The theory we actually want to discuss is $\mathcal{N} = 4$ Super-Yang-Mills theory with gauge group $SU(N)$, which has a double trace deformation [106]. In this section I explain the different diagrams having single trace and double trace interactions. The field $\phi$ is now an $N \times N$ matrix valued scalar and the two interaction terms in the Lagrangian look like

$$L_{\text{int}}(N, \lambda) = \frac{g^2}{4} \text{tr} (\phi^4) + \frac{f}{2} \left( \frac{1}{N} \text{tr} \phi^2 \right)^2$$

(B.1)

respectively. The different interactions correspond to different flow of the gauge group degrees of freedom. If the fields are written with their gauge group indices, e.g. $\phi^a_b$, the traces can be represented as appropriate contraction of indices and the two different interactions just differ in some $\delta$-functions. The single and double trace vertices then read

$$\text{Tr} \phi^4 = \phi^b_a \phi^d_c \phi^f_g \phi^h_i \delta^c_d \delta^f_g \delta^h_i \quad \text{and} \quad (\text{Tr} \phi^2)^2 = \phi^b_a \phi^d_c \phi^f_g \delta^a_d \delta^g_i \delta^f_h \delta^h_i,$$

(B.2)

respectively. The indices contracted run over gauge group degrees of freedom $a \ldots h = 1 \ldots N$ and do not affect the momentum flow in the diagrams calculated above.

For each interactions of the theory, there is a coupling, namely the single-trace $g$ and the double-trace couplings $f$. Each get renormalized by an appropriate set of diagrams. If a diagram renormalizes the one or the other is determined by the structure of the gauge group flow. If it corresponds to the flow of a single trace vertex, the diagram renormalizes $g$, if it corresponds to a double trace structure, it renormalizes $f$. It is not important, which kind the internal vertices of the diagram are and a diagram can even contain vertices of different kinds.

The diagram including its gauge group structure is evaluated by taking the result from section A and replacing the dummy coupling $\lambda$ by the appropriate form of the single and double trace interactions.
Appendix B. Diagrammatics of the double trace interaction

Figure B.1: A straight and a twisted propagator in double line notation.

coupling, \(g^2\) and \(\frac{2f}{N}\), respectively. After the appropriate order of the ’t Hooft couplings \(g^2N\)
and \(f\) are split off, the order of \(N\) gets extracted as a pre-factor and the symmetry factor gets
adjusted. Finally, we add up the diagrams order by order in \(N\) to get our final answer up to the
desired order in \(N\), i.e. the first sub-leading order.

**B.1 Remark about twisting**

If we use double line notation, we notice that all the propagators can be “twisted”, i.e. the line
connects the upper vertex on the left hand side with the lower index on the right hand side and
vice versa, as indicated in fig. B.1 Usually, this implies non-planarity and hence, these diagrams
are suppressed in the large \(N\) limit. Since, here, we are taking into account next to leading
order in \(1/N\) diagrams, we must carefully examine, if diagrams with a twist do contribute in
our approximation.

A single twist would not be compatible with the group structure, because arrows couldn’t
be placed in opposite directions on the same propagator any more. A Twist in two or more
propagators can, depending on the diagram at hand and the specific propagator, that is twisted,
or the combination thereof, have one or more of the following effects:

1. The trace structure is changed from single to double trace, e.g.

   ![Diagram](image)

   →

   ![Diagram](image)

2. Nothing changes, e.g.

   ![Diagram](image)

   →

   ![Diagram](image)

Because of the significance of the problem at hand, twists with the first effect are drawn as
separate diagrams, to visualize them clearly. Diagrams of the second kind, that just double an
untwisted diagram, are not drawn separately in order to maintain readability, but are taken into
account by placing a combinatorial factor in front of the untwisted diagram which is multiplied.
Appendix B. Diagrammatics of the double trace interaction

B.2 One-loop diagrams with trace structure

At the one-loop level, we have an overall symmetry factor of $\frac{1}{2}$ due to interchange of the two propagators. Then, for one channel, we obtain the group structure

$$= \begin{cases} \frac{1}{2} g^4 N + 4g^2 \left( \frac{2f}{N^2} \right) & \text{single trace} \\ 2 \left( \frac{2f}{N^2} \right)^2 N^2 + 4 \left( \frac{2f}{N^2} \right)^2 + 4 \left( \frac{2f}{N^2} \right)^2 + 2g^2 \left( \frac{2f}{N^2} \right) N + \frac{1}{2} g^4 & \text{double trace} \end{cases}$$

(B.3)

where the first two terms renormalize the single-trace coupling. Using ’t Hooft couplings, we obtain

$$= (6i)^2 i V(p^2) \frac{1}{2} \left\{ \frac{1}{4} (g^2 N)^2 \frac{1}{N} + \left[ 2f(g^2 N) + 2f^2 + \frac{1}{4} (g^2 N)^2 \right] \frac{1}{N^2} \right\} \left[ \frac{1}{4} (g^2 N)^2 \frac{1}{N} + \left[ 2f(g^2 N) + 2f^2 + \frac{1}{4} (g^2 N)^2 \right] \frac{1}{N^2} \right]$$

(B.4)

from which we keep diagrams up to $O(N^{-2})$. This group structure contains the momentum structure (A.19) and, hence, is inherited also by the one-loop counterterm, which we are going to determine subsequently.

To calculate the counterterm, we need to refine the renormalization conditions. For each of the two couplings we have a tree level diagram. Hence, the renormalization conditions now read

$$iM(p_1 p_2 \rightarrow p_3 p_4) = 6i \left( g^2 + \frac{2f}{N^2} \right)$$

at $s = t = u = \mu^2$. (B.5)
The one-loop counterterm now cancels the diagrams respecting their group structure in
\[
i\mathcal{M}^{(1)}_{f,g} = 6i \left( g^2 + \frac{2f}{N^2} \right) + (6i)^2 \text{[grp. struct.]} \cdot 3i \cdot 2V(\mu^2) + 6i \left( \delta_g^{(1)} + \delta_f^{(1)} \right), \tag{B.6}
\]
where we insert equation (B.4). The second and third term need to cancel, but only single trace and double trace terms contribute to \(\delta_g^{(1)}, \delta_f^{(1)}\), respectively. The factor 2 in front of the momentum factor removes the symmetry factor which is now absorbed into the group structure. We read off the counter terms up to second order in \(1/N\)
\[
\delta_g^{(1)} = -(6i)^2 (g^2 N)^2 \frac{1}{4N} V(\mu^2),
\]
\[
\delta_f^{(1)} = -(6i)^2 \left[ 2f(g^2 N) + 2f^2 + \frac{1}{4}(g^2 N)^2 \right] \frac{1}{N^2} V(\mu^2). \tag{B.8}
\]

### B.3 2-LOOP DIAGRAMS WITH DOUBLE-TRACE STRUCTURE

At the two-loop level, we have two diagrams. For \(\mathcal{O}(N^{-3})\), suppressing all diagrams with \(\mathcal{O}(N^{-3})\) in the 't Hooft limit, we obtain the group structure for one channel of this diagram
\[
= \mathcal{O}(N^{-3}) + 4 \cdot 4 + 3 \\
\text{single trace} + \text{double trace}
\]
\[
= (6i)^3 [iV(p^2)]^2 \left[ \frac{1}{8} (g^2 N)^3 \frac{1}{N} + \left( 3f(g^2 N)^2 + 6f^2 (g^2 N) + 4f^3 + \frac{3}{4} (g^2 N)^3 \right) \frac{1}{N^2} \right]. \tag{B.9}
\]
Appendix B. Diagrammatics of the double trace interaction

For the other diagram, we only have one single and two double trace diagrams contributing up to order $O(N^{-2})$

$$= \text{single trace} + 2 \text{double trace} \tag{B.10}$$

$$= (6i)^3 \left[ (g^2N)^3 \frac{1}{N} + \left( \frac{1}{2} f (g^2N)^2 + \frac{1}{8} (g^2N)^3 \right) \frac{1}{N^2} \right] \cdot \text{[momentum]} \cdot \tag{B.11}$$

The last type of diagram is a one-loop diagram with the second order counterterm replacing one vertex. Just as the first order counterterm, the second order counterterms inherits the above trace structure. The results (B.7) and (B.8) are now included into the calculation of the two-loop counterterm. Here, for the diagrams containing counterterms, its trace structure needs to fit the rest of the diagram, again. Such diagrams have an overall single or double trace structure coming from the one-loop diagrams, in which every one vertex is replaced by its corresponding counterterm, i.e. a single trace coupling with $\delta_\phi^{(1)}$ and a double trace coupling with $\delta_f^{(1)}$. The number of diagrams thus doubles but some of them are identical\footnote{Those diagrams count double, which is the same as removing one symmetry factor of $\frac{1}{2}$ because of the counterterm removing the symmetry between the two vertices. Such diagrams are only listed once.}. Since the index structure of the counterterm is the same as the one of the coupling, index loops and the
Appendix B. Diagrammatics of the double trace interaction

order in \( N \) don’t change. Hence, we find for a one-loop diagram with counterterm

\[
\begin{align*}
= & \quad \left(6i\right)^2 V(p^2) \left[ \frac{1}{2} \left( g^2 N \right) \delta_g^{(1)} + \frac{1}{2} \left( g^2 N \right) \frac{1}{N} \delta_f^{(1)} + f \frac{1}{N^2} \delta_g^{(1)} \right] \\
+ & \quad 2f \delta_f^{(1)} + \left( g^2 N \right) \delta_g^{(1)} + 2f \frac{1}{N} \delta_f^{(1)} + 3f \frac{1}{N^2} \delta_f^{(1)} + \frac{1}{2} \left( g^2 N \right) \frac{1}{N} \delta_g^{(1)} \\
= & \quad \left(6i\right)^2 V(p^2) \left[ \frac{1}{2} \left( g^2 N \right) \delta_g^{(1)} \right]
\end{align*}
\]

\[
\begin{align*}
& + \left( 2f + \left( g^2 N \right) \right) \delta_f^{(1)} + \left( 2f + \frac{1}{2} \left( g^2 N \right) \right) \frac{\delta_g^{(1)}}{N} + \mathcal{O}(N^{-3}) \\
= & \quad \left(6i\right)^3 V(p^2) V(\mu^2) \cdot 2 \left[ \frac{1}{8} \left( g^2 N \right)^3 \frac{1}{N} \right] \\
& + \left( 4f^3 + 6f^2 \left( g^2 N \right) + 3f \left( g^2 N \right)^2 + \frac{3}{8} \left( g^2 N \right)^3 \right) \frac{1}{N^2} .
\end{align*}
\]

\[\text{(B.12)}\]

In the second line, we have suppressed terms of higher order than \( \mathcal{O}(N^{-2}) \). After having filled in the expressions of the counterterms (B.7), (B.8), we see that diagrams contributing to the double trace structure are sub-leading in \( 1/N \) as expected earlier.

Finally, we have to determine the second order vertex counterterms \( \delta_{g,f}^{(2)} \) with the respective trace structure. These need to cancel all the contributions of the two-loop diagrams which can
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not be absorbed into the one loop counterterms $\delta^{(1)}_{g,f}$. Considering only single and double trace diagrams, respectively, equation (A.46) looks

$$\delta^{(2)}_{g,f} = -3 \cdot \left( d_{1}^{g,f} + 2d_{ii}^{g,f} + 2d_{iii}^{g,f} \right) \bigg|_{p^2 = \mu^2} . \quad (B.13)$$

We extract the corresponding terms from equations (B.9), (B.11) and (B.12) and sort them by their different couplings to obtain

$$\delta^{(2)}_{g} = -3(6i)^3 (g^2 N)^3 \frac{1}{N} \left[ \frac{8}{(6i\lambda)^3} \text{ eq. (A.28)} + \frac{2}{(6i\lambda)^3} \text{ eq. (A.36)} + \frac{1}{2} \text{ eq. (A.40)} \right] \quad (B.14)$$

and

$$\delta^{(2)}_{f} = -3(6i)^3 \frac{1}{N^2} \left[ f(g^2 N)^2 \left( \frac{3}{(6i\lambda)^3} \text{ eq. (A.28)} + \frac{1}{(6i\lambda)^3} \text{ eq. (A.36)} + 12 \text{ eq. (A.40)} \right) \right. \right.$$

$$+ f^2 (g^2 N)^2 \left( \frac{6}{(6i\lambda)^3} \text{ eq. (A.28)} + 24 \text{ eq. (A.40)} \right)$$

$$+ f^3 \left( \frac{4}{(6i\lambda)^3} \text{ eq. (A.28)} + 16 \text{ eq. (A.40)} \right)$$

$$\left. \left. \left. \left. \left. + (g^2 N)^3 \left( \frac{3}{4} \text{ eq. (A.28)} + \frac{1}{4} \text{ eq. (A.36)} + \frac{3}{2} \text{ eq. (A.40)} \right) \right) \right. \right. \right] .$$

Now, we have all information at hand, which we need to extract any contribution to either the single- or double-trace $\beta$-function at any order in $1/N$ present up to two loops. I will explain the schematics of this process in the following section.

### B.4 Extraction of the $\beta$-functions

#### B.4.1 Schematics

The $\beta$-function can be extracted from the Callan-Symanzik equation (2.157) applied to the 4-point Greens' function as noted earlier. As an ingredient thereof, we need to determine the anomalous dimension $\gamma_\phi$ of the scalar field, which is non-vanishing at the two-loop order. We obtain it from the Callan-Symanzik equation applied to the two-point Greens' function

$$\left( \mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} + \beta_f \frac{\partial}{\partial f} + 2\gamma_\phi \right) G^{(2)} = 0 , \quad (B.15)$$

which reduces to

$$\mu \frac{\partial}{\partial \mu} G^{(2),\text{2nd order}} + 2\gamma_\phi^{(2)} G^{(2),\text{0th order}} = 0 , \quad (B.16)$$

at second order, because the first contribution to the anomalous dimension comes at the second order and the first order contribution to the $\beta$-function vanishes. Here, $G^{(2),\text{0th order}} = \frac{1}{p^2}$
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denotes the uncorrected propagator and $G^{(2),\text{2nd order}}$ contains the terms of second order in the couplings of the Green’s function. From here, the anomalous dimension can readily be extracted to be
\[
\gamma_\phi = -\frac{48 f^2 + 48 g^2 N + g^4 N^4}{3072 (N^2 \pi^4 \mu)} .
\] (B.17)

We only regard 4 point vertices at the moment. It is therefore convenient to abbreviate the single trace coupling with $e = g^2$, which we will do in the following. The functions $\beta_{e,f}$ are extracted from the single and double trace 4-point Green’s functions, respectively,
\[
G^{(4)}_{s/d}(p_1, p_2, p_3, p_4) = \mathcal{M}_{s,d}(p_1, p_2 \to p_3, p_4) \prod_{k=1}^{4} \frac{-i}{p_k^2}.
\] (B.18)

by evaluating the Callan-Symanzik equation (2.157) successively order by order. It is important to distinguish between the single and double trace Greens’ functions, since this doubles the number of equations.

We identify the different terms by matching the coefficients of each term in the polynomial expansion of the Green’s function. The first contribution to the $\beta$ function comes in at the one-loop order, which is second order in the couplings for the four point Greens’ function
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta_e \frac{\partial}{\partial e} + \beta_f \frac{\partial}{\partial f} + 4 \gamma_\phi \right) \left( G^{e,e}_{4,s/d} + G^{e,f}_{4,s/d} + G^{e,f}_{4,s/d} + G^{f,f}_{4,s/d} \right) = 0 ,
\] (B.19)

where we denote the single and double coefficient of the term $x$ with $G^{x}_{4,s/d}$, respectively. Applying the operators to the Greens’ function, we note, that only the counterterms introduce a dependence on $\mu$ to the Greens’ function, such that the derivative with respect to $\mu$ kills the terms first order in the couplings. Hence, we find in terms of coefficients
\[
\mu \frac{\partial}{\partial \mu} \left( G^{e,e}_{4,s/d} + G^{e,f}_{4,s/d} + G^{f,f}_{4,s/d} \right) + \beta_g \left( G^{e}_{4,s/d} + 2G^{e}_{4,s/d} + G^{e}_{4,s/d} \right) + \beta_f \left( G^{f}_{4,s/d} + 2G^{f}_{4,s/d} + 2G^{f}_{4,s/d} \right) + 4 \gamma_\phi \left( G^{e,e}_{4,s/d} + G^{f}_{4,s/d} + G^{e,f}_{4,s/d} + G^{f,f}_{4,s/d} \right) = 0 ,
\] (B.20)

which needs to be satisfied term by term. The anomalous dimension doesn’t have any linear contribution as seen in (B.17), so at the one loop order, it just drops out. The $\beta$-functions contain a term for each order in the coupling constants, which we denote with $\beta_{e,f}$ in the following. Since there is no linear contribution to the $\beta$ function, those terms drop out. Furthermore, we observe that there are no tree level terms double and single trace terms with coupling $e$ and $f$, respectively. Hence, the relations simplify considerably. We read off the relations for the individual coefficients after having restored the trace structure of the Greens’
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functions

\[ \beta_e^{s2} = -\frac{\mu}{G_s} \partial_\mu G_s^{s2}, \quad \beta_f^{s2} = -\frac{\mu}{G_d} \partial_\mu G_d^{s2} \]  
\[ \beta_e^{f2} = -\frac{\mu}{G_s} \partial_\mu G_s^{f2}, \quad \beta_f^{f2} = -\frac{\mu}{G_d} \partial_\mu G_d^{f2} \]  
\[ \beta_e^{ef} = -\frac{\mu}{G_s} \partial_\mu G_s^{ef}, \quad \beta_f^{ef} = -\frac{\mu}{G_d} \partial_\mu G_d^{ef} \]  

where \( \beta_e^{f2} \) and \( \beta_e^{ef} \) vanish because single trace Greens’ functions can only contain single trace vertices, at the one loop level.

At the two loop order, the Callan Symanzik equation reads

\[ \mu \partial_\mu \left( G_e^s e + G_f^s f + G_e^{s2} e^2 + G_e^{ef} e f + G_f^{f2} f^2 
+ G_e^{s3} e^3 + G_e^{f2} e f^2 + G_e^{ef} f^2 + G_f^{f3} f^3 \right) 
+ \beta_e \left( G_e^s e + 2G_e^{s2} e^2 + G_e^{f} e f + 3G_e^{s3} e^3 + 2G_e^{f2} f e + G_e^{f3} f^3 \right) 
+ \beta_f \left( G_f^s f + G_f^{s2} f^2 + G_f^{f} e f + 2G_f^{s3} e f + 2G_f^{f3} f^3 \right) 
+ 4\gamma \left( G_e^s e + G_f^s f + G_e^{s2} e^2 + G_e^{f} e f + G_f^{s2} f^2 
+ G_e^{s3} e^3 + G_e^{f2} e f^2 + G_e^{f3} f^3 \right) = 0, \]  

from which we extract, again, the relations third order in the couplings for the \( \beta \) functions. Their second order coefficients should be inserted from the one loop equations. Again, after restoring the trace structure of the Greens’ functions, we can read off the eight third order coefficients. For that we note, that there are no single trace diagrams with only double trace couplings, so \( G_d^{s2} = 0 \) and alike. We note that in our approximation, taking into account only terms up to third order in \( \frac{1}{N} \), the single trace Greens’ functions contain single trace couplings, only. Hence, the expressions for the third order coefficients of the \( \beta \) functions simplify to

\[ \beta_e^{s3} = -\frac{1}{G_s} \left( \mu \partial_\mu G_s^{s3} + 2\beta_e^{s2} G_s^{s2} \right) - 4\gamma e^2 \]  
\[ \beta_f^{s3} = -\frac{1}{G_d} \left( \mu \partial_\mu G_d^{s3} + 2\beta_e^{s2} G_d^{s2} + \beta_f^{ef} G_d^{ef} \right) \]  
\[ \beta_e^{f2} = -4\gamma e^f \]  
\[ \beta_f^{f2} = -\frac{1}{G_d} \left( \mu \partial_\mu G_d^{f2} + \beta_e^{f} G_d^{s2} + 2\beta_f^{f} G_d^{f2} + \beta_f^{ef} G_d^{ef} \right) - 4\gamma f^2 \]  
\[ \beta_e^{ef} = -4\gamma e^f \]  
\[ \beta_f^{ef} = -\frac{1}{G_d} \left( \mu \partial_\mu G_d^{ef} + 2\beta_f^{f} G_d^{f2} + \beta_f^{ef} G_d^{ef} \right) - 4\gamma e^f \]  
\[ \beta_e^{s3} = 0 \]  
\[ \beta_f^{s3} = -\frac{1}{G_d} \left( \mu \partial_\mu G_d^{f3} + 2\beta_f^{f2} G_d^{f2} \right) - 4\gamma f^2 . \]
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The relations contained in this section represent a schematic way to extract all the information about the $\beta$-functions up to the two-loop order. Upon integration, the $\beta$-functions yield the renormalized couplings, which we use in section 2.5 to replace the RG-scale with the value of the scalar field by replacing the coupling in the effective potential.