Omega_4 \right) . \] (2.116)
The function $b$ can be determined in terms of $\phi$ from the field equations, which asymptotically read
\[ b^2 = \rho^2 + 1 + \frac{\alpha^2(\ln \rho)^2}{3\rho^2} + \frac{\alpha(4\beta - \alpha) \ln \rho}{3\rho^2} + \frac{8\beta^2 - 4\alpha\beta + \alpha^2}{12\rho^2} , \] (2.117)
whereas the scalar field $\phi$ is subject to
\[ b^2 \phi'' + \left( \frac{4b^2}{\rho} + bb' \right) \phi' - R^2_{\text{AdS}} V_\phi = 0 , \] (2.118)
where $' = \partial_\rho$. Furthermore, it is required that the solution is regular at the origin, $\phi'(0) = 0$, such that the instanton solutions can be labeled by $\phi_0 = \phi(0)$. An instanton can be constructed by integrating (2.118) for a given boundary condition $\phi_0$. One finds, indeed, that asymptotically
\[ \phi(\rho) = \frac{\alpha \ln \rho}{\rho^2} + \frac{\beta}{\rho^2} , \] (2.119)
where $\alpha, \beta$ are now constants.

This is used to construct time symmetric initial data for the Lorentzian solution by restricting to the equator of the $S^4$. The Euclidean radial coordinate $\rho$ thus becomes the radial distance $r$ on the initial data slice. For a given boundary condition $\alpha(\beta)$, one selects that point, which is also an instanton solution. For $f > 0$, there is precisely one such configuration. This solution is then analytically continued to a Lorentzian solution, which describes the evolution of such initial data under AdS-invariant boundary condition. Note that those slightly differ from $\alpha = f\beta$ and are expressed as
\[ \alpha \left( 1 - \frac{f}{2} \ln \alpha \right) = f\beta . \] (2.120)
For small $f$, this difference is negligible. The analytic continuation transforms the origin of the Euclidean instanton to the lightcone of the Lorentzian solution and the $O(5)$ symmetry to $SO(4, 1)$. This ensures, that inside the lightcone, the solution must behave like an open FRW universe
\[ ds^2 = -dt^2 + a^2(t) dH_4 , \] (2.121)
where $dH_4$ is the metric on the four-dimensional unit hyperboloid. As the field $\phi$ rolls down the negative, unbounded potential, the scale factor shrinks and vanishes in finite time. The associated degeneracy of the metric is the big crunch singularity. Outside the lightcone, the scalar field remains bounded and the solution is given by (2.116) with the sphere $d\Omega_4$ replaced by four-dimensional de Sitter space.

For initial scalar field profiles satisfying the generalized $\alpha = f\beta$ boundary conditions one can argue on general grounds that a big crunch singularity will develop and spread to the boundary in finite global time \[104,105\]. Approximate solutions can be found by analytically continuing
Chapter 2. The resolution of cosmic singularities

$SO(5)$ invariant Euclidean instanton solutions, describing the decay of the maximally supersymmetric AdS vacuum along the direction of the potential that is unbounded from below. Inside the light-cone that spreads from the origin the solution is described by a crunching FRLW cosmology, which hits the boundary in finite global time. Because the scalar field ends up rolling down an unbounded exponential potential the appearance of a big crunch singularity should not come as a surprise. This case should probably be considered much more severe than the relatively mild singularities appearing in Coleman-De Luccia instantons describing the decay of a false AdS vacuum into another stable AdS minimum, which have recently received renewed attention because of their potential holographic description in terms of a cut-off field theory at the spherical domain wall separating the two AdS vacua [114, 115]. It would certainly be of interest to see how any of these ideas apply in this, more extreme, case. Having briefly summarized the bulk story, we would now like to move on to the holographically dual gauge theory description.

2.3.3 The effective potential in $\mathcal{N} = 4$ SYM with a double trace deformation

In the maximally supersymmetric AdS vacuum the holographic dual is of course the $\mathcal{N} = 4$ Super-Yang-Mills theory with $SU(N)$ gauge group, whose action is [106]

$$S_0 = \int d^4x \, \text{tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \Phi^i D^\mu \Phi^i + \frac{1}{4} g^2 [\Phi^i, \Phi^j] [\Phi^i, \Phi^j] + \text{fermions} \right\}, \quad (2.122)$$

with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and covariant derivative $D_\mu \Phi^i = \partial_\mu \Phi^i + ig[A_\mu, \Phi^i]$ [7]. We deform this action by adding a double trace potential [82, 106]

$$V_{\text{Tr}^2} = -\frac{f}{2} \int d^4x \, \mathcal{O}^2, \quad f > 0, \quad (2.123)$$

where the operator $\mathcal{O}$ is chosen to be the half-BPS operator of dimension two, holographically dual to the $SO(5)$ invariant bulk scalar field $\varphi$,

$$\mathcal{O} = \frac{1}{N} \text{Tr} \left[ \Phi_1^2 - \frac{1}{5} \sum_{i=2}^{6} \Phi_i^2 \right], \quad (2.124)$$

where $\Phi_1 \ldots \Phi_6$ are the six scalars of the theory.

Fixing the $\frac{1}{N}$ counting

Since we want to calculate $\frac{1}{N}$ corrections to the $\beta$ function, we need to fix the counting of $N$ consistently. Observe that in our theory (2.122) a single trace 4-vertex comes with $g^2$ and a

\footnote{Actually, the coupling $g = g_{\text{YM}}$. Since we are only interested in the scalar sector of the theory for the moment, we keep it as $g$, though.}

40
Chapter 2. The resolution of cosmic singularities

double trace 4-vertex with \( \frac{f}{N^2} \). In the following, \( n \) denotes the number of loops of a given diagram.

Let us first fix the overall scaling by looking at diagrams with only single trace vertices. A general single vertex diagram scales as

\[
\sim g^{2(n+1)} N^n = (g^2 N)^{n+1} \frac{1}{N}.
\]  

(2.125)

For a ’t Hooft coupling for the single trace vertex

\[
\lambda = g^2 N^{N \to \infty} \text{ const.}
\]  

(2.126)

we find that diagrams scale at the same order in \( N \) for all loops as the tree-level diagram. Hence all diagrams scale at \( \frac{1}{N} \) in the large \( N \) limit and no diagram is allowed to scale at a power higher in \( N \).

We now fix the double trace ’t Hooft coupling. Because there are different ways to impose this group structure, the number of closed index loops can vary according to the specific diagram and hence the order in \( N \). We focus on the index structure which produces the highest order in \( N \). Such \( n \)-loop diagrams scale at

\[
\sim \frac{f^{n+1}}{N^2} = (f N^i)^{n+1} N^{-2-(n+1)i}.
\]  

(2.127)

Requiring that each diagram scales at the same order in \( N \) as the tree level diagram

\[
\sim \frac{f}{N^2} = (f N^i) N^{-(2+i)}
\]  

(2.128)

yields the condition

\[
-(2+i) = -2 - (n+1)i \Leftrightarrow i = 0.
\]  

(2.129)

Therefore the consistent ’t Hooft coupling for the double trace interaction is

\[
f^{N \to \infty} \text{ const.}
\]  

(2.130)

and every diagram with only double trace vertices scales at order \( \frac{1}{N^2} \), which is one order lower than the single trace diagrams above and hence consistent with the maximal scaling requirement.

We now check that this definition of ’t Hooft couplings is still consistent for diagrams with different kind of vertices. For such diagrams, we also assume the group structure that produces the highest possible order in \( N \). Such \( n \)-loop diagrams with \( j \) double trace and \( n+1-j \) single trace vertices have one index loop per boson loop and one additional index loop for a boson loop between two double trace couplings, hence for all up to one. The diagram scales as

\[
\sim \left( \frac{f}{N^2} \right)^j (g^2)^{n+j-1} = f^j \left(g N^2\right)^{n+1-j} N^{-2},
\]  

(2.131)

which is also one order less than single trace diagrams and consistent with the scaling requirement.
Renormalization in the large-N limit

The squaring of the trace results in a contraction of the gauge degrees of freedom differently than in the single trace case. The scalar field $\Phi_1$ is identified as the steepest negative direction in the effective potential (2.124) and we focus on the dynamics of $\Phi_1$ rolling down a fixed direction $\Phi_1(x) = \phi(x)U$\(^2\). We should add that strictly speaking the supersymmetric gauge theory is defined on a 3-sphere, in which case a mass term appears for the $\Phi_1$ scalar due to the conformal coupling to the curvature of the $S^3$. This quadratic mass term can be neglected in the UV, or equivalently for large enough field values.

The bare potential of this theory is negative and unbounded from below. The double trace coupling gets renormalized at one-loop level. The one-loop effective potential reads

$$V(O) = -\frac{f}{2} O^2 \left[ 1 - \frac{f}{2} \ln \left( \frac{O}{\mu^2} \right) \right], \quad (2.132)$$

where $\mu$ is a UV cut-off scale. Following the Coleman-Weinberg prescription \([116]\) we define the renormalized coupling $f_{\text{ren.}}(\mu)$ by the renormalization condition

$$V(\mu) = V(O)|_{O=\mu^2} = -\frac{f_{\text{ren.}}(\mu)}{4} \mu^4. \quad (2.133)$$

This corresponds to having the sliding scale $\mu$ to be set by the (homogeneous) expectation value of the operator $O$, rather than by an external momentum\(^3\) implementing dimensional transmutation. The beta function is most readily obtained by demanding that the effective potential (2.132) is independent of the scale $\mu$, leading to

$$\mu \frac{\partial f}{\partial \mu} = -f^2. \quad (2.136)$$

After identifying $\phi = \sqrt{O}$, this gives the following result for the renormalized coupling

$$f_{\text{ren.}}(\phi) = \frac{f(M)}{1 + f(M) \ln(\phi^2/M^2)}, \quad (2.137)$$

with an arbitrary scale $M$ acting as the scale at which the perturbative theory is defined. On physical (continuity) grounds it is natural to suppose that this infrared scale is close to the

---

\(^2\) $U$ is a constant Hermitian matrix satisfying $\text{Tr}U^2 = 1$, so that $\phi$ is a canonically normalized scalar field. We focus on the dynamics of $\Phi_1$, only.

\(^3\) Note that this means that the renormalization scale of the theory changes as $\phi \sim \sqrt{O}$ rolls down the potential. This can only be done adiabatically and therefore we have to obey the "slow-roll condition" that the time-scale at which the system is probed is small compared to the time-scale on which the system changes

$$\frac{1}{|\mu|} \ll \frac{1}{|\phi|}, \quad (2.134)$$

which yields with $\mu = \phi$

$$\frac{|\phi|}{|\phi|} \gg 1. \quad (2.135)$$

However, the larger $\phi$ becomes, the larger becomes its slope and perturbation theory might break down.
Chapter 2. The resolution of cosmic singularities

scale where the conformal masses will start dominating the effective potential. The Coleman-Weinberg potential at one-loop level now reads, for $\mu = \phi \gg M$,

$$V = -\frac{1}{4} \phi^4 \ln(\phi^2/M^2).$$  \hspace{1cm} (2.138)

This 1-loop result given in [106] is exact in the large-$N$ limit for $N_c = 4$ SYM theory. This means, that there are no higher order corrections to the coupling (in the large-N limit) and as a consequence the theory is asymptotically free. Therefore, the larger the field value the smaller the coupling and one concludes that perturbation theory should become an ever more accurate description for larger field values, i.e. at larger energies. On the other hand, the absence of an (approximately) defined vacuum state for large field values is obviously a source of concern. The (renormalized) field theory potential, regularized in the IR, is depicted in fig. 2.4.

In the absence of a ground state, the wave-function of the scalar field will spread to infinity in finite time. This means that unitarity will be lost. As a solution to this dilemma, it has been proposed [106] to employ specific boundary conditions at infinity, which reflect each mode back as it rolls down the potential. Such boundary conditions are known as a self-adjoint extension [117,118]. Rather than an extension, it is a restriction of the Hilbert space to contain only such modes for which the Hamiltonian

$$\hat{H} = -\frac{1}{2} \frac{d^2}{dx^2} - \frac{1}{4} \lambda x^p$$  \hspace{1cm} (2.139)

is self-adjoint. For large $x$, the WKB approximation is increasingly accurate and the WKB wavefunctions

$$\chi^\pm_E(x) = \left[ 2 \left( E + \frac{\lambda x^p}{4} \right) \right]^{-\frac{1}{4}} \exp \left( \pm i \int^x \sqrt{2 \left( E + \frac{\lambda y^p}{4} \right)} \, dy \right)$$  \hspace{1cm} (2.140)

for a given energy $E$ can be used as an ansatz to study the generic behavior of energy eigenfunctions. It can be seen that for a linear combination of these eigenfunctions

$$\psi^\alpha_E(x) \sim x^{-p/4} \cos \left( \frac{\sqrt{2\lambda x^p/2} + 1}{p + 2} + \alpha \right),$$  \hspace{1cm} (2.141)
the Hamiltonian is, indeed, self-adjoint. Such a procedure works in quantum mechanics, but near the singularity, the evolution of the field is ultra-local, which means that spatial gradients become unimportant for the field evolution. That implies that the quantum field theory can be seen as a collection of independent quantum mechanical oscillators at each point in space and it can be attempted to impose these conditions at every spatial point.

There are, however, serious doubts about the validity and motivation of such an approach. First of all, such a selection of a sub-Hilbert space is ad hoc and not justified by any symmetries or other properties of the Hamiltonian. They are rather imposed ex post to justify the theory. Secondly, although this procedure removes the unitarity violation, it still does not give the theory a ground state. Hence, it remains unclear on how to build a Fock space from the vacuum. In essence, the prescription applies to quantum mechanics only and is, here, extended to a limit of quantum field theory. In fact, it has been observed numerically [109], that very quickly, the energy of the initial configuration gets converted into gradient energy, which eventually diverges. The initial wave package evolves non-adiabatically and particles are produced. This energy is not converted back into a homogeneous mode. In particular, there is no transition from a Big Crunch to a Big Bang. On these grounds, I am not satisfied with the self-adjoint extension as a solution to the problem of having a dual quantum field theory without a ground state. Rather than imposing a solution, I want to examine if the theory itself regularizes its potential by taking into account all the quantum effects, in particular those suppressed by $1/N$.

We expect that this potential gets turned around by $1/N$ corrections, i.e. that including finite-$N$ diagrams in the calculation of the effective potential will render it finite and create a true vacuum. The minimal change of the coupling, which would achieve such a behavior, is

$$f(\phi) = \frac{\epsilon}{\ln \phi^2 + \alpha \phi^A},$$

(2.142)

where $A$ is a number of function determined by renormalization below, that needs to reach a value $A_4$ for some large value of $\phi$ in order to cancel the $\phi^4$ in the numerator. Note that the scalar field occurs here, because of the renormalization procedure, in which $\mu = \phi$ and thus the exponent $A$ doesn’t need to respect the invariance of the theory under $\phi \rightarrow -\phi$. In fact, since $A$ is determined by renormalization theory, it doesn’t even need to be integer and will also change it’s value, as the renormalization scale $\phi$ increases. The behavior of the potential indeed changes as desired as seen in fig. 2.5. Note that a singularity at $\phi_0 < 1$ remains and that the potential is unbounded there. This is a region in which perturbation theory is not valid. We remark that the sign of the added term should be $\alpha > 0$, because otherwise we do not create a minimum but a new maximum.

A few comments are probably in order. The deformed theory has a UV conformal fixed corresponding to the standard super Yang-Mills theory which is suggestive of a consistent and complete holographically dual description, i.e. no new degrees of freedom have to be introduced at or above some UV cut-off scale, which is clearly important if our ambition is to understand (or resolve) the appearance of space-like crunch singularities in the bulk. On the other hand, the bad news is that the theory does not have a well-defined vacuum state. One
Chapter 2. The resolution of cosmic singularities

Figure 2.5: The effective potential (2.142) which now features a vacuum.

could imagine “fixing” this by regulating the potential such that instead of being unbounded from below it features a new globally stable minimum at some large field value. One obvious way to do this would be to add higher-dimensional irrelevant corrections to the potential. This has one obvious drawback, namely that it requires the introduction of irrelevant operators that turn the theory incomplete beyond some UV cut-off. Instead of adding such operators by hand, one could even imagine that higher-order, non-planar, quantum corrections could change the anomalous dimension of the coupling \( f \) to become irrelevant at some high energy scale, disturbing the asymptotically free nature of the double trace deformation. Even though this might produce a stable vacuum, it would be disastrous from the point of view of having to rely on a theory with a (perturbative) UV fixed point. To avoid this one could consider stabilizing the potential with a positive and exactly marginal contribution, like the single trace quartic operator [108]. This is not what we are after in this work. Instead, the modest but difficult goal we have set is to investigate the behavior of the beta-functions for the double trace deformation at the non-planar two-loop order. The aim is to explicitly check whether the double trace deformed theory remains asymptotically free and if the effective potential remains unbounded from below. Because two-loop non-planar corrections will involve mixing between the gauge and the double trace coupling we will be forced to also study the running of the gauge coupling at higher order. Analysis of the coupled system of RG-flow equations can then reveal the behavior of the effective double trace coupling and potential. The necessary inclusion of a second, coupled, degree of freedom in the form of the gauge coupling will turn out to have important consequences for the UV behavior of the deformed theory.

After this introduction, let us now move on to an analysis of two-loop nonplanar corrections.

2.4 Limitations of the large-\( N \) limit

In the previous subsection, I have mentioned that the effective potential was one-loop exact and that the Yang-Mills coupling does not contribute to the RG-flow of the double trace coupling. In this subsection I want to explain these statements and justify why I expect them to change
2.4.1 Effects of the large-$N$ limit

When dealing with a double-trace vertex, it is important to realize, that as opposed to the single trace vertex, it can be contracted in two different ways. Depending on the contraction, the number of closed index loops changes. This means, that with a double trace vertex in a diagram, not only the number of vertices but also the specific contraction at each vertex determines the order in $1/N$ of a given diagram. Looking at the one-loop order first, depicted in figure 2.7, we see that only one of the possible three diagrams will contribute in the large $N$ limit. Each vertex contributes a factor $f/N^2$, whereas each closed index loop contributes a factor of $N$. Combined, we see that only the contraction which yields two closed index loops has the same order in $1/N$ as the vertex (cf. figure 2.6). We see that the large $N$ limit significantly reduces the number of diagrams, we have to take into account. I treat the contributions of the diagrams sub-leading in $1/N$ to the one-loop renormalization in detail in appendix B.2.

Let us now turn our attention to the two-loop level. There are two prototypical shapes of two-loop diagrams, which are represented in figure 2.8. To see the group structure, we have to thicken all the propagators to be double lines and replace the vertices by single trace and double trace contractions. Since there are quite a few of these diagrams, this is reserved for appendix B.3. It is, however, clear by inspection, that the diagrams with the two loops intertwined can at most have two closed index loops as opposed to the left diagram, for which the maximal number of index loops is four as depicted in figure 2.10. We see that for this contraction,
Figure 2.8: The momentum structure of two loop diagrams correcting the four-vertex. For our purposes, the propagators have to be doubled and the vertices have to be replaced by single and double trace contractions. If we draw all the diagrams and do the counting, we observe that only diagrams of the left shape survive in the large $N$ limit.

Figure 2.9: The diagram with the most closed index loops at the two-loop level, which is leading in the large $N$ limit. Note, that it has the same order in $1/N$ as the vertex. It factorizes into a square of the leading one-loop diagram in figure 2.7.

Figure 2.10: The non-factorizable two-loop diagram in double-line notation. The maximum number of index loops is 2 such that the diagrams of this type are always subleading in $1/N$ as compared to the diagram in figure 2.9.
a new vertex, which counts $f/N^2$ comes with two new index loops contributing $N^2$ so that these diagrams are the leading ones in the large $N$ limit. Besides, for any loop, the incoming momentum is the same, which means that the diagram can be factorized into a square of the leading one-loop diagram in figure 2.7.

The same argument can be repeated for any order of loops: The chain-type diagrams are leading in the large $N$ limit and are factorizable. This implies that, in the large $N$ limit, higher loops do not contribute any new divergences to the Green’s function and hence, there is no need to introduce new counterterms, which means that the $\beta$-function does not change. In other words, it is one-loop exact (cf. [82, 106, 107]). We see that the large $N$ limit leads to an amazing simplification of this theory.

The one-loop exactness of the theory has another astonishing consequence. So far, I have only talked about diagrams, that contain only one type of vertex, namely the double trace one. It is, however, conceivable, that a diagram contains single and double trace vertices at the same time. In figure 2.11 I depict such a one-loop diagram.

It can be shown that, actually, such diagrams do not contribute to the effective potential [106]. To see this, we expand the (real) scalar fields around a constant background $\Phi^1(x) = \phi(x)U$ for the steep direction, such that $\text{tr} \, U^2 = 1$.[4] Observe that $U$ can be chosen to be diagonal to simplify the calculations. We compute the masses of the various modes in this background

\[
\Phi^1 = \phi U + R^1
\]
\[
\Phi^i = 0 + R^i, \quad i = 2 \ldots 6.
\]

Taking the terms quadratic in $R$, we see that the masses of the scalars are

\[
(M_{ab}^1)^2 = g^2\phi^2(U_{aa} - U_{bb})^2 + \frac{2fa^2\phi^2}{N^2}(1 + \delta_{ab}U_{aa})^2
\]
\[
(M_{ab}^i)^2 = g^2\phi^2(U_{aa} - U_{bb})^2 - \frac{2fa^2\phi^2}{5N^2}.
\]

[4] See section 2.5 for details on the background field method and its extension to two loops.
Observe that those are already diagonal in field space. Each bosonic mode contributes
\[
\frac{1}{32\pi^2} \frac{1}{2} M^4 \ln \left( \frac{M^2}{\Lambda^2} \right)
\]
to the effective potential. Since we are looking for a term proportional to \( f \), we focus on the cross-terms of the two couplings. Those need to come from off-diagonal entries, since otherwise the single trace term vanishes. We find for \( \Phi_1 \)
\[
2 g^2 \phi^2 (U_{aa} - U_{bb})^2 \frac{2 f a^2 \phi^2}{N^2} \ln \left( g^2 \phi^2 (U_{aa} - U_{bb})^2 + \frac{2 f a^2 \phi^2}{N^2} \right)
\]
and for \( \Phi_2 \ldots \Phi_6 \)
\[
5 \cdot 2 g^2 \phi^2 (U_{aa} - U_{bb})^2 \frac{2 f a^2 \phi^2}{5 N^2} \ln \left( g^2 \phi^2 (U_{aa} - U_{bb})^2 + \frac{2 f a^2 \phi^2}{5 N^2} \right).
\]
To simplify this, we need to expand the logarithm around small \( f \), which is possible, because we perceive the double trace interaction as a perturbation of SYM. Then the contributions of the 6 scalar fields to the mixed term precisely cancel. Thus, this is a feature of the scalar sector and of the exact form of the scalar potential rather than a property due to supersymmetry. The commutator squared part for the single trace interaction is typical for SYM theory, whereas the form of the double trace operator, which preserves an \( SO(5) \) sub-symmetry is due to the specific choice of truncation in the dual supergravity theory.

### 2.4.2 Qualitative Changes Beyond Large-\( N \)

As I have been mentioning before, the one-loop renormalized theory, which is exact in the large-\( N \) limit, does not have a ground state. If a theory turns pathological in a certain approximation, this can be taken as a hint that the approximation was not a valid one to do. In this case, there is another indication that \( 1/N \) effects might be important. They correspond to quantum gravity or \( g_s \)-effects in the bulk, which certainly play a rôle around around singularities, where gravity is strong. For a field theory that is dual to a bulk with a cosmic singularity, it is therefore expected that finite-\( N \) effects will qualitatively change the behavior of the theory.

The reason why new features are conceivable, is because at sub-leading order in \( 1/N \), non-factorizable diagrams such as the one depicted on the right of figure 2.8 come into play at the two-loop level. With them, new divergences arise, which contribute new terms to the \( \beta \)-function. Details on the full two-loop renormalization of \( \phi^4 \) theory can be found in appendix B.3. In particular, it is possible that the RG-flows of the Yang-Mills coupling \( g \) and the double-trace coupling \( f \) mix. Recall that the cancellation of such diagrams was due to a conspiracy of supersymmetry and the resident \( SO(5) \) R-symmetry. Whereas the one-loop effective potential only depends on the masses of the scalars, the two-loop contributions also depend on the couplings directly in a non-trivial way.

The most drastic qualitative change of the theory would be that quantum corrections regularize the effective potential. Such an effect can happen, if there is a contribution to the \( \beta \)-function,
which is linear in the double trace-coupling \( f \), as I will show below. Recall that the argument, why such corrections where absent at the one-loop order is independent of the order in \( 1/N \). Therefore, we expect such contributions to arise at the two loop level, where they would be encoded in diagrams with two single and one double trace vertices.

### 2.4.3 Potential Self-correction from an RG Point of View

As mentioned above, the theory is one-loop exact in the large-\( N \) limit, such that higher-loop effects alone wouldn’t change the behavior of the potential. Hence, we expect that a “turnaround” as described could only be created by including finite-\( N \) effects in perturbation theory at 2-loop order. If quantum corrections were to take care of the unboundedness of the potential and would regularize it, this would show up as a higher order correction to the coupling constant \( f(\phi) \). In the following, I determine a necessary condition for a turnaround by presuming a favorable form of the coupling. I integrate its \( \beta \)-function to determine its dependence on the cutoff.

The \( \beta \)-function is the derivative of the coupling with respect to the field (multiplied by the field in order to render it dimensionless). Thus for (2.142)

\[
\beta(f) = \frac{\partial f}{\partial \phi} = -\frac{\epsilon}{(\ln \varphi^2 + \alpha \varphi A)^2} \left( \frac{1}{\varphi^2} 2\phi + \alpha A \varphi^{A-1} \right) \phi 
\]  

\[
= -\frac{f^2}{\epsilon} \left( 2 + \alpha A \varphi^A \right) 
\]  

\[
= -\frac{f^2}{\epsilon} \left( 2 + \epsilon A f + \ldots \right) 
\]  

\[
= -2\frac{f}{\epsilon}^2 - Af 
\]

where in the second last line we used that \( f \rightarrow \frac{\epsilon}{\alpha \varphi^A} \) for large values of \( \varphi \). Note that the prefactor of the linear term determines the scaling and needs to be \( > 4 \) in order to produce a turnaround. From the form of the \( \beta \)-function in (2.153) we see that the desired correction needs to be caused by diagrams

- involving only one double trace vertex (and thus two single trace vertices)
- contributing with the same sign as the 1-loop correction

We now integrate \( \beta(f) \) to get the RG-flow of \( f \) with respect to the cutoff \( \Lambda \), which is set up to equal the field \( \phi \) in (106).

\[
\frac{\partial f}{\partial \ln \Lambda} = -\frac{2}{\epsilon} f^2 - 6f \iff -\frac{\partial f}{\partial \ln(3 + \epsilon f)} = \partial \ln \Lambda 
\]

integrating which on both sides yields

\[
-\frac{1}{6} \left( \ln f + \ln(3 + \epsilon f) \right) = \ln \Lambda 
\]
Figure 2.12: The RG-flow of $f$ with the expected two-loop correction with $\epsilon = 1$. The theory remains asymptotically free.

which we can solve for $f$ to get

$$f = \frac{3}{-\epsilon + e^6 \ln \Lambda} = \frac{3}{-\epsilon + \Lambda^6} .$$

This is still an asymptotically free coupling as can be seen in fig. 2.12 where we have plotted $f$ with respect to $\ln \Lambda$ and $\Lambda$, respectively.

2.4.4 Coupled RG-flow of two scalar couplings

As explained above we are looking for corrections to the $\beta$-function which are sub-leading in $1/N$. Such corrections will show up at two-loop order. In the following we are laying out, how to extract the $\beta$-function from the two-loop counterterms.

It is important to notice that at the two-loop level, mass and field renormalizations kick in in all the theories under consideration. Thus the full renormalization of all parameters needs to be done and subsequently the Callan-Symanzik-equation needs to be solved. Note also, that the $\beta$-function for $g$ and $f$ are coupled. Furthermore, in a massive theory, the mass term is reparametrised to $m^2 \phi^2 = a \mu^2 \phi^2$ for the sake of dimensional regularization. The Callan-Symanzik-equation for an $n$-point Green’s function $G^{(n)}$ in a theory with only one field then reads

$$\left( \mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} + \beta_f \frac{\partial}{\partial f} + \beta_m \frac{\partial}{\partial a} + \sum_{k \in \text{fields}} n_k \gamma_k \right) G^{(n)}(\{x_i\}, \mu, g, f, a) = 0 .$$

where $\beta_m = (d - 6 + \gamma_m)a$. Here, $d$ is the (regularized) dimension of the field theory and $\gamma_m$ takes the scaling of the mass operator in the Green’s function into account. Later on we are working in a massless theory and hence, this part of the Callan-symanzik-equation drops out.

The correction of the $\beta$-function which regularizes the effective potential finite after including finite-$N$ corrections has been expected in (2.153) and the comments thereunder to come from a diagram involving one double trace and two single trace interactions as depicted in fig. 2.13.

However, there are more diagrams that could renormalize the double trace interaction coming from couplings to fermions and vectors as well as from mass renormalizations of bosons and
fermions. We neglect those effects, because they would not be different from standard, undeformed $N = 4$ super Yang Mills theory and cancel due to supersymmetry to yield a conformal theory. The deformation, which breaks supersymmetry, and hence, on a quantum level, also conformal invariance, applies to the scalar sector, only.

At the two-loop level, the $\beta$-functions of the two remaining couplings have the following terms

$$\frac{\partial g}{\partial \ln \mu} = \eta^{(g)}_{4,0} g^4 + \eta^{(g)}_{2,1} g^2 f + \eta^{(g)}_{0,2} f^2 + \eta^{(g)}_{6,0} g^6 + \eta^{(g)}_{4,1} g^4 f + \eta^{(g)}_{2,2} g^2 f^2 + \eta^{(g)}_{0,3} f^3$$  \hspace{1cm} (2.158)

$$\frac{\partial f}{\partial \ln \mu} = \eta^{(f)}_{4,0} g^4 + \eta^{(f)}_{2,1} g^2 f - \eta^{(f)}_{0,2} f^2 + \eta^{(f)}_{6,0} g^6 + \eta^{(f)}_{4,1} g^4 f + \eta^{(f)}_{2,2} g^2 f^2 + \eta^{(f)}_{0,3} f^3$$ \hspace{1cm} \text{1-loop} \hspace{1cm} \text{2-loop} \hspace{1cm} (2.159)

In principle, also terms $\sim e^{-f/g}$ and $\sim \ln f/g$ could occur in the $\beta$-function but because they would correspond to non-perturbative contributions we will neglect them here. These two $\beta$-functions form a system of coupled differential equations we would like to study more carefully. In particular, the actual presence and signs of the different terms will of course play a crucial role in determining the UV behavior of the couplings.

Obviously some of the terms appearing in (2.158) will have to vanish or are simply of no interest to us. First of all, in the absence of the double trace deformation the gauge theory is super-conformal and therefore all terms independent of $f$ should vanish in the beta-function for $g^2$. So we immediately conclude that

$$\eta^{(g)}_{4,0} = \eta^{(g)}_{6,0} = 0 .$$  \hspace{1cm} (2.160)

Furthermore, for $N = 4$ SYM theory, a non-renormalization argument excludes the diagram with one single- and one double-trace vertex at the one-loop level

$$\eta^{(f)}_{2,1} = 0 .$$  \hspace{1cm} (2.161)

Under the assumption that $f \ll g \ll 1$ we can also neglect terms of higher order in $f$, implying that $\eta^{(g)}_{2,2} = \eta^{(f)}_{2,2} = 0$. Finally, and importantly, we are not interested in the term proportional to $f^3$ because it cannot affect the UV behavior of the double trace coupling that we are interested in. Basically, the only terms that can have interesting effects on the couplings in the far UV are the leading contribution to the scale dependence of the gauge coupling and the next to leading, gauge coupling dependent, contribution to the running of the double trace coupling. Hence, under this assumption and using that we have already shown that $\eta^{(f)}_{0,2} = 1$, we are left with
Chapter 2. The resolution of cosmic singularities

0.2 0.4 0.6 0.8 1.0

Figure 2.14: Vector plot of the coupled system of $\beta$-functions for $g$ and $f$ assuming all coefficients to be 1. The point $(g = 0, f = 0)$ is an unstable fixed point. Including corrections to the 't Hooft limit drive the theory away from this point.

The following set of coupled differential equations,

\[
\frac{\partial g^2}{\partial \ln \mu} = \eta_{2,1}(g)^2 f + \eta_{4,1}(g)^4 f \\
\frac{\partial f}{\partial \ln \mu} = -f^2 + \eta_{4,1}(f)^4 f.
\]

Note that for now we kept two terms in the beta function for $g^2$. In the final analysis we will only be interested in the leading contribution (which is the $g^2f$ term, unless the coefficient vanishes). Determining the coefficients of these terms requires either a standard perturbative Feynman diagram analysis. The results for all the coefficients are given in appendix B.4.

The structure of these flow equations is such that a number of different things could happen, depending on the signs of the coefficients. Due to the mixing with the running gauge coupling one possibility is that both the double trace and gauge coupling increase towards the UV, implying the double trace deformed theory is actually ill-defined, contrary to the result at one-loop. Another option would keep the double trace coupling behavior relevant, but depending on the behavior of the gauge coupling in the UV limit, the effective potential could be turned around featuring a stable vacuum state.

For this reduced coupled system of $\beta$-functions we give the RG-flow in a vector plot in fig. 2.14. Here, we see that the point of the free theory for $f = g = 0$ is an unstable fixed point, from which the theory is driven away, if $g \neq 0$, hence, if we include corrections to the large $N$ limit in which effectively $g = 0$. As commented earlier, the term which has a chance to cause the turnaround is $\eta_{4,1}^2 g^4 f$. If its sign changes, the result might be different. However, the picture doesn’t change qualitatively as we see in fig. 2.15. It is worth to contemplate about the bulk
dual of this term. Since it is suppressed by $1/N$ and contains two ’t Hooft couplings,
\[ \eta^{(f)}_{4,1} g^4 f \leftrightarrow \eta^{(f)}_{4,1} g s \left( \frac{R}{\alpha'} \right)^8. \]

This should be matched with a corresponding term in the supergravity action. Note also that
the term $\eta^{(f)}_{4,0} g^4$ will lead to a logarithmic scaling of $f$.

### 2.5 Results

A convenient way to calculate the effective potential including all kinds of fields, such as scalars, fermions, vectors and ghosts is the background field method [119–122]. Here, one can fix a
gauge and compute quantum corrections without losing explicit gauge invariance. Although
the effective potential is gauge dependent, its physical properties are not [123 124]. In partic-
ular, we employ the setup proposed in [125], which performs the calculation in Landau gauge.
As opposed to calculating the 4-vertex Green’s function, this has the advantage, that the effec-
tive potential is calculated by summing only vacuum graphs without external momenta, which
simplifies the calculations enormously.

In this formalism, all the fields are separated into a classical background and its quantum fluc-
tuations about it. For instance, a (real) scalar field is represented as $\phi + R$ with its background
$\phi$ and the perturbations $R$ around it. The effective potential is the tree-level potential in the
classical background plus the sum of all connected one-particle-irreducible vacuum graphs
\[ V_{\text{eff.}} = V^{(0)} + \frac{1}{16\pi^2} V^{(1)} + \frac{1}{(16\pi^2)^2} V^{(2)} + \ldots, \]

*A complex scalar field would be represented as a background with two real fluctuations.*
Chapter 2. The resolution of cosmic singularities

where $V^{(n)}$ denotes the $n$-loop correction. When calculating those with Feynman rules, the couplings and masses acquire a dependence on the background.

Since the one-loop diagram does not contain a vertex, the one-loop correction only depends on the masses, induced by the background, and reads

$$V^{(1)} = \frac{1}{4} \sum_{\text{fields } i} (-1)^{2s_i} (2s_i + 1)(m_i^2)^2 \left( \ln \frac{m_i^2}{Q^2} - k_i \right), \quad (2.165)$$

where the index $i$ runs over the real scalars, two-component fermions and vector degrees of freedom in the theory. The renormalization scale is denoted by $Q$, $s_i = 0, 1/2, 1$ for scalars, fermions and vectors, respectively, and the constants $k_i$ depend on the renormalization scheme.

The two-loop contributions, which we are particularly interested in, here, are of the schematic form

$$V^{(2)} = \sum_{i,j} g^{ij} f_{ij}(m_i^2, m_j^2, Q) + \sum_{i,j,k} |g^{ijk}|^2 f_{ijk}(m_i^2, m_j^2, m_k^2, Q), \quad (2.166)$$

where $g^{ijkl}$ and $g^{ijk}$ are field dependent four- and three-particle couplings and the functions $f_{ij}(x, y, Q)$ and $f_{ijk}(x, y, z, Q)$ are the results of the two-loop integrals, which depend on the renormalization scale.

Turning on a background effectively produces masses for both the scalar and the gauge fields. It also leads to cross-terms between gauge and scalar fields., which need to be eliminated by adding a gauge fixing term to the Lagrangian

$$L_{gf} = \frac{g^2}{2} [R, \phi] [\phi, R], \quad (2.167)$$

which gives rise to a mass also for the scalar field along the steep direction [126]. Having masses for the fields means, that we first have to transform them to square-mass eigenstates. Since our theory comprises a deformation of the scalar sector, I will focus on the scalar part of the Lagrangian in the following. Its kinetic part contains terms like

$$-\mathcal{L} = \frac{1}{2} m_{ij}^2 R_i R_j, \quad (2.168)$$

where $i, j$ run over all the (real) scalar fields and $m_{ij}^2$ are real symmetric matrices, which depend on the classical background field. The primes denote, that the scalars are not yet in a squared-mass eigenstate. They are rotated like

$$R_i' = N^{(S)}_{ji} R_j, \quad (2.169)$$

by an orthogonal matrix $N^{(S)}$ defined by

$$N^{(S)}_{ik} m_{kl}^2 L^{(S)}_{jl} = \delta_{ij} m_i^2, \quad (2.170)$$

where $m_i^2$ are the scalar squared-mass eigenvalues. This basis rotation also has an effect on the interaction terms of the theory, which we denote, again for the scalar sector, as

$$\mathcal{L}_S = -\frac{1}{6} \lambda^{ijk} R_i R_j R_k - \frac{1}{24} \lambda^{ijkl} R_i R_j R_k R_l \quad (2.171)$$
For the calculation of the effective potential, the couplings have to be used in this basis as defined here. The interactions $\lambda^{ijk}$ and $\lambda^{ijkl}$ are symmetric in all their indices and real. They generically depend on the classical background field as they depend on the rotation matrix $N^{(S)}$. Note that we are suppressing interactions of the scalars with other fields, since those do not differ from the standard $\mathcal{N} = 4$ SYM theory.

In our case, following [106], we expand the (real) scalar fields around a constant background $\Phi^1(x) = \phi U$ for the steep direction, such that $\text{tr} U^2 = 1$. Observe that $U$ can be chosen to be diagonal to simplify the calculations. This background introduces effective masses which can be read off from the interaction terms. We have already mentioned the expressions for the masses of the various modes in this background

$$\Phi^1 = \phi U + R^1 \quad \text{(2.172)}$$

$$\Phi^i = 0 + R^i, \quad i = 2 \ldots 6 . \quad \text{(2.173)}$$

We should carefully check that with the double trace interaction the fields are still in a square mass eigenstate for the chosen basis. For the off-diagonal modes, the contribution to the squared masses from the double trace interaction are

$$(M_{ab}^1)^2 \sim \frac{f a^2}{2N^2} \left[ \text{tr} \left( \phi^2 U^2 + (R^1)^2 + 2\phi U R^1 - \frac{1}{5}(R^i)^2 \right) \right] \quad \text{(2.174)}$$

from which the masses only comprise the terms quadratic in the fields $R$, namely

$$(M_{ab}^1)^2 = \frac{f a^2}{2N^2} \left( \phi^2 U_{aa} U_{aa} + R_{ab}^1 R_{ba}^1 + 2\phi U_{aa} R_{aa}^1 \right) \quad \text{(2.175)}$$

$$= \frac{f a^2}{2N^2} \left( 2\phi^2 R_{ab}^1 R_{ab}^1 + 4\phi^2 U_{aa} R_{aa}^1 U_{cc} R_{cc}^1 \right) , \quad \text{(2.176)}$$

where $a$ and $c$ are different indices to be summed over. In the last term in the last line we see that there are cross-terms between different $SU(N)$ degrees of freedom. Hence, we still need to diagonalize these masses.

The way I propose to do this was by rotating the fields $R$ to a basis, with one element parallel and all others perpendicular to the background $U$

$$R^i_\parallel = R^i\alpha U^\alpha U \quad \text{(2.177)}$$
$$R^i_\perp = R^i - R^i\alpha U^\alpha U , \quad \text{(2.178)}$$

where we have used the scalar product on $SU(N)$, $X \cdot Y = \text{tr} (XY)$ and the normalization of the background $\text{tr} U^2 = 1$. In this basis, the mass squared term looks like

$$(M_{ab}^1)^2 = \frac{f a^2}{2N^2} \left( \frac{6\phi^2 (R_\parallel^1)^2}{m_\parallel^2} + \frac{2\phi^2}{\frac{1}{2} (m_\perp^2)^a} R_{\perp a}^1 R_{\perp a}^1 \right) , \quad \text{(2.179)}$$
where the index $a$ runs over the $N^2 - 1$ orthogonal components of $R$.

With all these data at hand, it is easy to verify the one-loop result for the effective potential. It is most concisely expressed in the $\overline{\text{DR}'}$ renormalization scheme as

$$V^{(1)} = \sum_i (-1)^{2s_i} (2s_i + 1) h(m_i^2) ,$$

with

$$h(x) = \frac{x^2}{4} \left( \ln \frac{x}{Q^2} - \frac{3}{2} \right) .$$

I have already commented above in subsection 2.4.1 that summing over all the scalar fields will produce the curious effect that any contributions to the effective potential with both single and double trace interactions vanish.

This is why I turn my attention to the two-loop level, again restricting myself to the sector with scalar interactions, only. The contributions to the effective potential are

$$V^{(2)}_{SSS} = \sum_{i,j,k} \frac{1}{12} (\lambda_{ijk})^2 f_{SSS}(m_i^2, m_j^2, m_k^2) ,$$

$$V^{(2)}_{SS} = \sum_{i,j} \frac{1}{8} \lambda_{iijj} f_{SS}(m_i^2, m_j^2) .$$

The loop-integral functions are given in terms of the standard functions

$$f_{SSS}(x, y, z, Q) = -I(x, y, z, Q) ,$$

$$f_{SS}(x, y, Q) = J(x, y, Q) .$$

which were introduced in [127] as

$$J(x, y, Q) = xy(\ln \frac{x}{Q^2} - 1)(\ln \frac{y}{Q^2} - 1) ,$$

$$I(x, y, z, Q) = \frac{1}{2} (x - y - z) \ln \frac{y}{Q^2} \ln \frac{z}{Q^2} + \frac{1}{2} (y - x - z) \ln \frac{x}{Q^2} \ln \frac{z}{Q^2}$$

$$+ \frac{1}{2} (z - x - y) \ln \frac{x}{Q^2} \ln \frac{y}{Q^2}$$

$$+ 2x \ln \frac{x}{Q^2} + 2y \ln \frac{y}{Q^2} + 2z \ln \frac{z}{Q^2} - \frac{5}{2} (x + y + z) - \frac{1}{2} \xi(x, y, z) ,$$

where the function $\xi(x, y, z)$ is expressed in terms of dilogarithms

$$\frac{\xi(x, y, z)}{R} = 2 \ln \frac{z + x - y - R}{2z} \ln \frac{z + y - x - R}{2z} - \ln \frac{x}{z} \ln \frac{y}{z}$$

$$- 2 \text{Li}_2 \frac{z + x - y - R}{2z} - 2 \text{Li}_2 \frac{z + y - x - R}{2z} + \frac{\pi^2}{3}$$

with

$$R = \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz} .$$

This information specifies the two-loop potential of the scalar sector completely, which can now be calculated systematically on a computer.
I report on the calculation performed with the Mathematica Software. To simplify the computational load, I have considered a model with $N_f = 2$ scalar fields and rank $N = 2$ of the gauge group. The qualitative result of this calculation can then be extended to large $N$ by performing a well-defined ’t Hooft limit. To account for the fact that we only consider two scalar fields in this example, the double trace operator is adjusted to be $O = \text{tr} \left[ (\Phi_1)^2 - (\Phi_2)^2 \right]$. I have chosen a diagonal background whose trace is normalized, $\text{tr} U^2 = 1$, such that

$$U = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$  

(2.193)

The result is lengthy and therefore not reproduced here in full. Instead, I only note its important properties. The crucial observation is that a term linear in the double trace coupling $f$ appears at the two loop level. This is precisely the term of the sort, which we have argued to be necessary to have the potential being regularized and turned around in section 2.4.3.

The effective potential obtained in this way still explicitly depends on the renormalization scale. To compare with the Coleman-Weinberg potential as obtained in [106], I want to identify the scale $Q$ with the value of the scalar field $\phi$. The correct way to do this is to ensure that the effective potential is invariant under RG-transformations, yielding $\beta$-functions for the couplings and anomalous dimensions for the fields (cf. section 7 in [125]) via

$$ Q \frac{dV}{dQ} = \left( Q \frac{\partial}{\partial Q} + \beta_g \frac{\partial}{\partial g} + \beta_f \frac{\partial}{\partial f} - \sum_{i=1}^{6} \gamma_i \Phi^i \frac{\partial}{\partial \Phi^i} \right) V_{\text{eff}} = 0. $$

(2.194)

The $\beta$ functions and anomalous dimensions can be extracted order by order (cf. appendix B.4)

$$ Q \frac{\partial}{\partial Q} V^{(1)} + \left( \beta_g^{(1)} \frac{\partial}{\partial g} + \beta_f^{(1)} \frac{\partial}{\partial f} - \sum_{i=1}^{4} \gamma_i^{(1)} \Phi^i \frac{\partial}{\partial \Phi^i} \right) V^{(0)} = 0, $$

(2.195)

$$ Q \frac{\partial}{\partial Q} V^{(2)} + \left( \beta_g^{(2)} \frac{\partial}{\partial g} + \beta_f^{(2)} \frac{\partial}{\partial f} - \sum_{i=1}^{4} \gamma_i^{(2)} \Phi^i \frac{\partial}{\partial \Phi^i} \right) V^{(1)} + \left( \beta_g^{(2)} \frac{\partial}{\partial g} + \beta_f^{(2)} \frac{\partial}{\partial f} - \sum_{i=1}^{4} \gamma_i^{(2)} \Phi^i \frac{\partial}{\partial \Phi^i} \right) V^{(0)} = 0. $$

(2.196)

(2.197)

The first order RG-equation for the double trace coupling is found to be

$$ - \frac{29}{4\pi^2} f(Q)^2 - Qf'(Q) = 0, $$

(2.198)

which we solve to find the renormalized coupling

$$ f(Q) = - \frac{4\pi^2}{4\pi^2 C - 29 \log(Q)}, $$

(2.199)

where $C$ is a constant of integration to be determined below. For the second order RG-equation, we neglect the logarithmic terms for simplicity and find

$$ - \frac{1}{4\pi^2} \phi^4 \left[ 84g^4 f(Q) + (130g^2 + 29\pi^2) f(Q)^2 + 4\pi^4 Q f'(Q) \right] = 0, $$

(2.200)
which is solved to obtain the two-loop renormalized double trace coupling

\[ f(Q) = -\frac{84g^4}{e^C (130g^2 + 29\pi^2)} - Q \frac{21g^4}{\pi^4}. \]  

(2.201)

We substitute these renormalized couplings back into the one- and two-loop expressions for the effective potential, respectively. Note that we have made an adiabatic approximation for the Yang-Mills coupling: Since there is a term in the effective potential, which contains both couplings, we know that also the single-trace coupling has a non-trivial \( \beta \)-function. However, we assume that \( g \) flows much more slowly than \( f \). In principle, we would have to solve the coupled set of RG-equations for both couplings. It is, however, not possible to extract the \( \beta \)-functions for both couplings and the anomalous dimensions from the effective potential. Those are rather needed as an extra input, which can for instance be extracted from the expansion in Feynman diagrams. As an approximation, this assumption is, however, justified, as we can also see from the coupled RG-flow as presented in figures 2.14 and 2.15.

I have argued in section 2.4.3 that for a turnaround to happen, the coefficient of the linear term in the \( \beta \)-function of \( f \) must be bigger than 4. To check this condition, I read off this coefficient from equation (2.200) to be

\[ -\frac{21g^4}{\pi^4} \]  

(2.202)

which generalizes to \(-\frac{84}{\pi^4} \frac{(g^2 N)^2}{N^4}\) for arbitrary \( N \). We see that this condition is only met if the Yang-Mills coupling is larger than

\[ g \geq \frac{\pi}{21^{1/4}} \approx 1.5. \]  

(2.203)

Although this is bigger than unity, we can still trust perturbation theory, because the expansion is actually not in the coupling \( g \), only, but rather in the ratio \( \frac{g^2}{\pi^2} \approx 0.12 \), which is still smaller than unity. To make sure that this condition is met, we can just tune the ’t Hooft coupling to be large enough. This, however, appears not to be necessary. As long as the Yang-Mills sector is free in the IR, \( g \) will be driven to a larger value at higher scales and the absolute value of the coefficient will eventually be large enough at some scale. The other condition, which has to be met for the linear term in \( f \) to make the potential turn around is that the sign of the coefficient is negative. This appears to be correct in the example calculation performed, but it should be robust on general grounds. We know that the double trace operator is marginally irrelevant. This means that the theory should be asymptotically free in the double-trace coupling, which a \( 1/N \) suppressed effect should not change. Therefore, the sign of this term in the \( \beta \)-function should be negative, as required.

To finally obtain the result, we need to determine the constant of integration \( C \) in (2.198) and (2.200). We do this by matching the one-loop and two-loop potentials at a point for a specific value for \( \phi \). Then, the one-loop and two-loop RG-equations are solved consistently. Since the double-trace coupling is asymptotically free, this matching can best be done at a large value of \( \phi \), infinity, say, where the two-loop effective potential approaches zero. However, the region, where we can best trust the perturbative treatment is for small values of \( \phi \).
Figure 2.16: A plot of the tree level (green), one-loop (red) and two-loop (blue) contributions to the effective potential. The value chosen for the Yang-Mills coupling is $g = 2.5$, which results in a coefficient for the term linear in $f$ in the $\beta$-function of about $8.4 > 4$, as required for a regularization of the potential. We see, that the potential, indeed, turns around and that quantum corrections generate a ground state for the scalar sector at the two-loop level. In this plot, I have ignored the contributions of the logarithms in the effective potential.

When examining the results, we encounter a hitch. Since we have ignored the logarithmic terms in the two-loop $\beta$-function it seems consistent to also ignore the logarithmic terms in the two-loop effective potential after replacing the double trace coupling by its renormalized version. The result of this procedure is illustrated in figures 2.16 and 2.17. In the former plot, I also show the one-loop and tree-level potentials. It was surprisingly not possible to find a value for the integration constant such that the one- and two-loop potentials would intersect. Apart from that, the effective potential shows the expected behavior: It starts out negative around the renormalization scale, turns around and asymptotes to zero owing to the asymptotic freedom.

One could conclude that neglecting the logarithmic terms was not appropriate. In figure 2.18 I plot the result keeping those contributions after replacing the bare with the renormalized coupling. Surprisingly, the logarithmic terms seem to have a large impact on the effective potential, which is now positive and diverges at infinity. Conveniently, it still shows a turnaround, though. A possible point of further investigation is, if it leads to a better control of the potential to include the logarithmic terms also when solving the two-loop RG-equations.

At the end of the day, the important criterion which determines whether the field theory constitutes a well-defined way to describe the dual cosmological singularity is whether it has a ground state. If the effective potential turns around, we can still draw useful conclusions from our result. We see, indeed, that the potential turns around and now features a minimum in both cases. This leads to a stable ground state for the field theory and defeats the problem of unitarity loss in the evolution of the scalar field, because the scalar will in fact thermalize around that ground state.
Chapter 2. The resolution of cosmic singularities

Figure 2.17: Same plot as 2.16 zoomed in to the region, where the two-loop potential turns around.

Figure 2.18: This plot shows the one-loop (red) and two-loop (blue) effective potentials including their logarithmic terms. We see that the two-loop potential is still bounded from below.
Chapter 2. The resolution of cosmic singularities

2.6 THE BULK INTERPRETATION

Now that we have found that taking into account $1/N$ corrections bound the effective potential of the boundary theory, which is well under calculational control, we are in a position to ask ourselves how the resolution of the cosmological, space-like singularity works in the gravitational bulk itself. It has been argued in [105] that a regularization of the effective potential by adding a higher dimensional operator in the boundary field theory will lead to the formation of a black hole with a horizon which covers up the singularity. Such a regularization has the disadvantage that the field theory becomes non-renormalizable. In our case, however, the regularization arises naturally without adding any irrelevant operators. Rather, the corrections contained in the limit of the marginal operator, which bound the potential. Therefore, the theory remains renormalizable and the situation remains well-understood all the way to the cutoff.

We review the behavior of the bulk in the present context (cf. [106]). The appropriate initial data for any boundary conditions is obtained by slicing an $O(5)$-invariant Euclidean instanton of the form

$$d\tilde{s}^2 = \frac{d\rho^2}{b^2(\rho)} + \rho^2 d\Omega_4 \quad (2.204)$$

with $\phi = \phi(\rho)$ through its center. The instanton field equations with boundary conditions $\alpha_f = f \beta$ determine $b$ to be

$$b^2(\rho) = \rho^2 + 1 + \frac{2\alpha^2}{2\rho^2} + \frac{\alpha(4\beta - \alpha)\log \rho}{3\rho^2} + \frac{8\beta^2 - 4\alpha\beta}{12\rho^2} + \alpha^2 \quad (2.205),$$

and the scalar field obeys

$$b^2 \phi'' + \left(\frac{4b^2}{\rho} + bb'\right) \phi' - R_{\text{AdS}}^2 V_{,\phi} = 0 \quad (2.206)$$

with $' = \partial_\rho$. The mass of this initial data for the Lorentzian solution is

$$M = -\frac{\pi^2 R_{\text{AdS}}^2 f^2 \beta^2}{4} \quad (2.207).$$

Hence, the instanton specifies negative mass initial data.

Numerically integrating the Einstein equations, one can show that the theory admits static, spherical black holes with the chosen boundary conditions [106]. In particular, there is precisely one black hole with scalar hair for any given horizon size. Its mass is given by

$$M_{hbh} = 2\pi^2 R_{\text{AdS}}^2 \left[\frac{3}{2} M_0 + \beta^2 \left(1 - \frac{1}{2} f\right)\right] \quad (2.208),$$

where $M_0$ is the mass of an equally large, bald, usual Scharzschild black hole corresponding to the standard vacuum with $\langle O \rangle = 0$. We see that the scalar hair adds some mass to it and the smaller $f$, the bigger the mass increase. In particular, the mass $M_{hbh}$ is always positive. If the potential is unbounded, there is no black hole to conceal the singularity and it extends all
Chapter 2. The resolution of cosmic singularities

the way to the boundary. The hairy black hole then represents an excitation about the local maximum of the field theory potential.

We can change the boundary conditions such that negative mass solutions become available. The example treated in [106] is to adjust the boundary conditions to

$$\alpha_{f, \epsilon} = f \beta - \epsilon \beta^3. \quad (2.209)$$

For a sufficiently small parameter $\epsilon$ a negative mass black hole exists with the same mass as the instanton $M \sim -\epsilon R_{AdS}^2$ and hence is the natural final state of the bulk evolution. Changing the boundary conditions for the bulk scalar field, however, will source a different operator in the field theory. Inspecting relation (2.83)

$$\alpha = -\frac{\delta W}{\delta \beta}, \quad (2.210)$$

we see that the corresponding deformation of the field theory is

$$W = -\frac{f}{2} \beta^2 + \epsilon \beta^4, \quad (2.211)$$

where $\beta = \langle \mathcal{O} \rangle$ is the expectation value of the single trace operator. This means that changing the boundary conditions in the bulk such that negative mass black holes are accessible corresponds to regularizing the field theory potential by an irrelevant quadruple trace operator. In the limit $\epsilon \to 0$, we recover the case of the unbounded potential, for which the singularity spreads to the boundary.

In the previous section 2.5, I have shown that it is not necessary to regularize the potential by hand. Rather, the contributions subleading in $1/N$ of the double trace operator automatically regularize the potential. In the large-$N$ limit, the potential, however, is unbounded just as in the case of (2.211) for $\epsilon \to 0$. The double trace operator is marginally irrelevant and as such resembles the regularization in (2.211). Yet, since the contribution which ensures the turnaround of the potential is suppressed by $1/N^2$, it should correspond to higher curvature and string loop corrections in the bulk, which are not captured by the supergravity approximation. Therefore, the expectation is that the singularity in the bulk is resolved by a "small" black hole, whose horizon is only supported by quantum corrections, with scalar hair.

This also matches with our expectation on the field theory side, in which the final state will be thermal. On the Poincaré patch, the geometry can only be Euclidean AdS with zero temperature or an AdS black hole, which is thermal. Indeed, we see in figure [2.19] that the Penrose diagrams of a black hole in AdS space and of a space-like singularity stretching all the way to the boundary are almost the same. The black hole case can be seen as the limit of the cosmological case in which the time at which the singularity hits the boundary becomes larger and larger until it is infinite for the case of an eternal black hole.

So far, our picture explains how a big crunch singularity is resolved into a black hole, which forms the thermal endpoint of the evolution. We should keep in mind that we were using a setup in supergravity, while the $1/N$ corrections in the dual field theory suggest that string
Chapter 2. The resolution of cosmic singularities

Figure 2.19: In their Penrose diagrams, we see that an AdS space with a black hole singularity and with a big bang singularity are very similar. Only the upper corners of the diagram are changed. Whereas the time around the singularity at infinity goes on forever, the big bang singularity hits the boundary at some finite time.

loop corrections are important and this picture is necessarily incomplete. In particular, the truncation to a theory with a fixed dilation breaks down as the singularity approaches the boundary. It remains an interesting question if it is possible to re-interpret this black hole in a more complete picture as the starting point of the evolution of the universe, the resolved big bang.

2.7 CONCLUSIONS

The aim of this study was to examine, if effects of quantum gravity resolve cosmic singularities using a holographic Super-Yang-Mills description, which is a well-defined quantum theory of type IIB supergravity on AdS$_5$. If a low-energy effective description of gravity descends from a UV complete theory, one expects in general that it does not contain any singularities. The supergravity theory I have examined is related to type IIB string theory in ten dimensions compactified on an S$^5$ and full string theory is considered to be a consistent theory of quantum gravity. Quantum gravity must resolve cosmological singularities.

In the specific example I have studied, the scalar field potential in the bulk is unbounded from below. As the bulk scalar field rolls down the potential to infinity, a space-like singularity forms. If quantum corrections do not bound this potential, the theory remains ill-defined and needs to be discarded. Here, this can be seen as follows. The scalar field describes one of the quadrupole distortions of the S$^5$ on which the bulk is compactified. Since it is driven down an unbounded potential, this means that the sphere becomes highly squashed, signaling a breakdown of the low-energy effective description. The mass of the background scalar considered here satisfies the Breitenlohner-Friedman bound and the background is perturbatively stable. Therefore it appears as if the gravity side is not fundamentally flawed and we think that quantum effects will play an important rôle.

The unbounded potential in the bulk is replicated by an unbounded potential in the dual field theory. The conformal field theory is deformed by a double trace deformation, which breaks
Chapter 2. The resolution of cosmic singularities

...the superconformal symmetries. This implies that the coupling constant of the double trace coupling flows. Although supersymmetry is broken by the double trace interaction, we have seen that the resident R-symmetry has prolonged the protection of the Yang-Mills coupling to the one-loop level, but it is hard to conceive, that it extends to even higher orders. Therefore, one expects that the RG-flows of the two couplings mix and also the Yang-Mills coupling gets renormalized. In the $\beta$-function of any one coupling, the other one appears as a coefficient. This means that, in fact, the RG-flow of the two couplings is described by a system of coupled differential equations and the flow of one coupling influences the flow of the other. The main point is that by neglecting the running of the gauge coupling the well-established results on the leading qualitative UV behavior of the double trace coupling can be misleading. Inclusion of the running gauge coupling can result in the gauge theory becoming an effective field theory, only valid up to some UV cut-off, by turning the double trace operator into an irrelevant deformation. Or it could lead to a turn-around in the effective potential, stabilizing the dynamics. What happens crucially depends on the coefficients in the beta-functions, which are determined by perturbative analysis of the deformed gauge theory.

We have seen in the calculation performed here on the field theory side, that at the two loop level, indeed, the renormalization of the single- and double-trace couplings mix. In particular, a term linear in the double trace coupling survives at the two-loop level, which can turn the effective potential around and, thus, provide the field theory with a ground state generated by quantum effects. The critical condition for this to happen is that the modulus of the coefficient of this term is big enough, namely bigger than four. This is not at all ensured a priori. As mentioned above, this coefficient contains in particular a power of the Yang-Mills coupling $g$, whose value is arbitrary at the conformal fixed point. As such, it is at first not possible to determine the value of this coefficient. However, conformal invariance is broken at the one-loop level for the double trace coupling and at the two-loop level for the single trace coupling. The latter is growing with the RG-flow until it will inevitably be big enough to ensure the turn-around of the effective potential to happen. Inclusion of $1/N$ corrections bounds the effective potential.

This could have been expected in retrospect, since the unavoidable running of the gauge coupling signals the presence of a dynamical dilaton field in the bulk, which one would indeed expect to become an important factor as one approaches the (spreading) crunch singularity. The background solution we have chosen has a dilaton fixed at its expectation value. This means, that the string coupling, which corresponds to the Yang-Mills coupling, is constant. As soon as this coupling flows and the dilaton becomes dynamical, the truncation used is too restrictive. As the dilaton grows it also influences the scalar field potential in the bulk. Hence, the truncation to supergravity with only one scalar field is rendered invalid. String loop corrections can no longer be neglected.

Since quantum corrections regularize the potential the theory has now a ground state. The evolution of the scalar fields ends in a thermalization around this well-defined minimum and unitarily loss in the field theory is avoided. For the bulk gravity, this means that the spreading...
of the singularity towards the boundary will stop and that it is covered up by a huge black hole, which conceals the singularity. Thus, I have explained using a specific example, how $1/N$ corrections of a field theory dual to a string theory in an unstable bulk can resolve a cosmological singularity.