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Socio-dynamic discrete choice: Theory and application

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In this chapter, we apply techniques from the mathematics of dynamical systems and bifurcation theory to re-visit the multinomial logit model with social interactions originally studied by Brock and Durlauf (2002, 2006). Doing so reveals an intuitively logical but previously unnoticed hysteresis regime in midrange parameter space when there are more than two choice alternatives.

In section 4.1, we describe the sociodynamic trinary multinomial logit model for choice between three alternatives as a planar autonomous system of differential equations. In section 4.2, we characterize the stability of solutions using the determinant and trace of the Jacobian matrix of the system, and find values of the parameter specifying the level of social feedback such that the solutions are bifurcation points. In section 4.3, we see that a qualitatively similar alternative system of equations can be expressed as a gradient system, yielding a straightforward visual way of directly determining the stability of the equilibria. In section 4.4, we visualize the analytical results derived in this chapter in terms of the solution trajectory with increasing social feedback, and in terms of a classical bifurcation diagram.

As the techniques in this chapter apply to a planar autonomous system of differential equations, the interested reader may find it especially helpful to *first* review Appendix A where a one parameter scalar autonomous differential equation is considered in the sociodynamic binary logit case. Understanding the analysis for one parameter scalar autonomous system, may namely increase the appreciation and understanding for the approach for the one parameter planar autonomous system of equations. *After* reading this chapter, the interested reader may find it helpful to next review Appendix B where a two parameter scalar autonomous differential equation is considered for the sociodynamic binary logit model with constant bias, and then subsequently Appendix C where a two parameter planar autonomous system of differential equations is considered for the sociodynamic trinary logit model with constant bias.

4.1 PLANAR AUTONOMOUS SYSTEM

In this section, we describe the sociodynamic trinary multinomial logit model for choice between three alternatives as a planar autonomous system of differential equations. For the trinary multinomial logit model we assume a sample of N decision-making agents indexed $(1, \dots, n, \dots, N)$ each faced with a single choice among mutu-

ally exclusive elemental alternatives i in each agent's choice subset C_n of the universal choice set $C = 0, 1, 2$. In general the choice set C_n will vary in size and content across agents: not all elemental alternatives i in the universal choice set may be available to all agents. For example, when considering trinary choice of transportation mode between say, car, bicycle and public transit, some decision-makers might not have a driver's license or access to a car whether as driver or passenger, some agents might not have access to a bicycle, some agents might not have a transit stop that is near their origin or their destination. For the purposes of the theoretical analysis in this chapter however, we will assume that all alternatives are available to all decision-making agents, that is:

$$C_n = C : A_{in} \equiv 1 \quad (4.1)$$

Recalling the formulation of the multinomial logit model in section 2.1, under the assumption of independent and identically Gumbel distributed error terms, the probability $P_n(i|C)$ that the individual decision-making entity n chooses alternative i within the choice set $C = 0, 1, 2$ is then given by:

$$P_n(i|C) = \frac{e^{\mu V_{in}}}{\sum_{j=0}^2 e^{\mu V_{jn}}} \quad (4.2)$$

where μ is a strictly positive scale parameter which we normalize to 1, following standard convention.

$$\mu \equiv 1 \quad (4.3)$$

If we now assume that the only term in the systematic utility is a field effect with utility parameter β real, finite:

$$V_{in} = \beta p_i \quad (4.4)$$

where p_i is the proportion of decision-making agents that have chosen alternative i , then in such a case, when the agents include their own choice with equal weight to others' choices in the calculation of the field effect for a given alternative, the agents' choice behavior is perfectly homogeneous across agents.

$$P(i|C) = \frac{e^{\beta p_i}}{\sum_{j=0}^2 e^{\beta p_j}} \quad (4.5)$$

For a large sample population, the rate of change of the proportions p_0, p_1, p_2 of decision-making agents that have chosen each alternative is given by the probabilities $P(i|C)$ of choosing respectively alternatives 0, 1, 2 among the possible elemental alternatives in the choice set, minus these proportions. This yields a system of three equations in three unknowns, with p_0, p_1, p_2 defined on $[0, 1]$. We refer to (4.6)

as the *sociodynamic trinary logit model*. Given β real, finite, we will be interested to find the steady-state solutions p_0, p_1, p_2 of the system.

$$\begin{aligned}\dot{p}_0 &= \frac{e^{\beta p_0}}{e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2}} - p_0 \\ \dot{p}_1 &= \frac{e^{\beta p_1}}{e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2}} - p_1 \\ \dot{p}_2 &= \frac{e^{\beta p_2}}{e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2}} - p_2\end{aligned}\quad (4.6)$$

$$p_0, p_1, p_2 \in [0, 1]$$

At equilibrium, we have $\dot{p}_i = 0$, thus we see that the proportions p_i of decision-making agents that have chosen alternative i must equal the homogeneous choice probabilities $P(i|C)$. This is the principle of *self-consistency*.

$$\dot{p}_0 = 0: p_0 = \frac{e^{\beta p_0}}{e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2}} \quad (4.7)$$

$$\dot{p}_1 = 0: p_1 = \frac{e^{\beta p_1}}{e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2}} \quad (4.8)$$

$$\dot{p}_2 = 0: p_2 = \frac{e^{\beta p_2}}{e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2}} \quad (4.9)$$

By definition, the proportions and the choice probabilities must both sum to unity. Indeed, a simple check shows that adding (4.7), (4.8), (4.9) gives the desired result:

$$p_0 + p_1 + p_2 = \frac{e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2}}{e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2}} = 1 \quad (4.10)$$

We can thus immediately re-write p_2 in terms of p_0, p_1 to eliminate one equation from the system (4.6)

$$p_2 = 1 - p_0 - p_1 \quad (4.11)$$

Furthermore from the definition of p_i on $[0, 1]$, (4.11) implies $0 \leq p_2 = 1 - p_0 - p_1$, so that

$$p_0 + p_1 \leq 1 \quad (4.12)$$

Substituting (4.11) back into the first two equations of (4.6) at equilibrium then yields the planar autonomous system of equations in p_0, p_1 under condition (4.12):

$$\dot{p}_0 = \frac{e^{\beta p_0}}{e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)}} - p_0 = 0 \quad (4.13)$$

$$\dot{p}_1 = \frac{e^{\beta p_1}}{e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)}} - p_1 = 0 \quad (4.14)$$

Multiplying through by the denominator of (4.5) which is always strictly positive for β real, finite with p_0, p_1 defined on $[0, 1]$, we have a qualitatively similar converted system of equations that are easier to work with:

$$g_0 \equiv e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 e^{\beta p_1} - p_0 e^{\beta(1-p_0-p_1)} = 0 \quad (4.15)$$

$$g_1 \equiv e^{\beta p_1} - p_1 e^{\beta p_0} - p_1 e^{\beta p_1} - p_1 e^{\beta(1-p_0-p_1)} = 0 \quad (4.16)$$

The planar system of equations (4.13) and (4.14), or alternatively (4.15) and (4.16), can be solved conveniently graphically, for example, by plotting the null clines of the surfaces \dot{p}_0 and \dot{p}_1 , or the surfaces g_0 and g_1 , on a graph and finding their intersection. Depending on the value of β , the system may have more than one solution. \diamond

4.1.1 Previous Results Established by Brock and Durlauf

Writing (4.7), (4.8) and (4.9) more compactly, where L is the number of choice alternatives, we have:

$$\dot{p}_i = 0: p_i = \frac{e^{\beta p_i}}{\sum_{j=0}^{L-1} e^{\beta p_j}} \quad (4.17)$$

For the case when $\beta = 0$ where there is no social feedback (and no explanatory variables at all) in the multinomial logit model, it is immediate that there will exist a single unique equilibrium solution such that $p_i = 1/L$ for the multinomial logit null model since:

$$p_i = \frac{e^{0 \cdot p_i}}{\sum_{j=0}^{L-1} e^{0 \cdot p_j}} = \frac{1}{\sum_{j=0}^{L-1} 1} = \frac{1}{L} \quad (4.18)$$

For the trinary multinomial logit model, this unique equilibrium will thus be at $p_i = 1/3$ for $\beta = 0$.

At the other extreme, as $\beta \rightarrow \infty$ when there is infinitely high social feedback in the model, Brock and Durlauf (2006) show that $p_i = 1/L$ is still an equilibrium in the multinomial logit model with social interactions (4.17), but it is no longer the only equilibrium. They demonstrate that in fact any configuration $p_i = 1/k$ for some subset k of the L choices, with other choices having zero probability will be an equilibrium. The number of possible equilibria as $\beta \rightarrow \infty$ is thus:

$$\sum_{k=1}^L \binom{L}{k} = 2^L - 1 \quad (4.19)$$

For the trinary multinomial logit model with social interactions, this implies three equilibria with $p_i = 1$ for each of the three possible alternatives with the other two alternatives with zero probability, three equilibria with $p_i = 1/2$ for each possible pair of two of the three

alternative with the other one alternative with zero probability, plus the one above mentioned equilibrium with $p_i = 1/3$, thus in total seven equilibria for $\beta \rightarrow \infty$.

Moreover Brock and Durlauf (2006) prove that there will exist multiple steady state equilibria in general if $\beta/L > 1$, where L is the number of choice alternatives. This implies $\beta/3 > 1$ for the trinary model, or otherwise said, $\beta > 3$, when the proportions p_0, p_1 are defined on $[0, 1]$. In what follows in this chapter, however we will soon see that multiple steady equilibria *also* occur for a narrow range of values of the utility parameter $\beta < 3$ in the sociodynamic trinary multinomial logit model. The narrow range of parameter space is an interesting hysteresis region. .

4.2 STABILITY ANALYSIS

In this section we proceed to characterize stability of solutions of the sociodynamic trinary multinomial logit model. In particular, we find values of the utility parameter β for the social feedback such that solutions p_0, p_1, p_2 are bifurcation points.

Aoki (1995) and Blume and Durlauf (2003) show that with a large population, the long-run behavior of the binary logit model with social interactions tends to congregate at the stable equilibria of the defining differential equation. By extension, we will thus be interested to characterize the stability of the solutions to (4.13) and (4.14), or alternatively (4.15) and (4.16) to determine where the long run mode shares will settle in our trinary example. In the binary model in Chapter 3 we have seen that the stability of a solution can be determined straightforwardly from evaluation of the derivative at the solution. A solution is locally stable if the derivative is negative, and solution is unstable if the derivative is positive. Cases where the derivative is zero are candidates for bifurcations.

Our planar autonomous system for the trinary multinomial logit model with social interactions is a direct extension. Since the stability type of an equilibrium point is a local property, the stability type of equilibrium points of planar autonomous systems can be determined under certain conditions from the approximation of the vector field $g = (g_0, g_1)$ with its derivative, which is a linear vector field. We can then study the eigenvalues of the Jacobian matrix J of first partial derivatives of the system to characterize the stability.

We first review some standard mathematical terminology and known theorems:

DEFINITION Suppose that $g = (g_0, g_1)$ is a C^1 function. Let the *Jacobian* of g at the point \mathbf{p} be the matrix:

$$\mathbf{J} \equiv Dg(\mathbf{p}) = \begin{bmatrix} \frac{\partial g_0}{\partial p_0}(\mathbf{p}) & \frac{\partial g_0}{\partial p_1}(\mathbf{p}) \\ \frac{\partial g_1}{\partial p_0}(\mathbf{p}) & \frac{\partial g_1}{\partial p_1}(\mathbf{p}) \end{bmatrix} \quad (4.20)$$

THEOREM (Asymptotic stability and instability from linearization)

Let $g = (g_0, g_1)$ be a C^1 function. If all the eigenvalues of the Jacobian matrix $\mathbf{J} \equiv Dg(\bar{\mathbf{p}})$ have negative real parts, then the equilibrium point $\bar{\mathbf{p}}$ of the differential equation $\dot{\mathbf{p}} = g(\mathbf{p})$ is *asymptotically stable*. If at least one of the eigenvalues of the Jacobian matrix $\mathbf{J} \equiv Dg(\bar{\mathbf{p}})$ has positive real part, then the equilibrium point $\bar{\mathbf{p}}$ of the differential equation $\dot{\mathbf{p}} = g(\mathbf{p})$ is *unstable*. (Hale and Koçak, 1996, p. 267, 272)

THEOREM (Flow equivalence near hyperbolic equilibria, *Grobman - Hartman*)

If $\bar{\mathbf{p}}$ is a hyperbolic equilibrium point of $\dot{\mathbf{p}} = g(\mathbf{p})$, that is, if all the eigenvalues of the Jacobian matrix $Dg(\bar{\mathbf{p}})$ have nonzero real parts, then there is a neighborhood of $\bar{\mathbf{p}}$ in which g is topologically equivalent to the linear vector field $\dot{\mathbf{p}} = Dg(\bar{\mathbf{p}})\mathbf{p}$. (Hale and Koçak, 1996, p. 301; Grobman, 1959; Hartman, 1963, 1964;)

An implication of the theorem of Grobman and Hartman is that the stability type of a hyperbolic equilibrium point is preserved under arbitrary but small nonlinear perturbations. To find possible values of the parameter β which lead to bifurcations in behavior, that is, change in number or stability of stationary points, we are thus interested cases when at least one eigenvalue of the Jacobian has zero real part.

The characteristic equation of the Jacobian is:

$$\begin{aligned} 0 &= \det \begin{bmatrix} j_{00} - \lambda & j_{01} \\ j_{10} & j_{11} - \lambda \end{bmatrix} = (j_{00} - \lambda)(j_{11} - \lambda) - j_{01}j_{10} \\ &= \lambda^2 - (j_{00} + j_{11})\lambda + (j_{00}j_{11} - j_{01}j_{10}) \\ &= \lambda^2 - (\text{tr } \mathbf{J})\lambda + (\det \mathbf{J}) \end{aligned} \quad (4.21)$$

Solving for eigenvalues using the quadratic formula we have:

$$\lambda_{1,2} = \frac{1}{2}\text{tr } \mathbf{J} \pm \frac{1}{2}\sqrt{(\text{tr } \mathbf{J})^2 - 4 \det \mathbf{J}} \quad (4.22)$$

Note from the form of (4.22) that there will exist one zero eigenvalue if the determinant of the Jacobian is equal to zero and the trace is nonzero. There will exist zero real part of a pair of complex eigenvalues, ie. purely imaginary eigenvalues, if the determinant is posi-

tive and trace is equal to zero. There will exist double zero eigenvalues if both the determinant and trace are equal to zero.

$$\det \mathbf{J} = 0 : \lambda_{1,2} = \frac{1}{2} \operatorname{tr} \mathbf{J} \pm \frac{1}{2} \sqrt{(\operatorname{tr} \mathbf{J})^2 - 4 \cdot 0} = \frac{1}{2} \operatorname{tr} \mathbf{J} \pm \frac{1}{2} \operatorname{tr} \mathbf{J} \quad (4.23)$$

$$\lambda_1 = 0, \lambda_2 = \operatorname{tr} \mathbf{J}$$

$$\det \mathbf{J} > 0, \operatorname{tr} \mathbf{J} = 0 : \lambda_{1,2} = \frac{1}{2} \cdot 0 \pm \frac{1}{2} \sqrt{(0)^2 - 4 \det \mathbf{J}} = \pm i \sqrt{\det \mathbf{J}} \quad (4.24)$$

4.2.1 Elements of the Jacobian matrix

For the system given by (4.15) and (4.16), the four terms in the Jacobian matrix (4.20) are computed directly as :

$$\frac{\partial g_0}{\partial p_0} = \beta e^{\beta p_0} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} - \beta p_0 e^{\beta p_0} + \beta p_0 e^{\beta(1-p_0-p_1)} \quad (4.25)$$

$$\frac{\partial g_0}{\partial p_1} = -\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta(1-p_0-p_1)} \quad (4.26)$$

$$\frac{\partial g_1}{\partial p_0} = -\beta p_1 e^{\beta p_0} + \beta p_1 e^{\beta(1-p_0-p_1)} \quad (4.27)$$

$$\frac{\partial g_1}{\partial p_1} = \beta e^{\beta p_1} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} - \beta p_1 e^{\beta p_1} + \beta p_1 e^{\beta(1-p_0-p_1)} \quad (4.28)$$

◇

Having computed the elements of the Jacobian matrix, we can now write expressions for the determinant and the trace of the Jacobian matrix for use in (4.23) and in (4.24). We will first consider the case (4.23) where the determinant is equal to zero in subsection 4.2.2. Then in subsection 4.2.3 we will consider the case (4.24) where the trace is equal to zero and the determinant is strictly positive.

4.2.2 Case I: Suppose $\det \mathbf{J} = 0$

Our goal is to find possible values of the utility parameter β such that the solutions to the system of equations (4.15) and (4.16) are bifurcation points. From (4.23) we know that there will exist at least one zero eigenvalue if the determinant of the Jacobian is equal to zero. This leads us to the following theorem which we prove in this subsection.

THEOREM (Characterization of β at bifurcation points of the socio-dynamic trinary multinomial logit model for which at least one

zero eigenvalue occurs) Suppose that individual choices in a large sample population are characterized by the probabilities (4.5) where the only contribution to the systematic utility of choices is a global field effect with utility parameter β real, finite on the proportion p_i of decision-making agents that have chosen alternative i . The equilibrium solutions p_0, p_1, p_2 defined on $[0, 1]$ of the sociodynamic trinary multinomial logit model (4.6) will be candidates for bifurcation points for which at least one zero eigenvalue occurs, if values of the parameter β satisfy:

$$\beta_{+,-} = \frac{(p_0 p_1 + p_0 p_2 + p_1 p_2) \pm \sqrt{(p_0 p_1 + p_0 p_2 + p_1 p_2)^2 - 3 p_0 p_1 p_2}}{3 p_0 p_1 p_2} \quad (4.29)$$

Proof. Let us consider case (4.23)

$$\begin{aligned} 0 = \det \mathbf{J} &= \frac{\partial g_0}{\partial p_0} \frac{\partial g_1}{\partial p_1} - \frac{\partial g_0}{\partial p_1} \frac{\partial g_1}{\partial p_0} = \\ & (\beta e^{\beta p_0} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} - \beta p_0 e^{\beta p_0} + \beta p_0 e^{\beta(1-p_0-p_1)}) \\ & \times (\beta e^{\beta p_1} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} - \beta p_1 e^{\beta p_1} + \beta p_1 e^{\beta(1-p_0-p_1)}) \\ & - (-\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta(1-p_0-p_1)}) (-\beta p_1 e^{\beta p_0} + \beta p_1 e^{\beta(1-p_0-p_1)}) \\ & = [-e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} + \beta(e^{\beta p_0} - p_0 e^{\beta p_0}) + \beta p_0 e^{\beta(1-p_0-p_1)}] \\ & \times [-e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} + \beta(e^{\beta p_1} - p_1 e^{\beta p_1}) + \beta p_1 e^{\beta(1-p_0-p_1)}] \\ & - (-\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta(1-p_0-p_1)}) (-\beta p_1 e^{\beta p_0} + \beta p_1 e^{\beta(1-p_0-p_1)}) \end{aligned} \quad (4.30)$$

Substituting (4.15) and (4.16) into (4.30)

$$\begin{aligned} 0 &= (-e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} + \beta p_0 e^{\beta p_1} + 2\beta p_0 e^{\beta(1-p_0-p_1)}) \\ & \times (-e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} + \beta p_1 e^{\beta p_0} + 2\beta p_1 e^{\beta(1-p_0-p_1)}) \\ & - (-\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta(1-p_0-p_1)}) (-\beta p_1 e^{\beta p_0} + \beta p_1 e^{\beta(1-p_0-p_1)}) \end{aligned} \quad (4.31)$$

Re-grouping terms:

$$\begin{aligned} 0 &= (e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)}) \\ & - (e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})(\beta p_0 e^{\beta p_1} + 2\beta p_0 e^{\beta(1-p_0-p_1)}) \\ & - (e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})(\beta p_1 e^{\beta p_0} + 2\beta p_1 e^{\beta(1-p_0-p_1)}) \\ & + (\beta p_0 e^{\beta p_1} + 2\beta p_0 e^{\beta(1-p_0-p_1)})(\beta p_1 e^{\beta p_0} + 2\beta p_1 e^{\beta(1-p_0-p_1)}) \\ & - (-\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta(1-p_0-p_1)}) (-\beta p_1 e^{\beta p_0} + \beta p_1 e^{\beta(1-p_0-p_1)}) \end{aligned} \quad (4.32)$$

Dividing through by the first term, which is always positive for β real, finite:

$$\begin{aligned}
0 &= 1 - \frac{\beta(p_0 e^{\beta p_1} + 2p_0 e^{\beta(1-p_0-p_1)} + p_1 e^{\beta p_0} + 2p_1 e^{\beta(1-p_0-p_1)})}{(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})} \\
&+ \frac{\beta(p_0 e^{\beta p_1} + 2p_0 e^{\beta(1-p_0-p_1)})}{(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})} \cdot \frac{\beta(p_1 e^{\beta p_0} + 2p_1 e^{\beta(1-p_0-p_1)})}{(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})} \\
&- \frac{\beta(p_0 e^{\beta p_1} - p_0 e^{\beta(1-p_0-p_1)})}{(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})} \cdot \frac{\beta(p_1 e^{\beta p_0} - p_1 e^{\beta(1-p_0-p_1)})}{(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})}
\end{aligned} \tag{4.33}$$

Substituting (4.7), (4.8), (4.9) and (4.11) into (4.33):

$$\begin{aligned}
0 &= 1 - \beta[p_0 p_1 + 2p_0(1-p_0-p_1) + p_1 p_0 + 2p_1(1-p_0-p_1)] \\
&+ \beta[p_0 p_1 + 2p_0(1-p_0-p_1)]\beta[p_1 p_0 + 2p_1(1-p_0-p_1)] \\
&- \beta[p_0 p_1 - p_0(1-p_0-p_1)]\beta[p_1 p_0 - p_1(1-p_0-p_1)]
\end{aligned} \tag{4.34}$$

Multiplying out and re-grouping terms:

$$\begin{aligned}
0 &= 1 - 2\beta[p_0 p_1 + p_0(1-p_0-p_1) + p_1(1-p_0-p_1)] \\
&+ \beta^2[(p_0 p_1)^2 + 2p_0^2 p_1(1-p_0-p_1) \\
&+ 2p_0 p_1^2(1-p_0-p_1) + 4p_0 p_1(1-p_0-p_1)^2] \\
&- \beta^2[(p_0 p_1)^2 - p_0^2 p_1(1-p_0-p_1) \\
&- p_0 p_1^2(1-p_0-p_1) + p_0 p_1(1-p_0-p_1)^2] \\
&= 1 - 2\beta[p_0 p_1 + p_0(1-p_0-p_1) + p_1(1-p_0-p_1)] \\
&+ \beta^2 p_0 p_1 [p_0 p_1 + 2p_0(1-p_0-p_1) \\
&+ 2p_1(1-p_0-p_1) + 4(1-p_0-p_1)^2 \\
&- p_0 p_1 + p_0(1-p_0-p_1) + p_1(1-p_0-p_1) - (1-p_0-p_1)^2] \\
&= 1 - 2\beta[p_0 p_1 + p_0(1-p_0-p_1) + p_1(1-p_0-p_1)] \\
&+ \beta^2 p_0 p_1 [3p_0(1-p_0-p_1) + 3p_1(1-p_0-p_1) + 3(1-p_0-p_1)^2] \\
&= 1 - 2\beta[p_0 p_1 + p_0(1-p_0-p_1) + p_1(1-p_0-p_1)] \\
&+ 3\beta^2 p_0 p_1 (1-p_0-p_1)[p_0 + p_1 + (1-p_0-p_1)] \\
&= 1 - 2\beta[p_0 p_1 + p_0(1-p_0-p_1) + p_1(1-p_0-p_1)] \\
&+ 3\beta^2 p_0 p_1 (1-p_0-p_1)
\end{aligned} \tag{4.35}$$

Substituting (4.11) and re-arranging terms:

$$0 = 3p_0 p_1 p_2 \beta^2 - 2(p_0 p_1 + p_0 p_2 + p_1 p_2)\beta + 1 \tag{4.36}$$

Finally, solving for β using the quadratic formula we have the condition:

$$\beta_{+,-} = \frac{(p_0 p_1 + p_0 p_2 + p_1 p_2) \pm \sqrt{(p_0 p_1 + p_0 p_2 + p_1 p_2)^2 - 3p_0 p_1 p_2}}{3p_0 p_1 p_2} \tag{4.37}$$

Hence, any bifurcation point of the sociodynamic trinary multinomial logit model for which at least one zero eigenvalue occurs must satisfy (4.37). \diamond

Given the *general* characterization of β at bifurcation points of the sociodynamic trinary multinomial logit model for which at least one zero eigenvalue occurs, we now proceed to determine the *specific* values of β for *specific* solutions p_0, p_1, p_2 defined on $[0, 1]$ that satisfy this characterization. We start by making an immediate educated guess given the special 3-way symmetry in the general characterization (4.37). This leads to the lemma directly below. Then afterwards we search systematically using a null clines approach, substituting (4.37) back into the planar autonomous system (4.15) and (4.16).

LEMMA (Bifurcation of the sociodynamic trinary multinomial logit model at $\beta = 3$) The Jacobian matrix of the planar autonomous system (4.15) and (4.16) has a double zero eigenvalue at $\beta = 3$ for the 3-way symmetric equilibrium solution $p_0 = p_1 = p_2 = 1/3$.

Proof. Given the special 3-way symmetry in (4.37), suppose the following symmetric solution:

$$p \equiv p_0 = p_1 = p_2 \quad (4.38)$$

Thus, by (4.10) we have:

$$p = 1/3 \quad (4.39)$$

Substituting (4.38) and (4.39) into (4.37) we have:

$$\begin{aligned} \beta_{+,-} &= \frac{(p^2 + p^2 + p^2) \pm \sqrt{(p^2 + p^2 + p^2)^2 - 3p^3}}{3p^3} \\ &= \frac{3p^2 \pm \sqrt{9p^4 - 3p^3}}{3p^3} = \frac{3p \pm \sqrt{3p(3p-1)}}{3p^2} \quad (4.40) \\ &= \frac{3(1/3) \pm \sqrt{3(1/3)[3(1/3)-1]}}{3(1/3)^2} = \frac{1 \pm \sqrt{1 \cdot 0}}{(1/3)} = 3 \end{aligned}$$

Now we confirm whether (4.38) and (4.40) are indeed a solution of (4.15) and (4.16):

$$\begin{aligned} g_0 &\equiv e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 e^{\beta p_1} - p_0 e^{\beta(1-p_0-p_1)} \\ &= e^{3(1/3)} - (1/3)e^{3(1/3)} - (1/3)e^{3(1/3)} - (1/3)e^{3[1-(1/3)-(1/3)]} \quad (4.41) \\ &= e^1 - 3(1/3)e^1 = e - e = 0 \end{aligned}$$

$$\begin{aligned} g_1 &\equiv e^{\beta p_1} - p_1 e^{\beta p_0} - p_1 e^{\beta p_1} - p_1 e^{\beta(1-p_0-p_1)} \\ &= e^{3(1/3)} - (1/3)e^{3(1/3)} - (1/3)e^{3(1/3)} - (1/3)e^{3[1-(1/3)-(1/3)]} \quad (4.42) \\ &= e^1 - 3(1/3)e^1 = e - e = 0 \end{aligned}$$

Thus we see that the symmetric solution (4.38) with (4.39) has at least one zero eigenvalue when the parameter $\beta = 3$. To determine the nature of the second eigenvalue, following (4.23) we calculate the trace of the Jacobian evaluated at this solution:

$$\begin{aligned}
 \text{trJ} &= \frac{\partial g_0}{\partial p_0} + \frac{\partial g_1}{\partial p_1} \\
 &= \beta e^{\beta p_0} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} - \beta p_0 e^{\beta p_0} + \beta p_0 e^{\beta(1-p_0-p_1)} \\
 &\quad + \beta e^{\beta p_1} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} - \beta p_1 e^{\beta p_1} + \beta p_1 e^{\beta(1-p_0-p_1)} \\
 &= 3e^{3(1/3)} - e^{3(1/3)} - e^{3(1/3)} - e^{3[1-(1/3)-(1/3)]} \\
 &\quad - 3(1/3)e^{3(1/3)} + 3(1/3)e^{3[1-(1/3)-(1/3)]} \\
 &\quad + 3e^{3(1/3)} - e^{3(1/3)} - e^{3(1/3)} - e^{3[1-(1/3)-(1/3)]} \\
 &\quad - 3(1/3)e^{3(1/3)} + 3(1/3)e^{3[1-(1/3)-(1/3)]} \\
 &= 3e - e - e - e - e + e + 3e - e - e - e - e + e = 0
 \end{aligned}
 \tag{4.43}$$

Since both the determinant and the trace are equal to zero, we have a bifurcation point with double zero eigenvalues at the 3-way symmetric solution (4.38) with (4.39) when the utility parameter $\beta = 3$. \diamond

To verify whether or not the 3-way symmetric solution (4.38) is the only solution when (4.23) holds, we substitute (4.37) back into (4.15) and (4.16) and solve graphically, plotting the null clines of the surfaces g_0 and g_1 on a graph and finding their intersection. See Figure 4.1 on page 45.

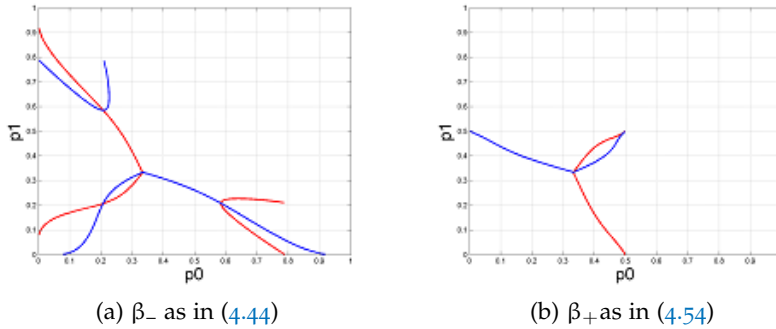


Figure 4.1: Null clines solution of system (4.15) and (4.16) for parameter values of β satisfying (4.37)

First we consider the case with a minus sign in (4.37) as shown in the left panel of Figure 4.1 on page 45. This leads directly to the following lemma.

LEMMA (Bifurcation of the sociodynamic trinary multinomial logit model at $\beta \approx 2.7456$) The Jacobian matrix of the planar autonomous system (4.15) and (4.16) has a single zero eigenvalue and a negative eigenvalue at $\beta \approx 2.7456$ for the three 2-way

symmetric equilibrium solutions $p_i = p_j \approx 0.5848$, $p_k \approx 0.5848$,
 $\forall i, j, k \in 0, 1, 2$.

Proof. For the case with a minus sign in (4.37) we have:

$$\beta_- = \frac{(p_0 p_1 + p_0 p_2 + p_1 p_2) - \sqrt{(p_0 p_1 + p_0 p_2 + p_1 p_2)^2 - 3p_0 p_1 p_2}}{3p_0 p_1 p_2} \quad (4.44)$$

In addition to the expected 3-way symmetric solution (4.38) with double zero eigenvalue, we find there are also three 2-way symmetric solutions in the left panel of Figure 4.1 on page 45 indicating bifurcations:

$$p_0 = p_1 \approx 0.2076, p_2 = 1 - p_0 - p_1 \approx 0.5848 \quad (4.45)$$

$$p_0 \approx 0.2076, p_1 \approx 0.5848, p_2 = 1 - p_0 - p_1 \approx 0.2076 \quad (4.46)$$

$$p_0 \approx 0.5848, p_1 \approx 0.2076, p_2 = 1 - p_0 - p_1 \approx 0.2076 \quad (4.47)$$

Substituting into (4.44), we see by symmetry that the solutions (4.45), (4.46) and (4.47) all yield at least one zero eigenvalue at the same parameter value of β_-

$$\begin{aligned} p_0 p_1 + p_0 p_2 + p_1 p_2 &= (0.2076)(0.2076) + 2(0.2076)(0.5848) = 0.2859 \\ 3p_0 p_1 p_2 &= 3(0.2076)(0.2076)(0.5848) = 0.07561 \\ \beta_- &= \frac{(0.2859) - \sqrt{(0.2859)^2 - 0.07561}}{0.07561} \approx 2.7456 \end{aligned} \quad (4.48)$$

To determine the nature of the other eigenvalues, following (4.23) we calculate the trace of the Jacobian evaluated at solutions. We see by symmetry of the trace that the solutions (4.45), (4.46) and (4.47) all yield

$$\begin{aligned} \text{tr} \mathbf{J} &= \frac{\partial g_0}{\partial p_0} + \frac{\partial g_1}{\partial p_1} \\ &= \beta e^{\beta p_0} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta p_2} - \beta p_0 e^{\beta p_0} + \beta p_0 e^{\beta p_2} \\ &\quad + \beta e^{\beta p_1} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta p_2} - \beta p_1 e^{\beta p_1} + \beta p_1 e^{\beta p_2} \\ &= (\beta - 2 - \beta p_0) e^{\beta p_0} + (\beta - 2 - \beta p_1) e^{\beta p_1} \\ &\quad + (-2 + \beta p_0 + \beta p_1) (e^{\beta p_2}) \\ &= (\beta(1 - p_0) - 2) e^{\beta p_0} + (\beta(1 - p_1) - 2) e^{\beta p_1} + (\beta(1 - p_2) - 2) e^{\beta p_2} \\ &= 2(2.7456 \cdot (1 - 0.2076) - 2) \exp(2.7456 \cdot 0.2076) \\ &\quad + (2.7456 \cdot (1 - 0.5848) - 2) \exp(2.7456 \cdot 0.5848) \\ &= -3.6627 \end{aligned} \quad (4.49)$$

Since the trace is negative, solutions (4.45), (4.46) and (4.47) yield bifurcations with one zero eigenvalue and one negative eigenvalue at the value of the utility parameter β given by (4.48). \diamond

In addition to the above bifurcations, we see there are furthermore three other 2-way symmetric solutions in the left panel of Figure 4.1 on page 45 indicating bifurcations:

$$p_0 = p_1 = 0, p_2 = 1 - p_0 - p_1 = 1 \quad (4.50)$$

$$p_0 = 0, p_1 = 1, p_2 = 1 - p_0 - p_1 = 0 \quad (4.51)$$

$$p_0 = 1, p_1 = 0, p_2 = 1 - p_0 - p_1 = 0 \quad (4.52)$$

Substituting into (4.44), we see by symmetry that the solutions (4.50), (4.51) and (4.52) all occur in the limit when the parameter β_- is undefined.

$$\begin{aligned} p_0 p_1 + p_0 p_2 + p_1 p_2 &= (0)(0) + 2(0)(1) = 0 \\ 3p_0 p_1 p_2 &= 3(0)(0)(1) = 0 \\ \beta_- &= \frac{(0) - \sqrt{(0)^2 - 0}}{0} = \frac{0}{0} \end{aligned} \quad (4.53)$$

Since by definition we are only concerned with values of the utility parameter β real, finite, we discard these solutions.

Next we consider the case as shown in the right panel of Figure 4.1 on page 45. For the case with a plus sign in (4.37) we have:

$$\beta_+ = \frac{(p_0 p_1 + p_0 p_2 + p_1 p_2) + \sqrt{(p_0 p_1 + p_0 p_2 + p_1 p_2)^2 - 3p_0 p_1 p_2}}{3p_0 p_1 p_2} \quad (4.54)$$

In addition to the expected 3-way symmetric solution (4.38) with double zero eigenvalues, we find there are additionally three 2-way symmetric solutions in the right panel of Figure 4.1 on page 45:

$$p_0 = p_1 = 1/2, p_2 = 1 - p_0 - p_1 = 0 \quad (4.55)$$

$$p_0 = 1/2, p_1 = 0, p_2 = 1 - p_0 - p_1 = 1/2 \quad (4.56)$$

$$p_0 = 0, p_1 = 1/2, p_2 = 1 - p_0 - p_1 = 1/2 \quad (4.57)$$

Substituting into (4.54), we see by symmetry that the solutions (4.55), (4.56) and (4.57) all occur in the limit when the parameter $\beta_+ \rightarrow +\infty$.

$$\begin{aligned} p_0 p_1 + p_0 p_2 + p_1 p_2 &= (1/2)(1/2) + 2(1/2)(0) = 1/4 \\ 3p_0 p_1 p_2 &= 3(1/2)(1/2)(0) = 0 \\ \beta_+ &= \frac{(1/4) + \sqrt{(1/4)^2 - 0}}{0} = \frac{1/2}{0} \rightarrow +\infty \end{aligned} \quad (4.58)$$

Since by definition we are only concerned with values of the utility parameter β real, finite, we discard these solutions.

This concludes our search for bifurcations for the case (4.23) where the determinant is equal to zero, leading to at least one zero eigenvalue. In the next subsection we will consider case (4.24) where the trace is equal to zero and the determinant is strictly positive.

4.2.3 Case II: Suppose $\text{tr} J = 0$ and $\det J > 0$

We continue our search for possible values of the utility parameter β such that the solutions to the system of equations (4.15) and (4.16) are bifurcation points. From (4.24), we know there will exist zero real part of a pair of complex eigenvalues, i.e. purely imaginary eigenvalues, if the determinant is positive and trace is equal to zero. In this subsection we will prove that there are no such purely imaginary eigenvalues for the sociodynamic trinary multinomial logit model. This leads us to the following theorem.

THEOREM (No bifurcation points of the sociodynamic trinary multinomial logit model with purely imaginary eigenvalues) Suppose that individual choices in a large sample population are characterized by the homogeneous probabilities (4.5) where the only contribution to the systematic utility of choices is a global field effect with utility parameter β real, finite on the proportion p_i of decision-making agents that have chosen alternative i . The equilibrium solutions p_0, p_1, p_2 defined on $[0, 1]$ of the sociodynamic trinary multinomial logit model (4.6) exhibit no bifurcation points with purely imaginary eigenvalues.

Proof. Let us consider case (4.24)

$$\begin{aligned}
0 = \text{tr} J &= \frac{\partial g_0}{\partial p_0} + \frac{\partial g_1}{\partial p_1} = \beta e^{\beta p_0} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} \\
&\quad - \beta p_0 e^{\beta p_0} + \beta p_0 e^{\beta(1-p_0-p_1)} \\
&\quad + \beta e^{\beta p_1} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta(1-p_0-p_1)} - \beta p_1 e^{\beta p_1} + \beta p_1 e^{\beta(1-p_0-p_1)} \\
&= \beta(e^{\beta p_0} - p_0 e^{\beta p_0} + e^{\beta p_1} - p_1 e^{\beta p_1}) \\
&\quad - 2(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)}) + \beta(p_0 + p_1)e^{\beta(1-p_0-p_1)}
\end{aligned} \tag{4.59}$$

Substituting (4.15) and (4.16) into (4.59):

$$\begin{aligned}
0 &= 2\beta(p_0 + p_1)e^{\beta(1-p_0-p_1)} + \beta(p_0 e^{\beta p_1} + p_1 e^{\beta p_0}) \\
&\quad - 2(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})
\end{aligned} \tag{4.60}$$

Dividing through by the last term in the sum, which is always positive for β real, finite:

$$0 = \beta(p_0 + p_1) \frac{e^{\beta(1-p_0-p_1)}}{(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})} + \frac{\beta}{2} \frac{(p_0 e^{\beta p_1} + p_1 e^{\beta p_0})}{(e^{\beta p_0} + e^{\beta p_1} + e^{\beta(1-p_0-p_1)})} - 1 \quad (4.61)$$

Substituting (4.7), (4.8), (4.9) and (4.11) into (4.61):

$$0 = \beta(p_0 + p_1)(1 - p_0 - p_1) + \frac{\beta}{2}(p_0 p_1 + p_1 p_0) - 1 \quad (4.62)$$

$$= \beta(p_0(1 - p_0 - p_1) + p_1(1 - p_0 - p_1) + p_0 p_1) - 1$$

Solving for β and substituting (4.11):

$$\beta = \frac{1}{p_0(1 - p_0 - p_1) + p_1(1 - p_0 - p_1) + p_0 p_1} \quad (4.63)$$

$$= \frac{1}{p_0 p_2 + p_1 p_2 + p_0 p_1}$$

Given the special 3-way symmetry in (4.63), suppose again the symmetric solution (4.38). Substituting (4.38) and (4.39) into (4.63), we retrieve the same result as in (4.40):

$$\beta = \frac{1}{p^2 + p^2 + p^2} = \frac{1}{3p^2} = \frac{1}{3(1/3)^2} = 3 \quad (4.64)$$

We have already seen from the analysis of (4.23) that the determinant of the Jacobian evaluated at $p_0 = p_1 = p_2 = 1/3$ and $\beta = 3$ is zero, and thus we have double zero eigenvalues.

To verify whether or not (4.38) is the only solution when (4.24) holds, we substitute (4.63) back into (4.15) and (4.16) and solve graphically, plotting the null clines of the surfaces g_0 and g_1 on a graph and finding their intersection. See Figure 4.2 on page 50.

In addition to the 3-way symmetric solution (4.38) where we have double zero eigenvalues, we find there are additionally three 2-way symmetric solutions in Figure 4.2 on page 50 :

$$p_0 = p_1 = 0, \quad p_2 = 1 - p_0 - p_1 = 1 \quad (4.65)$$

$$p_0 = 0, \quad p_1 = 1, \quad p_2 = 1 - p_0 - p_1 = 0 \quad (4.66)$$

$$p_0 = 1, \quad p_1 = 0, \quad p_2 = 1 - p_0 - p_1 = 0 \quad (4.67)$$

Substituting into (4.63), we see by symmetry that the solutions (4.65), (4.66) and (4.67) all occur in the limit when the parameter $\beta \rightarrow +\infty$.

$$\beta = \frac{1}{0 \cdot 0 + 0 \cdot 1 + 0 \cdot 1} = \frac{1}{0} \rightarrow +\infty \quad (4.68)$$

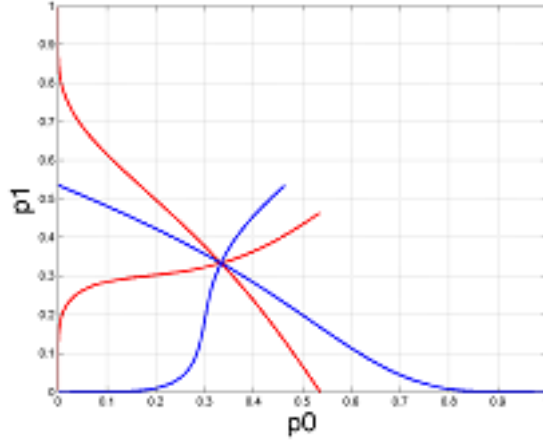


Figure 4.2: Nullclines solution of system (4.15) and (4.16) for parameter values of β satisfying (4.63)

Hence, there are no bifurcation points of the sociodynamic trinary multinomial logit model with utility parameter β real and finite for which purely imaginary eigenvalues occur. \diamond

We conclude that the system given by (4.15) and (4.16) has two bifurcation values of the parameter β real, finite. One bifurcation occurs at $\beta = 3$ when the Jacobian (4.20) has double zero eigenvalues. Another bifurcation occurs near $\beta \approx 2.7456$ when the Jacobian has one zero eigenvalue and one negative, real eigenvalue. The Jacobian of the system given by (4.15) and (4.16) with β real, finite has no purely imaginary eigenvalues. This latter fact guides our search for a possible potential function.

4.3 GRADIENT SYSTEM

In section 4.2 we have studied the eigenvalues of the Jacobian matrix J of first partial derivatives of the system to characterize the stability. However if we can characterize the system as a so-called *gradient system*, there is a more straightforward visual way of directly determining the stability of the equilibria.

We first review some standard mathematical terminology and a known lemma.

DEFINITION If $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^2 function, the *gradient vector field* is

$$-\nabla G(\mathbf{p}) \equiv -\left(\frac{\partial G}{\partial p_0}(\mathbf{p}), \frac{\partial G}{\partial p_1}(\mathbf{p}) \right) \quad (4.69)$$

and the corresponding *gradient system* of differential equations is

$$\dot{\mathbf{p}} = -\nabla G(\mathbf{p}) \quad (4.70)$$

(Hale and Koçak, 1996, p. 433)

DEFINITION Suppose that $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^2 function. Let the *Hessian* of G at the point \mathbf{p} be the matrix of the second partial derivatives:

$$\mathbf{H} \equiv \begin{bmatrix} \frac{\partial^2 G}{\partial p_0^2}(\mathbf{p}) & \frac{\partial^2 G}{\partial p_0 \partial p_1}(\mathbf{p}) \\ \frac{\partial^2 G}{\partial p_1 \partial p_0}(\mathbf{p}) & \frac{\partial^2 G}{\partial p_1^2}(\mathbf{p}) \end{bmatrix} \quad (4.71)$$

DEFINITION A point $\bar{\mathbf{p}}$ is said to be a *critical point* of G if the gradient vector field of G is zero. A critical point $\bar{\mathbf{p}}$ is called *nondegenerate* if the eigenvalues of the Hessian matrix at $\bar{\mathbf{p}}$ are nonzero. (Hale and Koçak, 1996, p. 433)

Note that the Jacobian matrix in the linear variational equation about an equilibrium point of a gradient system (4.70) is the Hessian matrix (4.71) of G evaluated at that point. Furthermore, the eigenvalues of the Hessian matrix are *real* because the Hessian is by definition a symmetric matrix. Consequently, nondegenerate critical points of a gradient system correspond to hyperbolic equilibrium points.

LEMMA (Dynamics from the local geometry of G) An equilibrium point of a gradient system (4.70) is hyperbolic if and only if the corresponding critical point of G is nondegenerate. If $\bar{\mathbf{p}}$ is a hyperbolic equilibrium of (4.70), then: $\bar{\mathbf{p}}$ is an unstable node if and only if G has an isolated maximum at $\bar{\mathbf{p}}$; $\bar{\mathbf{p}}$ is asymptotically stable if and only if G has an isolated minimum at $\bar{\mathbf{p}}$; $\bar{\mathbf{p}}$ is a saddle point if and only if G has a saddle at $\bar{\mathbf{p}}$. (Hale and Koçak, 1996, p. 433)

Solutions of a gradient vector field cross the level sets of the function G orthogonally and inward, except at critical points. Another implication, as a consequence, is that a gradient system cannot have any periodic or homoclinic orbits. Equilibrium points are the only possible limit sets of gradient systems.

From the geometry of G , we can thus infer key aspects of the behavior of a gradient system. The analogy of G in the case of the scalar autonomous differential equation which characterizes the binary logit model is a so-called potential function. Aoki (1995) is indeed able to derive a potential function for the binary logit model with social interactions. Blume and Durlauf (2003) conjecture that their results for binary choice with social interactions should be extendable to choice between multiple alternatives when the underlying population game described by the expected utilities has a “potential” (ie. can be written as gradient system).

THEOREM (The sociodynamic trinary multinomial logit model as a gradient system) Suppose that individual choices in a large sample population are characterized by the homogeneous probabilities (4.5) where the only contribution to the systematic utility

of choices is a global field effect with utility parameter β real, finite on the proportion p_i of decision-making agents that have chosen alternative i . The *qualitative* stability of hyperbolic equilibrium solutions p_0, p_1, p_2 defined on $[0, 1]$ of the sociodynamic trinary multinomial logit model (4.6) can be determined from the isolated maxima, isolated minima and saddles of the function G given by:

$$G = p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 - \frac{\beta}{2}(p_0^2 + p_1^2 + p_2^2) + \tilde{C} \quad (4.72)$$

Proof. Since we are primarily interested in qualitative behavior, for computational convenience we derive an alternative system of equations. Divide (4.7) and (4.8) by (4.9) to obtain:

$$\frac{p_0}{p_2} = \frac{e^{\beta p_0}/(e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2})}{e^{\beta p_2}/(e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2})} = e^{\beta(p_0 - p_2)} \quad (4.73)$$

$$\frac{p_1}{p_2} = \frac{e^{\beta p_1}/(e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2})}{e^{\beta p_2}/(e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2})} = e^{\beta(p_1 - p_2)} \quad (4.74)$$

Taking the natural logarithm of both sides:

$$\ln \frac{p_0}{p_2} = \beta(p_0 - p_2) \quad (4.75)$$

$$\ln \frac{p_1}{p_2} = \beta(p_1 - p_2) \quad (4.76)$$

Re-arranging terms and substituting in (4.11) at equilibrium:

$$\begin{aligned} \check{g}_0 &\equiv -\ln p_0 + \ln p_2 + \beta(p_0 - p_2) \\ &= -\ln p_0 + \ln(1 - p_0 - p_1) + \beta(p_0 - (1 - p_0 - p_1)) = 0 \end{aligned} \quad (4.77)$$

$$\begin{aligned} \check{g}_1 &\equiv -\ln p_1 + \ln p_2 + \beta(p_1 - p_2) \\ &= -\ln p_1 + \ln(1 - p_0 - p_1) + \beta(p_1 - (1 - p_0 - p_1)) = 0 \end{aligned} \quad (4.78)$$

Now try to find potential function G by integrating \check{g}_0 with respect to p_0 and integrating \check{g}_1 with respect to p_1 , then comparing terms.

$$\begin{aligned} \int \check{g}_0 dp_0 &= \int (-\ln p_0 + \ln(1 - p_0 - p_1) + \beta(p_0 - (1 - p_0 - p_1))) dp_0 \\ &= -\int \ln p_0 dp_0 + \int \ln(1 - p_0 - p_1) dp_0 + \beta \int p_0 - (1 - p_0 - p_1) dp_0 \\ &= -(p_0 \ln p_0 - p_0) - (1 - p_0 - p_1) \ln(1 - p_0 - p_1) + (1 - p_0 - p_1) \\ &\quad + \beta \left(\frac{p_0^2}{2} - p_0 + \frac{p_0^2}{2} + p_0 p_1 \right) \\ &= -p_0 \ln p_0 - (1 - p_0 - p_1) \ln(1 - p_0 - p_1) \\ &\quad + \frac{\beta}{2} p_0^2 + \frac{\beta}{2} (1 - 2p_0 + p_0^2 + 2p_0 p_1) + (1 - p_1 - \frac{\beta}{2}) \\ &= -p_0 \ln p_0 - (1 - p_0 - p_1) \ln(1 - p_0 - p_1) \\ &\quad + \frac{\beta}{2} p_0^2 + \frac{\beta}{2} (1 - 2p_0 + p_0^2 + 2p_0 p_1) + C_0 \end{aligned} \quad (4.79)$$

$$\begin{aligned}
 \int \check{g}_1 dp_1 &= \int (-\ln p_1 + \ln(1 - p_0 - p_1) + \beta(p_1 - (1 - p_0 - p_1))) dp_1 \\
 &= -\int \ln p_1 dp_1 + \int \ln(1 - p_0 - p_1) dp_1 + \beta \int p_1 - (1 - p_0 - p_1) dp_1 \\
 &= -(p_1 \ln p_1 - p_1) - (1 - p_0 - p_1) \ln(1 - p_0 - p_1) + (1 - p_0 - p_1) \\
 &\quad + \beta\left(\frac{p_1^2}{2} - p_1 + \frac{p_1^2}{2} + p_0 p_1\right) \\
 &= -p_1 \ln p_1 - (1 - p_0 - p_1) \ln(1 - p_0 - p_1) \\
 &\quad + \frac{\beta}{2} p_1^2 + \frac{\beta}{2} (1 - 2p_1 + p_1^2 + 2p_0 p_1) + (1 - p_1 - \frac{\beta}{2}) \\
 &= -p_1 \ln p_1 - (1 - p_0 - p_1) \ln(1 - p_0 - p_1) \\
 &\quad + \frac{\beta}{2} p_1^2 + \frac{\beta}{2} (1 - 2p_1 + p_1^2 + 2p_0 p_1) + C_1
 \end{aligned} \tag{4.80}$$

In order to satisfy

$$\check{g}_0 = -\frac{\partial G}{\partial p_0}, \quad \check{g}_1 = \frac{\partial G}{\partial p_1} \tag{4.81}$$

we must have

$$G = -\int \check{g}_0 dp_0 = -\int \check{g}_1 dp_1 \tag{4.82}$$

Without loss of generality, the terms in (4.80) that appear exclusively in terms of p_1 can be absorbed into the additive constant term in (4.79). Likewise, the terms in (4.79) that appear exclusively in terms of p_0 can be absorbed into the additive constant term in (4.80). Furthermore, the only mixed terms that contain both p_0 and p_1 appear in both in (4.79) and (4.80). Thus we can write:

$$\begin{aligned}
 G &= -(-p_0 \ln p_0 - p_1 \ln p_1 - (1 - p_0 - p_1) \ln(1 - p_0 - p_1) \\
 &\quad + \frac{\beta}{2} p_0^2 + \frac{\beta}{2} p_1^2 + \frac{\beta}{2} (1 - 2p_0 + p_0^2 - 2p_1 + p_1^2 + 2p_0 p_1)) + \check{C} \\
 &= p_0 \ln p_0 + p_1 \ln p_1 + (1 - p_0 - p_1) \ln(1 - p_0 - p_1) \\
 &\quad - \frac{\beta}{2} (p_0^2 + p_1^2 + (1 - p_0 - p_1)^2) + \check{C}
 \end{aligned} \tag{4.83}$$

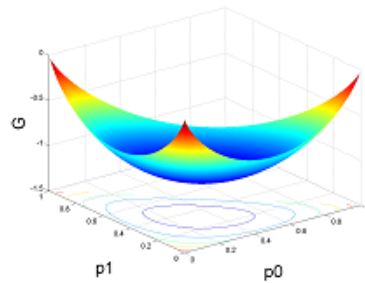
Or, substituting (4.11) to see the symmetry in p_0 , p_1 , and $p_2 = 1 - p_0 - p_1$ more immediately:

$$G = p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 - \frac{\beta}{2} (p_0^2 + p_1^2 + p_2^2) + \check{C} \tag{4.84}$$

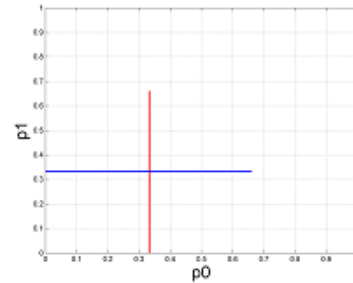
See Figures 4.3 through 4.5 on pages 54-56. \diamond

4.4 BRINGING IT ALL TOGETHER

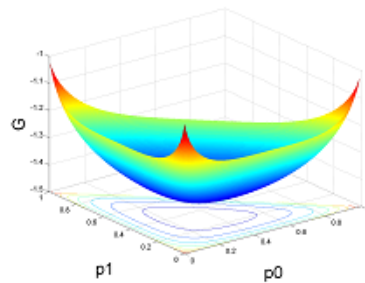
In this section, we visualize the analytical results derived in this chapter in two convenient graphical ways. First, we depict the solution trajectory in the (p_0, p_1) -plane over parameter β for sociodynamic



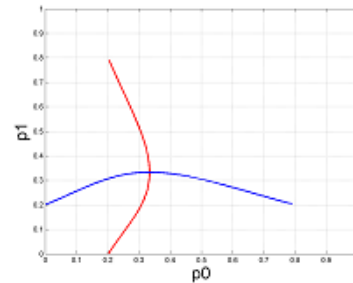
(a) Potential Function
 $\beta = 0$ [one stable equilibrium]



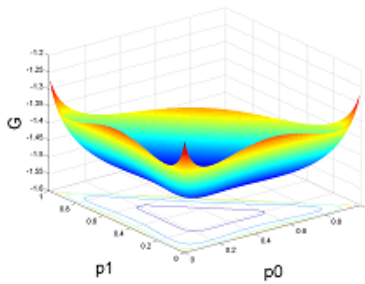
(b) Null Clines Solution of System
 $\beta = 0$ [one stable equilibrium]



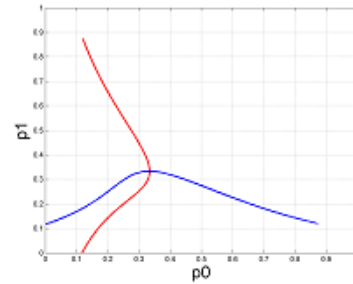
(c) Potential Function
 $\beta = 2$ [one stable equilibrium]



(d) Null Clines Solution of System
 $\beta = 2$ [one stable equilibrium]

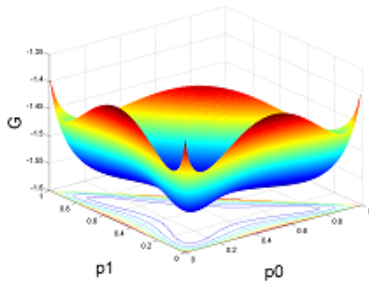


(e) Potential Function
 $\beta = 2.5$ [one stable equilibrium]

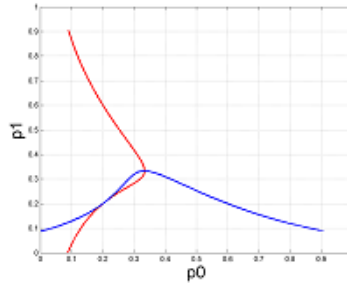


(f) Null Clines Solution of System
 $\beta = 2.5$ [one stable equilibrium]

Figure 4.3: Potential function and null clines solution of the sociodynamic trinary multinomial logit model for a parameter sweep from $\beta = 0$ to $\beta = 10$, showing two bifurcation points at $\beta \approx 2.7456$ and at $\beta = 3$

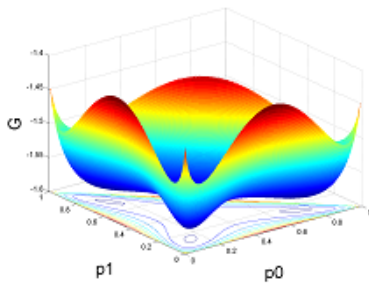


(a) Potential Function

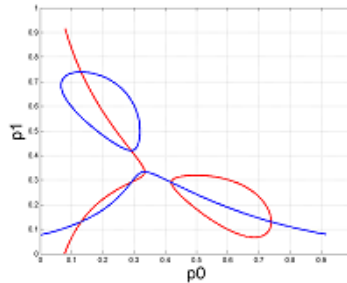


(b) Null Clines Solution of System
 $\beta \approx 2.7456$

[four equilibria: 1 stable nodes, 3 bifurcation points]

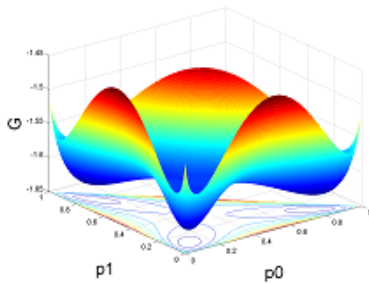


(c) Potential Function

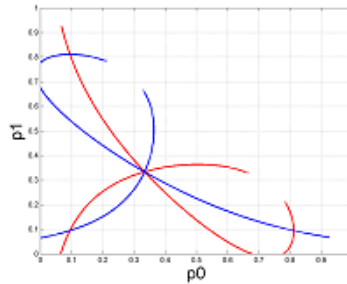


(d) Null Clines Solution of System
 $\beta = 2.85$

[seven equilibria: 4 stable nodes, 3 saddle points]



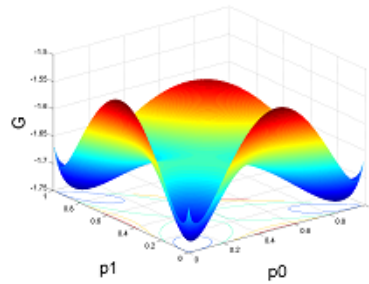
(e) Potential Function



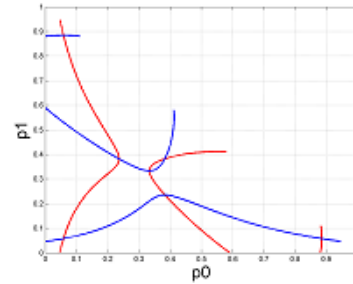
(f) Null Clines Solution of System
 $\beta = 3$

[four equilibria: 1 bifurcation point, 3 stable nodes]

Figure 4.4: Potential function and null clines solution of the sociodynamic trinary multinomial logit model for a parameter sweep from $\beta = 0$ to $\beta = 10$, showing two bifurcation points at $\beta \approx 2.7456$ and at $\beta = 3$, continued



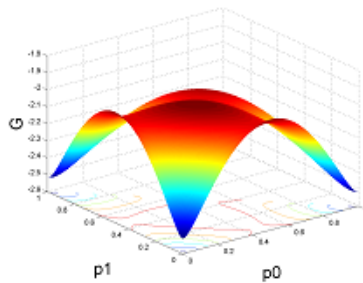
(a) Potential Function



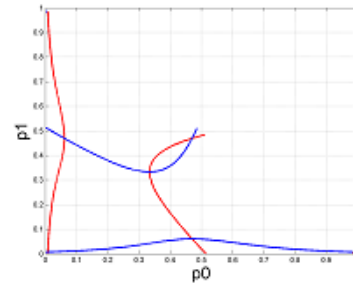
(b) Null Clines Solution of System

$\beta = 3.3$

[seven equilibria: 1 unstable node, 3 stable nodes, 3 saddle points]



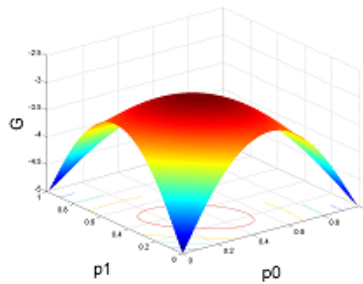
(c) Potential Function



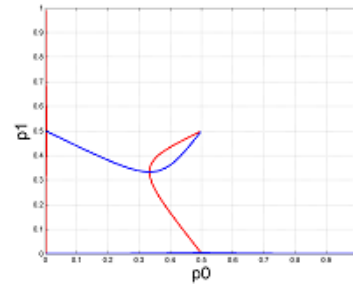
(d) Null Clines Solution of System

$\beta = 5$

[seven equilibria: 1 unstable node, 3 stable nodes, 3 saddle points]



(e) Potential Function



(f) Null Clines Solution of System

$\beta = 10$

[seven equilibria: 1 unstable node, 3 stable nodes, 3 saddle points]

Figure 4.5: Potential function and null clines solution of the sociodynamic trinary multinomial logit model for a parameter sweep from $\beta = 0$ to $\beta = 10$, showing two bifurcation points at $\beta \approx 2.7456$ and at $\beta = 3$, continued

ternary multinomial logit model, indicating the bifurcation points separating three regimes. Second, we depict the bifurcation diagram in the (β, p_0) -plane for sociodynamic ternary multinomial logit model, where we see very clearly the two different types of bifurcations separating the three regimes.

4.4.1 Solution Trajectory

See Figure 4.6 on page 57.

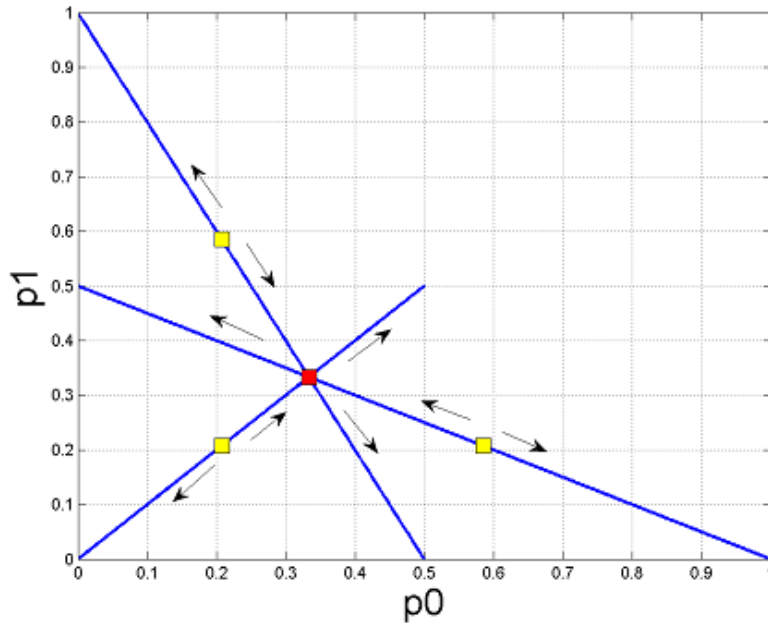


Figure 4.6: Solution trajectory in the (p_0, p_1) -plane over β for sociodynamic ternary multinomial logit model showing bifurcation points separating three regimes. Up until $\beta \approx 2.7456$, there is one stable equilibrium at $(1/3, 1/3)$. At $\beta \approx 2.7456$ we have the sudden appearance of 3 saddle-node bifurcations, marked in yellow. With increasing β the stable nodes then proceed outwards respectively towards $(0, 0)$, $(0, 1)$ and $(1, 0)$, and the saddle points all proceed inwards towards $(1/3, 1/3)$. At $\beta = 3$ we have a transcritical bifurcation, marked in red: the saddle points coalesce at $(1/3, 1/3)$ and flip the directionality of their saddles, and the equilibrium at $p = 1/3$ becomes unstable. As $\beta \rightarrow \infty$ the flipped saddle points then continue their trajectory onwards respectively towards $(1/2, 1/2)$, $(1/2, 0)$ and $(0, 1/2)$.

4.4.2 Bifurcation Diagram

See Figure 4.7 on page 58.

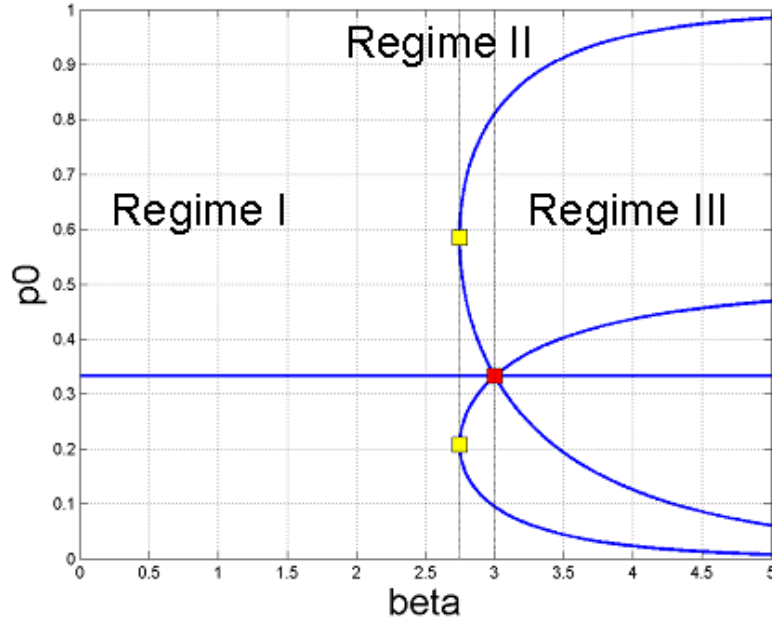


Figure 4.7: Bifurcation diagram for sociodynamic trinary multinomial logit model showing bifurcation points at $\beta \approx 2.7456$ and at $\beta = 3$, separating three regimes. In Regime I, there is one stable node at $p = 1/3$; in Regime II we have the appearance of new pairs of equilibria, each pair consisting of one stable node and one saddle point equilibrium; in Regime III the saddle points have changed directionality and the original node at $p = 1/3$ is unstable.

4.5 CONCLUSIONS AND REFLECTIONS

In this chapter we have derived a planar autonomous system of equations describing the dynamics of the socio-dynamic multinomial logit model. We have shown how this system of equations can be solved conveniently graphically using null clines. We have also derived the stability of the steady-state solutions at equilibrium. Our analysis reveals three qualitatively distinct solution regimes, including an intuitively logical, but previously undetected hysteresis regime in midrange parameter space. In addition, a gradient system is derived which allows us to easily visually characterize the stability of the solutions. Two types of bifurcations are observed which describe the transitions between the regimes. We will use these theoretical results in this chapter as a benchmark for our further exploration of the socio-dynamic multinomial logit model later in Chapter 7. Remarkably, we will find in our case study application in Chapter 7, that the system lies in this interesting newly observed hysteresis regime.

In the next chapter we will incrementally extend the results in this chapter for the case of the socio-dynamic trinary nested logit model. In the nested logit model, there will be two key parameters that char-

acterize the behavior of the planar autonomous system of equations. We will again of course encounter the parameter β describing the importance of the social feedback in the model as in this chapter, but we will also encounter a new parameter describing the scale μ_L of the correlation for the nested alternatives.

For a deeper appreciation of the research presented in the next chapter involving a two parameter bifurcation, the interested reader is highly encouraged at this point to first review Appendix B where a simpler two parameter scalar autonomous differential equation is considered for the sociodynamic binary logit model with constant bias, and then subsequently Appendix C where a two parameter planar autonomous system of differential equations is considered for the sociodynamic trinary logit model with constant bias.