The universe on edge: Limits of the effective field theory approach in the very early universe

Oberreuter, J.M.

Citation for published version (APA):
Oberreuter, J. M. (2013). The universe on edge: Limits of the effective field theory approach in the very early universe.

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We are looking at a theory with a single scalar field $\phi$ and a potential $-\frac{\lambda}{4}\phi^4$. This is the harmonic oscillator potential upside down, so the configuration is unstable and the field rolls down this potential to infinity. The counterterm necessary to renormalize this interaction is $\frac{\delta\lambda}{4}\phi^4$ and the Lagrangian for the renormalized fields and parameters is

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 - \frac{1}{2} \delta Z (\partial_\mu \phi)^2 - \frac{1}{2} \delta m \phi^2 + \frac{\delta \lambda}{4} \phi^4.$$  \hspace{1cm} (A.1)

Note that with this definition of the interaction term, the vertex Feynman rule should be $6i\lambda$. The metric used is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$  \hspace{1cm} (A.2)

In the following we are calculating the contribution of one- and two-loop diagrams to $\delta \lambda$, which we denote as $\delta^{(1)} \lambda$ and $\delta^{(2)} \lambda$, respectively.

### A.1 One-loop Renormalization

On the level of one-loop interactions the following diagrams contribute to the four-point function:

The three loop diagrams just represent different channels corresponding to the Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$, respectively. The last diagram is the vertex counterterm defined to cancel the divergences of the first three diagrams.
Appendix A. Two-loop renormalization of $\phi^4$-theory

Figure A.1: One-loop diagrams and the counterterm for the four-point function.

The first three diagrams can be evaluated at once. We obtain their amplitude in 4 dimensions by integrating over the free momentum $k$ and multiplying with the two interaction vertices

$\langle 6i\lambda \rangle^2 iV(p^2) \equiv \frac{(6i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + m^2} \frac{-i}{(k+p)^2 + m^2}$,

(A.3)

where $\frac{1}{2}$ is the symmetry factor of the diagram. We denote the momentum flow in the loop with two insertions of momentum $p$ and $-p$, respectively, with $V(p^2)$. For each diagram, $p^2$ should be replaced by the corresponding Mandelstam variable and the three contributions are added. The vertex is worth $6i\lambda$ and this is the contribution we have to add on top for the tree level diagram to get the entire amplitude at one-loop level:

$iM^{(1)} = 6i\lambda + (6i\lambda)^2 [iV(s) + iV(t) + iV(u)] + 6i\delta^{(1)}_\lambda$,

(A.4)

where the last term $\delta^{(1)}_\lambda$ denotes the order $\lambda$ contribution to the counterterm.

We impose the following renormalization conditions:

$iM(p_1p_2 \rightarrow p_3p_4) = 6i\lambda \text{ at } s = t = u = \mu^2$,

(A.5)

which relates the physical coupling to a renormalization scale $\mu$, which later on will be related to the value of the scalar field. Due to this condition the second and third terms should cancel each other if the incoming momenta equal the renormalization scale. There, the value of the square bracket just becomes $3iV(\mu^2)$ and the counterterm can be read of as

$\delta^{(1)}_\lambda = (6i\lambda)^2 \cdot \left( -\frac{1}{2} V(\mu^2) \right) = 18\lambda^2 V(\mu^2)$.

(A.6)

We now evaluate (A.3) using dimensional regularization. This implies that every dimensionful parameter must be expressed in terms of a dimensionless number times the appropriate power of the renormalization scale, in order to ensure that the overall dimension of the Lagrangian works out. So in the following, we replace $m^2 = a\mu^2$\footnote{We actually also would have to adjust the interaction term by including a factor of $\mu^{4-d}$, however, this factor will vanish in the expansion around $d = 4$ to first order in $\epsilon$, since $\lambda\mu^{4-d} \simeq \lambda(1 + \ln \mu\epsilon)$.} Combining the two denominators by use of a Feynman-parameter (A.3) looks like

$iV(p^2) = -\frac{1}{2} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 + 2xkp + xp^2 + a\mu^2]^2}$.

(A.7)
Appendix A. Two-loop renormalization of $\phi^4$-theory

We shift the integration variable to $l = k + xp$ and obtain

$$iV(p^2) = -\frac{1}{2} \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 + x(1-x)p^2 + a\mu^2]^2}, \quad (A.8)$$

rotate to Euclidean space by performing a Wick rotation

$$l_0^0 \to -i l_0^0, \quad \vec{l}_E = \vec{l}, \quad (A.9)$$

which leads to

$$iV(p^2) = -i \frac{1}{2} \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l_0^2 + x(1-x)p^2 + a\mu^2]^2} \quad (A.10)$$

$$= -i \frac{1}{2} \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left[ \frac{1}{x(1-x)p^2 + a\mu^2} \right]^{2 - \frac{d}{2}} \quad (A.11)$$

$$= -i \frac{1}{2} \int_0^1 dx \frac{\Gamma(\frac{2}{2})}{(4\pi)^{2-\frac{d}{2}}} \frac{1}{[a\mu^2 + x(1-x)p^2]^{\frac{d}{2}}} \quad (A.12)$$

where in the last line we have replaced $\epsilon = 4 - d$. In the limit $d \to 4$ or $\epsilon \to 0$, the Gamma-function has a pole and its approximation looks

$$\Gamma(\epsilon) \approx \frac{1}{\epsilon - \gamma + \frac{1}{2} \left( \gamma^2 + \frac{\pi^2}{6} \right) \epsilon + O(\epsilon^2)}, \quad (A.13)$$

where $\gamma$ is the Euler-Mascheroni constant. Altogether we therefore get

$$\ldots \approx -\frac{i}{32\pi^2} \left( \frac{2}{\epsilon} - \gamma + O(\epsilon) \right) \left( 1 + \frac{\log(4\pi)}{2} \epsilon \right) \quad (A.14)$$

where the logarithmic terms come from a Taylor expansion of the denominators around $d = 4$. The integral evaluates to

$$-2 + \frac{2\sqrt{-4a\mu^2 - p^2}}{p} \arctan \frac{p}{\sqrt{-4a\mu^2 - p^2}} + \log a\mu^2. \quad (A.15)$$

Assembling this into (A.14) we obtain the one-loop amplitude in dimensional regularization up to first order in $\epsilon$

$$iV(p^2) = -\frac{i}{32\pi^2} \left( \frac{2}{\epsilon} + \log 4\pi - \gamma + 2 - \log a\mu^2 \right) \quad (A.16)$$

$$-2\sqrt{-4a\mu^2 - p^2} \arctan \frac{p}{\sqrt{-4a\mu^2 - p^2}} + O(\epsilon)$$

Inserting this result into equation (A.6) we obtain for the shift of the coupling constant at one-loop order

$$\delta^{(1)}_\lambda = -\frac{9\lambda^2}{16\pi^2} \left( \frac{2}{\epsilon} + \log 4\pi - \gamma + 2 - \log a\mu^2 - 2\sqrt{-4a - 1} \arctan \frac{1}{\sqrt{-4a - 1}} \right). \quad (A.17)$$
Note that the result agrees with the one given in \[106\], appendix B\[2\]

\[
\delta^{(1)}_{\lambda} = -\frac{9\lambda^2}{16\pi^2} \left( \frac{2}{\epsilon} - \log m^2 + \text{finite} \right). \tag{A.18}
\]

Eventually, we will be interested in the massless theory. We can then discard off the mass term in the denominator in (A.12), which simplifies the evaluation. We obtain

\[
iV_{m=0}(p^2) = -\frac{i}{32\pi^2} \left( \frac{2}{\epsilon} + 2 + \log 4\pi - \gamma - \log p^2 \right). \tag{A.19}\]

In order to calculate the two-loop renormalization, we also need the mass and field renormalization constants. Those we obtain from the one loop corrections to the two-point functions. We define the sum of all one-particle-irreducible insertions into the propagator as

\[
1\text{PI} = -iM^2(p^2). \tag{A.20}
\]

defines the pole of the propagator to be at \(m^2 = a\mu^2\) and having residue 1. On the other hand, this propagator is defined by a geometric series as

\[
1\text{PI} + 1\text{PI}^2 + \cdots = \frac{-i}{p^2 - a\mu^2 - M^2(p^2)}. \tag{A.21}
\]

With this at hand the renormalization condition can be restated as

\[
M^2(p^2) \bigg|_{p^2 = \mu^2} = 0 \quad \text{and} \quad \frac{d}{dp^2} M^2(p^2) \bigg|_{p^2 = \mu^2} = 0, \tag{A.22}
\]

respectively. The one-loop divergence of the two-point function is cancelled by the \(\delta^{(1)}_Z\) and \(\delta^{(1)}_m\) terms in the action, which leads to the following relation for the two-point function:

\[
1\text{PI} = -iM^2(p^2) = (6i\lambda) \frac{1}{2} \int d^d k \frac{-i}{(2\pi)^d k^2 + a\mu^2} - i(p^2 \delta^{(1)}_Z + \delta^{(1)}_m), \tag{A.23}
\]

with \(\frac{1}{2}\) being the symmetry factor of the one-loop diagram. Note that the sign of the \(\delta_Z\)-term differs from standard renormalization procedure due to the different sign of the term in the action (A.1) chosen. We perform the integral and get

\[
\cdots = -\frac{3i\lambda}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1 - \frac{d}{2})}{(-a\mu^2)^{1-\frac{d}{2}}} - i(p^2 \delta^{(1)}_Z + \delta^{(1)}_m). \tag{A.24}
\]

Since the first term is independent of \(p^2\) we conclude that

\[
\delta^{(1)}_Z = 0 \quad \text{and} \quad \delta^{(1)}_m = -\frac{3\lambda}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1 - \frac{d}{2})}{(-a\mu^2)^{1-\frac{d}{2}}}, \tag{A.25}
\]

ensuring that \(M^2(p^2)\) vanishes for all momenta as required by the renormalization conditions (A.22). So there is no contribution to \(M^2(p^2)\) at the one-loop level. The vanishing of the field renormalization at one-loop order is a common feature of \(\phi^4\)-theories.

\[\text{Note that } \arctan z = \frac{1}{2} \left( \log(1 + iz) - \log(1 - iz) \right) \text{ and thus imaginary for imaginary } z \text{ and well defined for } |z| < 1. \text{ Hence, the combination } \sqrt{-4a - 1} \arctan \frac{-1}{\sqrt{4a - 1}} \text{ is real and well defined for } a > -\frac{1}{2}.\]
Appendix A. Two-loop renormalization of $\phi^4$-theory

A.2 TWO-LOOP RENORMALIZATION

A.2.1 RENORMALIZATION OF THE COUPLING

We are now going one loop further. The contributing diagrams are listed in [128], page 339, Fig. 10.5. We divide the two loop diagrams into three groups. So that the two-loop counterterm has three contributions

$$\delta^{(2)}_\lambda \sim d_1 + d_{11} + d_{111}. \quad (A.26)$$

To begin with, we only calculate the $s$-channel diagrams. The $t$ and $u$-channels are then just the same under our renormalization conditions.

The first double loop diagram has the value

$$d_1 = \left( 6i\lambda \right)^3 \cdot \left[ iV(p^2) \right]^2, \quad (A.27)$$

which, using the result (A.16), reads

$$d_1 = -\left( 6i\lambda \right)^3 \frac{1}{1024\pi^4} \left( \frac{4}{\epsilon^2} + \frac{4}{\epsilon} \left( 2 + \log 4\pi - \gamma - \log p^2 \right) + \left( 2 + \log 4\pi - \gamma - \log p^2 \right)^2 \right). \quad (A.28)$$

The second diagram we have to calculate is

$$d_{11} = \left( 6i\lambda \right)^3 \frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \frac{-i}{k^2 + a\mu^2} \frac{-i}{(k + p)^2 + a\mu^2} V((k + p_3)^2), \quad (A.29)$$

in which we combine the first two propagator terms by multiplying and simplifying them after dropping the mass term and use our previous one-loop result (A.12).

$$d_{11} = \left( 6i\lambda \right)^3 \frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + kp)^2} \int_0^1 dx \int \frac{d^d r_E}{(2\pi)^d} \frac{1}{r_E^2 + x(k^2 + p_3^2 + 2kp_3) - x^2(k + p_3)^2} \quad (A.30)$$

where we do the integral and obtain

$$d_{11} = \left( 6i\lambda \right)^3 \frac{i}{2(4\pi)^{\frac{d}{2}}} \Gamma(2 - \frac{d}{2}) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + kp)^2} \left( \frac{1}{(k + p_3)^2} \right)^{2 - \frac{d}{2}} \int_0^1 dx \left( \frac{1}{x(1-x)} \right)^{2 - \frac{d}{2}}. \quad (A.31)$$

The last integral is just an Euler beta function. We combine the two denominators by use of

$$d_{11} = \left( 6i\lambda \right)^3 \frac{i}{2(4\pi)^{\frac{d}{2}}} \Gamma(2 - \frac{d}{2}) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + kp)^2} \frac{1}{(k + p_3)^2} \int_0^1 dx \left( \frac{1}{x(1-x)} \right)^{2 - \frac{d}{2}}. \quad (A.32)$$
We find a nonlocal divergence in the latter expression which should be cancelled by the corresponding counterterm, which we calculate in the diagrams containing the one-loop counterterms.

\[
d_{11} = (6i\lambda) i V(p^2) 6i \delta^{(1)}_{\lambda} = (6i\lambda)^3 3V(p^2) V(\mu^2) = -648i\lambda^3 V(p^2) V(\mu^2),
\]
in which we insert the expression \((\text{A.19})\) but this time expanded to second order in \(\epsilon\), since there are two factors of \(1/\epsilon\) to render those finite and obtain

\[
d_{1\text{II}} = (6i\lambda)^3 \frac{3}{(4\pi)^4} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} (2 - \gamma + \log 4\pi - \log p - \log \mu) + 3 - 2\gamma + \frac{\gamma^2}{2} - \frac{\pi^2}{24} + \log 16 + 2 \log \pi - \gamma \log 4\pi + \frac{1}{2} (\log 4\pi)^2 \\
+ (-2 + \gamma - \log 4\pi) (\log p + \log \mu) + \frac{1}{2} [(\log p)^2 + (\log \mu)^2] + \log p \log \mu \right] \tag{A.40}
\]

This diagram, indeed, also contains a nonlocal divergence from the logarithm of \(p^2\) in the \(\frac{1}{\epsilon}\)-term, which just cancels with the corresponding term in \((\text{A.36})\). The remaining divergences can be included in the tree level counterterm.

Note that the diagrams of type \(d_{1\text{II}}\) and \(d_{1\text{II}}\) also need to be included “upside down”, i.e. with initial and final momenta interchanged (group III of \(\text{[128]}\)). This means in particular

\[
p \rightarrow -p \quad \text{and} \quad p_3 \rightarrow -p_3 . \tag{A.41}
\]

We see from \((\text{A.29})\) that \(d_{1\text{I}}\) remains invariant under this exchange if we also reparametrise \(k \rightarrow -k\), which we can do since this just changes the arbitrary orientation of the internal momentum. Then

\[
\begin{align*}
  k^2 & \rightarrow (-k)^2 = k^2 , \\
  (k + p)^2 & \rightarrow (-k - p)^2 = (k + p)^2 \quad \text{and} \tag{A.42} \\
  (k + p_3)^2 & \rightarrow (-k - p_3)^2 = (k + p_3)^2 . \tag{A.43}
\end{align*}
\]

For \(d_{1\text{II}} \tag{A.37}\) we see that it only depends on \(p^2\), anyway. Hence, the contributions \((\text{A.36})\) and \((\text{A.40})\) of these diagrams can just be doubled. Then we have accounted for all the s-channel diagrams on the two-loop level. In order to include also the t- and u-channels we have to take this contribution three times, so that schematically

\[
i\mathcal{M}^{(2)} = i\mathcal{M}^{(1)} + 3 \cdot (d_1 + 2d_{1\text{II}} + 2d_{1\text{III}}) + \delta_{\lambda}^{(2)} . \tag{A.45}
\]

We first investigate how the interplay between the three types of diagrams removes all the divergences and the dependence of the regulator from the final result. For that, we need to split the contribution from the diagram \(d_{1\text{II}}\) into the contribution from one channel, \(s\), say, for the first order counterterm, and the two others. Then the two loop result for one channel is

\[
(6i\lambda)^{-3}(d_1 + 2d_{1\text{II}} + 2d_{1\text{III}}) = (iV(p^2))^2 + 2d_{1\text{II}} + 2 \cdot 3V(p^2)V(\mu^2) \\
= -V^2(p^2) + 2V(p^2)V(\mu^2) + 4V(p^2)V(\mu^2) + 2d_{1\text{II}} \\
\underbrace{\frac{4}{3} d_{1\text{II}}}_{\text{finite}} \\
= - (V(p^2) - V(\mu^2))^2 + V^2(\mu^2) + 2d_{1\text{II}} + \frac{4}{3} d_{1\text{III}} .
\]
The combination $V(p^2) - V(\mu^2)$ has been made finite at the one loop level. In the following, we extract only the divergent terms

$$ (d_I + 2d_{II} + 2d_{III})_{\text{divergent}} = \frac{(6i\lambda)^3}{(4\pi)^4} \left[ \frac{3}{\epsilon^2} + \frac{9 - 3\gamma + (5 - \frac{1}{2}\gamma) \log[4\pi] - \log \left[ \frac{256\pi^2}{\epsilon} \right] - 3 \log \mu^2}{\epsilon} \right] $$

This is the momentum independent divergent part, which we include in the third order vertex counterterm $\delta^{(2),\text{vertex}}$.

Furthermore, we need to determine the counterterm for the remaining divergences. Since the renormalization conditions were already fulfilled at the one-loop level $i\mathcal{M}^{(1)}$, the two-loop counterterm needs to cancel the two-loop contribution completely and hence

$$ \delta^{(2)}_\lambda = -3 \cdot (d_I + 2d_{II} + 2d_{III} - \delta^{(2),\text{vertex}}) \bigg|_{p^2 = \mu^2} \quad (A.46) $$

### A.2.2 Two-loop Vacuum Diagram

In order to do the resummation of the perturbation ordered by the number of loops à la Coleman-Weinberg [116], we need to evaluate the two-loop “prototype diagrams”. Those are shown in fig. A.2. The first one is evaluated as follows:

$$ \begin{align*}
&= \frac{(6i\lambda)^2}{3!} \int \frac{d^dl}{(2\pi)^d} \frac{-i}{l^2 + a\mu^2} iV ((p + l)^2) \\
&= \frac{(6i\lambda)^2}{3!} \frac{-1}{2} \int \frac{d^dl}{(2\pi)^d} \frac{1}{l^2 + a\mu^2} \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{(x(1-x)(p+l)^2 + a\mu^2)^{\frac{d}{2}}} \\
&= \frac{(6i\lambda)^2}{3!} \frac{-1}{2} \frac{\Gamma(3 - \frac{d}{2})}{(4\pi)^2} B \left( \frac{d}{2} - 1, \frac{d}{2} - 1 \right) \int_0^1 dy y^{1 - \frac{d}{2}} i \int \frac{d^ds E}{(2\pi)^d} \frac{1}{[s^2 + p^2 y(1-y)]^{3 - \frac{d}{2}}} \\
&= -i \frac{(6i\lambda)^2}{3!} \frac{3\pi \csc((4 - \epsilon)\pi)}{(4\pi)^{4-\epsilon}} \frac{\Gamma(2 - \frac{d}{2}) \Gamma(1 - \frac{d}{2})}{\Gamma(4 - \frac{d}{2}) \Gamma(2 - \epsilon)} (p^2)^{1-\epsilon} \\
&= i \frac{(6i\lambda)^2}{3!} \frac{p^2}{2(4\pi)^4} \left[ \frac{1}{\epsilon} + \frac{13}{4} - \gamma + 2 \log 2 + \log \pi - \log p^2 \right] \quad (A.47)
\end{align*} $$

where we have introduced a shifted momentum variable $s = l + y p$ and Wick rotated afterwards.
We now turn to the second diagram, which evaluates as follows.

\[
\begin{align*}
\infty & = \frac{6i\lambda}{2^d} \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \frac{-i}{k^2 + a\mu^2} \frac{-i}{l^2 + a\mu^2} \quad (A.48) \\
& = \frac{6i\lambda}{2^d} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2 - a\mu^2} \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{l_E^2 - a\mu^2} \quad (A.49) \\
& = \frac{6i\lambda d^2}{16(4\pi)^d} \Gamma^2 \left( -\frac{d}{2} \right) \left( \frac{1}{-a\mu^2} \right)^{2-d} \quad (A.50)
\end{align*}
\]