Deformations of CFTs and holography
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Chapter 1

Introduction

The framework of quantum field theory has proven to be extremely useful in describing a large number of diverse phenomena, ranging from elementary interactions in high-energy physics to collective excitations of condensed matter systems. It is also the underlying theme of this thesis, whose aim is to study various aspects of quantum field theories and their applications to physical systems. In the first part of the thesis we will take some small steps towards the development of a holographic framework to describe strongly coupled non-relativistic field theories. As we will see, such theories are thought to be relevant for various setups appearing in condensed matter and statistical physics, such as cold atoms at unitarity and smectic liquid crystals. We will then move on to study non-renormalization theorems in quantum field theories that enjoy supersymmetry. Such theorems are concerned with the coupling constant dependence of certain field theory observables, and are potentially relevant for phenomenology. Moreover, they can be used to compute physical quantities in the strongly coupled regime from the knowledge of the weakly coupled results, so they provide ways to test various dualities between strongly coupled and weakly coupled theories. Lastly, we will study some recent proposals relating black hole entropy to two dimensional field theories, which may shed some light on the microscopic description of certain asymptotically flat black holes. The relevance of such a description comes from the fact that correctly reproducing the entropy formula from a sum over microstates provides a stringent consistency check on theories of quantum gravity such as string theory.

The ideas that we develop in this thesis can at first appear quite heterogeneous; it is the purpose of this introduction to show that in fact they all fit together into a coherent picture. For this, we need to describe the basic ingredients that will
be used throughout the thesis, in particular the idea of deformation of a quantum field theory and the concept of holography.

Quantum field theories are typically very difficult to solve exactly, and several perturbative methods have been developed over the years. The underlying idea is to consider first a related theory that can be solved exactly, such as a theory with no interactions, or a theory with a lot of symmetry. If the deviation of such a theory from the original one we wish to study is sufficiently “small”, the observables of the latter can be recovered as a power series in some small parameter, where the leading term is given by the answer of the exactly solvable model.

A large class of quantum field theories, including the Standard Model, can be analyzed (at least in some energy regime, as we will discuss in the next section) by treating the interactions as perturbations of a free theory. In this case the classical Lagrangian $\mathcal{L}$ is split in two parts

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}},$$  

where $\mathcal{L}_{\text{free}}$ is the free (quadratic) part of the Lagrangian and $\mathcal{L}_{\text{int}}$ contains the interaction terms. Such terms typically involve the product of various fundamental fields at the same point. For example, the celebrated interaction term of quantum electrodynamics reads

$$\mathcal{L}_{\text{int}} = e \bar{\psi} \gamma^\mu \psi A_\mu,$$  

where $e$ is the electric charge, which determines the strength of the electromagnetic interactions and therefore plays the role of a coupling constant, while $\psi$ and $A_\mu$ represent the electron and photon fields respectively. When $e = 0$, the electrons do not interact with the electromagnetic field. Solving the theory in this case is quite easy: electrons and photons freely travel through space without seeing each other. In order to compute the corrections to the free result as a power-series in the coupling constant, we can use so-called Feynman diagrams: a physical observable $\mathcal{O}$ can be expanded as

$$\mathcal{O} = o_0 + o_1 e^2 + \sum_{n=2}^{\infty} o_n (e^2)^n,$$  

and the $o_n$’s can be determined by computing a finite number of Feynman diagrams, which amounts to performing a finite number of integrals. However, both the number and the complexity of these diagrams rapidly increases with the order. Luckily enough, the physical electric charge of the electrons in our universe is small

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1To be more precise, the dimensionless coupling constant of QED is the fine-structure constant, which is related to the electric charge by $\alpha = \frac{e^2}{4\pi\varepsilon_0 c} \approx 1/137$.

2In the following, we will sometimes refer to such terms as operators, since they do become operators acting on the Hilbert space of the theory upon quantization.
1.1 Relevant, marginal and irrelevant deformations

enough so that higher-order corrections give increasingly smaller contributions to the final result; keeping only few terms in the series above leads to extremely accurate predictions. For example, the measurement of the dimensionless magnetic moment (or g-factor) can be compared to the QED predictions obtained by truncating the series to fourth order, and an agreement to within ten parts in a billion has been found.\(^3\)

More generally, \( \mathcal{L}_{\text{int}} \) will contain a number of interaction terms whose strength is determined by “coupling constants”, which we will collectively denote by \( g \), and observables in the interacting theory can in principle be computed as power-series in \( g \). From a physical point of view, we can think of these terms as deformations of the exactly solvable starting point, which in our case is a free theory. This will in fact be the main theme of this thesis: in a nutshell, we will tackle a variety of physical problems by mapping them into deformations of quantum field theories, and we will try to extract meaningful results by employing perturbative techniques.

In the rest of this brief introduction we broadly describe the main common ingredients that will be employed in the rest of the thesis. We first introduce the renormalization group (RG), which allows us to classify deformations in terms of their behavior under change of the energy scale of the process under consideration. Particular attention will be given to the RG fixed points, which describe scale invariant theories. Then we move on to the description of the concept of holography, which is a well-established framework to describe strongly coupled field theories on the one hand, and quantum theories of gravity on the other. Finally, we give an outline of the thesis and review the main original results that we derive in the subsequent chapters, and see how they all fit into the general framework described in this introduction. More specific details and background material will be given at the beginning of each chapter.

1.1 Relevant, marginal and irrelevant deformations

Even when the coupling constants are small, the calculation of Feynman diagrams involves loop integrals over all the possible momenta of intermediate virtual states. These integrals are typically divergent, so in order to extract physically meaningful results we need to find a way to regulate them. One very powerful idea that has emerged to cure these divergences is the renormalization group, which is based on

\(^3\)However, the power-series of QED is believed to be divergent, and can be at best considered an asymptotic series. The truncated answer will give increasingly accurate results only up to the order \( n \sim 137 \), and then it will start to diverge. We will discuss this in more detail in chapter 2.
the observation that quantum field theories should not be regarded as fundamental theories valid at all energy scales, rather they come with an intrinsic energy cutoff. This cutoff can be thought of as the (inverse) lattice spacing in statistical physics or the energy scale at which new particle interactions become important in high-energy physics. In any case, it is possible to isolate the high-energy (ultraviolet) degrees of freedom and describe their effects on low-energy observables through effective interactions between the low-energy degrees of freedom.

The most important outcome of the renormalization group analysis is that the coupling constants appearing in the effective Lagrangian should be thought of as being dependent on the energy scale of the process under consideration. It is indeed possible to derive differential equations controlling the energy scale dependence of the coupling constants and correlation functions of the theory, known as Callan–Symanzik equations. In this sense, different Lagrangians do not just describe different physical systems, but sometimes the same system at different energy scales. As a consequence, the renormalization group produces a flow across different effective descriptions as we change the typical energy scale of the processes under consideration.

In general, the renormalization group tends to produce all the possible interaction terms compatible with the symmetries of the problem, even when these terms do not appear in the Lagrangian of the fundamental microscopic theory. Typically, there are infinitely many such terms, so the problem of computing observable quantities would seem to be intractable at first sight, since we would need to take into account arbitrarily complicated interactions between the fields of the effective theory, and we would need an infinite number of experiments to fix all the coupling constants. However, the contribution of most of these interaction terms can be shown to be under control whenever the energy scale of the process under consideration is much smaller than the cutoff scale.

In fact, the possible interaction terms can be divided in two main classes: those whose coupling grows as the energy is decreased are called relevant, while those whose coupling decreases are called irrelevant. A typical example of a relevant coupling is the mass term for a scalar field in four dimensions, while an important example of an irrelevant coupling is given by the four-fermion interaction of Fermi’s theory of weak interactions. In the case where the coupling remains constant under the renormalization group flow, the corresponding operators are called marginal. Such operators are typically very difficult to come by, because the condition that the coupling remains constant as a function of the energy scale requires highly non-trivial cancellations of various quantum effects. It is in fact common for operators that look marginal at the classical level to become either marginally relevant (such as the interaction term in QCD) or marginally irrele-
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vant when quantum corrections are included. As we will see, theories that enjoy supersymmetry do have exactly marginal operators.

It turns out that in most cases the number of relevant and marginal operators compatible with the symmetries of the problem is finite. From the point of view of the low-energy effective theory, this means that it is sufficient to measure a finite number of parameters (such as the electric charge at zero momentum) in order to extract meaningful predictions from the theory. We refer to such theories as renormalizable, and the renormalization group explains why they are so ubiquitous in the description of physical systems at low energy. Irrelevant couplings give rise to corrections that are suppressed by positive powers of $E/M_{\text{cutoff}}$, where $E$ is the energy scale of the process considered, while $M_{\text{cutoff}}$ is the mass scale appearing in the irrelevant coupling constant. One very important example is general relativity, where gravitational interactions around flat space are described by an irrelevant operator. In this case, the cutoff scale is the Planck mass $M_P \approx 1.22 \times 10^{19}$ GeV, which is larger than any energy scale that can be directly probed by present day experiments.

1.1.1 Conformal field theories and their deformations

In this thesis, quantum field theories with conformal symmetry will play a prominent role. These theories, on top of the usual symmetries such as Lorentz symmetry (or some non-relativistic counterpart), are also invariant under rescalings of the coordinates. One of the consequences is that these theories do not change under the renormalization group flow, in the sense that the various coupling constants present in the theory do not depend on the energy scale.

The world around us seems very far from being scale invariant, and in fact many of the things we measure have a characteristic size. Scaling symmetry might thus appear very peculiar and unphysical. However, it turns out that scale invariant phenomena are actually quite common. In fact, many macroscopic systems tend to exhibit “critical” behavior, in the sense that certain observable quantities are described by probability distributions that do not have an intrinsic scale. Quantities as diverse as citations in scientific papers and sizes of earthquakes are in fact described by power-law distributions, which look the same at any scale. Scale invariance is also extremely important in statistical physics, where it is at the heart

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4It is believed that relativistic theories with scale invariance are also invariant under the bigger conformal group, which includes special conformal transformations. This has been proven to be the case in two dimensions, and some progress has been recently made in four dimensions. For this reason and for the sake of simplicity, in this introduction we decided to use the term “conformal” rather loosely, and it will refer both to theories with only scaling symmetry and to proper conformal field theories.
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of the effective description of physical systems undergoing a phase transition: at
the transition points, also known as critical points, the correlation functions of the
system exhibit power-law behavior, signaling scale invariant physics.

Conformal field theories provide a very powerful framework to account for various
properties of these critical points in the continuum limit (that is when we consider
length scales that are much bigger than the typical lattice spacing). They are
also typically found at the endpoints (UV or IR) of the renormalization group
flow. Furthermore, two-dimensional conformal field theories play a fundamental
role in perturbative string theory, where they describe excitations propagating on
the string. More surprisingly, recent developments have shown that they also play
a prominent role in quantum gravity, via various holographic dualities that will be
described in the next section.

In this thesis, conformal field theories will provide the starting point for many of
the perturbative analyses that we will perform. In fact, while these theories are
not necessarily free, their large symmetry group make them relatively simple to
study compared to non-conformal theories. The applications that we will consider
are however very diverse, and range from RG flows in theories with non-relativistic
scaling symmetry to black hole physics. We refer to the last section of this intro-
duction for a more detailed explanation of the role of conformal field theories in
the context of this work.

1.2 The AdS/CFT duality

The Bekensten–Hawking formula for the entropy of black holes has led to the
proposal [5, 6] that the degrees of freedom of gravitational theories can be encoded
“holographically” in a theory with one dimension less. The gauge/string dualities
are a concrete realization of this idea, where certain quantum theories of gravity
in \( d + 1 \) dimensions are claimed to be equivalent (or dual) to ordinary quantum
field theories (typically gauge theories) in \( d \) dimensions. The power of the duality
lies in the fact that it exchanges the weakly coupled regime of one description with
the strongly coupled regime of the other, providing therefore a powerful tool to
study strong-coupling problems both in field theory and quantum gravity. They
have in fact advanced our understanding of various classic problems in theoretical
physics, such as confinement, transport properties of strongly coupled QFTs and
black hole physics.

The best understood example is probably the duality between type IIB superstring
theory on \( \text{AdS}_5 \times S^5 \) and \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory in four dimen-
sions [7]. The latter enjoys conformal symmetry, so the correspondence is often
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called AdS/CFT duality. The amount of evidence substantiating this duality that has been accumulated so far is overwhelming.

Various generalizations involving different backgrounds and CFTs have also been proposed. The most well-understood cases are those that arise from the decoupling limit of certain brane solutions in string theory. These branes can be described in terms of closed strings, where they correspond to gravitational objects similar to black holes, and the decoupling limit dynamically isolates the near-horizon region. They can also be described in terms of open strings, in which case the decoupling limit corresponds to a low-energy limit of the effective field theory describing open string interactions on the brane. The two descriptions are equivalent, and by carefully tracking the decoupling of various modes it is possible to derive explicit gauge/gravity dualities.

In order to build some intuition and set the terminology that we will use in the thesis, we briefly describe the correspondence between \( \mathcal{N} = 4 \) super Yang–Mills theory and type IIB string theory on \( \text{AdS}_5 \times S^5 \) in more detail. We use the notation of [8] and we refer to it for further details. Let us start by describing the free parameters of the two theories. In the field theory side, we can obviously choose the gauge group, which we take to be \( SU(N) \). The rank \( N \) then corresponds to one free parameter. The only other free parameter is the gauge coupling constant \( g_{YM} \).

On the string theory side we also have only two independent dimensionless numbers, the (integer) flux \( N \) of the self-dual 5-form \( F_5 \) across the internal sphere, which essentially determines the ratio between the AdS radius and the Planck length, and the string coupling constant \( g_s \).

The 5-form flux corresponds to the rank of the gauge group of the CFT, while the gauge and string couplings are related by

\[
g_{YM}^2 = 4\pi g_s \tag{1.4}\]

The power of the duality becomes apparent when we discuss the regime of validity of various descriptions. First of all, loop diagrams in the field theory contribute with powers of \( \lambda = g_{YM}^2 N \), \( \tag{1.5} \)

where we introduced the ’t Hooft coupling \( \lambda \). The perturbative field theory regime, where the expansion in terms of Feynman diagrams becomes meaningful, then corresponds to \( \lambda \ll 1. \)

On the gravity side, the ratio between the AdS radius \( L \)

\[\text{In principle, we can also add a } \theta \text{ term to the theory, which corresponds to the axion in the bulk, but for the sake of simplicity and brevity we will not include it in our discussion.}\]

\[\text{If } N \text{ is large, there are some further simplifications that we will not discuss here. Once again, the interested reader is encouraged to look at [8] for further details and references.}\]
and the string length $\ell_s$ is given by
\[ \frac{L}{\ell_s} = \lambda^{1/4}. \tag{1.6} \]

It turns out that the contribution of string states is negligible when the radius of curvature of the background is much bigger than the string scale, that is when $\lambda \gg 1$. However, quantum corrections are not suppressed unless the ratio between $L$ and the Planck length, which as we said is controlled by $N$, is large. This leads us to consider the so called large $N$ limit. If both $N$ and $\lambda$ are much greater than 1, classical supergravity computations become reliable.

We reach the conclusion that classical type IIB supergravity on AdS$_5$ should provide a good perturbative description of the large $N$ limit [9] of $\mathcal{N} = 4$ super Yang–Mills in the strongly coupled regime. On the other hand, the computation of observables in the CFT at weak coupling can in principle be used to study type IIB string theory at strong coupling. The duality therefore exchanges the weakly coupled and strongly coupled regimes, a fact that makes it extremely useful and very difficult to prove at the same time.

We also notice that the isometry group of AdS$_5$, which is $SO(2,4)$, precisely matches the conformal group in 3+1 dimensions. This fact will become particularly important in the first part of this thesis, where we consider generalizations of the correspondence to non-relativistic theories. Also in that case, the symmetry group in the field theory side will correspond to the isometry group of the dual gravitational background.

### 1.2.1 Relation between the observables

The basic observables in a quantum field theory are the correlation functions of (gauge invariant) local operators $O_I(x)$
\[ \langle O_{I_1}(x_1) \ldots O_{I_n}(x_n) \rangle. \tag{1.7} \]
All the information contained in these correlation functions can be conveniently collected in a generating functional called the \textit{partition function}. The idea is to consider a classical source $\phi^I$ for each operator $O_I$. The object
\[ Z_{\text{QFT}}[\phi^I] = \left\langle \exp \left( - \int d^d x \sqrt{-g} \phi(x)^I O_I(x) \right) \right\rangle \tag{1.8} \]
allows us to recover the correlation functions by means of functional differentiation:
\[ \langle O_{I_1}(x_1) \ldots O_{I_n}(x_n) \rangle = \left. \left( - \frac{\delta}{\delta \phi^I_1(x_1)} \right) \cdots \left( - \frac{\delta}{\delta \phi^I_n(x_n)} \right) \log Z_{\text{QFT}}[\phi^I] \right|_{\phi^I = 0}. \tag{1.9} \]
As we discussed before, the computation of these correlation functions (or equivalently the partition function) in interacting theories relies on perturbative methods such as Feynman diagrams. The holographic dualities provide us with an alternative means to construct the generating functional. The prescription is as follows: we start from a gravitational theory that has AdS\(_{d+1}\) spacetime as a classical solution.\(^7\) Since AdS\(_{d+1}\) has a (conformal) boundary, the quantization of such a theory must be supplemented by “boundary conditions". For each classical field in the gravitational theory \(\Phi^I\) we need to specify the behavior at the boundary \(\Phi^I|_{\text{boundary}}(x) = \phi^I(x)\), where \(x\) is a coordinate on the boundary.\(^8\) The quantization of the theory then leads to a partition function \(Z_{\text{string}}[\phi^I]\) that is a functional of such boundary conditions. The main statement of the gauge/string duality is that this partition function is precisely the generating functional of a quantum field theory living in \(d\) dimensions \([10, 11]\):

\[
Z_{\text{string}}[\phi^I] = Z_{\text{QFT}}[\phi^I].
\]

(1.10)

The formula above implies a one-to-one correspondence between classical gravity fields and operators of the dual CFT, also known as field/operator correspondence. The power of this formula comes from the fact that since the duality, as explained before, exchanges the strongly coupled regime with the weakly coupled regime of the dual description, we can use perturbative methods on one side of the duality to compute observables on the other side of the duality in the strongly coupled regime.

In this thesis we mostly use the duality in the regime where classical gravity computations are reliable. In this case, the string theory partition function is dominated by the contributions coming from the saddle points, which are nothing else than the classical solutions. As a consequence, we have

\[
Z_{\text{string}}[\phi^I] \approx \exp(-S[\phi^I]),
\]

(1.11)

where \(S[\phi^I]\) is the gravitational on-shell action as a functional of the boundary conditions. We will see that the object \(S[\phi^I]\) is divergent and, before it can be used to compute correlation functions, needs to be renormalized. This is the analog of the renormalization procedure in standard quantum field theory, and will be described in more detail in chapter 2.

When we have correctly renormalized the on-shell action and properly identified the boundary conditions at the conformal boundary, correlation functions of the

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\(^7\)In many cases involving supergravity, such spacetimes can be argued to be solutions of the full string theory.

\(^8\)To be more precise, we would need to take into account the radial behavior of the various fields, since they typically either diverge or vanish as we go to the boundary. For the sake of simplicity, we ignore these issues in the introduction and we will be more careful when we actually perform computations in specific examples in the next chapter.
dual strongly coupled field theory can be computed by taking functional derivatives of $S[\phi^I]$ as follows:

$$\langle O_{I_1}(x_1)\ldots O_{I_n}(x_n) \rangle = - \left( -\frac{\delta}{\delta \phi^{I_1}(x_1)} \right) \cdots \left( -\frac{\delta}{\delta \phi^{I_n}(x_n)} \right) S[\phi^I] \bigg|_{\phi^I=0}.$$  \hspace{1cm} (1.12)

### 1.2.2 The holographic renormalization group

As we said before, while conformal field theories are interesting in their own right, in the end we want to learn something about theories that are not conformal. The techniques illustrated above also allow us to discuss possible conformal field theory deformations in the holographic context. Suppose that we want to introduce a deformation by the operator $O$, which is dual to the gravitational field $\Phi$ via the field/operator correspondence. Since coupling constants are nothing else than constant sources for their operators, we can simply set the source (or boundary condition) for $\Phi$ in (1.12) to some constant value at the end of the computation, instead of taking it to zero. In this way we obtain the correlators of a (strongly coupled) field theory deformed by the operator $O$.

In this setting, irrelevant operators correspond to those deformations that grow as we approach the boundary. This typically means that the spacetime is no longer asymptotically AdS. In this situation, renormalizing the on-shell action is particularly difficult, since one typically needs an infinite number of counter-terms. Relevant deformations on the other hand vanish at the boundary, and they only change the solution in the interior. Marginal deformations are of course also present, and the comments that we made in the field theory context apply here as well, including the distinction between marginally relevant and irrelevant operators. In any case, if the deformation is not exactly marginal, conformal invariance is broken, so we expect a non-trivial renormalization group flow. While the renormalization group parameter in field theory is the energy scale, in holography it is the extra (radial) coordinate that we get by going from $d$ dimensions to $d+1$. By looking at the radial behavior of the fields,\footnote{Assuming of course that the classical gravity approximation remains reliable across the flow.} we can learn something about the renormalization group flow of the dual field theory. In particular, in the presence of relevant deformations, the solution might flow to another AdS solution in the interior, which then corresponds to a non-trivial IR fixed point of the dual field theory. In the course of this thesis, we will also encounter irrelevant deformations. This is a much more problematic situation, because it calls for a new UV description of the theory. Unfortunately, the RG flow is irreversible in going from the UV to the IR, so the information provided by the knowledge of the IR theory and the irrelevant deformation is typically not enough to pinpoint a UV completion.
We will illustrate some of the implications of this observation in the last chapter, where we discuss irrelevant deformations in the context of black holes.

1.3 Outline and summary of the main results

In this last section we briefly give an outline of the thesis and describe the main results.

1.3.1 Lifshitz holography

In chapter 2 and 3, we describe some developments in extending the holographic dictionary outside the realm of asymptotically AdS spacetimes.

Chapter 2 describes the techniques developed in [1], in which we took some small steps towards the definition of holographic dualities where the background gravitational geometry is Lifshitz instead of AdS. We decided to study these constructions because they are conjectured to describe strongly coupled non-relativistic field theories, which are potentially interesting for phenomenological applications in condensed matter and statistical physics.

In particular, we describe how the problem of holographic renormalization can be tackled in this case and show in one particular example how these techniques allow us to determine whether certain boundary conditions are allowed or not. While we do not provide a complete framework for Lifshitz holography, we present some evidence that such a framework does indeed exist. For example, simple field theory toy models with non-relativistic scaling symmetry can be deformed by relevant operators that induce an RG flow to a relativistic theory in the IR. We will show that such a possibility is also present for strongly coupled non-relativistic field theories with Lifshitz duals. We will in fact show that it is possible to holographically turn on a marginally relevant deformation of the Lifshitz UV fixed point that flows to AdS (which is believed to be dual to a relativistic field theory) in the IR. In doing so, we will come across some interesting physical phenomena familiar in standard field theory, such as certain non-analyticities in the free energy as a function of the coupling constant. While the general formalism was published in [1], most of the results that we describe in this chapter are yet unpublished.

From a condensed matter perspective, deformations like the one studied in chapter 2 are interesting because they allow us to move away from the (quantum) critical point. In particular, they can be used to probe the finite temperature phase
diagram in the vicinity of the critical point and study phenomena such as finite temperature crossovers. In the holographic context, this is tantamount to studying black hole solutions that approach the marginally relevant deformation of Lifshitz spacetime that we have discussed.

In chapter 3 we give further evidence that gravity theories on Lifshitz backgrounds define non-relativistic field theories by computing the analog of the Weyl anomaly in this anisotropic setting. In particular, we provide a full characterization of the anisotropic scaling anomaly for a general (parity invariant) theory with Lifshitz scaling symmetry and \( z = 2 \), and show that it is given by two possible structures, one containing only time derivatives and the other containing only spatial derivatives. We then compute the anomaly of two non-relativistic models, one defined by standard field theory methods, the other defined holographically. We will show that a striking phenomenon occurs: while the two theories are in principle completely unrelated, one being free and the other being strongly coupled, they both produce only the term in the anomaly containing time derivatives. This hints at the possibility that gravity duals described by Einstein gravity cannot produce the second structure in the anomaly, a state of affairs similar to the \( a = c \) result in standard AdS/CFT. On the other hand, it has recently been shown [12] that Horava–Lifshitz gravity duals do produce this structure. Furthermore, anomalies typically control universal field theory properties that can be in principle measured, such as the Casimir energy in two-dimensional conformal field theories. If these results could be carried over to Lifshitz field theories, the anomaly could be used to constrain (or rule out) gravity models for non-relativistic strongly coupled field theories.

### 1.3.2 Non-renormalization theorems

In chapter 4 we discuss certain marginal deformations of conformal field theories with a large amount of supersymmetry. The corresponding coupling constants can be thought of as living on a geometrical space, where each point represents a different conformal field theory. We will study how certain quantities vary as we move in this geometrical space. More specifically, we will show that the structure constants of chiral operators are covariantly constant on this space.

In more technical terms, we provide a non-renormalization theorem [3] for the structure constants of chiral operators for general \( \mathcal{N} = (4,4) \) supersymmetric conformal field theories in two dimensions and \( \mathcal{N} = 4 \) supersymmetric conformal field theories in four dimensions. We will also discuss some extensions to less supersymmetric multiplets.

Non-renormalization theorems have proven to be extremely useful in various phe-
nomenological setups, both in field theory and string theory, where they constrain perturbative and non-perturbative contributions to numerous physical observables. For example, a powerful non-renormalization theorem in $\mathcal{N} = 1$ supersymmetric field theories protects the radiative corrections to the Higgs mass, making supersymmetry a possible resolution of the hierarchy problem. Non-renormalization theorems also allow the precise counting of microstates of various supersymmetric black holes, the most notable example being probably the Strominger–Vafa black hole. In our case, they provide a powerful tool to test the AdS/CFT correspondence. In fact, the computations on the gravity side and field theory side cannot be compared in general, since they are performed at different points in the moduli space. However, protected quantities do not depend on such moduli, and should be the same in both descriptions.

1.3.3 Black hole entropy

In chapter 5 we show how certain black hole constructions that recently appeared in the literature can be understood in terms of irrelevant deformations. The motivation comes in this case from the problem of explaining the entropy of black holes microscopically. For many supersymmetric black holes in string theory, the entropy has been successfully accounted for by studying the degeneracy of certain protected states in dual CFTs. In that case supersymmetry played an extremely important role, because the degeneracy of such states is a topological quantity that is constant in the moduli space of the theory, so that it can be computed exactly in the weakly coupled regime of the CFT.

Surprisingly enough, the entropy of generic black holes, even far away from extremality, seems to be of a CFT form, that is it strongly resembles Cardy’s asymptotic growth of states in a two-dimensional CFT. This “numerological” observation has sparked a lot of interest in trying to find some CFT description for the microstates of general black holes. This has led to some interesting developments, such as the Kerr/CFT correspondence, which among other things uncovered a “hidden” $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry of the scalar wave equation in the near-horizon region, which is reminiscent of conformal symmetry in two dimensions.

Later, this “hidden conformal symmetry” has been given a geometric interpretation by replacing the original (asymptotically flat) black hole with a different geometry that, while preserving the near-horizon properties, has a different asymptotic behavior. It is the purpose of this last chapter to elucidate the relation between the original black hole and this “subtracted” geometry: we will show that there is a RG flow between the two geometries, which can be interpreted in CFT language as being driven by an irrelevant deformation [4]. As we discussed before, irrelevant
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deformations are generically problematic for predictability, so the results derived in this last chapter might indicate a problem with the Kerr/CFT proposal.