Deformations of CFTs and holography

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Chapter 3

Lifshitz Anomaly

In this chapter we continue our study of Lifshitz models with anisotropic scaling symmetry. We now consider a different type of deformation, that is we change the geometry on which the theory lives. Studying the theory on a non-trivial manifold is in principle a daunting task, but here we will focus our attention on a specific problem that turns out to be tractable, namely we ask ourselves to what extent anisotropic scale invariance is broken.

We have already hinted in the previous chapter that the non-relativistic analog of the Weyl anomaly, derived in the context of ordinary AdS/CFT in [28], might be present for 3+1 dimensional Lifshitz spacetimes with dynamical exponent $z = 2$. Conformal anomalies play an important role in relativistic field theories, especially in two dimensions, where various physical quantities display a universal behavior that is governed by the central charge only, such as the Casimir energy or the logarithmic contribution to the entanglement entropy. As we illustrated in the previous chapter, many interesting physical systems, such as smectic liquid crystals or cold atoms at unitarity, exhibit non-relativistic scaling symmetry. As a consequence, if similar results carried over to non-relativistic field theories, they could be used as a guiding principle to construct (or rule out) bottom-up holographic models for such systems. In this chapter we therefore take the first steps in this direction, showing that $z = 2$ Lifshitz field theories in 2+1 dimensions do indeed lead to anomalies.

The purpose of this chapter is manifold. In section 3.1, we give a complete characterization of the possible structures that can appear in the non-relativistic scaling anomaly. We describe how this quantity can be extracted in field theory, via the heat-kernel method, and in holography, using the Hamilton–Jacobi approach.
In section 3.2 we revisit the quantum Lifshitz model introduced in the previous chapter; we first couple it to a non-trivial background and we proceed to determine the anomaly by computing various coefficients in the heat-kernel expansion. In section 3.3 we describe the computation of the anomaly for strongly coupled non-relativistic theories that admit bulk duals described by gravity coupled to a massive vector field.

The main result of this chapter is that, while there are two structures that can in principle appear in the anomaly, the two models we consider here only contribute to structure containing time derivatives. In the final section we will offer some possible interpretations of this result.

3.1 The Anisotropic Scaling Anomaly

In this section we introduce the concept of anisotropic scaling anomaly, which we will sometimes refer to as the “Lifshitz anomaly”. We first describe general concepts valid for any theory with non-relativistic scaling symmetry. We then illustrate how this anomaly can be computed in the standard path integral quantization of quantum field theory using the heat-kernel method. Finally, we describe how the anomaly shows up in the Hamilton–Jacobi approach to holographic renormalization described in the previous chapter.

3.1.1 Generalities

As we explained in the previous chapter, theories with Lifshitz scaling symmetry are characterized by the dynamical critical exponent $z$ that appears in the transformation rule (2.2). The theories we are concerned with in this chapter will couple to a non-trivial background metric, so it is useful to describe these transformations in terms of an auxiliary three-dimensional metric

$$ds^2 = N^2 dt^2 + h_{ij} dx^i dx^j.$$  \hfill (3.1)

As we discussed previously, non-relativistic theories have a notion of time, so our metric above exhibits a preferred time foliation.$^1$ In fact, the metric (3.1) keeps its form under diffeomorphisms in time, $t \mapsto \tau(t)$, and in space, $x^i \mapsto \xi^i(\vec{x})$. The anisotropic scale transformation is implemented in this language by the following transformation rules for the metric

$$N \rightarrow e^{z\omega} N, \quad h_{ij} \rightarrow e^{2\omega} h_{ij}.$$  \hfill (3.2)

$^1$We do not include a shift $N^i$, because it can be removed locally by a foliation preserving diffeomorphism, e.g. $t \mapsto \tau(t)$ and $x^i \mapsto \xi^i(t, \vec{x})$. 

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If we allow \( \omega \) to be an arbitrary function of \( t \) and \( x^i \), the transformation above describes a local anisotropic scale transformation.

We now consider a general classical field theory, which is coupled to the metric (3.1). We can achieve this for example by replacing spatial derivatives with covariant derivatives with respect to the spatial metric \( h_{ij} \) and analogously for time derivatives. However, there might be different, non minimal ways to do so, and we will discuss one such possibility in the conclusions section. In any case, if we perform an infinitesimal rescaling (3.2), where \( \omega = \delta \rho \) is an infinitesimal quantity (so that we keep only terms that are linear in \( \delta \rho \)), the classical action \( S \) transforms as

\[
\delta S = \int dtd^2x \left( \delta N \frac{\delta S}{\delta N} + \delta h_{ij} \frac{\delta S}{\delta h_{ij}} \right) = \int dtd^2x \delta \rho \left( zN \frac{\delta S}{\delta N} + 2h_{ij} \frac{\delta S}{\delta h_{ij}} \right). \tag{3.3}
\]

If \( \delta S = 0 \), then the theory is invariant under local anisotropic scale transformations at the classical level. This condition has the following physical interpretation: if we define the energy density \( \mathcal{E} \) and momentum flux (spatial stress tensor) \( \Pi_{ij} \) as

\[
\mathcal{E} = \frac{2}{N\sqrt{h}} N^2 \frac{\delta S}{\delta N^2}, \quad \Pi_{ij} = \frac{2}{N\sqrt{h}} \frac{\delta S}{\delta h_{ij}}, \tag{3.4}
\]

we see that local anisotropic scale invariance implies

\[
z \mathcal{E} + \Pi_{ii} = 0, \tag{3.5}
\]

which is the non-relativistic analog of the tracelessness condition \( T_{aa} = 0 \).

So far our discussion has been classical. At the quantum level, the classical action \( S \) is replaced by a quantum effective action, which we call \( W \). Since the metric (3.1) is an external background field, that is we do not path-integrate over it, \( W \) will be a functional of this field \( W[N, h_{ij}] \). Therefore it still makes sense to consider the variation \( \delta W \) under (3.2), which will still given by (3.3); the only difference is that we now interpret the variation with respect to \( N \) and \( h_{ij} \) as the quantum expectation values of the energy density \( \langle \mathcal{E} \rangle \) and the spatial stress tensor \( \langle \Pi_{ij} \rangle \) respectively, so that we have

\[
\delta W = \int dtd^2x N\sqrt{h} \delta \rho \left( z \langle \mathcal{E} \rangle + \langle \Pi_{ii} \rangle \right). \tag{3.6}
\]

Even when \( \delta S = 0 \), this does not imply that \( \delta W = 0 \); in fact, in the process of quantizing the theory, one often encounters divergences that need to be renormalized. As discussed in chapter 1, such a procedure typically involves the introduction of a scale in the problem, and it is not guaranteed that scale-independent quantities at the classical level will remain so after the renormalization procedure has been
carried out. As we will see both in field theory and in holography, the right-hand side of (3.6) can be non-zero, and we have

\[ 2 \langle \mathcal{E} \rangle + \langle \Pi^i_i \rangle = A \]  

(3.7)

where \( A \) is by definition the \textit{anomaly}. In principle, the anomaly depends on the renormalization scheme, for example it is affected by local counterterms that can be added to the classical action. However, it cannot be completely removed by such counterterms, and as we will see a “part of it” is indeed renormalization-scheme independent.

Contrary to the relativistic case, where anomalies are present only for even dimension, in the non-relativistic setting anomalies can also be generated in odd dimension. That this is in principle possible can be seen quite easily by dimensional analysis. The analogue of the trace of the stress-tensor for non-relativistic field theories has dimension \( z + d - 1 \). A term with \( a \) time derivatives and \( b \) spatial derivatives, on the other hand, has dimension \( az + b \). For generic \( z \), there can only be contributions to the conformal anomaly with \( a = 1 \) and \( b = d - 1 \). However, such terms have an odd number of time derivatives, so they break time reversal invariance; such an anomaly may appear only in theories that break this symmetry. The theories we consider in this chapter are time-reversal invariant, so they will not be anomalous for generic \( z \). However there are special values of \( z \) for which other values of \( a, b \) are allowed: for example, if \( d = z + 1 \), terms with either \( (a, b) = (2, 0) \) or \( (a, b) = (0, 2z) \) can appear.

In the remaining part of this chapter we will consider theories with \( z = 2, d = 3 \). In this case the argument above shows that in principle we can have terms with either two time or four spatial derivatives. While there are many such terms that one can write down, the more detailed analysis that we give below shows that the anomaly is generated by a total of two linearly independent and non-trivial structures.

### 3.1.2 Analysis of the possible terms

Just by dimensional analysis (i.e. by requiring invariance under constant rescalings) we can see that there are many terms of the right dimension that can appear in the anomaly. When \( z = 2 \), \( \partial_t \) has dimension two and \( \partial_i \) has dimension one, and we are interested in terms of dimension four that are covariant under reparametrizations of \( t \) and reparametrizations of \( x^i \). These reparametrizations should not mix \( t \) and \( x^i \) since as we explained that would ruin the form of \( D \).
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Examples of terms of the right dimension are\(^2\)

\[
N^{-2} \partial_t h_{ij} G^{ijkl} \partial_t h_{kl},
\]

\[
R^2, \quad N^{-1} \Delta N R, \quad (N^{-1} \Delta N)^2,
\]

\[
h_{ij} h_{kl} (N^{-1} \partial_i N)(N^{-1} \partial_k N)(N^{-1} \nabla_j \partial_l N).
\]

... (3.8)

where \(G^{ijkl} = \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl}\) is the DeWitt metric and \(\lambda\) an arbitrary real number (in General Relativity \(\lambda = \frac{1}{D-2}\), where \(D\) is the spacetime dimension). Terms with one time derivative and two space derivatives cannot appear because of time reversal symmetry. All these terms are scale invariant, but the anomaly should also obey the Wess–Zumino consistency condition. This comes about from the observation that the anomaly is obtained by functionally differentiating the functional \(W\). The requirement that functional derivatives commute then imposes some constraints on the anomaly. Concretely, if we denote the variation with parameter \(\omega\) by \(\delta_{\omega}\), we have

\[
\delta_{\omega_1} \delta_{\omega_2} W = \delta_{\omega_2} \delta_{\omega_1} W,
\]

which in terms of the anomaly reads

\[
\delta_{\omega_1} \int dt d^2 x N \sqrt{h} \omega_2 A = \delta_{\omega_2} \int dt d^2 x N \sqrt{h} \omega_1 A.
\] (3.10)

This condition imposes strong restrictions on the possible terms that can appear in \(A\). The computation is straightforward: one writes down all the terms of the right dimension and determines which combinations solve the condition above by explicitly taking functional variations. This is done explicitly in appendix 3.A.1, and here we only report the final result:\(^3\) the anomaly is given by the following

\[^2\]A complete classification can be found in Appendix 3.A.1.

\[^3\]As shown in [45], the problem of finding Wess–Zumino consistent structures can be phrased in terms of cohomology, very much like the relativistic case: one defines a nilpotent operator \(s\) acting on local functionals that implements local scale transformations. In this language, the Wess–Zumino condition turns out to be equivalent to the requirement that the anomaly is \(s\)-closed (that is annihilated by \(s\)). Trivial terms that can be removed by local counterterms turn out to be \(s\)-exact (that is they are of the form \(sQ\) for some local functional \(Q\)), so that the non-trivial structures are computed by the cohomology of the complex defined by \(s\). It should be noted that the analysis of [45] appeared earlier than the one in [2], which is presented here, because the latter preprint contained a mistake that was corrected in the published version.
two independent non-trivial structures

$$A = C_1 \frac{1}{N^2} \left( h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right)$$

$$+ C_2 \left( R + \frac{1}{N} \Delta N - h^{ij} \left( \frac{1}{N} \partial_i N \right) \left( \frac{1}{N} \partial_j N \right) \right)^2$$

$$+ \text{(trivial total derivatives)}.$$ (3.11)

Here the trivial total derivatives can be canceled by appropriate local counterterms. The values of $C_1$ and $C_2$ are model dependent, and will be computed in two different models later in the chapter.

### 3.1.3 Heat-kernel expansion

In this section we will employ the notation and results of [46]. We consider the quantization of a general Lifshitz field theory. If the action is quadratic in the canonical momenta, a path-integral quantization can be carried out without additional complications, so that the effective action $W$ is simply given by

$$e^{-W} = \int D\phi e^{-S},$$

(3.12)

where $\phi$ is a shorthand notation for all the dynamical fields that are present in $S$. Even though the action $S$ is classically scale invariant, the functional measure $D\phi$ might not be so. While determining this anomalous behavior might seem a very complicated task in principle, it turns out that a one-loop computation is sufficient (see [46] and references therein). As a consequence, we only consider quadratic terms in the classical action

$$S = \int d^3 x N \sqrt{h} \phi D \phi,$$

(3.13)

where $D$ is a self-adjoint operator with respect to some scalar product (implied by the expression above) and with some suitable domain of definition. Here we are not considering sources for our fields $\phi$, but it is not difficult to show that their presence would not affect the analysis below. The integral is Gaussian and can be computed explicitly. It is in fact given by the formal expression

$$W = \frac{1}{2} \log \det(D),$$

(3.14)

where $\det(D)$ is the determinant of the operator $D$ appearing in equation (3.13). The expression above is only formal because the operator acts on an-infinite di-
mensional space, so the product over the eigenvalues need not converge.\footnote{We are using the term “eigenvalue” a bit loosely, since the operator does not necessarily have a discrete spectrum. However, the precise nature of the spectrum will not be important in what follows.} In order to give meaning to the determinant, we employ $\zeta$-function regularization. We will illustrate the idea on a toy example and then extend it to more general operators.

**A toy example**

Consider an operator $A$ whose spectrum is $\lambda_n = n$, $n = 1, 2, \ldots$. The “determinant” of this operator, following the intuition that comes from the finite-dimensional case, should be given by the product of the eigenvalues

$$\det(A) = \prod n,$$

which is however divergent. In order to give meaning to the previous expression, we first transform the infinite product into an infinite sum by taking the logarithm. This corresponds to taking the trace of $\log A$, that is

$$\log \det(A) = \text{Tr}(\log A) = \sum \log n.$$ (3.16)

The trace is divergent and consequently the operator $\log A$ is not trace-class. The idea is to introduce a family of operators parametrized by a complex parameter $s$ and whose trace is well-defined at least in a region of the complex $s$–plane. We then define the trace of the original operator by analytically continuing the answer and then taking appropriate limits. Concretely, we consider the operators $A^{-s}$, which are well-defined due to the self-adjointness of $A$\footnote{Note that the self-adjointness of $A$ is crucial for the definition of the functional determinant.}. If we take a sufficiently large positive $s$ (in our case $\text{Re}(s) > 1$), the trace is well-defined and is given by

$$\text{Tr}(D^{-s}) = \sum n^{-s} = \zeta(s),$$ (3.17)

where $\zeta(s)$ is the usual Riemann zeta function. The series $\sum n^{-s}$ converges only when $\text{Re}(s) > 1$, but it defines an analytic function of $s$ that can be analytically continued to all complex values of $s \neq 1$. If we consider $\zeta'(s)$, and we use the series $\sum n^{-s}$ even when $\text{Re}(s) < 1$, we immediately see that

$$\zeta'(0) = \sum \log n,$$ (3.18)

where the equality is only formal because we are not allowed to use the sum representation when $\text{Re}(s) \leq 1$. Since the left-hand side is well-defined and finite, and it formally reproduces the series we would like to give meaning to, it is natural to define our functional determinant as

$$\log \det(A) = -\zeta'(0) = \log \sqrt{2\pi}.$$ (3.19)
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At this point, the reader might be a little worried about the uniqueness of the previous result. It is in fact not unique, as we could have used for example many different inequivalent analytic continuations. However, quantum field theories are more constrained than our toy example, in that the regularization procedure should respect the various symmetries of the system, such as reparametrization invariance and gauge symmetry. Furthermore, divergences should be removed by local counterterms. Respecting all these requirements at once puts strong restrictions on the possible regularization prescriptions that can be employed, and \( \zeta \)-function regularization certainly succeeds in this respect. Ultimately we are interested in computing the non-relativistic anomaly and, as shown before, different renormalization schemes related by finite local counterterms can change only its trivial part. As a consequence, \( \zeta \)-function regularization appears entirely adequate for the purpose of computing the anomaly.

The general construction

We extend the construction above to general operators by defining the generalized \( \zeta \)-function as

\[
\zeta(s, f, D) = \text{Tr}_{L^2}(f D^{-s}),
\]

where \( s \) is an arbitrary positive number and \( L^2 \) an appropriate function space on which \( D^{-s} \) is trace-class, at least for sufficiently large \( \text{Re}(s) \). The regularized effective action is given by

\[
W = -\frac{1}{2} \zeta'(0, 1, D) - \frac{1}{2} \log(\mu^2) \zeta(0, 1, D),
\]

where \( \zeta'(0, f, D) = \partial_s \zeta(s, f, D)|_{s=0} \) and the term proportional to \( \log(\mu^2) \) can be shown to be local and is related to a renormalization scheme ambiguity. The effective action as defined above can be shown to be finite, but it is still very difficult to compute. However, we are only interested in how this object transforms under anisotropic rescalings, and as we will see this piece of information is encoded in local quantities that can be computed exactly.

The first step is to define the so-called heat kernel

\[
K(\epsilon, f, D) = \text{Tr}_{L^2}(f e^{-\epsilon D}),
\]

where \( f \) is an arbitrary function of \( t \) and \( x^i \), and \( \epsilon \) is an arbitrary positive parameter. It is not difficult to show that the zeta function \( \zeta(s, f, D) \) is related to the object defined above by a Mellin transformation:

\[
\zeta(s, f, D) = \Gamma(s)^{-1} \int_0^\infty d\epsilon \epsilon^{s-1} K(\epsilon, f, D),
\]

\[
K(\epsilon, f, D) = \frac{1}{2\pi i} \oint ds e^{-s} \Gamma(s) \zeta(s, f, D).
\]
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In principle $K$ depends on the global behavior of the operator $D$ (the trace can be written as a sum over the spectrum of the operator, which is determined by global properties); however there is an asymptotic series of the form

$$K(\epsilon, f, D) \sim \sum_{k=0}^{\infty} \epsilon^{\frac{k}{2}-1} \tilde{a}_k(f, D),$$

(3.25)

where $\tilde{a}_k(f, D)$ can be computed locally from $N$ and $h_{ij}$. By repeating the analysis of [46] section 7.1, it is possible to show that the variation of the renormalized effective action under an infinitesimal anisotropic local scale transformation $h \rightarrow (1 + 2\delta \rho)h$, $N \rightarrow (1 + 2\delta \rho)N$, is given by

$$\delta W = -2 \tilde{a}_2(\delta \rho, D).$$

(3.26)

As explained above, this will be a local functional of $N$ and $h$; we will therefore write

$$\tilde{a}_2(f, D) = \int dt d^2x N \sqrt{f} a_2(N, h_{ij}),$$

(3.27)

where $a_2(N, h_{ij})$ is a local function that depends on $N$ and $h_{ij}$.

Using the previous expression in (3.26) and comparing with (3.6), we immediately see that

$$A = -2 a_2(N, h_{ij}).$$

(3.28)

Therefore in order to compute the anomalous transformation of $W$ under local anisotropic scale transformations, we need to extract the coefficient of order $\epsilon^0$ in the heat-kernel expansion of the operator $D$, that is $a_2$.

3.1.4 Hamilton–Jacobi analysis

We explained in the previous section how one can compute the anomaly in the field theory side using the path-integral formulation. In this section we describe how this same anomaly can be computed holographically in the Hamilton–Jacobi framework.

Recall from section 2.2 that we want to extract the divergent part $S_{loc}$ of the (bare) on-shell action by solving the functional differential equation

$$\{S_{loc}, S_{loc}\} - L = 0,$$

(3.29)

where the notation is as in chapter 2. However, it might happen that there is no solution to the problem above compatible with the requirement that $S_{loc}$ is indeed

5The factor 2 comes from the factor 4 in $D \rightarrow e^{-4\rho}D$ under scale transformations.
local. However, recall that our purpose is to remove divergences in the on-shell action; as a consequence, we will impose the weaker condition that at least the “divergent” terms in (3.29) be canceled. A term $F$ is called divergent if

$$\lim_{r \to \infty} \sqrt{\gamma} F = +\infty. \quad (3.30)$$

It might still happen (and it does happen as we will see) that we get a finite remainder $H_{\text{rem}}$ that cannot be removed by local counterterms. As a consequence, the Hamilton–Jacobi equation for the effective action $\Gamma$ in the large $r$ limit, where as argued before we can ignore $\{\Gamma, \Gamma\}$, must be corrected by

$$2\{S_{\text{loc}}, \Gamma\} \approx -H_{\text{rem}}. \quad (3.31)$$

The symbol “$\approx$” means an equality in the large $r$ limit.

We will show in the following that if the boundary conditions are chosen appropriately, the expression above for large $r$ becomes

$$2\sqrt{\gamma}\{S_{\text{loc}}, \Gamma\} \approx zN \frac{\delta \Gamma}{\delta N} + 2h_{ij} \frac{\delta \Gamma}{\delta h_{ij}}. \quad (3.32)$$

By comparing with (3.7), we find that the Lifshitz anomaly is given by

$$A = -\lim_{r \to \infty} \sqrt{\gamma} H_{\text{rem}}. \quad (3.33)$$

In other words, the holographic computation of the anisotropic scale anomaly involves the determination of the remainder term in the Hamiltonian constraint, just as in the relativistic case.

### 3.2 Field-theoretic Calculation

In this section we turn to the computation of the anomaly of the quantum Lifshitz model described in the previous chapter. First we need to couple this field theory to the non-trivial metric background (3.1). We therefore consider the following generalization of the action (2.3):

$$S = \int dt d^2x N\sqrt{h} \left( \frac{1}{2} N^{-2} (\partial_t \phi)^2 + \frac{1}{2} (\Delta \phi)^2 \right). \quad (3.34)$$

In this expression, $\Delta = \nabla_i \nabla^i$ is constructed out of the covariant derivatives of the spatial metric $h_{ij}$. By integrating by parts and ignoring boundary terms$^6$, the action can be written as

$$S = \int dt d^2x N \sqrt{h} \phi D\phi, \quad (3.35)$$

$^6$In the following, we will assume that the theory is defined on a manifold without boundary.
where \( D \) is given by
\[
D = -\frac{1}{N \sqrt{h}} \partial_t N^{-1} \sqrt{h} \partial_t + \frac{1}{N} \Delta N \Delta
\] (3.36)

The model defined in (3.34) is classically invariant under local anisotropic scale transformations (3.2), with the additional requirement that the scalar field \( \phi \) transform trivially
\[
\phi \rightarrow \phi
\] (3.37)

It is very easy to show that the operator \( D \) transforms as
\[
D \rightarrow e^{-4\omega} D
\] (3.38)

so that the action (3.34) is indeed invariant under local anisotropic rescalings.

We will prove that the classical anisotropic invariance of this model is broken at the quantum level. In particular, we will use the heat-kernel method described in the previous section to show that
\[
\mathcal{A} = \frac{1}{1536 \pi N^2} \left( 16 h^{ij} N \partial_t (N^{-1} \dot{h}_{ij}) + 5 (h^{ij} \dot{h}_{ij})^2 - 10 \dot{h}_{ij} \dot{h}_{jk} h^{kl} \dot{h}_{li} \right) + \frac{1}{480 \pi N} \nabla_i J^i,
\] (3.39) (3.40)

where a dot indicates \( \partial_t \) and \( \nabla_i J^i \) is a “trivial” total derivative, by which we mean that it can be removed by adding appropriate local counterterms. In fact the anomaly can be written in the simpler form
\[
\mathcal{A} = \frac{1}{128 \pi N^2} \left( h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right)
\] (3.41)

when appropriate counterterms are added to the action (3.34).

As an aside, one of the reasons for the particular interest in this model is that the ground-state wave functional is invariant under time-independent conformal transformations in space. All equal-time correlators can be computed using the machinery of a two-dimensional field theory [25, 24]. One may thus naively expect that the anomalous breaking of anisotropic scaling symmetry (2.2) is somehow related to the two-dimensional Weyl anomaly \( \langle T^i_i \rangle \propto R \), see e.g. §5.A of [48].

We find, however, that this is not the case: the anomaly involves only derivatives with respect to the time coordinate, whereas the two-dimensional Ricci scalar \( R \) obviously contains only spatial derivatives.

As we explained above, the heat kernel can be expanded as
\[
K(\epsilon, f, D) = \sum_{k \geq 0} \epsilon^{k-1} \int dt d^2 x \sqrt{h} f a_k(N, h_{ij}),
\] (3.42)
where $a_k(N,h_{ij})$ is a local function of $N$ and $h_{ij}$. To evaluate this we need a suitable basis; it is customary to use the rescaled Fourier modes so that they are orthonormal with respect to the measure that includes the $N\sqrt{h}$ factor. Nevertheless, as pointed out in [49], the cyclicity of the trace allows us to use the usual flat Fourier modes. We thus find

$$K = \int \frac{d\omega d^2k}{(2\pi)^3} \int dt d^2x \ e^{-i\omega t - ikx} f e^{-\epsilon D} e^{i\omega t + ikx}. \quad (3.43)$$

We can conjugate the Fourier mode to the left to get the expression

$$K = \int \frac{d\omega d^2k}{(2\pi)^3} \int dt d^2x \ e^{-\epsilon D_2}, \quad (3.44)$$

where $D_2$ is obtained from $D$ by shifting the derivatives as follows:

$$\partial_t \rightarrow \partial_t + i\omega \quad \partial_i \rightarrow \partial_i + ik_i. \quad (3.45)$$

The most singular term in the heat kernel is the one where we keep only the terms in $D_2$ without derivatives, leading to

$$\frac{1}{\epsilon} \tilde{a}_0(f,D) = \int \frac{d\omega d^2k}{(2\pi)^3} \int dt d^2x \ e^{-\epsilon(N - \omega^2 + (k^2)^2)}, \quad (3.46)$$

where $k^2 \equiv h^{ij}k_ik_j$. This expression is readily evaluated to yield the first term in the heat kernel expansion:

$$\tilde{a}_0(f,D) = \frac{1}{16\pi} \int dt d^2x N\sqrt{h} f(t,x). \quad (3.47)$$

Computing the subleading terms is now straightforward, though somewhat involved. We shall write

$$D_2 = D_2^0 + D_2^{\text{int}} \quad (3.48)$$

where $D_2^0$ is the piece we isolated above that contains $\omega^2$ and $k^4$, and $D_2^{\text{int}}$ the remainder. We then expand the exponential of $D_2^{\text{int}}$. This contains an explicit factor of $\epsilon$, but $\omega$ counts as $\epsilon^{-1/2}$ and $k$ as $\epsilon^{-1/4}$ in the Gaussian integral, as it is obvious from the fact that

$$\int d\omega dk \ k e^{-\epsilon(\omega^2 + (k^2)^2)} \omega^s k^r \propto \epsilon^{1-s-\frac{r}{2}}. \quad (3.49)$$

This means that $D_2^{\text{int}}$ has a term which scales as $\epsilon^{-1/4}$, and to get to the finite term one needs to expand $D_2^{\text{int}}$ up to fourth order, so that we get terms up to $k^{12}$. This computation might thus appear extremely daunting; however, the problem becomes tractable if we consider the time-derivative and space-derivative sectors separately. This is consistent because the anomaly can only have structures involving either two time derivatives or four spatial derivatives.
The two-derivative anomaly

In order to compute the two-derivative contribution to the anomaly, and in turn $C_1$, it is sufficient to consider metrics that only depend on $t$, and not on $x^a$. Thus we can drop all the terms with spatial derivatives $\partial_i$ in $D_2^{\text{int}}$. Moreover, by changing the coordinate $t$ if necessary, we can take $N = 1$. With these assumptions, we have

\[
D_2^0 = \omega^2 + (k^2)^2, \tag{3.50}
\]
\[
D_2^{\text{int}} = -i\omega(\partial_t + \frac{1}{\sqrt{h}} \partial_t \sqrt{h}) - \frac{1}{\sqrt{h}} \partial_t \sqrt{h} \partial_t. \tag{3.51}
\]

The power-counting argument above shows that we need to expand to second order in $D_2^{\text{int}}$. Since $D_2^{\text{int}}$ and $D_2^0$ do not commute, we use the following formula:

\[
e^{A + B} = e^A + \int_{0 \leq \alpha \leq 1} d\alpha e^{\alpha A} B e^{(1-\alpha)A}
+ \int_{0 \leq \alpha + \beta \leq 1} d\alpha d\beta e^{\alpha A} B e^{\beta A} B e^{(1-\alpha-\beta)A} + \mathcal{O}(B^3). \tag{3.52}
\]

We find the following contribution to $a_2$:\footnote{This computation is in principle quite lengthy. However, since there are only few terms that can appear, one can work this out for a diagonal $h_{ij}$ and then reconstruct the full answer.}

\[
\tilde{a}_2(f, D) = \frac{-1}{1536\pi} \int dt \int dx \sqrt{h} f \left\{ 16 h^{ij} \dot{h}_{ij} + 5(h^{ij} \dot{h}_{ij})^2 - 10 h^{ij} \dot{h}_{jk} h^{kl} \dot{h}_{li} \right\} + \ldots \tag{3.53}
\]

where the ellipses denote possible four-derivative contributions. To reinstate $N$, we simply need to change $dt \to dtN$ and $\partial_t \to N^{-1} \partial_t$. We can remove the first term in the right-hand side of the expression above by adding local counterterms, as explained in detail in appendix 3.A.1. We thus obtain the two-derivative contribution to the anomaly

\[
\frac{1}{128\pi} \frac{1}{N^2} \left( \dot{h}^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right). \tag{3.54}
\]

Using (3.11) we see that

\[
C_1 = \frac{1}{128\pi}. \tag{3.55}
\]

The four-derivative anomaly

We now determine the four derivative contribution, and in turn $C_2$. As explained in Appendix 3.A.1, there are 6 possible terms that can appear, 5 of which are total derivatives. These structures are distinguished by a metric of the form $h_{ij} = e^{2f(x)} \delta_{ij}$ and $N = e^{g(x)}$, which can be used to greatly simplify the computation.
The expression for $D^{\text{int}}_2$ is considerably more involved, but it is straightforward to derive it by conjugating the Fourier modes to the left as explained above for the second-derivative case. Furthermore we need to use the appropriate generalization of (3.52) to fourth order in $B$. Showing these expressions would not provide any conceptual clarification, so we decided to present directly the final result of the computation. The four-derivative contribution to the anomaly is thus

$$A = \frac{1}{480\pi} \frac{1}{N} \nabla_i \left( -5(\partial^i N)R + 3(\partial^i N) \left( \frac{1}{N} \Delta N \right) + 2(\partial^j N) \left( \frac{1}{N} \nabla_j \partial^i N \right) - 5\partial^i \Delta N \right).$$

(3.56)

It is interesting to note that this result is a total derivative and, as predicted by the Wess–Zumino consistency condition, it is orthogonal\textsuperscript{8} to the non-trivial total derivative $J$ defined in equation (3.89). As a consequence, this term can be removed by a local counterterm and we conclude that

$$C_2 = 0.$$  

(3.57)

In Appendix 3.A.3 we present an alternative derivation of $C_2 = 0$.

The anomaly

In summary, the Lifshitz model (3.34) exhibits an anomaly under anisotropic local scale transformations, which after the addition of appropriate counterterms is given by

$$A = \frac{1}{128\pi} \frac{1}{N^2} \left( h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right).$$

(3.58)

It is striking that the anomaly involves only time derivatives. So far, it is unclear to us why this happens. It is also in contrast to the naive expectation that the anomaly is somehow related to the trace anomaly of a two-dimensional conformal field theory, as we mentioned at the beginning of this section.

### 3.3 Holographic Calculation

In the previous section we showed that a theory with anisotropic scaling symmetry has an anisotropic scaling symmetry anomaly parametrized by two central charges, denoted by $C_1$ and $C_2$. We computed these central charges for a particular model defined by the action (3.34). In this section we show that these central charges can be computed holographically for the Lifshitz spacetime considered in [16, 1].

\textsuperscript{8}To see what we mean by ‘orthogonal’, we refer to the appendix 3.A.1.
3.3. Holographic Calculation

3.3.1 Boundary conditions and anomaly

From the field theory side, we know that the volume form has a definite scaling weight \([\text{Vol}_d] = [dt^{d-1}x] = z + d - 1\). In the dual gravitational picture this weight is translated to a radial scaling, such that

\[
z + d - 1 = [\text{Vol}_d] = \frac{\partial_r \sqrt{\gamma}}{\sqrt{\gamma}} = \partial_r \log N + \frac{1}{2} \partial_r \log \det h
\]  

(3.59)

Here, \(N\) is the lapse function and \(h_{ij}\) is the induced metric on a spatial slice of \(\Sigma_r\). We assume that the spatial metric is of the form \(h_{ij} = e^{2r} \hat{h}_{ij}\), where \(\hat{h}_{ij}\) has a finite limit as \(r \to \infty\). It then follows that the lapse scales as \(N \sim e^{zr}\). This puts a restriction on the degrees of freedom contained in the metric. It implies in particular that we must turn off the off-diagonal mode in \(\gamma_{ti}\) that scales as \(e^{2zr}\); in terms of the linearized modes discussed in \([40, 16, 1]\), one needs to kill the \(c_{1i}\) mode. This naturally leads us to consider only deformations with a preferred time foliation, as suggested also in \([16]\).

Let us redefine \(N\) and \(h_{ij}\) to be the renormalized lapse and induced metric,\(^9\)

\[
N = \lim_{r \to \infty} e^{-zr} (-\gamma^{tt})^{-1/2},
\]

(3.60)

\[
h_{ij} = \lim_{r \to \infty} e^{-2r} \gamma_{ij},
\]

(3.61)

such that the renormalized volume form is given by \(N \sqrt{h} = \lim_{r \to \infty} e^{-(z+d-1)r} \sqrt{\gamma}\).

With these conditions, it is straightforward to see that:

\[
2\sqrt{\gamma} \{S_{\text{loc}}, \Gamma\} \approx 2z N^2 \frac{\delta \Gamma}{\delta N^2} + 2h_{ij} \frac{\delta \Gamma}{\delta h_{ij}} + z \hat{A}_t \frac{\delta \Gamma}{\delta \hat{A}_t} \approx -\sqrt{\gamma} H_{\text{rem}},
\]

(3.62)

where \(\hat{A}_t = \lim_{r \to \infty} e^{-zr} A_t\). As noted in the previous chapter, the “vector field mode” \(c_3\) requires an infinite set of counterterms to be properly renormalized holographically. Furthermore, this mode introduces logarithmic divergences in the metric sector, spoiling our definition of anisotropic conformal infinity \((3.60)\).

For this reason, we will turn off this mode\(^10\) by setting \(c_3 = 0\). In particular notice the important relation:

\[
A_t = e^{2r} N + \text{subleading},
\]

(3.63)

where the subleading terms scale as \(e^{-4r}\). In the frame field language of \([16]\), this corresponds to \(\delta(A_{\hat{A}}) = \delta(A_a e_a^\hat{A}) = 0\). Notice that with these boundary conditions,

\(^9\)Again, we do not consider a shift \(N^i\), as it can locally be removed by a foliation-preserving diffeomorphism.

\(^10\)As shown in \([1]\), it is possible to consistently impose this condition when higher order non-linear corrections are considered, and we believe there is no obstruction at the full non-linear level.
\( \Gamma \) becomes a functional of \( N \) and \( h \) only. Therefore we have:

\[
\frac{\partial \Gamma}{\partial N} \bigg|_{h=\text{const}} = \frac{\delta \Gamma}{\delta N} + \frac{\delta \Gamma}{\delta \hat{A}_t} \frac{\partial \hat{A}_t}{\partial N},
\]

where the variations on the right are unconstrained, while the variation on the left represents the total variation of \( \Gamma \) with respect to \( N \). Therefore (3.62) becomes

\[
2N \frac{\partial \Gamma}{\partial N} + 2h_{ij} \frac{\delta \Gamma}{\delta h_{ij}} = -\sqrt{-\gamma} H_{\text{rem}}.
\]

By comparing with (3.7), we find that the anomaly is indeed given by

\[
\mathcal{A} = -\lim_{r \to \infty} \epsilon^{4r} H_{\text{rem}}.
\]

3.3.2 Holographic anomaly

The computation of \( H_{\text{rem}} \) is as follows. We first write the most general ansatz for \( S_{\text{loc}} \), such as

\[
S_{\text{loc}} = \int d^d x \sqrt{\gamma} \left\{ U(\alpha) + \mathcal{F}_1 R + \mathcal{F}_2 D_a A_b D^a A^b + \mathcal{F}_3 D_a A_b D^b A^a + \ldots \right\},
\]

where \( U(\alpha) \) was discussed in the previous chapter, and each coefficient \( \mathcal{F}_i \) is a function of \( \alpha \). The ellipses denote additional two-derivative terms that can be constructed out of covariant derivatives of \( A_a \) and higher derivative terms. More details can be found in [2].

Solving the Hamilton constraint recursively by expanding the various functions of \( \alpha \) as a power-series in \( \alpha - \alpha_0 \) as explained in the previous section, one finds a remainder of the form

\[
H_{\text{rem}} = -D_a A_b D^b A^a + \frac{1}{2} (D_a A^a)^2 + \frac{1}{8} R^2 - \frac{1}{4} R_{ab} R^{ab} + \ldots,
\]

where the ellipses denote four-derivative terms involving the vector field \( A_a \). Using the definitions in 3.3.1 to write the two-derivative piece in terms of \( h_{ij} \) and \( N \), and plugging the result in (3.66), we get the following contribution to the anomaly

\[
\mathcal{A} = \frac{\ell^2}{64\pi G} \frac{1}{N^2} \left( h_{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h_{ij} \dot{h}_{ij})^2 \right) + \ldots,
\]

and using (3.11) we conclude that

\[
C_1 = \frac{1}{128\pi} \frac{2\ell^2}{G}.
\]
We reinstated the four-dimensional Newton’s constant $G$ and the curvature length scale $\ell$. The four-derivative piece is more complicated to analyze, but we can extract $C_2$ by extracting the coefficient of the square of the two-dimensional Ricci scalar $(\mathcal{R})^2$. Writing the three dimensional Ricci tensor in terms of the two-dimensional one gives

$$\mathcal{H}_{\text{rem}} = \ldots + \frac{1}{4} R_{ij} R^{ij} - \frac{1}{8} (\mathcal{R})^2 + \ldots,$$

where we have not shown terms that involve derivatives acting on $N$. Then we can use the off-shell identity that relates the Ricci tensor to the Ricci scalar, $\mathcal{R}_{ij} = \frac{1}{2} \mathcal{R} h_{ij}$, which is specific to two dimensions. When we plug this into (3.71), we find that the Ricci-squared terms cancel and we have (equation (3.11)):

$$C_2 = 0,$$

which agrees with the field theory computation.

Notice that for the purpose of computing $C_2$, which was the aim of this section, we did not have to compute the complete answer that includes trivial total derivative. In fact, a full analysis of the counterterm action and remainder at the four-derivative level, while conceptually straightforward, would be rather involved. Nevertheless, the complete answer has been computed using the results of [16] in [45], in perfect agreement with our result $C_2 = 0$. In conclusion, the holographic anomaly is given by

$$\mathcal{A} = \frac{\ell^2}{64\pi G} \frac{1}{N^2} \left( h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} h_{ij})^2 \right).$$

3.4 Discussion and conclusions

In this chapter we computed the anisotropic scaling anomaly of two Lifshitz theories, one defined using a standard field theory quantization of an explicit classical action (3.34), the other defined using the holographic correspondence. A precise definition of Lifshitz holography is still lacking, and a microscopic characterization of the strongly coupled field theories dual to Lifshitz spacetime is not known. It is therefore a priori not very meaningful to compare the two anomalies. Nevertheless, we found that the anomalies are quite similar. In both cases one of the two possible central charges vanishes, and as a consequence the two anomalies are directly proportional to each other. The ratio of the two anomalies, in the conventions used in this chapter, is $2\ell^2/G$, with $\ell$ the curvature radius of the Lifshitz spacetime and $G$ the 4d Newton constant. It would be interesting to evaluate this quantity in explicit string theory embeddings of Lifshitz spacetimes to see how
it scales with the various integer fluxes, as this will provide some measure of the effective number of degrees of freedom of the dual field theory.

It is quite mysterious that the conformal anomaly only involves time derivatives, it is even mysterious that there exists a conformal anomaly at all. According to [37], the dynamical critical exponent is in general renormalized, and as soon as \( z = 2 + \epsilon \) a (time-reversal invariant) conformal anomaly can no longer be written down. So either there is some unknown mechanism that protects the value of \( z = 2 \), or the conformal anomaly can somehow be removed in the full quantum theory. Further work will be required to clarify this issue.

It is also clearly of interest to explore other systems with anisotropic scale invariance to examine whether the conformal anomaly is still of the same form. In particular, whenever one has a Lifshitz solution in a theory with Chern-Simons type terms, time reversal symmetry is broken and it is logically possible to have contributions with an odd number of time derivatives to the conformal anomaly. It is in principle straightforward to extend the analysis in appendix 3.A.1 to determine whether there are non-trivial terms of this type and we leave this as an exercise.

Some progress on these issues was recently reported in [12], building on the results of [45] that partly overlapped with our [2]. The authors showed that the second possible structure in the anomaly, the one involving spatial derivatives, can be generated in field theory by adding the following non-minimal coupling between the scalar field \( \phi \) and the metric

\[
S = \ldots + \int dt d^2 x N \sqrt{h} \left( R + \frac{1}{N} \Delta N - h^{ij} \left( \frac{1}{N} \partial_i N \right) \left( \frac{1}{N} \partial_j N \right) \right)^2 \phi^2. \quad (3.74)
\]

Employing the \( \zeta \)-function regularization discussed above, the authors argued that \( C_2 \neq 0 \) in this model. In view of this result, it is natural to wonder what kind of gravity models in the bulk can reproduce the non-trivial four derivative term in the anomaly. In [12] it was shown that various nonprojectable versions of Horava–Lifshitz gravity [50, 51] in the bulk do indeed generate non-zero \( C_2 \).

With these partial results, we may speculate that only theories with \( C_2 = 0 \) can be formulated in terms of bulk gravity duals with full local relativistic invariance, while models with \( C_2 \neq 0 \) require non-relativistic theories in the bulk such as Horava–Lifshitz gravity. It would be extremely interesting to explore this relationship further, and especially determine how the purported "hidden relativistic symmetry" of theories with \( C_2 = 0 \) can be understood from the field theory side.

As mentioned before, one of the main uses of the conformal anomaly is that it is a relatively simple quantity of a field theory which sometimes controls certain
universal properties. For example, in the relativistic case, in $d = 2$ the conformal anomaly completely fixes the free energy at high temperatures, and it also controls the logarithmic contributions in the entanglement entropy in $d = 2, 4$. Whether similar universal properties also exist for non-relativistic field theories is an interesting open problem that we hope to come back to in the future.
3. Appendices

3.A Classification of possible terms in the anomaly

In this appendix we explore to what extent it is possible to remove total derivatives from the anomaly. This is achieved by adding appropriate scale invariant counterterms to the action that are not invariant under local scale transformations.

Clearly, we can discuss the two-derivative and the four-derivative terms separately. Let us start with the former; there are only three possible scale-invariant terms that we can construct with two time derivatives:

\[ h^{ij} \frac{1}{N} \partial_t (\frac{1}{N} \partial_t h_{ij}), \quad \frac{1}{N^2} (h^{ij} \dot{h}_{ij})^2, \quad h^{ij} \dot{h}_{jk} h^{kl} \dot{h}_{li}. \]  

(3.75)

It is straightforward to see that the two combinations

\[ h^{ij} \frac{1}{N} \partial_t (\frac{1}{N} \partial_t h_{ij}), \quad \frac{1}{N^2} \left( h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right), \]  

(3.76)

are invariant under local scale transformations (up to total derivatives). These two terms are related by partial integration, and we now show that it is indeed possible to “partially integrate” inside the anomaly by adding an appropriate counterterm to the action. The most general form of the anomaly at the two derivative level is:

\[ \delta W = \int \delta \rho \left\{ a \frac{1}{N} h^{ij} \partial_t (\frac{1}{N} \partial_t h_{ij}) + b \frac{1}{N^2} \left( h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right) \right\}. \]  

(3.77)

The presence of the factor \( \delta \rho \) prevents us from doing partial integration directly. Let us add the following counterterm to the action:

\[ W' = W + c \int N \sqrt{h} \frac{1}{N^2} (h^{ij} \dot{h}_{ij})^2. \]  

(3.78)

It is then easy to check that

\[ \delta W' = \int \delta \rho \left\{ (a - 8c) \frac{1}{N} h^{ij} \partial_t \frac{1}{N} \dot{h}_{ij} + (b + 8c) \frac{1}{N^2} \left( h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right) \right\}. \]  

(3.79)

Therefore we can pick \( c = a/8 \) and get rid of the first term, which is tantamount to integrating by parts, or discarding total derivatives in the anomaly. For instance, in the field theory analysis, we went from (3.53) to (3.54) using this procedure. In particular, we had \( a = 1/48 \pi \) and \( b = -5/384 \pi \), such that \( b + 8c = a + b = 1/128 \pi. \)\(^{11}\)

\(^{11}\)Bear in mind that there was an extra factor \(-2\) coming from the relation between \( a_2(\delta \rho, D) \) and the integrated anomaly, cf. (3.26).
3.A. Appendices

Let us now consider the four derivative level. In this case we are interested in terms of the form $\nabla_i J^i$ in the anomaly. We ask ourselves to what extent it is possible to remove them by adding local counterterms $G$ to the action. Both the total derivatives and the local counterterms must be scale invariant, therefore there is only a finite number of them. Let us choose a basis:

\[ J^i_a \quad a = 1, \ldots, N \]  
\[ G_b \quad b = 1, \ldots, M. \]  

(3.80)  
(3.81)

The Weyl variation of a linear combination $\sum_b q_b G_b$ can be written, after partial integration, as:

\[ \delta \sum_b q_b G_b = \frac{\omega}{N} \sum_{ab} M_{ab} q_b \nabla_i J^i_a, \]  

(3.82)

If the variation of the effective action reads

\[ \delta W = \int N \sqrt{h} \omega \left( A + \sum_a c_a \nabla_i J^i_a \right), \]  

(3.83)

we can get rid of the total derivatives if we can solve the system of linear equations:

\[ M_{ab} q_b = c_a. \]  

(3.84)

If we are to remove all the possible total derivatives that can appear, the number of rows $N$ of the matrix $M_{ab}$ must be less than or equal to the number of columns $M$, and the rank of the matrix should be maximal. It is easy to check that there are 6 possible functionally independent scale invariant currents $J^i$, and we choose the following basis:

\[ J^1_i = N \partial^i R \]  
\[ J^2_i = (\partial^i N) R \]  
\[ J^3_i = (\partial^i N)(\frac{1}{N} \Delta N) \]  
\[ J^4_i = (\partial^i N)(\frac{1}{N} \partial_i \Delta N) \]  
\[ J^5_i = (\partial^i N)(\frac{1}{N} \partial_j \partial^j N) \]  
\[ J^6_i = \partial^i \Delta N. \]  

(3.85)

Analogously, there are 12 functionally independent scale invariant counterterms, and we choose the basis:

\[ G_1 = R^2 \]  
\[ G_2 = \Delta R \]

\[ G_3 = (\frac{1}{N} \Delta N) R \]  
\[ G_4 = (\frac{1}{N} \partial_i N)(\frac{1}{N} \partial^i N) R \]

\[ G_5 = ((\frac{1}{N} \partial_i N)(\frac{1}{N} \partial^i N))^2 \]  
\[ G_6 = (\frac{1}{N} \partial_i N)(\frac{1}{N} \partial^i N)(\frac{1}{N} \Delta N) \]

\[ G_7 = (\frac{1}{N} \Delta N)^2 \]  
\[ G_8 = (\frac{1}{N} \partial^i N)(\frac{1}{N} \partial^j N)(\frac{1}{N} \nabla_i \partial_j N) \]

\[ G_9 = (\frac{1}{N} \partial^i N)(\frac{1}{N} \partial_i \Delta N) \]  
\[ G_{10} = \frac{1}{N} \nabla_i \partial_j N \frac{1}{N} \nabla^i \partial^j N \]

\[ G_{11} = \frac{1}{N} \Delta^2 N \]  
\[ G_{12} = \frac{1}{N} \partial^i N \partial_i R \]  

(3.86)
While we have many more possible counterterms than currents, it is important to notice that not all the counterterms are independent, since we can always partially integrate inside the action. This means that some linear combinations of counterterms will have the same Weyl transformation. Furthermore, there can be Weyl invariant combinations of counterterms that do not help in removing total derivatives from the anomaly.

By taking the Weyl variation of the 12 terms $G_b$, it is straightforward to compute the matrix $M$, which is given by:

$$M_{ab} = \begin{pmatrix}
-4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
-4 & -2 & -2 & -4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
0 & 0 & 0 & 2 & -8 & -6 & 0 & -5 & -6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & -8 & 2 & 4 & -2 & 0 & 0 \\
0 & 0 & 0 & -4 & 0 & 4 & 0 & -2 & 4 & -4 & 0 & 0 \\
0 & -2 & -2 & 0 & 0 & 4 & 0 & -4 & 4 & 0 & 2 & 0
\end{pmatrix}$$ (3.87)

It is easily checked that $M$ does not have maximal rank (which would be 6), but it has rank 5. In fact, $M$ has a 7-dimensional space of null vectors, which is spanned by the 6 total derivatives $\nabla_i J^i$ and a Weyl invariant term:

$$\delta \int \sqrt{h} \nabla_i J^i_a = 0, \quad \delta \int N \sqrt{h} \left( R + \frac{1}{N} \Delta N - \frac{1}{N^2} \partial_i N \partial^i N \right)^2 = 0. \quad (3.88)$$

Since the rank of $M$ is 5, the Weyl variation of the most general counterterm spans a 5-dimensional subspace of the 6-dimensional space generated by $c_a \nabla_i J^i_a$. That means that we can find an orthonormal basis (with respect to the usual Euclidean scalar product $\delta_{ab}$) for the currents where 5 are trivial (i.e. removable by counterterms) and 1 is non-trivial. In other words, we look for 5 vectors $e_a$ such that $e_a = M_{ab} q_b$ admits a solution. If we now take $u_a$ to be the null vector of the transpose of $M_{ab}$, it is obviously orthogonal to all the $e_a$ since $u_a e_a = e_a M_{ab} q_b = 0$. We define the non-trivial current $J^i$ to be:

$$J^i = u_a J^i_a = J^i_1 - J^i_2 + J^i_4 + J^i_5 + 2 J^i_6. \quad (3.89)$$

However, we will presently show that this current does not obey the Wess–Zumino consistency condition, therefore it cannot appear in the anomaly.

**Wess–Zumino consistency condition and $J^i$**

The goal of this section is to figure out whether all possible terms that we found above satisfy the Wess–Zumino consistency conditions. To this end, we shall
compute the quantities

\[ \Omega_a \equiv \delta_1 \int \sqrt{h} \omega_2 \nabla_i J_i^a - \delta_2 \int \sqrt{h} \omega_1 \nabla_i J_i^a \] (3.90)

\[ = \int \delta_2 (\sqrt{h} J_i^a) \partial_i \omega_1 - \int \delta_1 (\sqrt{h} J_i^a) \partial_i \omega_2 \] (3.91)

for each \( a = 1, \ldots, 6 \). The main idea of this analysis is to find all possible linear combinations of the \( \Omega \)'s such that

\[ \sum_{a=1}^{6} c_a \Omega_a = 0 \] (3.92)

If the vector space spanned by the vectors \( \{ \vec{c} \} \) is six dimensional, all \( J_i^a \)'s are Wess–Zumino-consistent. If, on the other hand, this vector space is five-dimensional then we must conclude that one of the \( J_i^a \)'s is inconsistent. Since we already know that five currents can be generated by varying appropriate local scale invariant terms, these are manifestly consistent. Therefore the inconsistent current, if present, must be the non-trivial current of equation (3.89).

The way we shall carry out this computation is by first computing the first term in (3.91). The second term in (3.91) is then obtained from the first one by replacing the derivatives that act on \( \omega_1 \) for derivatives that act on \( \omega_2 \) by means of partial integration.

We shall start with \( \Omega_1 \). The first term in (3.91) is

\[ \delta_2 (\sqrt{h} J_1^i) \partial_i \omega_1 = \sqrt{h} (-\partial^i \omega_2 NR - \partial^i \Delta \omega_2 N) \partial_i \omega_1 \] (3.93)

The second term is then

\[ \delta_1 (\sqrt{h} J_1^i) \partial_i \omega_2 = \sqrt{h} (-\partial^i \omega_1 NR - \partial^i \Delta \omega_1 N) \partial_i \omega_2 \] (3.94)

\[ = \sqrt{h} (-\partial^i \omega_2 NR - \nabla^i \nabla^j (\partial_j \omega_2 N)) \partial_i \omega_1 \] (3.95)

\[ = \sqrt{h}(-\partial^i \omega_2 NR - \partial^i \Delta \omega_2 N \]

\[ - \Delta \omega_2 \partial^i N - \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \] (3.96)

so that

\[ \Omega_1 = \int \sqrt{h} \left( \Delta \omega_2 \partial^i N + \partial^i (\partial_j \omega_2 \partial^j N) \right) \partial_i \omega_1 \] (3.98)

Similarly, from \( J_2^i \):

\[ \delta_2 (\sqrt{h} J_2^i) \partial_i \omega_1 = \sqrt{h} (\partial^i \omega_2 NR - \Delta \omega_2 \partial^i N) \partial_i \omega_1 \] (3.99)

\[ \delta_1 (\sqrt{h} J_2^i) \partial_i \omega_2 = \sqrt{h} (\partial^i \omega_1 NR - \Delta \omega_1 \partial^i N) \partial_i \omega_2 \]

\[ = \sqrt{h} (\partial^i \omega_2 NR + \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \] (3.100)

\[ \Omega_2 = -\int \sqrt{h} \left( \Delta \omega_2 \partial^i N + \partial^i (\partial_j \omega_2 \partial^j N) \right) \partial_i \omega_1 \] (3.101)
From $J_3^1$:

\[
\delta_2(\sqrt{h} J^1_3) \partial_i \omega_1 = \sqrt{h} \left( \partial^i \omega_2 \partial_j N \partial^j N + 2 \partial_2 \omega_2 \partial^j N \partial^i N \right) \partial_i \omega_1
\]

(3.102)

\[
\delta_1(\sqrt{h} J^1_3) \partial_i \omega_2 = \sqrt{h} \left( \partial^i \omega_1 \partial_j N \partial^j N + 2 \partial_1 \omega_1 \partial^j N \partial^i N \right) \partial_i \omega_2
\]

(3.103)

\[
\Omega_3 = 0
\]

(3.104)

From $J_4^1$:

\[
\delta_2(\sqrt{h} J^1_4) \partial_i \omega_1 = \sqrt{h} \left( \partial^i \omega_2 \Delta N + 2 \partial_2 \omega_2 \frac{1}{h} \partial^i N \partial^j N + \Delta \omega_2 \partial_i N \right) \partial_i \omega_1
\]

(3.105)

\[
\delta_1(\sqrt{h} J^1_4) \partial_i \omega_2 = \sqrt{h} \left( \partial^i \omega_1 \Delta N + 2 \partial_1 \omega_1 \frac{1}{h} \partial^i N \partial^j N + \Delta \omega_1 \partial_i N \right) \partial_i \omega_2
\]

(3.106)

\[
\Omega_4 = \int \sqrt{h} \left( \Delta \omega_2 \partial_i N + \partial^i \left( \partial_2 \omega_2 \partial^i N \right) \right) \partial_i \omega_1
\]

(3.107)

From $J_5^1$:

\[
\delta_2(\sqrt{h} J^1_5) \partial_i \omega_1 = \sqrt{h} \left( \partial_j \omega_2 \nabla^i \partial^j N + \partial^i \omega_2 \frac{1}{h} \partial^j N \partial_j N + \nabla_j \partial^i N \partial^j N \right) \partial_i \omega_1
\]

(3.108)

\[
\delta_1(\sqrt{h} J^1_5) \partial_i \omega_2 = \sqrt{h} \left( \partial_j \omega_1 \nabla^i \partial^j N + \partial^i \omega_1 \frac{1}{h} \partial^j N \partial_j N + \nabla_j \partial^i N \partial^j N \right) \partial_i \omega_2
\]

(3.109)

\[
\Omega_5 = \int \sqrt{h} \left( \Delta \omega_2 \partial_i N + \partial^i \left( \partial_2 \omega_2 \partial^i N \right) \right) \partial_i \omega_1
\]

(3.110)

From $J_6^1$:

\[
\delta_2(\sqrt{h} J^1_6) \partial_i \omega_1 = \sqrt{h} \left( \partial^i \left( 2 \partial_2 \omega_2 \partial^j N + \Delta \omega_2 N \right) \partial_i \omega_1
\]

(3.111)

\[
\delta_1(\sqrt{h} J^1_6) \partial_i \omega_2 = \sqrt{h} \left( \partial^i \left( 2 \partial_2 \omega_1 \partial^j N + \Delta \omega_1 N \right) \partial_i \omega_2
\]

(3.112)

\[
\Omega_6 = 2 \int \sqrt{h} \left( \Delta \omega_2 \partial_i N + \partial^i \left( \partial_2 \omega_2 \partial^i N \right) \right) \partial_i \omega_1
\]

(3.113)

We thus find that each $\Omega_a$ is a multiple of

\[
\int \sqrt{h} \left( \Delta \omega_2 \partial_i N + \partial^i \left( \partial_2 \omega_2 \partial^i N \right) \right) \partial_i \omega_1,
\]

(3.114)
which means that there is one linear combination that does not satisfy the Wess–Zumino consistency conditions. In other words, all but one of the six $J^i_a$’s can be made consistent. Since we have already found that five of the six $J^i_a$’s can be canceled by variations of local terms, the one that cannot be canceled (which we called $J^i$) must be inconsistent. We can make this more precise by noticing that the consistency equation

$$c_1 - c_2 + c_4 + c_5 + 2c_6 = 0 \quad (3.115)$$

describes a five-dimensional hypersurface of consistent linear combinations $c_a J^i_a$. The set of all such $c_a$-vectors can be defined as those that are orthogonal to the inconsistent vector, $v_a$ say, such that $c_a v_a = 0$. The inconsistent vector is

$$\vec{v} = (1, -1, 0, 1, 1, 2) \quad (3.116)$$

As a consistency check on our computations, notice that this is precisely the five-dimensional hypersurface that we mentioned above, which may be defined as all vectors that are orthogonal to $u_a$ (as defined in (3.89)). Namely, the vector $u_a$ is the same as the inconsistent vector, i.e. $u_a = v_a$. The fact that $J^i$ does not satisfy the Wess–Zumino condition means that it cannot appear as the variation of either local or non-local terms. The fact that there are precisely five total-derivative terms in the anomaly, all of which can be canceled by variations of local terms, was also noted in [45].

### 3.A.2 Role of the massive vector on the field theory side

In this section we explore the conformal invariance of the field theory Lifshitz model from a different perspective. In particular, we will show that a preferred timelike vector $n^\mu$ plays a very similar role to the vector field $A^\mu$ appearing in the bulk.

Our set-up is the following three-dimensional scalar model with critical exponent $z = 2$ [24],

$$S = \int d^2x \, dt \, \mathcal{L} = \frac{1}{2} \int d^2x \, dt \left( \dot{\phi}^2 - (\Delta \phi)^2 \right). \quad (3.117)$$

The operator $\Delta$ is the spatial Laplacian $\Delta = \delta^{ij} \partial_i \partial_j$ and the dot denotes differentiation with respect to (imaginary) time, $\dot{\phi} = \partial_t \phi$. The Noether current density $(J_a)^b$ corresponding to the infinitesimal diffeomorphism $x^a \mapsto x^a + \varepsilon^a$ is given via the usual definition\(^{12}\)

$$\delta_\varepsilon S = \int d^2x \, dt \, (J_a)^b \partial_b \varepsilon^a, \quad (3.118)$$

\(^{12}\) We use the notation $x^t = t$, i.e. the index $a$ runs over $a = t, 1, 2$.  

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3. Lifshitz Anomaly

The current \((J_t)^a\) generates time reparametrizations and \((J_i)^a\) generates the spatial ones; their components are given by

\[
(J_t)^t = -\frac{1}{2} \ddot{\phi}^2 - \frac{1}{2} (\Delta \phi)^2
\]
\[
(J_i)^i = \partial^i \dot{\phi} \Delta \phi - \dot{\phi} \partial^i \Delta \phi
\]
\[
(J_t)^t = -\dot{\phi} \partial_t \phi
\]
\[
(J_i)^j = \delta_j^i \mathcal{L} + \partial_i \partial^j \phi \Delta \phi - \partial_i \phi \partial^j \Delta \phi
\]

where \(\partial^i = \delta^{ij} \partial_j\). One thing we see here is that \((J_t)^t = -E\), where \(E\) is the Hamiltonian/energy density. The conservation law reads

\[
\partial_b (J_a)^b = -\partial_a \phi (\ddot{\phi} + \Delta^2 \phi) \approx 0
\]

The symbol \(\approx\) denotes weak equality, i.e. equality up to terms that vanish on shell. The ‘gauge’ parameter that generates the Lifshitz scaling is \(\epsilon^t = 2 \epsilon t\) and \(\epsilon^i = \epsilon x^i\) (\(\epsilon\) is just a small real number). The condition for scale invariance is

\[
2 (J_t)^t + (J_i)^i = \partial_i (-2 \partial^i \phi \Delta \phi)
\]

whose right-hand side is not zero but a total divergence. The conserved current \(S^a\) associated to scale invariance of the theory is

\[
S^t = 2 t (J_t)^t + x^i (J_i)^t \quad S^i = 2 t (J_i)^i + x^j (J_j)^i
\]

Note that we cannot interpret the \(J\)'s as comprising an energy momentum tensor, since it would be far from being symmetric.

If we couple the Lifshitz model to \(N\) and \(h_{ij}\), we can easily write down the condition for conformal invariance, however the relation between the bulk (with its complete metric and the extra gauge field) and the field theory model is rather obscure. Clearly, the bulk metric does not couple to the energy momentum tensor of the field theory as defined through \(J_t\) and \(J_i\), since that tensor is not even symmetric.

So we will now make a more precise proposal about the relation between the two. We introduce a three-dimensional metric \(g_{\mu \nu}\) and a unit timelike vector \(n_a\) so that \(n_a n^a = -1\). Define the projector \(h_a^b = g_a^b + n_a n^b\), which is orthogonal to \(n^a\), and

\[
\Delta \phi \equiv \partial_a (h_{ab} \partial_b \phi) + \frac{1}{2} h_{ab} h^{cd} \partial_a \phi \partial_b h_{cd}
\]

then we can couple the Lifshitz model to \(h_{ab}\) and \(n_a\) via the covariant action

\[
S = \int d^2 x \sqrt{-g} \left( (n^a \partial_a \phi)^2 - (\Delta \phi)^2 \right).
\]

This action is conformally invariant under

\[
\delta n_a = 2 \omega \ n_a, \quad \delta h_{ab} = 2 \omega \ h_{ab}
\]
This is why it is useful to introduce $h_{ab}$, since the three-dimensional metric itself would transform as $\delta g_{ab} = -4\omega n_an_b + 2\omega h_{ab}$ (using the completeness relation $g_{ab} = -n_an_b + h_{ab}$). Of course, all of this is not very profound. We have merely replaced the spatial metric $h_{ij}$ by the projection of the metric in the plane perpendicular to unit normal $n_a$.

The claim is that (3.127) describes the coupling of the Lifshitz model to a metric and a gauge field in exactly the same way as one would expect from the bulk description.

With this fully covariant action, we can define a symmetric ”stress tensor” by varying it with respect to $g_{ab}$. Due to the presence of $n_a$, this stress-tensor is not conserved though. The precise equation that expresses general covariance of the theory reads

$$2D_b \frac{\delta S}{\delta g_{ab}} = (D^a n_b) \frac{\delta S}{\delta n_b} - D_b \left( n^a \frac{\delta S}{\delta n_a} \right),$$

(3.129)

where on the left hand side we recognize the covariant derivative of the stress-tensor. The background field $n_a$ is the quantity that breaks the general covariance of the theory, which explains why this equation has a right-hand side.

In view of (3.128), to write the conformal anomaly in covariant variables, we also need a variation in terms of $n_a$

$$\sqrt{-g} A = (-4n_an_b + 2h_{ab}) \frac{\delta S}{\delta g_{ab}} + 2n_a \frac{\delta S}{\delta n_a}.$$

(3.130)

This is exactly the same as the bulk equation with $n_a$ playing the role of the asymptotic value of $A_a$. When we choose $n_t = N$, $g_{tt} = -N^2$, $g_{ti} = 0$ and $g_{ij} = h_{ij}$, the conformal anomaly becomes the expression we have been using all along.

### 3.A.3 Alternative computation of $C_2$

In this section we provide an alternative computation of $C_2$. Since the structure multiplying $C_2$ contains $R$, we can take $N = 1$ and assume that $h_{ij}$ does not depend on $t$ but does depend on $x^i$. With these assumptions, the $\omega$ integral separates out, yielding a factor of $\sqrt{\pi}/\epsilon$. What is left is to study the operator $\exp(-\epsilon \nabla^2)$. Now we can roughly think of the standard heat kernel expansion as the Laplace transform of the spectral density. So by taking the inverse Laplace transform we can reconstruct the spectral density. The inverse transform of $\epsilon^a$ is $s^{-1-a}/\Gamma(-a)$. Next, we can integrate this against $\exp(-\epsilon s^2)$ to obtain

$$\frac{\epsilon^{a/2} \Gamma(-a/2)}{2\Gamma(-a)}.$$

(3.131)
This suggests that if the operator $\nabla$ has heat kernel expansion
\[
\sum_{n \geq -1} \epsilon^n L_n
\] 
then $\nabla^2$ has the expansion
\[
\sum_{n \geq -1} \epsilon^{n/2} \frac{\Gamma(-n/2)}{2\Gamma(-n)} L_n.
\]

The term with $n = 1$, which would contribute to the anomaly, vanishes due to the Gamma function. Therefore the coefficient of the $R^2$ term vanishes\(^\text{13}\). We conclude that the coefficient $C_2$ in (3.11) vanishes as well. Notice that while this method is simpler than a direct computation, it is not powerful enough to determine the total derivative terms.

3.A.4 Divergent terms in the heat-kernel expansion

The methods of section 3.2 allow us to compute the divergent part of the heat-kernel expansion for the model considered in equation (3.34). We present the results here for completeness:
\[
K(\epsilon, f, D) \sim \frac{1}{\epsilon} \tilde{a}_0(f, D) + \frac{1}{\sqrt{\epsilon}} \tilde{a}_1(f, D) + O(\epsilon^0),
\]

where
\[
\tilde{a}_0(f, D) = \frac{1}{16\pi} \int dt d^2x N\sqrt{h} f(t, x),
\]
\[
\tilde{a}_1(f, D) = \frac{1}{48\pi^{3/2}} \int dt d^2x N\sqrt{h} f(t, x) \left( R - \frac{1}{N} \Delta N \right).
\]

\(^\text{13}\)See also [52] for a rigorous proof of this statement.