Socio-dynamic discrete choice: Theory and application

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Citation for published version (APA):
SOCIO-DYNAMIC TRINARY NESTED LOGIT: THEORY

By considering the nested logit model, a possibility to account for unobserved heterogeneity between choice alternatives is allowed via the nesting of alternatives that are assumed to be correlated. The analysis of the theoretical equilibrium behavior of the trinary nested logit model with global social interactions in this chapter yields rich bifurcation diagrams demonstrating several major additional new emergent steady-state regimes where symmetry is broken by the scale parameter for the level of correlation between alternatives. These results will be subsequently used as a benchmark for the application in Chapter 7.

First in section 5.1, we describe the trinary nested logit model with social interactions as a two parameter planar autonomous system in the utility parameter $\beta$ for the level of aggregate social influence and in the scale parameter $\mu_L$ for the level of correlation between alternatives. Then in section 5.2, by applying a graphical null clines approach, the number of solutions is counted and charted in the $(\beta, \mu_L)$-plane across a sweep of the utility parameter and a sweep of the scale parameter, paying particular attention to the inherent (broken) symmetries in the trinary nested logit model. This reveals seven qualitatively distinct solution regimes. In section 5.3, a gradient system is derived which allows us to easily visually characterize the stability of the solutions. In section 5.4 we discuss the qualitative nature of each of the seven regimes in terms of the number and stability of solutions. In section 5.5 five different types of bifurcations are discussed which describe the transitions between the regimes: hysteresis, double saddle-node, saddle-node, pitchfork and transcritical. In 5.6 we derive expressions for the defining bifurcation curves in the $(\beta, \mu_L)$-plane rigorously analytically by characterizing the stability of solutions via the eigenvalues of the Jacobian matrix of the system. In order to do this, we draw heavily on our earlier qualitative observations regarding the number of 2-way symmetric steady-state solutions in each regime where the mode shares of the correlated elemental alternatives in the lower level nest are equal. In section 5.7, we visualize the analytical results derived in this chapter in terms of the bifurcation curves in the $(\beta, \mu_L)$-plane, in terms of the solution trajectories in the $(p_0, p_1)$-plane over utility parameter $\beta$ for a sweep of the scale parameter $\mu_L$, and in terms of classical bifurcation diagrams in the $(\beta, p_0)$-plane and in the $(\beta, p_1)$-plane for a sweep of the scale.
parameter $\mu_L$. Finally, section 5.8 suggests a few important directions for elaborations and variations on this work.

As the techniques in this chapter apply to a two parameter planar autonomous system of differential equations, the interested reader may find it especially helpful to first review Appendix B where a two parameter scalar autonomous differential equation is considered for the sociodynamic binary logit model with constant bias, and then subsequently to read Appendix C where a two parameter planar autonomous system of differential equations is considered for the sociodynamic trinary logit model with constant bias. The development of the theory in this chapter is namely parallel with the development of the theory in Appendix C.

In both the sociodynamic nested logit model and the sociodynamic multinomial logit model with constant bias, the additional parameter in the system allows symmetry-breaking. The mathematics however in the case of observed heterogeneity involving constant bias for one of the choice alternatives are significantly simpler than the mathematics in the case of unobserved heterogeneity involving correlation in the error terms between two of the choice alternatives, and as a consequence perhaps easier to follow. Furthermore, understanding the analysis for the two parameter planar autonomous system in the case of the sociodynamic multinomial logit model with constant bias may enrich the appreciation and understanding of the solution regimes and the patterns of bifurcations between them that occur in the sociodynamic trinary nested logit model.

5.1 TRINARY NESTED LOGIT MODEL WITH SOCIAL INTERACTIONS

In this section, we describe the sociodynamic nested logit model for choice between three alternatives as a planar autonomous system of differential equations. In subsection 5.1.1 we write the individual choice probabilities. In subsection 5.1.2 we write the rate of change of the proportions of decision-making agents that have chosen each alternative in a large sample population as a two parameter system of differential equations, yielding the principle of self-consistency at equilibrium. In subsection 5.1.3 by way of reference, we consider the special case of a null trinary nested logit model with no social interactions.
Recall the general definition of the nested logit model presented in Chapter 2. Substituting (2.12) through (2.15) into (2.11) gives

\[ P(i|C_n) = \frac{A_{in} e^{\mu_m V_{in}}}{\sum_{j \in C_{mn}} A_{jn} e^{\mu_m V_{jn}}} \cdot \frac{e^{\vartheta_m + \frac{1}{\mu_m} \ln \sum_{j \in C_{mn}} A_{jn} e^{\mu_m V_{jn}}}}{e^{\vartheta_{m'} + \frac{1}{\mu_{m'}} \ln \sum_{j \in C_{m'n}} A_{jn} e^{\mu_{m'} V_{jn}}}} \]

(5.1)

It is the allowance for the possibility of shared unobserved heterogeneity at the nest level which is the signature of the nested logit model. A key feature of the nested logit model is thus that the symmetry of choice behavior is inherently broken by the assumed correlations among elemental alternatives. The nested logit model will reduce to the multinomial logit model if the scale parameter \( \mu_m \) for the lower nests is equal to the scale parameter \( \mu \) for the upper nest.

\[ \mu_m = \mu \]

\[ P(i|C_n) = \frac{A_{in} e^{\mu V_{in}}}{\sum_{j \in C_{mn}} A_{jn} e^{\mu V_{jn}}} \cdot \frac{e^{\vartheta_m + \ln \sum_{j \in C_{mn}} A_{jn} e^{\mu V_{jn}}}}{e^{\vartheta_{m'} + \ln \sum_{j \in C_{m'n}} A_{jn} e^{\mu V_{jn}}}} \]

(5.2)

Note also that without loss of generality, any shared observable attributes at the nest level may be defined at the elemental alternative level. This is what is done in practice in typical model estimation software packages, such as the freely available optimization toolkit Biogeme developed by Bierlaire (2003).

\[ \vartheta_{mn} \equiv 0 \]

(5.3)

### 5.1.1 Choice Probabilities

In order to be able to derive an analytical benchmark, we will assume that all alternatives are available to all decision-making agents, that is:

\[ C_n = C : A_{in} \equiv 1 \]

(5.4)
and the only term in the systematic utility at the elemental alternative level is a feedback effect with utility parameter $\beta$ real, finite:

$$V_{in} = \beta p_i$$  \hspace{1cm} (5.5)

where $p_i$ is the proportion of decision-making agents that have chosen alternative $i$. In such a case, when the agents include their own choice with equal weight to others’ choices in the calculation of the field effect for a given alternative, the agents’ choice behavior is perfectly homogeneous across agents. If we furthermore normalize the upper level scale parameter to unity, and set the scale parameter $\mu_m$ for the lower nests equal to each other for all nests:

$$\mu \equiv 1; \mu_m \equiv \mu_L, \forall m$$  \hspace{1cm} (5.6)

we have a two-parameter system in terms of $\beta$ and $\mu_L$ real, finite. Note furthermore that the scale parameter $\mu_L$ is constrained to be greater than or equal to unity, since by definition $\mu_L \geq \mu$ and we have set $\mu \equiv 1$.\(^1\)

For the case of trinary choice between three elemental alternatives we will have a system of three equations in three unknowns written in terms of $p_0, p_1, p_2$ defined on $[0, 1]$. To make this explicit, consider a scenario as in Figure 5.1 where elemental alternative 0 is an isolate in its own “nest,” and elemental alternatives 1 and 2, assumed to be correlated, are nested together.

![Diagram](image)

Figure 5.1: Depiction of the nesting structure of a trinary discrete choice model with unobserved heterogeneity that is shared between elemental alternatives 1, 2.

The probability of choosing alternative 0 within nest 0, conditional on having chosen nest 0 is:

$$P(i = 0|\text{nest 0}) = \frac{e^{\mu_m V_{in}}}{\sum_{j \in \text{nest 0}} e^{\mu_m V_{jn}}} = \frac{e^{\mu_L \beta p_0}}{e^{\mu_L \beta p_0}} = 1$$  \hspace{1cm} (5.7)

\(^1\) As noted earlier in section 2, it is not possible to identify both the scale parameter of the upper level nest and the lower level nests; following convention, the upper level nest is normalized to 1.
The inclusive value for nest 0 is then:

\[ I_{\text{nest }0} = \ln \sum_{j \in \text{nest }0} e^{\mu_m V_{jn}} = \ln e^{\mu_L \beta p_0} = \mu_L \beta p_0 \] (5.8)

The probabilities of choosing respectively alternative 1 and 2 within nest 1, conditional on having chosen nest 1, are the binary choice probabilities:

\[ P(i = 1|\text{nest }1) = \frac{e^{\mu_m V_{in}}}{\sum_{j \in \text{nest }1} e^{\mu_m V_{jn}}} = \frac{e^{\mu_L \beta p_1}}{e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2}} \] (5.9)

\[ P(i = 2|\text{nest }1) = \frac{e^{\mu_m V_{in}}}{\sum_{j \in \text{nest }1} e^{\mu_m V_{jn}}} = \frac{e^{\mu_L \beta p_2}}{e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2}} \] (5.10)

The inclusive value for nest 1 is then:

\[ I_{\text{nest }1} = \ln \sum_{j \in \text{nest }1} e^{\mu_m V_{jn}} = \ln(e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2}) \] (5.11)

The probabilities of choosing respectively nests 0 and 1 among the set of nests are:

\[ P(\text{nest }0|C) = \frac{e^{\mu(V_{mn} + \frac{1}{m} I_{mn})}}{\sum_{m' \in M_n} e^{\mu(V_{m'n} + \frac{1}{m'} I_{m'n})}} \]

\[ = \frac{e^{\mu L \beta p_0} (e^{\mu L \beta p_1} + e^{\mu L \beta p_2})}{e^{\mu L \beta p_0} + (e^{\mu L \beta p_1} + e^{\mu L \beta p_2})} \] (5.12)

Thus the probabilities of choosing respectively alternatives 0, 1, 2 among all possible elemental alternatives in the choice set are:

\[ P(i = 0|C) = P(i = 0|\text{nest }0) \cdot P(\text{nest }0|C) \]

\[ = \frac{e^{\mu L \beta p_0}}{e^{\mu L \beta p_0} + (e^{\mu L \beta p_1} + e^{\mu L \beta p_2})} \] (5.14)
At equilibrium, we have \( \dot{p}_i = 0 \) for each alternative \( i \), thus we see that the proportions \( p_i \) of decision-making agents that have chosen alternative \( i \) will equal the choice probabilities \( P(i|C) \), yielding the principle of self-consistency:

\[
P_0 = 0 : \quad p_0 = \frac{e^{\beta p_0}}{e^{\beta p_0} + (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2}) \frac{1}{\pi}}
\]

\[
P_1 = 0 : \quad p_1 = \frac{e^{\mu_1 \beta p_1}}{e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2}} \cdot \frac{e^{\beta p_0} + (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2}) \frac{1}{\pi}}
\]

\[
P_2 = 0 : \quad p_2 = \frac{e^{\mu_1 \beta p_2}}{e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2}} \cdot \frac{e^{\beta p_0} + (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2}) \frac{1}{\pi}}
\]
Furthermore, by definition, the proportions and the choice probabilities must both sum to unity. Indeed, a simple check shows that adding (5.18), (5.19), (5.20), gives the desired result by design.

\[ p_0 + p_1 + p_2 = \frac{e^{\beta p_0}}{e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_1 \beta p_2}) \frac{1}{\mu L}} + \frac{e^{\mu_L \beta p_1} + e^{\mu_1 \beta p_2}}{e^{\mu_L \beta p_1} + e^{\mu_1 \beta p_2}} \cdot \frac{(e^{\mu_L \beta p_1} + e^{\mu_1 \beta p_2}) \frac{1}{\mu L}}{e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_1 \beta p_2}) \frac{1}{\mu L}} = 1 \]  

(5.21)

We can thus immediately re-write \( p_2 \) in terms of \( p_0, p_1 \):

\[ p_2 = 1 - p_0 - p_1 \]  

(5.22)

Furthermore from the definition of \( p_i \) on \([0, 1]\), (5.22) implies \( 0 \leq p_2 = 1 - p_0 - p_1 \), so that

\[ p_0 + p_1 \leq 1 \]  

(5.23)

Substituting (5.22) back into (5.18) and (5.19) at equilibrium then yields a system of two equations in two unknowns \( p_0, p_1 \) under condition (5.23). Given \( \beta \) and \( \mu_L \) real, finite with \( \mu_L \geq 1 \), we want to find solutions \( p_0, p_1 \):

\[ 0 = p_0 = \frac{e^{\beta p_0}}{e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_1 \beta (1-p_0-p_1)}) \frac{1}{\mu L}} - p_0 \]  

(5.24)

\[ 0 = p_1 = \frac{e^{\mu_L \beta p_1}}{e^{\mu_L \beta p_1} + e^{\mu_1 \beta (1-p_0-p_1)}} \times \frac{(e^{\mu_L \beta p_1} + e^{\mu_1 \beta (1-p_0-p_1)}) \frac{1}{\mu L}}{e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_1 \beta (1-p_0-p_1)}) \frac{1}{\mu L}} - p_1 \]  

(5.25)

Finally, multiplying through by the denominator of the first term of (5.24) which is always strictly positive for \( \beta \) and \( \mu_L \) real, finite with \( p_0, p_1 \) defined on \([0, 1]\), we have a converted system of equations which will be analytically easier to work with:

\[ 0 = g_0 \equiv e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 (e^{\mu_L \beta p_1} + e^{\mu_1 \beta (1-p_0-p_1)}) \frac{1}{\mu L} \]  

(5.26)

\[ 0 = g_1 \equiv e^{\mu_L \beta p_1} (e^{\mu_L \beta p_1} + e^{\mu_1 \beta (1-p_0-p_1)}) \frac{1}{\mu L} - p_1 e^{\beta p_0} - p_1 (e^{\mu_L \beta p_1} + e^{\mu_1 \beta (1-p_0-p_1)}) \frac{1}{\mu L} \]  

(5.27)

The planar system of equations (5.24) and (5.25), or alternatively (5.26) and (5.27), can be solved conveniently graphically, for example, by plotting the null clines of the surfaces on a graph and finding their
Figure 5.2: Graphical derivation of the null clines solution of the planar autonomous system \((5.24)\) and \((5.25)\) for parameter values \(\beta = 1, \mu_L = 2\). The top panels show the intersection of the surfaces \(dp_0/dt\) and \(dp_1/dt\) respectively with the null plane. The middle panels show the projection of the surfaces, with their respective null clines. The lower panel shows the intersection of the two null clines in the \((p_0, p_1)\)-plane, indicating one unique solution whereby the mode shares of the correlated elemental alternatives in the lower level nest are equal, \(p_1 = p_2\). We will call this qualitative pattern of solution behavior Regime I.
Taking the natural logarithm of both sides:

\[
0 = g_0 = e^{\beta p_0} - p_0 \left( e^{\beta p_0} + (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2})^{\frac{1}{\mu_L}} \right)
\]  

(5.28)

\[
0 = g_1 = e^{\mu_1 \beta p_1} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2} \right)^{\frac{1}{\mu_L}} - p_1 \left( e^{\beta p_0} + (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2})^{\frac{1}{\mu_L}} \right)
\]  

(5.29)

\]

Lemma (Characterization of solutions of the sociodynamic trinary nested logit model) Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_1 \) of decision-making agents that have chosen alternative \( i \), and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter \( \mu_L \geq 1 \) real, finite. Then there will exist at least one equilibrium solution of the sociodynamic trinary nested logit model (5.17) with \( p_1 = p_2 \) defined on \([0, \frac{1}{2}]\) for all values of \( \beta \) and \( \mu_L \) real, finite. Any other solutions with \( p_1 \neq p_2 \) will be characterized by a transcendental relation between \( p_1 \) and \( p_2 \) that is dependent on \( \beta \) and \( \mu_L \):

\[
\ln \frac{p_1}{p_2} = \mu_L \beta(p_1 - p_2)
\]

(5.30)

Proof. Making use of the symmetry between \( p_1 \) and \( p_2 \), it is convenient to divide (5.19) by (5.20) to obtain:

\[
\frac{p_1}{p_2} = e^{\mu_1 \beta p_1} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2} \right)^{\frac{1}{\mu_L}} / e^{\beta p_0} + (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2})^{\frac{1}{\mu_L}}
\]  

(5.31)

Taking the natural logarithm of both sides:

\[
\ln \frac{p_1}{p_2} = \mu_L \beta(p_1 - p_2)
\]

(5.32)

From (5.32) it is immediate that there will exist at least one solution with \( p_1 = p_2 \) for all values of \( \beta \) and \( \mu_L \) real, finite. In such case, from the definition of \( p_1 \) on \([0, 1]\), (5.22) implies \( 0 \leq p_0 = 1 - p_1 - p_2 = 1 - 2p_1 \), so that we have also

\[
p_1 \leq \frac{1}{2}
\]

(5.33)
Any other solutions with $p_1 \neq p_2$ will be characterized by a transcendental relation between $p_1$ and $p_2$ that is dependent on $\beta$ and $\mu_L$, thus proving the lemma. ♦

**Lemma (Limiting solutions of the sociodynamic trinary nested logit model)** Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter $\beta$ on the proportion $p_1$ of decision-making agents that have chosen alternative $i$, and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter $\mu_L \geq 1$. Then any solutions of the sociodynamic trinary nested logit model (5.17) with $p_1 \to 0$ and $p_2 > 0$, or with $p_2 \to 0$ and $p_1 > 0$, or with $p_1 \to 1$ or $p_2 \to 0$ implies either $\beta \to +\infty$ or $\mu_L \to +\infty$ or both.

**Proof.** From (5.32) we can derive convenient relations for $\beta$ and $\mu_L$ in terms of the mode shares $p_1$, $p_2$ and respectively $\mu_L$ and $\beta$ for a solution with $p_1 \neq p_2$:

\[
\beta = \frac{\ln (p_1/p_2)}{\mu_L (p_1 - p_2)}
\]

(5.34)

\[
\mu_L = \frac{\ln (p_1/p_2)}{\beta (p_1 - p_2)}
\]

(5.35)

We see thus that a solution with $p_1 \to 0$ implies either $\beta \to +\infty$ or $\mu_L \to +\infty$ or both for $p_2 > 0$. Likewise a solution with $p_2 \to 0$ implies either $\beta \to +\infty$ or $\mu_L \to +\infty$ or both for $p_1 > 0$. Furthermore by (5.22) for mode shares $p_0$, $p_1$, $p_2$ defined on $[0, 1]$, a solution with $p_1 \to 1$ implies $p_2 \to 0$, and vice versa, a solution with $p_2 \to 1$ implies $p_1 \to 0$, and thus we require again $\beta \to +\infty$ or $\mu_L \to +\infty$ or both. In summary, theoretically there can exist no true corner solutions with $p_1 = 1$ or $p_2 = 1$ as well as no true boundary solutions in general with $p_1 = 0$ or $p_2 = 0$ and $p_1 \neq p_2$ for finite values of $\beta$ and $\mu_L$. Such solutions if they exist, can only exist when at least one or both of the parameters $\beta$ and $\mu_L$ goes to infinity. In practice however, for $\beta$ and/or $\mu_L$ “sufficiently large” we may see the approximation of a “corner” solution where $p_1 \approx 1$ or $p_2 \approx 1$ or a “boundary” solution $p_1 \approx 0$ or $p_2 \approx 0$ and $p_1 \neq p_2$. ♦

5.1.3 *Special Case: $\beta = 0$ (Null Model with No Social Interactions)*

**Lemma (Characterization of solutions the null trinary nested logit model with no social interactions)** Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) under the special condition $\beta = 0$, so that there is null contribution to the systematic utility of choices, and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter
\( \mu_L \geq 1 \) real, finite. Then there will exist only one unique equilibrium solution of this null trinary nested logit model (5.17) with \( \beta = 0 \). This solution will be characterized by \( p_1 = p_2 \) defined on \([1/4, 1/3]\). The scale parameter \( \mu_L \geq 1 \) will be given by:

\[
\mu_L = \frac{\ln 2}{\ln p_1 - \ln \left( \frac{1}{2} - p_1 \right)} \tag{5.36}
\]

**Proof.** Consider the following special case:

\( \beta = 0 \) \( \tag{5.37} \)

Substituting (5.37) into (5.26) and (5.27), we have the simplification:

\[
g_0|_{\beta=0} = e^0 - p_0 e^0 - p_0 (e^0 + e^0)^{\frac{1}{\mu_L}} = 1 - p_0 - p_0 \cdot 2^{\frac{1}{\mu_L}} = 0 \tag{5.38}
\]

\[
g_1|_{\beta=0} = e^0 (e^0 + e^0)^{\frac{1-\mu_L}{\mu_L}} - p_1 e^0 - p_1 (e^0 + e^0)^{\frac{1}{\mu_L}} = 2^{\frac{1-\mu_L}{\mu_L}} - p_1 - p_1 \cdot 2^{\frac{1}{\mu_L}} = 0 \tag{5.39}
\]

Solving (5.38) and (5.39) respectively for \( p_0 \) and \( p_1 \):

\[
p_0 = \frac{1}{1 + 2^{\frac{1}{\mu_L}}} \tag{5.40}
\]

\[
p_1 = \frac{2^{\frac{1-\mu_L}{\mu_L}}}{1 + 2^{\frac{1}{\mu_L}}} = \frac{1}{2} \cdot \frac{2^{\frac{1}{\mu_L}}}{1 + 2^{\frac{1}{\mu_L}}} \tag{5.41}
\]

Substituting into (5.22):

\[
p_2 = 1 - p_0 - p_1 = 1 - \frac{1}{1 + 2^{\frac{1}{\mu_L}}} - \frac{1}{2} \cdot \frac{2^{\frac{1}{\mu_L}}}{1 + 2^{\frac{1}{\mu_L}}} \tag{5.42}
\]

\[
= \frac{1}{2} \cdot \frac{2^{\frac{1}{\mu_L}}}{1 + 2^{\frac{1}{\mu_L}}}
\]

Thus comparing (5.41) and (5.42), we have

\[
p_1 = p_2 \tag{5.43}
\]

This result is not surprising since the only thing that we have specified in this trinary nested logit null model with no social interactions is an assumed correlation between choice alternatives 1 and 2. It follows that there is then nothing to distinguish between choice alternatives 1 and 2 and these two alternative must be chosen with an equal likelihood, \( p_1 = p_2 \). The actual value of these choice probabilities and how different they are from \( p_0 \) will be dependent on the extent of the correlation as given by the scale \( \mu_L \) of Gumbel distributed error terms within the nest relative to the scale \( \mu \equiv 1 \) of choosing the nest.
Solving \((5.40)\) or \((5.41)\) for \(\mu_L\):

\[
\ln \frac{1 - p_0}{p_0} = \frac{1}{\mu_L} \ln 2 \quad \Rightarrow \quad \mu_L = \frac{\ln 2}{\ln(1 - p_0) - \ln p_0} \quad (5.44)
\]

\[
\ln \left(\frac{1}{2} - p_1\right) + \frac{1}{\mu_L} \ln 2 = \ln p_1 \quad \Rightarrow \quad \mu_L = \frac{\ln 2}{\ln p_1 - \ln \left(\frac{1}{2} - p_1\right)} \quad (5.45)
\]

For the specific case \(\mu_L = 1\) when \(\beta = 0\), we have by \((5.40)\) and \((5.41)\):

\[
p_1 = p_2 = \frac{1}{2}, \quad \frac{2^1}{1 + 2^1} = \frac{1}{3} \quad (5.46)
\]

and thus we regain the result established by Brock and Durlauf (2006) in \((4.18)\) in subsection 4.1.1. This is intuitive since when there is nothing in the model to influence choices, neither in the systematic utility for choice alternatives nor via presumed correlations in the error structure, the outcome of decision-making can only be purely random. When there are three choice alternative, each will be chosen with equal probability \(1/3\).

For the case \(\mu_L \to +\infty\) when \(\beta = 0\), similarly we have:

\[
p_1 = p_2 = \frac{1}{2}, \quad \frac{2^0}{1 + 2^0} = \frac{1}{4} \quad (5.47)
\]

and

\[
p_0 = 1 - p_1 - p_2 = \frac{1}{2}
\]

This result too is intuitive. When the choice alternatives 1 and 2 are infinitely correlated and there is nothing else in the systematic utility to distinguish between alternatives, the choice problem becomes a random binary problem at the upper level of choosing between alternative 0, the isolate in its own nest on one hand, and alternatives 1 and 2 nested together on the other hand. These two options will be chosen with equal probability: that is, probability \(1/2\) for alternative 0, and probability \(1/2\) for alternatives 1 and 2 together. Then since there is nothing to distinguish between alternatives 1 and 2, these will be chosen individually with probability \(1/4\).

In conclusion, it is apparent that any finite value of \(\mu_L > 1\) with \(\beta = 0\) will yield a unique solution to our system according to \((5.40)\) and \((5.41)\) that lies on the line with \(p_1 = p_2\) between the two solutions \((5.46)\) and \((5.47)\).

5.2 \textit{Equilibrium regimes in }\((\beta, \mu_L)\)-parameter space

Following the approach indicated in Figure 5.2 on page 68, we now proceed to explore the nature of the solutions of \((5.24)\) and \((5.25)\). We repeat the null cline procedure many times over a sweep of values
of $\beta$ and values of $\mu_L$, counting the number of equilibria each time and identifying qualitatively equivalent patterns of the null clines in terms of how they cross. Figure 5.3 on page 74 and Figure 5.4 on page 75 display a summary of this exploration which capture the major equilibrium regimes. Each cell in Figures 5.3 and 5.4 corresponds to a null cline portrait. The number indicated in the cell gives the number of solutions.

See Figure 5.6 on page 77 and Figure 5.7 on page 78.

Figure 5.3 on page 74 shows a sweep of values of $\beta$ from 0 to 8 and values of $\mu_L$ from 1 to 10, showing six solution regimes. Figure 5.4 on page 75 zooms in on the range of $\beta$ from 2.6 to 2.8 and values of $\mu_L$ from 1 to 1.1, yielding an additional solution regime seen only at finer resolution, lying in between Regime I and Regime II, except at the multinomial case. All of the individual null clines portraits in Regime I are qualitatively similar to the null cline shown in Figure 5.2 on page 68. Figure 5.5 on page 76 shows representative null clines for the rest of the regimes in Figure 5.3 and Figure 5.4. The coloring of the cells in Figures 5.3 and 5.4 is such that yellow indicates one unique solution, orange indicates three solutions, green indicates five solutions and purple indicates seven solutions. It is important to note however that even though Regime II, Regime III and Regime VI all have seven solutions, the patterns of the how the null clines cross are structurally different from each other in each of these cases as can be seen in the representative null clines shown in Figure 5.5. Likewise, even though Regime V in Figure 5.3 and Regime VII (only visible at finer resolution) in Figure 5.4 both have three solutions, the patterns of how the null clines cross are again structurally different from each other as can be seen in Figure 5.5.

The intention of the coloring is to guide the eye in noticing the patterns of the transitions between the regimes in $(\beta, \mu_L)$-parameter space. Such a transition between regimes is called a “bifurcation.” At a bifurcation point in $(\beta, \mu_L)$-parameter space, there is a structural change in the pattern of the number and/or stability of equilibria. We will study the stability of the equilibria later in this chapter, however the way that the null clines cross already provides clues.

The cell at $\beta = 3$ and $\mu_L = 1$ is a special point with four solutions, indicated in red in Figure 5.3. It is a bifurcation point existing only at this specific set of values of $\beta = 3$ and $\mu_L = 1$. We will come back to this special bifurcation point again when we characterize the types of bifurcations, after we have studied the stability of the solutions. It is sufficient to note at this stage that this point separates Regime II from Regime III, both with seven solutions but with qualitatively different null clines patterns.

Before proceeding to study stability however, it is useful to further appreciate some qualitative findings thus far in order to better interpret our stability results later. We expect from the characterization
Figure 5.3: Computationally-derived bifurcation curves in the $(\beta, \mu_L)$-plane showing major regimes indicated with number of solutions of the system at various points in parameter space. In Regime I there is one solution in Regimes II and III, there are seven solutions in Regime IV and VII and there are three solutions in Regime V. In the framed region for small $\mu_L$ at finer resolution, see Figure 5.4 on page 75. The first row with $\mu_L = 1$ corresponds to the multinomial logit case in Chapter 4.
5.2 Equilibrium Regimes in $(\beta, \mu_L)$-Parameter Space

Figure 5.4: Detail of Figure 5.3 on page 74 showing computationally-derived bifurcation curves at an interesting region in parameter space near to the bifurcation to several different regimes, where empirical results in the case study in Chapter 7 occur ($\beta = 2.7595$, $\mu_L = 1.0339$). Zooming in on this region, we see that the actual estimated coefficients fall in a region with five steady-state solutions. The values however are within one standard error of regions in parameter space with one steady-state solution, three steady-state solutions and seven steady-state solutions.

of solutions in (5.32) that there will exist at least one solution with $p_1 = p_2$ for all values of $\beta$ and $\mu_L$ real, finite. We find this in indeed true. For the trinary nested logit null model with no social interactions ($\beta = 0$) we expect to see only one unique solution as shown in subsection 5.1.3. This is indeed true as well. We see furthermore that Regime I characterized by one unique solution extends from $\beta = 0$ over a range of values $\beta > 0$ dependent on $\mu_L$.

Now, let us reflect on the multinomial case where $\mu_L = 1$. As shown in (4.18) in subsection 4.1.1, we expect to find a single unique equilibrium at $p_1 = 1/3$ for $\beta = 0$ and $\mu_L = 1$, and indeed we do. The pattern of a unique solution continues at $\mu_L = 1$ for values of $\beta$ up until about $\beta = 2.75$ as can be seen in Figure 5.3 and Figure 5.4. For $\mu_L = 1$ and $\beta \to \infty$, we expect to find seven solutions as in (4.19). In practice for $\mu_L = 1$ and $\beta = 8$, the values of the solutions are already approaching the limiting solutions expected from (5.34) and (5.35). The qualitative pattern of having precisely seven solutions with the null clines crossing in this way, albeit with different solutions values of $p_L$, continues for at $\mu_L = 1$ for values of $\beta$ down until $\beta = 3$. Something special happens in the midrange between about $\beta = 2.75$ and $\beta = 3$ when $\mu_L = 1$.

When we look closely at Figure 5.4 near the point where Regime I transitions to Regime II, we can see with increasing scale $\mu_L$ the appearance of two different bifurcation curves seeming to come together at this point. On the one hand there is a transition from Regime I with one unique solution to Regime VII with three solutions thus changing the number of solutions by two. Let us call this Curve A.
Figure 5.5: Example null clines solutions of the system (5.24) and (5.25) in the major regimes in parameter space shown in Figure 5.3 on page 74 and Figure 5.4 on page 75 which display multiplicity of equilibria, indicated with the number of 2-way symmetric steady-state solutions where the mode shares of the correlated elemental alternatives in the lower level nest are equal, so that we have $p_1 = p_2 = 1 - p_0 - p_1$; the 2-way symmetric solutions thus lie on the line $p_1 = (1 - p_0)/2$: (a) seven equilibria, of which three with $p_1 = p_2$; (b) seven equilibria, of which three with $p_1 = p_2$; (c) five equilibria, of which one with $p_1 = p_2$; (d) three equilibria, of which one with $p_1 = p_2$; (e) seven equilibria, of which one with $p_1 = p_2$; and (f) three equilibria, of which three with $p_1 = p_2$ (inset shows lines do not cross in the region near $p_0 = 0.21$, $p_1 = 0.195$).
5.2 Equilibrium Regimes in $(\beta, \mu_L)$-Parameter Space

Figure 5.6: Graphical derivation of the null clines solution of the planar autonomous system (5.24) and (5.25) in the major regimes in parameter space which display multiplicity of equilibria. Compare with panels (a) through (c) in Figure 5.5 on page 76.
(a) $\hat{p}_0$ surface

$\mu_L = 8; \beta = 1$: Regime V

(b) $\hat{p}_1$ surface

$\mu_L = 8; \beta = 2.2$: Regime VI

(c) $\hat{p}_0$ surface

$\mu_L = 1.025; \beta = 2.72$: Regime VII

Figure 5.7: Graphical derivation of the null clines solution of the planar autonomous system (5.24) and (5.25) in the major regimes in parameter space which display multiplicity of equilibria, continued from Figure 5.6. Compare with panels (d) through (f) in Figure 5.5 on page 76.
On the other hand there is a transition from Regime VII with three solutions to Regime II with seven solutions thus changing the number of solutions by four. Let us call this Curve B. The first Curve A with the change of two solutions, appears to reach a sharp point with increasing $\mu_L$ and decreasing $\beta$ and then hook back again now with decreasing $\mu_L$ and increasing $\beta$ separating Regime II with seven solutions and Regime IV with five solutions, retaining the change in two solutions, until finally reaching the special bifurcation point at $\beta = 3$ and $\mu_L = 1$ as seen in Figure 5.3. The Curve B with the change of four solutions appears to simply keep on going with increasing $\mu_L$ and decreasing $\beta$, to separate Regime I with one solution from Regime IV with five solutions, retaining the change in four solutions, and onwards in Figure 5.3 to separate Regime V with three solutions from Regime VI with seven solutions, again retaining the change in four solutions, appearing to level off at $\beta = 2$ with increasing $\mu_L$.

Returning back to our special bifurcation at $\beta = 3$ and $\mu_L = 1$, this appears to be the endpoint of another bifurcation curve as well, separating Regime IV from Regime III with increasing $\mu_L$ and increasing $\beta$, appearing eventually level off near $\beta = 6$. Let us call this Curve C. As with the earlier mentioned Curve A separating Regime II from Regime IV, this Curve C also is characterized by a change in two solutions, but the bifurcation is qualitatively different since the patterns of the null clines in Regime II and Regime III are qualitatively different.

Finally for $\mu_L > 3$ we can observe a curve separating Regime I with one solution from Regime V with three solutions, thus changing the number of solutions by two. Let us call this Curve D. The Curve D continues around to separate Regime IV with five solutions from Regime VI with seven solutions, retaining the change in two solutions.

With these insights in mind, we now proceed to investigate the stability of the solutions.

### 5.3 Gradient System

Previously in section 4.3 we have seen that if we can characterize a planar autonomous system as a gradient system, there is a straightforward visual way of directly determining the stability of the equilibria from the geometry of the "potential function" surface. In this section we will try to express the qualitative behavior of the sociodynamic ternary nested logit model as a gradient system. In subsection 5.3.1 we will derive an alternative system of equations with the same qualitative solution behavior for analytical convenience. In subsection 5.3.2 we will derive the expression of the "potential function", proving the central theorem of this section. In subsection 5.3.3 we will visualize and discuss the three competing terms that comprise the "potential function" for subsequent reference in section 5.4 in our characterization of the various equilibrium regimes in $(\beta, \mu_L)$-parameter space.
Consider the gradient system of differential equations:

\[
p = -\nabla G(p) \equiv - \left( \frac{\partial G}{\partial p_0}(p), \frac{\partial G}{\partial p_1}(p) \right)
\]  

(5.48)

Let the Hessian of \( G \) at the point \( p \) be the matrix of the second partial derivatives:

\[
H \equiv \begin{bmatrix}
\frac{\partial^2 G}{\partial p_0^2}(p) & \frac{\partial^2 G}{\partial p_0 \partial p_1}(p) \\
\frac{\partial^2 G}{\partial p_0 \partial p_1}(p) & \frac{\partial^2 G}{\partial p_1^2}(p)
\end{bmatrix}
\]  

(5.49)

Recall from section 4.3 that an equilibrium point \( \bar{p} \) of a gradient system (5.48) is hyperbolic if and only if the eigenvalues of the Hessian matrix at \( \bar{p} \) are nonzero. If \( \bar{p} \) is a hyperbolic equilibrium of (5.48), then:

- \( \bar{p} \) is an unstable node if and only if \( G \) has an isolated maximum at \( \bar{p} \);
- \( \bar{p} \) is asymptotically stable if and only if \( G \) has an isolated minimum at \( \bar{p} \);
- \( \bar{p} \) is a saddle point if and only if \( G \) has a saddle at \( \bar{p} \).

5.3.1 An Alternative System

Since we are primarily interested in qualitative behavior, for computational convenience we derive an alternative system of equations. Making use of the symmetry between \( p_1 \) and \( p_2 \), divide (5.19) and (5.20) respectively by (5.18) to obtain:

\[
\frac{p_1}{p_0} = \frac{e^{\mu_1 \beta p_1} (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2})^{1 - \frac{1}{\mu_1}}}{e^{\beta p_0} (e^{\beta p_0} + (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2})^{1 - \frac{1}{\mu_1}})}
\]

(5.50)

\[
\frac{p_2}{p_0} = \frac{e^{\mu_1 \beta p_2} (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2})^{1 - \frac{1}{\mu_1}}}{e^{\beta p_0} (e^{\beta p_0} + (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2})^{1 - \frac{1}{\mu_1}})}
\]

(5.51)

Taking the natural logarithm of both sides:

\[
\ln \frac{p_1}{p_0} = \beta (\mu_1 p_1 - p_0) + \frac{1 - \mu_1}{\mu_1} \ln (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2})
\]

(5.52)

\[
\ln \frac{p_2}{p_0} = \beta (\mu_1 p_2 - p_0) + \frac{1 - \mu_1}{\mu_1} \ln (e^{\mu_1 \beta p_1} + e^{\mu_1 \beta p_2})
\]

(5.53)
Re-arranging terms and substituting in (5.22) at equilibrium:

\[
0 = \tilde{g}_1 \equiv \ln p_0 - \ln p_1 + \beta (\mu_L p_1 - p_0) + \frac{1 - \mu_L}{\mu_L} \ln (e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2}) \\
= \ln(1 - p_1 - p_2) - \ln p_1 + \beta (\mu_L p_1 - (1 - p_1 - p_2)) \\
+ \frac{1 - \mu_L}{\mu_L} \ln (e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2})
\]

(5.54)

\[
0 = \tilde{g}_2 \equiv \ln p_0 - \ln p_2 + \beta (\mu_L p_2 - p_0) + \frac{1 - \mu_L}{\mu_L} \ln (e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2}) \\
= \ln(1 - p_1 - p_2) - \ln p_2 + \beta (\mu_L p_2 - (1 - p_1 - p_2)) \\
+ \frac{1 - \mu_L}{\mu_L} \ln (e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2}) = 0
\]

(5.55)

The 3-way symmetry between \( p_0, p_1 \) and \( p_2 \) is in general broken by the term \( \beta \mu_L p_1 \) in (5.54), the term \( \beta \mu_L p_2 \) in (5.55) and the final term in both equations.

If we consider the case \( \mu_L = 1 \), the final term in both (5.54) and (5.55) drops out, and the symmetry is restored in the terms \( \beta \mu_L p_1 \) and \( \beta \mu_L p_2 \) so that we recover an analogous system as we have in Chapter 4.

\[
\tilde{g}_1|_{\mu_L = 1} \equiv \ln p_0 - \ln p_1 + \beta (p_1 - p_0) \\
= \ln(1 - p_1 - p_2) - \ln p_1 + \beta (p_1 - (1 - p_1 - p_2)) = 0
\]

(5.56)

\[
\tilde{g}_2|_{\mu_L = 1} \equiv \ln p_0 - \ln p_2 + \beta (p_2 - p_0) \\
= \ln(1 - p_1 - p_2) - \ln p_2 + \beta (p_2 - (1 - p_1 - p_2)) = 0
\]

(5.57)

For \( \beta = 0 \), we have the special case:

\[
\tilde{g}_1|_{\beta = 0} = \ln(1 - p_1 - p_2) - \ln p_1 + 0 + \frac{1 - \mu_L}{\mu_L} \ln (e^0 + e^0) \\
= \ln(1 - p_1 - p_2) - \ln p_1 + \frac{1 - \mu_L}{\mu_L} \ln 2 = 0
\]

(5.58)

\[
\tilde{g}_2|_{\beta = 0} = \ln(1 - p_1 - p_2) - \ln p_2 + 0 + \frac{1 - \mu_L}{\mu_L} \ln (e^0 + e^0) \\
= \ln(1 - p_1 - p_2) - \ln p_2 + \frac{1 - \mu_L}{\mu_L} \ln 2 = 0
\]

(5.59)

Setting (5.58) equal to (5.59), we require:

\[
p_1 = p_2
\]

(5.60)

Thus substituting (5.60) back into (5.58) or into (5.59) gives the relation:

\[
0 = \ln(1 - 2p_1) - \ln p_1 + \frac{1 - \mu_L}{\mu_L} \ln 2 = \ln \left( \frac{(1 - 2p_1)}{p_1} \cdot 2 \frac{1 - \mu_L}{\mu_L} \right)
\]

(5.61)
Exponentiating both sides of (5.61):

\[
1 = \left(\frac{1 - 2p_1}{p_1}\right)^{\frac{1}{1 - \mu_L}} \cdot 2^{\frac{1 - \mu_L}{1 - \mu_L}}
\]  

(5.62)

Solving (5.62) for \(p_1\):

\[
p_1 = (1 - 2p_1)^{\frac{1 - \mu_L}{1 - \mu_L}} = 2^{\frac{1 - \mu_L}{1 - \mu_L}} - 2^{\frac{1}{1 - \mu_L}}p_1
\]

(5.63)

Or alternatively, solving (5.61) for \(\mu_L\):

\[
\ln \frac{p_1}{1 - 2p_1} = \frac{1 - \mu_L}{\mu_L} \ln 2 = \frac{1}{\mu_L} \ln 2 - \ln 2
\]

\[
\ln \frac{p_1}{1 - p_1} = \frac{1}{\mu_L} \ln 2
\]

\[
\mu_L = \frac{\ln 2}{\ln p_1 - \ln(\frac{1}{2} - p_1)}
\]

(5.64)

Thus we recover the relation (5.36) for the scale parameter \(\mu_L\) in our lemma in subsection 5.1.3.

For the case \(\mu_L \to +\infty\) when \(\beta = 0\), we likewise recover the limiting solution (5.47) in subsection 5.1.3:

\[
p_1 = p_2 = \frac{1}{2} \cdot \frac{2^0}{1 + 2^0} = \frac{1}{4}
\]

\[
p_0 = 1 - p_1 - p_2 = \frac{1}{2}
\]

(5.65)

5.3.2 Expression of the Gradient System

Theorem (The sociodynamic trinary nested logit model as a gradient system) Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter \(\beta\) real, finite on the proportion \(p_i\) of decision-making agents that have chosen alternative \(i\), and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter \(\mu_L \geq 1\) real, finite. The qualitative stability of hyperbolic equilibrium solutions \(p_0, p_1, p_2\) defined on \([0, 1]\) of the sociodynamic trinary nested logit model (5.17) can be determined from the isolated maxima, isolated minima and saddles of the function \(G\) given by:
\[ G = p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 - \frac{\beta}{2}(p_0^2 + p_1^2 + p_2^2) - \frac{1 - \mu_L}{\mu_L} \ln((p_1 + p_2) \ln(p_1 + p_2) - p_1 \ln p_1 - p_2 \ln p_2) + \tilde{C} \]  

(5.66)

**Proof.** Let us try to write the planar autonomous system (5.54) and (5.55) as a gradient system:

\[ -\nabla G = -\left( \frac{\partial G}{\partial p_1}, \frac{\partial G}{\partial p_2} \right) = 0 \]  

(5.67)

In order to satisfy

\[ \dot{g}_1 = -\frac{\partial G}{\partial p_1}, \quad \dot{g}_2 = -\frac{\partial G}{\partial p_2} \]  

(5.68)

we must have

\[ G = -\int \dot{g}_1 dp_1 = -\int \dot{g}_2 dp_2 \]  

(5.69)

Try to find potential function \( G \) by integrating \( \dot{g}_1 \) with respect to \( p_1 \) and integrating \( \dot{g}_2 \) with respect to \( p_2 \), then comparing terms. First, integrating \( \dot{g}_1 \) with respect to \( p_1 \):

\[ -\int \dot{g}_1 dp_1 = \int \ln p_1 dp_1 - \int \ln(1 - p_1 - p_2) dp_1 - \beta \int (\mu_L p_1 - (1 - p_1 - p_2)) dp_1 \]

(5.70)

We evaluate the first two terms in (5.70) integrating by parts:

\[ \int \ln p_1 dp_1 - \int \ln(1 - p_1 - p_2) dp_1 = (p_1 \ln p_1 - p_1) + (1 - p_1 - p_2) \ln(1 - p_1 - p_2) - (1 - p_1 - p_2) = p_1 \ln p_1 + (1 - p_1 - p_2) \ln(1 - p_1 - p_2) - (1 - p_2) \]

(5.71)

We evaluate the third term in (5.70) similarly as follows:

\[ -\beta \int (\mu_L p_1 - (1 - p_1 - p_2)) dp_1 = -\beta(\frac{\mu_L p_1^2}{2} - p_1 + \frac{p_1^2}{2} + p_1 p_2) \]

\[ = -\frac{\mu_L \beta}{2} p_1^2 - \frac{\beta}{2} (2p_1 + p_1^2 + 2p_1 p_2) \]

\[ = -\frac{\mu_L \beta}{2} p_1^2 - \frac{\beta}{2} (1 - 2p_1 + p_1^2 + 2p_1 p_2 - 2p_2 + p_2^2) \]

\[ + \frac{\beta}{2} (1 - 2p_2 + p_2^2) \]

\[ = -\frac{\mu_L \beta}{2} p_1^2 - \frac{\beta}{2} (1 - p_1 - p_2)^2 + \frac{\beta}{2} (1 - 2p_2 + p_2^2) \]

(5.72)
We evaluate the fourth term in (5.70) by applying (5.31)

\[- \frac{1-\mu_L}{\mu_L} \int \ln(e^{\mu_L \beta} p_1 + e^{\mu_L \beta} p_2) dp_1 \]

\[- \frac{1-\mu_L}{\mu_L} \int \ln \left( (e^{\mu_L \beta} p_1) (1 + e^{\mu_L \beta} (p_2 - p_1)) \right) dp_1 \]

\[- \frac{1-\mu_L}{\mu_L} \int \ln e^{\mu_L \beta} p_1 + \ln (e^{\mu_L \beta} (p_2 - p_1) + 1) dp_1 \]

\[- \frac{1-\mu_L}{\mu_L} \int \mu_L \beta p_1 + \ln (e^{\mu_L \beta} (p_2 - p_1) + 1) dp_1 \]

\[- \frac{(1-\mu_L)}{2} \beta p_1^2 \]

\[- \frac{1-\mu_L}{\mu_L} \int \ln(e^{\mu_L \beta} (p_1 - p_2) + 1) dp_1 \]

\[- \frac{(1-\mu_L)}{2} \beta p_1^2 \]

\[- \frac{1-\mu_L}{\mu_L} \int \ln(p_2 + p_1) - \ln p_1 dp_1 \]

\[- \frac{(1-\mu_L)}{2} \beta p_1^2 \]

\[- \frac{1-\mu_L}{\mu_L} \int \ln(p_2 + p_1) - (p_2 + p_1) - p_1 \ln p_1 + p_1 \]

(5.73)

Finally, substituting (5.71), (5.72) and (5.73) back into (5.70), and absorbing terms exclusively in \(p_2\) into an additive constant:

\[- \int \hat{g}_1 dp_1 = p_1 \ln p_1 + (1 - p_1 - p_2) \ln(1 - p_1 - p_2) - (1 - p_2) \]

\[- \frac{\mu_L \beta}{2} p_1^2 - \frac{\beta}{2} (1 - p_1 - p_2)^2 + \frac{\beta}{2} (1 - 2p_2 + p_2^2) - \frac{(1-\mu_L)}{2} \beta p_1^2 \]

\[- \frac{(p_2 + p_1) \ln(p_2 + p_1) - (p_2 + p_1) - p_1 \ln p_1 + p_1) \]

\[= p_1 \ln p_1 + (1 - p_1 - p_2) \ln(1 - p_1 - p_2) - \frac{\beta}{2} p_1^2 - \frac{\beta}{2} (1 - p_1 - p_2)^2 \]

\[- \frac{1-\mu_L}{\mu_L} \int \ln(p_2 + p_1) - p_1 \ln p_1 + C_1 \]

(5.74)

Similarly, integrating \(\hat{g}_2\) with respect to \(p_2\),

\[- \int \hat{g}_2 dp_2 = \int \ln p_2 dp_2 - \int \ln(1 - p_1 - p_2) dp_2 \]

\[- \beta \int (\mu_L p_2 - (1 - p_1 - p_2)) dp_1 \]

(5.75)

\[- \frac{1-\mu_L}{\mu_L} \int \ln(e^{\mu_L \beta} p_1 + e^{\mu_L \beta} p_2) dp_2 \]
We evaluate (5.75) following similar steps as (5.71), (5.72) and (5.73), and absorb terms exclusively in \( p_1 \) into an additive constant:

\[
- \int \tilde{g}_2 dp_2 = p_2 \ln p_2 + (1 - p_1 - p_2) \ln(1 - p_1 - p_2) - (1 - p_1)
- \frac{\mu L p_2^2}{2} - \frac{\beta}{2} (1 - p_1 - p_2)^2 + \frac{\beta}{2} (1 - 2p_1 + p_1^2) - \frac{(1 - \mu L)}{2} \beta p_2^2
- \frac{1 - \mu L}{\mu L} ((p_1 + p_2) \ln(p_1 + p_2) - (p_1 + p_2) - p_2 \ln p_2 + p_2)
= p_2 \ln p_2 + (1 - p_1 - p_2) \ln(1 - p_1 - p_2) - \frac{\beta}{2} p_2^2 - \frac{\beta}{2} (1 - p_1 - p_2)^2
- \frac{1 - \mu L}{\mu L} ((p_1 + p_2) \ln(p_1 + p_2) - p_1 \ln p_1 - p_2 \ln p_2) + C_2
\]

(5.76)

Without loss of generality, the terms in (5.76) that appear exclusively in terms of \( p_2 \) can be absorbed into the additive constant term in (5.74). Likewise, the terms in (5.74) that appear exclusively in terms of \( p_1 \) can be absorbed into the additive constant term in (5.76). Furthermore, the mixed terms that contain both \( p_1 \) and \( p_2 \) appear in both in (5.74) and (5.76). Thus we can write:

\[
G = p_1 \ln p_1 + p_2 \ln p_2 + (1 - p_1 - p_2) \ln(1 - p_1 - p_2)
- \frac{\beta}{2} p_1^2 - \frac{\beta}{2} p_2^2 - \frac{\beta}{2} (1 - p_1 - p_2)^2
- \frac{1 - \mu L}{\mu L} (p_1 + p_2) \ln(p_1 + p_2) - p_1 \ln p_1 - p_2 \ln p_2 + \tilde{C}
\]

(5.77)

See Figure 5.10 on page 90. Or, substituting (5.22) to see the role of \( \mu L \) in symmetry-breaking between \( p_1, p_2 \) and \( p_0 = 1 - p_1 - p_2 \) more immediately:

\[
G = p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 - \frac{\beta}{2} (p_0^2 + p_1^2 + p_2^2)
- \frac{1 - \mu L}{\mu L} (p_1 + p_2) \ln(p_1 + p_2) - p_1 \ln p_1 - p_2 \ln p_2 + \tilde{C}
\]

(5.78)

Comparing with results with (4.72) in Chapter 4, we see that the potential function has the same form as the multinomial logit case, with the addition now of the final term which is equal to zero when \( \mu L = 1 \).

\[
\therefore
\]

5.3.3 Understanding the Competing Terms

To understand (5.78) better, notice that this function can be written neatly as the sum of three parts, where the first part is expressed purely in terms of functions of \( p_1, p_2 \) and \( p_0 \) and is not parametrized, the second part is parametrized linearly by \( \beta/2 \) and the third part is parameterized by a function of \( \mu L \).
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\[ G = G(1) + \frac{\beta}{2} G(2) - \frac{1 - \mu_L}{\mu_L} G(3) + \tilde{C} \]

\[ G(1) \equiv p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 \]
\[ G(2) \equiv -(p_0^2 + p_1^2 + p_2^2) \]
\[ G(3) = (p_1 + p_2) \ln(p_1 + p_2) - p_1 \ln p_1 - p_2 \ln p_2 \]

The first three logarithmic terms in (5.78) together form a bowl-shaped surface with a single unique minimum point at the bottom of the basin. See the upper panel of Figure 5.8 on page 87. The following term in \( \beta \) on the other hand forms an upside-down bowl for \( \beta > 0 \), since the sign in front of \( \beta \) is negative. See the middle panel of Figure 5.8. The term in \( \beta \) has three corner minima. We can expect thus that depending on the magnitude of \( \beta \), there will be interesting counteracting tendencies between the influence from the upward bowl surface and the influence from the downward bowl surface with \( \beta > 0 \) when added together. The presence of this downward bowl in the sum in (5.78) will give a possibility to generate a multiplicity of equilibria if \( \beta > 0 \) is large enough to counter the tendency from the bowl from the first three logarithmic terms. When \( \beta < 0 \), however, the unique minimum of the bowl from the first three terms will only be further accentuated, since this second bowl would then also be upward facing. Both bowl-shaped surfaces are however symmetric in \( p_0, p_1 \) and \( p_2 \).

The 3-way symmetry between \( p_0, p_1 \) and \( p_2 \) is in general broken by the term in \( \mu_L \) in (5.78). This term forms a half cone-shaped surface lying on its side with the tip of the half cone at \( p_0 = 1 \). See the lower panel of Figure 5.8. Note that this cone is two-way symmetric across the line \( p_1 = (1 - p_0)/2 \). The minima of this surface are given by the lines \( p_1 = 0 \) and \( p_2 = 1 - p_0 - p_1 = 0 \). The symmetry of the surface in \( p_1 = p_2 \) reflects the presumed correlation between alternatives 1 and 2. If we consider the case \( \mu_L = 1 \) as in the multinomial logit model, the term in \( \mu_L \) in (5.78) indeed drops out, and the 3-way symmetry is restored:

\[ G \big|_{\mu_L=1} = p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 - \frac{\beta}{2}(p_0^2 + p_1^2 + p_2^2) + \tilde{C} \]
\[ = G(1) + \frac{\beta}{2} G(2) + \tilde{C} \]

For \( \beta = 0 \), the quadratic terms \( p_0^2, p_1^2 \) and \( p_2^2 \) drop out of (5.78) leaving only the logarithmic terms
Figure 5.8: Competing terms in the “potential” function $G$ (5.79) for the sociodynamic trinary nested logit model: (a) Upward bowl with one unique minimum; (b) Downward bowl with three corner minima; (c) Half cone with minima along the sides $p_1 = 0$, $p_2 = 1 - p_0 - p_1 = 0$. 

(a) $G(1) = p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2$

(b) $G(2) = -(p_0^2 + p_1^2 + p_2^2)$

(c) $G(3) = (p_1 + p_2) \ln(p_1 + p_2) - p_1 \ln p_1 - p_2 \ln p_2$
In the special case $\beta = 0$ and $\mu_L = 1$, where there is social interaction and no correlation between alternatives, we are left with only the first three terms of (5.78), yielding the symmetric upward bowl with a unique minimum at $p_0 = p_1 = p_2 = 1/3$.

$$G |_{\beta=0, \mu_L=1} = p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 + \tilde{C} = G(1) + \tilde{C} \quad (5.82)$$

For $\beta$ very large, the quadratic terms $p_0^2$, $p_1^2$ and $p_2^2$ dominate, yielding effectively the downward bowl with three corner minima: $p_0 = 1$, $p_1 = 1$, $p_2 = 1$.

$$\lim_{\beta \to \infty} G \approx -\frac{\beta}{2}(p_0^2 + p_1^2 + p_2^2) = \frac{\beta}{2}G(2) + \tilde{C} \quad (5.83)$$

The dominance of the term in $\beta$ in (5.78) for $\beta \to \infty$ is thus irrespective of the value of the scale parameter $\mu_L$. This is because the parametrization of the term is linear in $\beta/2$, thus this term can become infinitely large. The parametrization of the term in $\mu_L$ in (5.78) however is such that this term approaches $G(3)$ as $\mu_L \to \infty$ since

$$\lim_{\mu_L \to \infty} \left(-\frac{1-\mu_L}{\mu_L}\right) \approx 1 \quad (5.84)$$

With these limiting cases in mind, let us now revisit the regimes in $(\beta, \mu_L)$-parameter space identified in Figure 5.3 on page 74 and Figure 5.4 on page 75.

### 5.4 Equilibrium Regimes in $(\beta, \mu_L)$-Parameter Space, Revisited

In this section we will characterize the stability of equilibria in each of the regimes identified in Figure 5.3 on page 74 and Figure 5.4 on page 75. To make this explicit we will briefly consider again in turn the parameter values of $\beta$, $\mu_L$ as given in each of the representative null clines portraits in Figure 5.2 on page 68 and Figure 5.5 on page 76.
Regime I. In Regime I, we expect to find one equilibrium as shown in Figure 5.2. From our preliminary analysis we expect this equilibrium to be stable. Indeed on the surface in Figure 5.9 on page 89, there is one isolated minimum at the bottom of the basin. This result is intuitive because at the corner values $\beta = 0$, $\mu_L = 1$, we have seen already that we will have the bowl-shaped surface in the upper panel of Figure 5.8 given by the term $G(1)$. In the region in $(\beta, \mu_L)$-parameter space near this point it is plausible that the contributions of the term $G(3)$ will move the position of the bottom of the basin away from the three-way symmetric solution $p_0 = p_1 = p_2 = 1/3$, but the contributions of $G(2)$ and $G(3)$ will not be sufficient to change the number and/or stability of equilibria. Furthermore from the preliminary analysis we expect there to be only one unique equilibrium all along the axis $\beta = 0$. The delineation of Regime I in Figure 5.3 thus makes sense.

![Figure 5.9: Example “potential” function $G$ as given by (5.66) in Regime I of the $(\beta, \mu_L)$-parameter space shown in Figure 5.3 on page 74, indicating one unique solution which is a stable node whereby the mode shares of the correlated elemental alternatives in the lower level nest are equal $p_1 = p_2$. Compare with the null clines solution of the system in Figure 5.2 on page 68.](image)

Regime II. In Regime II we expect to find seven equilibria as shown in the upper left panel of Figure 5.5. On the corresponding surface in Figure 5.10 on page 90, and using the contour plots below the surface to guide the eye, we see that the centermost equilibrium is an isolated minimum, thus stable, and also the three outermost equilibria are isolated minima, thus stable. The three equilibria in between the centermost and outermost ones are saddle points. The number and stability of equilibria are thus 3-way symmetric. This result too is intuitive. We see a visual interplay here between the contribution of the term $G(1)$ in the one stable equilibrium near the center and the contribution of the term $G(2)$ as shown in the middle panel of
Figure 5.10: Example potential function $G$ as given by (5.66) in the major regimes in the $(\beta, \mu_L)$-parameter space shown in Figure 5.3 on page 74 and Figure 5.4 on page 75 which display multiplicity of equilibria, indicated with the number of 2-way symmetric steady-state solutions where the mode shares of the correlated elemental alternatives in the lower level nest are equal: (a) four stable nodes (of which two with $p_1 = p_2$, two with $p_1 \neq p_2$), and three saddle points (of which one with $p_1 = p_2$, two with $p_1 \neq p_2$); (b) three stable nodes (of which one with $p_1 = p_2$, two with $p_1 \neq p_2$), three saddle points (of which one with $p_1 = p_2$, two with $p_1 \neq p_2$), and one unstable node ($p_1 = p_2$); (c) three stable nodes (of which one with $p_1 = p_2$, two with $p_1 \neq p_2$), and two saddle points ($p_1 \neq p_2$); (d) two stable nodes ($p_1 \neq p_2$), and one saddle point ($p_1 = p_2$); (e) four stable nodes ($p_1 \neq p_2$), and three saddle points (of which one with $p_1 = p_2$, two with $p_1 \neq p_2$); and (f) two stable nodes ($p_1 = p_2$), and one saddle point ($p_1 = p_2$). Compare with corresponding panels (a) through (f) for the null clines solution of the system in Figure 5.5 on page 76.
Figure 5.8 in the three outermost stable equilibria. The contribution of the term $G(3)$ slightly distorts the three-way symmetry of the contour plot for $\mu_L > 1$, but since the parameter $\mu_L$ is not so large in Regime II, the contribution of the term $G(3)$ isn’t strong enough to change the number and/or stability of the equilibria away from the 3-way symmetry.

**Regime III.** In Regime III we expect to find seven equilibria as shown in the upper right panel of Figure 5.5. On the corresponding surface in Figure 5.10, we see that the three corner solutions are isolated minima, thus stable, the three solutions midway along the boundaries are saddle points and the solution at the center is an isolated maximum, thus unstable. This result is intuitive since we know from the preliminary analysis that the term $G(2)$ will prevail for large $\beta$. The surface for Regime III in Figure 5.10 is qualitatively similar to $G(2)$ in the middle panel of Figure 5.8. The contribution of the term $G(3)$ for large $\mu_L$ distorts the three-way symmetry of the surface and contour plot for Regime III in Figure 5.10, but is not strong enough to change the number and/or stability of the equilibria away from that determined by $G(2)$ for large enough $\beta$.

**Regime IV.** In Regime IV we expect to find five equilibria as shown in the left panel of the middle row in Figure 5.5. On the corresponding surface in Figure 5.10, we see that the three corner solutions are isolated minima, thus stable, and the two solutions midway along and a little bit back from the edge of the boundaries at $p_1 = 0$ and $p_2 = 1 - p_0 - p_1 = 0$ are saddle points. We see that the surface for Regime IV in Figure 5.10 is visually the result of the surface in Regime II combined with the added effect of the half-cone shaped influence of the term $G(3)$ in the lower panel of Figure 5.8. In Regime IV, the parameter $\mu_L$ is indeed large enough to break the 3-way symmetry. Compared with Regime II, the centermost stable node and the saddle point with $p_1 = p_2$ have disappeared as a result of the half-cone shaped influence of the term $G(3)$.

**Regime V.** In Regime V we expect to find three equilibria as shown in the right panel of the middle row in Figure 5.5. On the corresponding surface in Figure 5.10 we see that the two solutions midway along the boundaries at $p_1 = 0$ and $p_2 = 1 - p_0 - p_1 = 0$ are isolated minima, thus stable, and the center solution between them is a saddle point. Here we see that the surface for Regime V in Figure 5.10 is visually the result of the bowl in Regime I in Figure 5.5 combined with the half-cone shaped influence of the term $G(3)$ in the lower panel of Figure 5.8. As we saw in Regime VI, in Regime V now too, the parameter $\mu_L$ is again large enough to break the 3-way symmetry. In Regime V, the “basin” of the bowl in Regime I has been pushed upward along the line $p_1 = (1 - p_0)/2$, and indeed so much so that the stability of that initial point at the center of the basin has changed and two new equilibria have appeared.
Regime VI. In Regime VI we expect to find seven equilibria as shown in the lower left panel of Figure 5.5. On the corresponding surface in Figure 5.10, we see that the two solutions nearest to the corners \( p_2 = 1 - p_0 - p_1 = 1 \) (ie. \( p_0 = p_1 = 0 \)) and \( p_1 = 1 \) and are isolated minima, thus stable, and the two solutions midway along the boundaries at \( p_1 = 0 \) and \( p_2 = 1 - p_0 - p_1 = 0 \) are saddle points. Near the nose of the surface at \( p_0 = 1 \), there are two isolated minima along the boundaries at \( p_1 = 0 \) and \( p_2 = 1 - p_0 - p_1 = 0 \), and a solution between them along the line \( p_1 = (1-p_0)/2 \) that is a saddle point. We see that the surface for Regime VI in Figure 5.10 is visually the result of the half-cone shaped influence of the term \( G(3) \) in the lower panel of Figure 5.8 pushing the surface for Regime IV upward even further so that changes in number and stability of equilibria now also occur near the nose of the surface at \( p_0 = 1 \). Furthermore we can observe the qualitative type of change that has occurred due the effect of the parameter \( \mu_L \) between Regime I and Regime V midway along the cone, is the same that occurs between Regime IV and Regime VI near the nose of the cone.

Regime VII. We see in Figure 5.4 that Regime VII only occurs for a narrow range of values in between Regime I and Regime II. From the lower right panel of Figure 5.5 we expect to find three solutions. The corresponding surface for Regime VII in the lower right panel of Figure 5.10 seems at first glance similar to the surface for Regime II in the upper left panel of Figure 5.10. The contour plots beneath the surfaces however reveal the key change. In Regime II there are two mirror stable nodes and two mirror saddle points where \( p_1 \neq p_2 \), but in Regime VII the only solutions are along the line \( p_1 = (1-p_0)/2 \). The equilibrium nearest the center of the surface is an isolated minimum, thus stable. The equilibrium nearest the corner \( p_0 = 1 \) is also an isolated minimum, thus stable. The equilibrium between them is a saddle point. As with the Regime II, in Regime VII we see here too the visual interplay here between the contribution of the term \( G(1) \) in the one stable equilibrium near the center and the contribution of the term \( G(2) \) as shown in the middle panel of Figure 5.8 in the three outermost stable equilibria. However in Regime VII the parametrization by \( \beta \) is weaker, and importantly, in fact weak enough to allow the contribution from \( \mu_L \) parametrizing the term \( G(3) \) to have enough effect to change the number and stability of the equilibria away from the 3-way symmetry. There is thus a fragile balance in Regime VII between the contributions from \( G(2) \) and \( G(3) \).

5.4.1 Qualitative Behavior: \( \mu_L \to \infty \)

In this subsection we will derive the qualitative behavior of the sociodynamic trinary nested logit model in the limit where the scale parameter \( \mu_L \to \infty \).
Thus we see that for very strong correlation between alternatives provides counteraction to this tendency for $\beta$, in the special case $\beta = 0$ and $\mu_L \to \infty$, where there is social interaction and infinite correlation between alternatives, we have a random binary choice problem at the upper nest level, yielding a minimum at $p_0 = p_1 + p_2 = 1/2$

$$\lim_{\mu_L \to \infty} G_{|\beta=0} \approx p_0 \ln p_0 + (p_1 + p_2) \ln(p_1 + p_2) + \tilde{C}$$

$$= p_0 \ln p_0 + (1 - p_0) \ln(1 - p_0) + \tilde{C}$$
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\[ \beta = 0, \mu_L \to \infty : \text{Bifurcation} \]

\[ \beta = 1, \mu_L \to \infty : \text{Regime V} \]

\[ \beta = 2, \mu_L \to \infty : \text{Bifurcation} \]

\[ \beta = 3, \mu_L \to \infty : \text{Regime VI} \]

\[ \beta \approx 6.226, \mu_L \to \infty : \text{Bifurcation} \]

\[ \beta = 10, \mu_L \to \infty : \text{Regime VIII} \]

Figure 5.11: Example potential functions in the limit \( \mu_L \to \infty \): (a) one bifurcation point at \( p_0 = 0.5, p_1 = 0.25 \); (b) one saddle point \( (p_1 = p_2) \), and two stable nodes \( (p_1 \neq p_2) \); (c) two bifurcation points at \( p_0 = 0.5, p_1 = 0 \) and 0.5, one saddle point \( (p_1 = p_2) \), and two stable nodes \( (p_1 \neq p_2) \); (d) three saddle points (of which one with \( p_1 = p_2 \), two with \( p_1 \neq p_2 \)), and four stable nodes \( (p_1 \neq p_2) \); (e) one bifurcation point at \( p_0 \approx 0.122, p_1 \approx 0.439 \), three saddle points (of which one with \( p_1 = p_2 \), two with \( p_1 \neq p_2 \)), and four stable nodes \( (p_1 \neq p_2) \); (f) one unstable node \( (p_1 = p_2) \), four saddle points (of which two with \( p_1 = p_2 \), two with \( p_1 \neq p_2 \)), and four stable nodes \( (p_1 \neq p_2) \).
Having studied the number and stability of equilibria in the each of the regimes, we are now able to characterize the bifurcations between them. We will do this in the next section.

5.5 BIFURCATION TYPES

In this section we will interpret in turn each of the Curves A through D noted initially in section 5.2 as well as the special bifurcation point at $\beta = 3$ and $\mu_L = 1$ (colored in red in Figure 5.3) in light of our results on stability of equilibria in each of the regimes.

From the preliminary analysis in section 5.2, we found that crossing Curves A through D in $(\beta, \mu_L)$-parameter space is associated with a change in the number of equilibria. However, knowing the stability of any new equilibria and whether or not any already existing equilibria have changed their stability in the process is also critically important for physical interpretation. If we let $p_0, p_1$ and $p_2 = 1 - p_0 - p_1$ represent mode shares, then any point $p = (p_0, p_1)$ on the surfaces in Figures and 5.10 represents a particular modal split. We can visually imagine the natural progression of the system as a ball that rolls on this surface. The system will typically naturally tend to roll toward a stable solution, i.e. an isolated minimum in Figures and 5.10. If there is only one stable solution, the system will tend to stay there. If there are multiple stable solutions and if stochastic fluctuations are large enough and/or if a given isolated minimum shallow, the system may be able to jump away from the path towards a particular isolated minimum towards another one. On the other hand, with only just a little nudge, the system will naturally tend to move away from an unstable solution, i.e. an isolated maximum in Figures and 5.10. With a saddle point there is a tendency to move toward it along one axis but move away from it along the perpendicular axis. Both the number and stability types of equilibria are thus necessary to characterize the expected states in which we expect to find the system.

In the model that we have discussed, what type of surface we have depends on the value of the utility parameter $\beta$ for the weight of the social feedback in the system (i.e. how important other decision-makers’ choices are in impacting a given decision-maker’s choice) and the value of the scale parameter $\mu_L$ telling us the extent of the correlation between alternatives within a nest. We have seen that different values of $\beta$ and $\mu_L$ produce seven qualitatively different regimes for the number and stability types of equilibria.

Curve A. From the preliminary analysis in section 5.2, we found that crossing Curve A between Regime I and Regime VII and between II and Regime IV is associated with a change in the number of equilibria by two. Knowing the stability of the new equilibria and whether or not any other existing equilibria also change stability during the
process will allow us to more fully characterize the transition. Let us consider a case where we start in Regime I and enter Regime VII with increasing $\beta$, i.e. increasing magnitude of the utility parameter for the social feedback. Our original stable equilibrium at the bottom of the basin remains unchanged in its stability, and two new equilibria suddenly appear along the line $p_1 = (1 - p_0)/2$, one stable node and one a saddle point. The presence of the new stable equilibrium gives the system a theoretical possibility to jump to a different state, namely the new stable equilibrium, where the modal split given by $p_0$, $p_1$ and $p_2 = 1 - p_0 - p_1$ is quite different from the original modal split. As $\beta$ increases further, the new (outermost) stable node proceeds along the line $p_1 = (1 - p_0)/2$ toward the nose of the surface near $p_0 = 1$. The saddle point on the other hand, proceeds in the opposite direction along the line $p_1 = (1 - p_0)/2$ with increasing $\beta$ towards the original (centermost) stable node. At the border of Regime II and Regime IV, the saddle node and the original stable node merge and disappear, leaving the new outermost stable node. Vice versa, considering a case where we start in Regime IV and enter Regime II with decreasing $\beta$, this process is reversed. This phenomenon is called *hysteresis*. In the hysteresis regime there are multiple equilibria. Which stable equilibrium the system has upon entering and existing, centermost or outermost in $p_0$, $p_1$ and $p_2 = 1 - p_0 - p_1$, depends on the direction of approach, increasing or decreasing $\beta$.

**Curve B.** From the preliminary analysis in section 5.2, we found that crossing Curve B between Regime VII and Regime II, between Regime I and Regime IV, and between Regime V and Regime VI is associated with a change in the number of equilibria by four. Let us consider again a case where we cross with increasing $\beta$. The original equilibria in Regime VII, Regime I and Regime V remain unchanged, but four new equilibria suddenly appear in Regime II, Regime IV and Regime VI, two stable nodes and two saddle points. These new equilibria are mirror solutions in $p_1$ and $p_2$ diametrically across the line $p_1 = (1 - p_0)/2$. The transition is similar in character to the above discussed transition from Regime I to Regime VII, except that where earlier the change occurred for mode shares tending toward $p_0$, the isolated choice alternative, here the changes occur for mode shares tending toward $p_1$ and $p_2$, the two nested choice alternatives. Also, importantly, whereas Curve A hooks around in $(\beta, \mu_L)$-parameter space so that hysteresis exists, Curve B does not. Curve B instead forms a clean split of $(\beta, \mu_L)$-parameter space extending from $\beta \approx 2.75$ at $\mu_L = 1$ and tapering off at $\beta = 2$ as $\mu_L$ gets very large. The four equilibria that are generated when crossing Curve B with increasing $\beta$, remain for all values of $\beta$ on the other side of the Curve B, and retain their same stability. The location however of these four equilibria varies in terms of $p_0$, $p_1$ and $p_2 = 1 - p_0 - p_1$ as $\beta$ and $\mu_L$ vary. Because a new saddle point and a new stable node suddenly appear.
5.5 bifurcations types

Together at one point in terms of \( p_0, p_1 \) and \( p_2 = 1 - p_0 - p_1 \) (before separating and further migrating apart with increasing \( \beta \)), this is called a saddle-node bifurcation. Furthermore this phenomena occurs everywhere along Curve B at two mirror points in \( p_1 \) and \( p_2 \) diametrically across the line \( p_1 = (1 - p_0)/2 \). Curve B marks thus a double saddle-node bifurcation.

Curve C. From the preliminary analysis in section 5.2, we found that crossing Curve C between Regime IV and Regime III is associated with a change in the number of equilibria by two. Examining the stability of equilibria in Figure 5.10 we see as was the case with crossing Curve A inwards to the hysteresis region and the case crossing Curve B with increasing \( \beta \), that crossing Curve C with increasing \( \beta \) is also a saddle-node bifurcation, but here one of qualitatively different nature. Whereas the former bifurcations yielded pairs of saddle points and stable nodes, the bifurcation when crossing Curve C with increasing \( \beta \) yields a saddle point and an unstable node. In Figure 5.10 we see this effect of the utility parameter \( \beta \) for the weight of the social feedback that parametrizes \( G(2) \) visually: compared with Regime IV, in Regime III the top of the bowl from \( G(2) \) has fully poked up creating the isolated maximum (i.e. unstable node) and with it a saddle point that then moves along the line \( p_1 = (1 - p_0)/2 \) toward the boundary at \( p_0 = 0 \) with increasing \( \beta \). As was the case with Curve B, Curve C also forms a clean split of \( (\beta, \mu_L) \)-parameter space from its limit at \( \beta = 3 \) at \( \mu_L = 1 \) and tapering off near \( \beta \approx 6 \) as \( \mu_L \) gets very large. The unstable equilibrium and the corresponding saddle point that are generated when crossing Curve C with increasing \( \beta \), remain for all values of \( \beta \) on the other side of the Curve C, and retain their same stability, although the location of these equilibria varies in terms of \( p_0, p_1 \) and \( p_2 = 1 - p_0 - p_1 \) as \( \beta \) and \( \mu_L \) vary.

Putting the role of Curves A, B and C into perspective in \( (\beta, \mu_L) \)-parameter space we see thus that the marked behavioral influence of the utility parameter \( \beta \) for the weight of the social feedback which parametrizes \( G(2) \) occurs in two distinct stages with increasing \( \beta \). In the first stage, two stable nodes suddenly appear that favor respectively each of the mode shares \( p_1 \) and \( p_2 \) of the nested alternatives as we cross Curve B. For small values of \( \mu_L \) this may be preceded by the sudden appearance of a stable node that favor the mode share \( p_0 \) of the non-nested alternative, followed by the disappearance of the original centermost stable node. Or for higher values of \( \mu_L \) the original centermost stable node will simply migrate along the line \( p_1 = (1 - p_0)/2 \) toward the corner mode share at \( p_0 = 1 \). Either way regardless of the value of \( \mu_L \) whether the hysteresis region delineated by Curve A is crossed or not, the result together with the unavoidable crossing of Curve B is that we have now the existence of three stable solutions that respectively favor the mode share of one of the each of the three choice alternatives. Later with higher \( \beta \) an unstable node
appears that specifically drives the system away from the center of the surface in towards these corner solutions. This effect gives us the social multiplier behavior.

*Curve D.* From the preliminary analysis in section 5.2, we found that crossing Curve D inwards between Regime I and Regime V and between Regime IV and Regime VI is associated with a change in the number of equilibria by two. Examining the stability of equilibria in Figure 5.10 however shows a very different bifurcation than we saw with Curves A, B and C. Crossing the bifurcation curve D inwards yields two new isolated minima, thus two new stable nodes. In addition, unlike the saddle-node bifurcation which has no effect on any other existing equilibria, crossing Curve D does indeed have an effect on the initially stable equilibrium along the line $p_1 = (1 - p_0)/2$ that favors the mode share $p_0$ of the non-nested alternative. This initially stable equilibrium in Regime I and Regime IV becomes a saddle point in Regime V and Regime VI in Figure 5.10 with a ridge forming along the line $p_1 = (1 - p_0)/2$. Furthermore, the location of the sudden occurrence of two new stable nodes is precisely that initially stable equilibrium which becomes a saddle. Moving inwards across Curve D, the stability of the initial equilibrium is transferred at that point to the two new stable equilibrium. Moving further towards the interior of Curve D in $(\beta, \mu_L)$-parameter space, the two stable equilibria proceed further away from the line $p_1 = (1 - p_0)/2$ each respectively towards the boundaries at $p_1 = 0$ and $p_2 = 1 - p_0 - p_1 = 0$. This is called a *pitchfork bifurcation* because of the geometry of the solutions in terms of $p_0$ and $p_1$. Crossing the Curve D outwards forms a reverse pitchfork bifurcation. The two stable nodes merge together at the saddle node along the ridge and disappear, in the process transferring their stability back to the solution on the line $p_1 = (1 - p_0)/2$.

The significance of Curve D in $(\beta, \mu_L)$-parameter space becomes apparent when we recognize that crossing Curve D inwards is qualitatively equivalent to crossing Curve D with increasing scale parameter $\mu_L$ of the of the lower level nest. The pitchfork bifurcation that we have just seen is qualitatively what occurs in the binary choice logit model with social interactions studied by Aoki (1995), Brock and Durlauf (2001a) and Blume and Durlauf (2003). We see that with high enough correlation between alternatives 1 and 2 in the trinary nested logit model, and a certain level of social feedback in the model, an equilibrium point where $p_1 = p_2$ will no longer be stable and the modal split will be driven towards either $p_1$ or $p_2$ for the given value of $p_0$. Here we see thus the influence of parameters $\beta$ and $\mu_L$ working together in determining the binary choice probabilities for alternatives 1 and 2 within the nest given at the outset in (5.9) and (5.10) in subsection (5.1.1).

On the contrary, when $\beta = 0$, the binary choice probabilities for alternatives 1 and 2 within the nest in (5.9) and (5.10) reduce to...
5.6 Stability Analysis

In this section we characterize the stability of solutions of the sociodynamic trinary nested logit model via the eigenvalues of the Jacobian matrix of the two-parameter planar autonomous system (5.26) and (5.27), and derive expressions for the defining bifurcation curves in the $(\beta, \mu_L)$-plane rigorously analytically.

As we have seen initially in section 5.2 and discussed further in section 5.4, Figure 5.3 on page 74 and Figure 5.4 on page 75 can be visually decomposed into several distinct bifurcation curves in the $(\beta, \mu_L)$-plane separating major regimes. In this section we will derive the analytical relations for these curves one by one. To

$$P(i = 1|\text{nest1}) = P(i = 2|\text{nest1}) = 1/2$$ for all $\mu_L$, thus purely random choice, as is logical when the systematic utility for elemental alternatives within the lower level nest is null. This leaves the parametrized influence from $\mu_L$ to occur only in the upper level when choosing nest0 or nest1, due to the factor $1/\mu_L$ in (5.12) and (5.13) which acts independently of $\beta$.

Point at $\beta = 3$, $\mu_L = 1$. Finally we return to the special bifurcation point at $\beta = 3$ and $\mu_L = 1$ colored in red in Figure 5.3. It is apparent that this point marks the joining of Curve A for $\beta < 3$ with Curve C for $\beta > 3$. However unlike the crossing of other curves in ($\beta, \mu_L$)-parameter space where the different bifurcations occur coincidently at completely different solutions in terms of $p_0$, $p_1$, and $p_2 = 1 - p_0 - p_1$, here for the point at $\beta = 3$ and $\mu_L = 1$ both the saddle-node bifurcation marking the exit of the hysteresis region with increasing $\beta$ and the saddle-node bifurcation marking the sudden appearance of the unstable node with increasing $\beta$ occur precisely at the same one very special three-way symmetric solution point $p_0 = p_1 = p_2 = 1/3$. Approaching the point $\beta = 3$ and $\mu_L = 1$ with increasing $\beta$, the three saddle nodes in Regime II in Figure 5.10 all proceed inwards towards $p_0 = p_1 = p_2 = 1/3$. At the point $\beta = 3$ and $\mu_L = 1$ the three saddle nodes come together with the existing centermost equilibrium in the basin. Then in Regime III with further increasing $\beta$, the saddle nodes continue their respective trajectories proceeding outwards toward the respective boundaries $p_0 = 0$, $p_1 = 0$ and $p_2 = 1 - p_0 - p_1 = 0$, but with flipped directionality of the saddles in Regime III than was the case in Regime II. Furthermore in the process of coming together with the saddle nodes at the point $\beta = 3$ and $\mu_L = 1$, the existing centermost equilibrium in Regime II which had been an isolated minimum, thus stable, for $\beta < 3$ pops up to become an isolated maximum, thus unstable, for $\beta > 3$ in Regime III. Such a bifurcation where the number of equilibria is the same before and after the bifurcation, but the stability is transferred between them during a process of coming together at a single equilibrium point, is called transcritical.
do this, we draw on our earlier observations regarding the number of 2-way symmetric steady-state solutions in each regime with \( p_1 = p_2 \) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, as noted in the example null clines solutions of the system (5.24) and (5.25) in Figure 5.2 on page 68 and Figure 5.5 on page 76 and in the example potential functions (5.66) in Figure 5.9 on page 89 and Figure 5.10 on page 90. In particular it can be observed that: (i) the saddle-node bifurcations exhibiting hysteresis behavior for Curve A and the saddle-node bifurcation for Curve C are all cases where the resulting pairs of saddle points and nodes are all characterized by \( p_1 = p_2 \); (ii) the pitchfork bifurcation for Curve D is a case where one equilibrium forming the "handle" of the pitchfork on one side of the bifurcation satisfies \( p_1 = p_2 \) but the two new equilibria in the new "prongs" of the pitchfork on the other side of the bifurcation satisfy \( p_1 \neq p_2 \); (iii) the double saddle-node bifurcation for Curve B is a case where the resulting pairs of saddle points and nodes are all characterized by \( p_1 \neq p_2 \). See Figure 5.12 on page 100.

In subsection 5.6.1 we compute the elements of the Jacobian matrix. In subsection 5.6.2 we give a brief example of how the computation of the eigenvalues of the Jacobian can be used to determine the stability of hyperbolic equilibrium solutions. In subsection 5.6.3 we study the solutions of the sociodynamic nested logit model with \( p_1 = p_2 \) where the mode shares of the correlated elemental alternatives in the lower level nest are equal. In subsection 5.6.4 we derive an expression for Curves A and C as shown in panel (a) of Figure 5.12. In subsection 5.6.5 we derive an expression for Curve D as shown in panel (b) of Figure 5.12. In subsection 5.6.6 we re-visit the special case of a null trinary nested logit model with no social interactions studied earlier in subsection 5.1.3 and show that a bifurcation exists in the limit \( \mu_L \to +\infty \) when \( \beta = 0 \). In subsection 5.6.7 we derive an expression for Curve B as shown in panel (c) of Figure 5.12.

![Figure 5.12: Bifurcation curves in the \((\beta, \mu_L)\)-plane. Compare with Figure 5.3 on page 74 and Figure 5.4 on page 75.](image)
5.6.1 Elements of the Jacobian matrix

Recall from Section 4.2 that the stability type of equilibrium points of planar autonomous systems can be determined under certain conditions from the approximation of the vector field \( g = (g_0, g_1) \) with its derivative, which is a linear vector field. Let the Jacobian of \( g \) at the point \( p \) be the matrix:

\[
J \equiv Dg(p) = \begin{bmatrix}
\frac{\partial g_0}{\partial p_0}(p) & \frac{\partial g_0}{\partial p_1}(p) \\
\frac{\partial g_1}{\partial p_0}(p) & \frac{\partial g_1}{\partial p_1}(p)
\end{bmatrix}
\]  

(5.88)

To find possible values of the parameter \( \beta \) and \( \mu_L \) which lead to bifurcations in behavior, we are interested in cases when at least one eigenvalue \( \lambda \) of the Jacobian has zero real part.

\[
\lambda_{1,2} = \frac{1}{2} \text{tr} J \pm \frac{1}{2} \sqrt{\text{tr} J^2 - 4 \text{det} J}
\]  

(5.89)

Note from the form of (5.89) that there will exist at least one zero eigenvalue if the determinant of the Jacobian is equal to zero. There will exist zero real part of a pair of complex eigenvalues, i.e. purely imaginary eigenvalues, if the determinant is positive and trace is equal to zero.

\[
\text{det} J = 0 : \lambda_{1,2} = \frac{1}{2} \text{tr} J \pm \frac{1}{2} \sqrt{(\text{tr} J)^2 - 4 \cdot 0} = \frac{1}{2} \text{tr} J \pm \frac{1}{2} \text{tr} J
\]

(5.90)

\[
\lambda_1 = 0, \lambda_2 = \text{tr} J
\]

\[
\text{det} J > 0, \text{tr} J = 0 : \lambda_{1,2} = \frac{1}{2} \cdot 0 \pm \frac{1}{2} \sqrt{0^2 - 4 \text{det} J} = \pm i \sqrt{\text{det} J}
\]

(5.91)

For the system given by (5.26) and (5.27), the four terms in the Jacobian matrix of \( g = (g_0, g_1) \) can be computed using the sum rule, the product rule and the chain rule:

\[
\frac{\partial g_0}{\partial p_0} = \frac{\partial}{\partial p_0} \left( e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 (e^{\mu_L \beta (1-p_0-p_1)}) \frac{1}{\mu_L} \right)
\]

\[
= \beta e^{\beta p_0} - e^{\beta p_0} - p_0 e^{\beta p_0} - (e^{\mu_L \beta (1-p_0-p_1)}) \frac{1}{\mu_L}
\]

\[
- p_0 (\mu_L) (e^{\mu_L \beta (1-p_0-p_1)}) \frac{1}{\mu_L} - (\mu_L) e^{\mu_L \beta (1-p_0-p_1)}
\]

\[
= \beta e^{\beta p_0} - e^{\beta p_0} - p_0 e^{\beta p_0} - (e^{\mu_L \beta (1-p_0-p_1)}) \frac{1}{\mu_L}
\]

\[
+ \beta p_0 e^{\mu_L \beta (1-p_0-p_1)} (e^{\mu_L \beta (1-p_0-p_1)}) \frac{1}{\mu_L}
\]

(5.92)
\[
\frac{\partial g_0}{\partial p_1} = \frac{\partial}{\partial p_1} \left( e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \right)
\]
\[
= -\frac{1}{\mu_L} p_0 \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} - 1
\times \left( \mu_L \beta e^{\mu_1 \beta p_1} - \mu_L \beta e^{\mu_1 (1-p_0-p_1)} \right)
\]
\[
= -\beta p_0 \left( e^{\mu_1 \beta p_1} - e^{\mu_1 (1-p_0-p_1)} \right) \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \left( \frac{1}{\mu_L} - 1 \right)
\]
(5.93)

\[
\frac{\partial g_1}{\partial p_0} = \frac{\partial}{\partial p_0} \left( e^{\mu_1 \beta p_1} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \right)
\]
\[
+ \frac{\partial}{\partial p_0} \left( -p_1 e^{\beta p_0} - p_1 \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \right)
\]
\[
= \frac{1}{\mu_L} e^{\mu_1 \beta p_1} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \left( -\mu_L \beta e^{\mu_1 (1-p_0-p_1)} \right)
\]
\[
- \beta p_1 e^{\beta p_0}
\]
\[
- \frac{1}{\mu_L} p_1 \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \left( -\mu_L \beta e^{\mu_1 (1-p_0-p_1)} \right)
\]
\[
= -\beta (1-\mu_L) e^{\mu_1 \beta p_1} e^{\mu_1 (1-p_0-p_1)} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \left( \frac{1}{\mu_L} - 1 \right)
\]
\[
- \beta p_1 e^{\beta p_0} + \beta p_1 e^{\mu_1 \beta (1-p_0-p_1)} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \left( \frac{1}{\mu_L} - 1 \right)
\]
(5.94)

\[
\frac{\partial g_1}{\partial p_1} = \frac{\partial}{\partial p_1} \left( e^{\mu_1 \beta p_1} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \right)
\]
\[
+ \frac{\partial}{\partial p_1} \left( -p_1 e^{\beta p_0} - p_1 \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \right)
\]
\[
= \mu_L \beta e^{\mu_1 \beta p_1} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L}
\]
\[
+ \frac{1}{\mu_L} \left( e^{\mu_1 \beta p_1} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \right) \left( -\mu_L \beta e^{\mu_1 (1-p_0-p_1)} \right)
\]
\[
- e^{\beta p_0} - \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L}
\]
\[
- \frac{1}{\mu_L} p_1 \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \left( -\mu_L \beta e^{\mu_1 (1-p_0-p_1)} \right)
\]
\[
= \mu_L \beta e^{\mu_1 \beta p_1} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L}
\]
\[
+ (1-\mu_L) e^{\mu_1 \beta p_1} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \left( \frac{1}{\mu_L} - 1 \right)
\]
\[
- e^{\beta p_0} - \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L}
\]
\[
- \beta p_1 \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right) \frac{1}{\mu_L} \left( e^{\mu_1 \beta p_1} + e^{\mu_1 (1-p_0-p_1)} \right)
\]
(5.95)
Note that if $\mu_L = 1$, then (5.92) - (5.95) reduces to the elements of the Jacobian matrix for the trinary multinomial case as given in subsection 4.2.1:

$$\left. \frac{\partial g_0}{\partial p_0} \right|_{\mu_L = 1} = \beta e^{\beta p_0} - e^{\beta p_0} - \beta p_0 e^{\beta p_0} - (e^{\beta p_1} + e^{\beta (1-p_0-p_1)})^{-1}$$

$$+ \beta p_0 e^{\beta (1-p_0-p_1)}(e^{\beta p_1} + e^{\beta (1-p_0-p_1)})^0$$

$$= \beta e^{\beta p_0} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta (1-p_0-p_1)} - \beta p_0 e^{\beta p_0} + \beta p_0 e^{\beta (1-p_0-p_1)}$$

(5.96)

$$\left. \frac{\partial g_0}{\partial p_1} \right|_{\mu_L = 1} = -\beta p_0 (e^{\beta p_1} - e^{\beta (1-p_0-p_1)}) (e^{\beta p_1} + e^{\beta (1-p_0-p_1)})^0$$

$$= -\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta (1-p_0-p_1)}$$

(5.97)

$$\left. \frac{\partial g_1}{\partial p_0} \right|_{\mu_L = 1} = \beta (1-1) e^{\beta (p_1 + (1-p_0-p_1))} (e^{\beta p_1} + e^{\beta (1-p_0-p_1)})^{-1}$$

$$- \beta p_0 e^{\beta p_0} + \beta p_1 e^{\beta (1-p_0-p_1)} (e^{\beta p_1} + e^{\beta (1-p_0-p_1)})^0$$

$$= -\beta p_1 e^{\beta p_0} + \beta p_1 e^{\beta (1-p_0-p_1)}$$

(5.98)

$$\left. \frac{\partial g_1}{\partial p_1} \right|_{\mu_L = 1} = \beta e^{\beta p_1} (e^{\beta p_1} + e^{\beta (1-p_0-p_1)})^0$$

$$+ (1-1) \beta e^{\beta p_1} (e^{\beta p_1} + e^{\beta (1-p_0-p_1)})^{-1} (e^{\beta p_1} - e^{\beta (1-p_0-p_1)})$$

$$- e^{\beta p_0} - (e^{\beta p_1} + e^{\beta (1-p_0-p_1)})^1$$

$$- \beta p_1 (e^{\beta p_1} + e^{\beta (1-p_0-p_1)})(e^{\beta p_1} - e^{\beta (1-p_0-p_1)})$$

$$= \beta e^{\beta p_1} - e^{\beta p_0} - e^{\beta p_1} - e^{\beta (1-p_0-p_1)} - \beta p_1 e^{\beta p_1} + \beta p_1 e^{\beta (1-p_0-p_1)}$$

(5.99)

Having computed the elements of the Jacobian matrix, we can now write expressions for the determinant and the trace of the Jacobian for use in computing the eigenvalues as given by (5.89). The determinant of the Jacobian matrix is:
The trace of the Jacobian matrix is:

\[
\text{tr} J = \frac{\partial g_0}{\partial p_0} + \frac{\partial g_1}{\partial p_1} + \frac{\partial g_0}{\partial p_0} + \frac{\partial g_1}{\partial p_1} - \frac{\partial g_0}{\partial p_0} - \frac{\partial g_1}{\partial p_1} - \frac{\partial g_0}{\partial p_0} - \frac{\partial g_1}{\partial p_1}
\]

\[
= \beta e^{p_0} - e^{p_0} - \beta p_0 e^{p_0} - (e^{g_0} e^{p_0} + e^{g_1} e^{(1-p_0-p_1)})^1_{\mu L}
+ \beta p_0 e^{g_1} g_1 \beta(1-p_0-p_1) (e^{g_1} e^{p_0} + e^{g_1} e^{(1-p_0-p_1)})^1_{\mu L} + \mu L \beta(e^{g_1} e^{p_0} + e^{g_1} e^{(1-p_0-p_1)})^1_{\mu L}
+ (1 - \mu L) e^{g_1} \beta(e^{g_1} e^{p_0} + e^{g_1} e^{(1-p_0-p_1)})^1_{\mu L} - \beta p_0 e^{g_1} g_1 \beta(1-p_0-p_1) (e^{g_1} e^{p_0} + e^{g_1} e^{(1-p_0-p_1)})^1_{\mu L}
+ (1 - \mu L) e^{g_1} \beta(e^{g_1} e^{p_0} + e^{g_1} e^{(1-p_0-p_1)})^1_{\mu L} - \beta p_0 e^{g_1} g_1 \beta(1-p_0-p_1) (e^{g_1} e^{p_0} + e^{g_1} e^{(1-p_0-p_1)})^1_{\mu L}
\]

(5.101)

In the next subsection we will give an example of how the computation of the elements of the Jacobian matrix, and in turn the computation of the determinant and trace of the Jacobian, and finally the
eigenvalues of the Jacobian are used to determine the stability of hyperbolic equilibrium solutions, that is, when all the eigenvalues of the Jacobian matrix have nonzero real parts.

5.6.2 An Example

Looking ahead to the estimation results in Chapter 7, we find that the estimated coefficients for the simple nested logit model with fully connected network given in Table 7.21 fall in an interesting region in parameter space, near to the bifurcation to several different regimes. We will determine the stability of the equilibria in each of these regimes.

Case I. Suppose \( \beta = 2.7595, \mu_L = 1.0339 \) (estimated values)

See Figure 5.13 on page 105. See Table 5.1 on page 106.

![Figure 5.13](image)

(a) \( \beta = 2.7595, \mu_L = 1.0339 \): Regime IV

Figure 5.13: Null clines solutions of the sociodynamic trinary nested logit model with estimated coefficients in the case study in Chapter 7

Case II. Suppose \( \beta = 2.7595, \mu_L = 1.015 \)

Regime II with seven solutions: 4 asymptotically stable nodes, 3 saddle points. See panel (a) in Figure 5.14 on page 107. See Table 5.2 on page 109.
Table 5.1: Stability analysis of the socio-dynamic trinary nested logit model with estimated coefficients in the case study in Chapter 7.

<table>
<thead>
<tr>
<th>Stability</th>
<th>( \gamma )</th>
<th>( \lambda )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saddle point</td>
<td>0.46</td>
<td>0.88-1.08</td>
<td>0.34</td>
<td>1.2</td>
<td>0.237</td>
</tr>
<tr>
<td>Asymptotically stable node</td>
<td>0.46</td>
<td>0.88-1.08</td>
<td>0.34</td>
<td>1.2</td>
<td>0.237</td>
</tr>
<tr>
<td>Asymptotically stable node</td>
<td>0.46</td>
<td>0.88-1.08</td>
<td>0.34</td>
<td>1.2</td>
<td>0.237</td>
</tr>
<tr>
<td>Asymptotically stable node</td>
<td>0.46</td>
<td>0.88-1.08</td>
<td>0.34</td>
<td>1.2</td>
<td>0.237</td>
</tr>
<tr>
<td>Saddle point</td>
<td>0.46</td>
<td>0.88-1.08</td>
<td>0.34</td>
<td>1.2</td>
<td>0.237</td>
</tr>
</tbody>
</table>

\[ \hat{p} = 1.0399 \]
5.6 Stability Analysis

(a) $\beta = 2.7595, \mu_L = 1.015$: Regime II

(b) $\beta = 2.72, \mu_L = 1.025$: Regime VII

(c) $\beta = 2.71, \mu_L = 1.0339$: Regime I

Figure 5.14: Null clines solutions of the sociodynamic trinary nested logit model with parameters within one standard error of the estimated coefficients in Chapter 7: (a) seven solutions; (b) three solutions (inset shows null clines do not cross in the region near $p_0 = 0.21, p_1 = 0.195$); (c) one solution (inset shows null clines do not cross in the region near $p_0 = 0.21, p_1 = 0.19$)
Case III. Suppose $\beta = 2.72$, $\mu_L = 1.025$

Regime VII with three solutions: 2 asymptotically stable nodes, 1 saddle point. See panel (b) in Figure 5.14 on page 107. See Table 5.3 on page 109.

Case IV. Suppose $\beta = 2.71$, $\mu_L = 1.0339$

Regime I with one solution: 1 asymptotically stable node. See panel (c) in Figure 5.14 on page 107. See Table 5.4 on page 109.

5.6.3 Special Solution: $p_1 = p_2$

We have seen in subsection 5.1.2 that there exists at least one equilibrium solution of the sociodynamic trinary nested logit model (5.17) with $p_1 = p_2$ defined on $[0, \frac{1}{2}]$ for all values of $\beta$ and $\mu_L$ real, finite. Let us now further examine the properties of this solution.

**Theorem** (Solution surface for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model) Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter $\beta$ real, finite on the proportion $p_i$ of decision-making agents that have chosen alternative $i$, and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter $\mu_L \geq 1$ real, finite. The solution surface for the solution of the sociodynamic trinary nested logit model (5.17) with $p_1 = p_2$ defined on $(0, \frac{1}{2})$ where the mode shares of the correlated elemental alternatives in the lower level nest are equal, is characterized by:

$$\left. \frac{1}{\mu_L} \right|_{p_1 = p_2} = \frac{\ln 2 + \ln p_1 + \beta(1 - 3p_1) - \ln(1 - 2p_1)}{\ln 2}$$  (5.102)

**Proof.** Given the special two-way symmetry at equilibrium between $p_1$ and $p_2$ in (5.19) and (5.20), suppose the following symmetric solution:

$$p_1 = p_2$$  (5.103)

Thus by (5.21) we can write:

$$p_0 = 1 - p_1 - p_2 = 1 - 2p_1$$  (5.104)

or alternatively:

$$p_1 = \frac{1}{2}(1 - p_0)$$  (5.105)
### Table 5.2: Stability analysis of the sociodynamic trinary nested logit model 
\((\beta = 2.7595, \mu_L = 1.015)\)

<table>
<thead>
<tr>
<th>NR</th>
<th>(p_0)</th>
<th>(p_1)</th>
<th>(p_2)</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.676</td>
<td>0.162</td>
<td>0.162</td>
<td>Asymptotically stable node</td>
</tr>
<tr>
<td>2</td>
<td>0.167</td>
<td>0.160</td>
<td>0.673</td>
<td>Asymptotically stable node</td>
</tr>
<tr>
<td>3</td>
<td>0.167</td>
<td>0.673</td>
<td>0.160</td>
<td>Asymptotically stable node</td>
</tr>
<tr>
<td>4</td>
<td>0.253</td>
<td>0.240</td>
<td>0.507</td>
<td>Saddle point</td>
</tr>
<tr>
<td>5</td>
<td>0.253</td>
<td>0.507</td>
<td>0.240</td>
<td>Saddle point</td>
</tr>
<tr>
<td>6</td>
<td>0.378</td>
<td>0.311</td>
<td>0.311</td>
<td>Asymptotically stable node</td>
</tr>
<tr>
<td>7</td>
<td>0.443</td>
<td>0.279</td>
<td>0.279</td>
<td>Saddle point</td>
</tr>
</tbody>
</table>

### Table 5.3: Stability analysis of the sociodynamic trinary nested logit model 
\((\beta = 2.72, \mu_L = 1.025)\)

<table>
<thead>
<tr>
<th>NR</th>
<th>(p_0)</th>
<th>(p_1)</th>
<th>(p_2)</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.638</td>
<td>0.181</td>
<td>0.181</td>
<td>Asymptotically stable node</td>
</tr>
<tr>
<td>2</td>
<td>0.426</td>
<td>0.287</td>
<td>0.287</td>
<td>Asymptotically stable node</td>
</tr>
<tr>
<td>3</td>
<td>0.430</td>
<td>0.285</td>
<td>0.285</td>
<td>Saddle point</td>
</tr>
</tbody>
</table>

### Table 5.4: Stability analysis of the sociodynamic trinary nested logit model 
\((\beta = 2.71, \mu_L = 1.0339)\)

<table>
<thead>
<tr>
<th>NR</th>
<th>(p_0)</th>
<th>(p_1)</th>
<th>(p_2)</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.643</td>
<td>0.179</td>
<td>0.179</td>
<td>Asymptotically stable node</td>
</tr>
</tbody>
</table>
Substituting (5.103) into (5.26):

\[
0 = g_0|_{p_1=p_2} = e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 (e^{\mu_L} \beta p_1 + e^{\mu_L} \beta (1-p_0-p_1)) \frac{1}{\mu_L}
\]

\[
= e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 (e^{\mu_L} \beta p_1 + e^{\mu_L} \beta (1-p_0-p_1)) \frac{1}{\mu_L}
\]

\[
= e^{\beta p_0} - p_0 e^{\beta p_0} - \frac{2}{\mu_L} p_0 e^{\beta p_1} \quad \text{(5.106)}
\]

Substituting in (5.104) to write (5.106) in terms of \( p_1 \):

\[
0 = g_0|_{p_1=p_2} = e^{\beta (1-2p_1)} - (1-2p_1) e^{\beta (1-2p_1)} - \frac{2}{\mu_L} (1-2p_1) e^{\beta p_1}
\]

\[
= 2p_1 e^{\beta (1-2p_1)} - \frac{2}{\mu_L} (1-2p_1) e^{\beta p_1} \quad \text{(5.107)}
\]

Dividing through by \( \exp(\beta p_1) \) which is always nonzero for \( \beta \) real, finite:

\[
0 = 2p_1 e^{\beta (1-3p_1)} - \frac{2}{\mu_L} (1-2p_1) \quad \text{(5.108)}
\]

Re-arranging terms:

\[
\frac{2p_1 e^{\beta (1-3p_1)}}{1-2p_1} = 2 \frac{1}{\mu_L} \quad \text{(5.109)}
\]

Taking the natural logarithm of both sides, we have

\[
\ln 2 + \ln p_1 + \beta (1-3p_1) - \ln (1-2p_1) = \frac{1}{\mu_L} \ln 2 
\]

where the values of \( p_1 \) are restricted for \( \mu_L \) real, finite:

\[
p_1 \neq 0 \; ; \; p_1 \neq \frac{1}{2} \quad \text{(5.111)}
\]

Solving (5.110) for \( \beta \) we have:

\[
\beta = \frac{1}{1-3p_1} \left( \frac{1}{\mu_L} \ln 2 + \ln (1-2p_1) - \ln 2 - \ln p_1 \right)
\]

\[
= \frac{1}{1-3p_1} \ln \frac{2}{\mu_L} (1-2p_1) \quad \text{(5.112)}
\]

Alternatively, solving (5.110) for \( 1/\mu_L \) we have:

\[
\frac{1}{\mu_L} = \frac{\ln 2 + \ln p_1 + \beta (1-3p_1) - \ln (1-2p_1)}{\ln 2} 
\]

\[
\text{(5.113)}
\]

And solving (5.110) for \( \mu_L \) we have:

\[
\mu_L = \frac{\ln 2}{\ln 2 + \ln p_1 + \beta (1-3p_1) - \ln (1-2p_1)} 
\]

\[
\text{(5.114)}
\]

See Figure 5.15 on page 111 and Figure 5.16 on page 111.

It is visually apparent from the upper ridge of solutions at \( (1/\mu_L) = 1 \) along the line \( p_1 = p_2 = 1/3 \) for all \( \beta \) in the left and right panels.
Figure 5.15: Solution surface (5.102) for the solution $p_1 = p_2$ of the socio-dynamic trinary nested logit model where the mode shares of the correlated elemental alternatives in the lower level nest are equal. Theoretical level sets for various values of the inverse of the lower level nest scale parameter ($1/\mu_L$) are projected in color below the solution surface. Note however that in practice, for correct empirical meaning when the scale of the upper level nest is normalized to unity, the parameter $\mu_L$ will be defined to be real and finite with $\mu_L \geq 1$, so that the empirically allowed values of $(1/\mu_L)$ will be limited to the range from zero to one.

Figure 5.16: Detail of the solution surface (5.102) in Figure 5.15, with the inverse of the lower level nest scale parameter ($1/\mu_L$) restricted to the empirically meaningful range from 0 to 1. Level sets for various values of $(1/\mu_L)$ are projected in color below the solution surface.
of Figure 5.16 on page 111, that the three-way symmetric equilibrium solution of the sociodynamic trinary nested logit model (5.17) exists for all values of the utility parameter $\beta$ real, finite. This comes as no surprise since we have already observed the special behavior of the three-way symmetric equilibrium solution in Chapter 4 when studying the sociodynamic trinary multinomial logit model. The three-way symmetric solution can be seen consistently in all panels of Figures 4.3 through 4.5 on pages 54-56. The persistence of the three-way symmetric equilibrium solution is also depicted clearly visually in the summary bifurcation diagram for the sociodynamic trinary multinomial logit model in Figure 4.7 on page 58. From Figure 5.16 on page 111 however it becomes visually apparent that this three-way symmetric equilibrium solution only exists in the limit of the sociodynamic trinary multinomial logit model. We formalize this useful fact in the following lemma.

**Lemma** (Three-way symmetric solution of the sociodynamic trinary nested logit model exists for all $\beta$) There will exist at least one equilibrium solution of the sociodynamic trinary nested logit model (5.17) with 3-way symmetric mode shares $p_0 = p_1 = p_2 = 1/3$ for all values of the utility parameter $\beta$ real, finite if and only if the scale parameter $\mu_L = 1$, i.e. in the sociodynamic trinary multinomial logit case.

**Proof.** Suppose the following 3-way symmetric solution:

$$ p \equiv p_0 = p_1 = p_2 \quad (5.115) $$

Thus, by (5.21) we have:

$$ p = 1/3 \quad (5.116) $$

Substituting (5.115) and (5.116) into (5.106) we have:

$$ 0 = g_0|_{p_1=p_2} = e^{\beta p_0} - p_0 e^{\beta p_0} - 2^{\frac{1}{3}} p_0 e^{\beta p_1} $$
$$ = e^{\beta/3} - (1/3)e^{\beta/3} - 2^{\frac{1}{3}} (1/3)e^{\beta/3} $$
$$ = \frac{(2 - 2^{\frac{1}{3}})}{3} e^{\beta/3} \quad (5.117) $$

Since the utility parameter $\beta$ is assumed to be real, finite, the term $e^{\beta/3}$ will be strictly positive. Thus to satisfy (5.117) we require strictly

$$ 0 = (2 - 2^{\frac{1}{3}}) \quad (5.118) $$

and therefore the 3-way symmetric solution (5.115) and (5.116) is valid if and only if

$$ \mu_L = 1 \quad (5.119) $$

Finally, since (5.118) is independent of $\beta$, we observe that the value of the scale parameter $\mu_L = 1$ furthermore satisfies (5.117) for all values of the utility parameter $\beta$ real, finite. ∎
**Lemma** (Allowed values of $\beta$ for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model) The allowed values of the utility parameter $\beta$ real, finite for the solution of the sociodynamic trinary nested logit model (5.17) with $p_1 = p_2$ defined on $(0, 1/2)$, where the mode shares of the correlated elemental alternatives in the lower level nest are equal, are characterized for values of the scale parameter $\mu_L \geq 1$ real, finite by:

\[
\frac{1}{3p_1 - 1} \ln \frac{2p_1}{1 - 2p_1} < \beta \leq \frac{1}{3p_1 - 1} \ln \frac{p_1}{1 - 2p_1} ; \ 0 < p_1 < \frac{1}{3}
\]

\[-\infty < \beta < +\infty ; \ p_1 = \frac{1}{3}
\]

\[
\frac{1}{3p_1 - 1} \ln \frac{p_1}{1 - 2p_1} \leq \beta < \frac{1}{3p_1 - 1} \ln \frac{2p_1}{1 - 2p_1} ; \ \frac{1}{3} < p_1 < \frac{1}{2}
\]

**Proof.** From the previous lemma we know that the 3-way symmetric solution $p_0 = p_1 = p_2 = 1/3$ is valid for all values of the utility parameter $\beta$ real, finite, thus proving the middle line of (5.120).

Now, since we have defined $\beta$ and $\mu_L$ real, finite with $\mu_L \geq 1$, then it follows from (5.114)

\[-\ln 2 < -\ln(1 - 2p_1) + \ln p_1 + \beta - 3\beta p_1 \leq 0 \tag{5.121}
\]

where $\beta$ and $\mu_L$ real, finite requires:

\[-\ln 2 < -\ln(1 - 2p_1) + \ln p_1 + \beta - 3\beta p_1
\]

\[\beta(3p_1 - 1) < -\ln(1 - 2p_1) + \ln p_1 + \ln 2 \tag{5.122}
\]

so that we have

\[\beta < \frac{1}{3p_1 - 1} \ln \frac{2p_1}{1 - 2p_1} ; \ p_1 > \frac{1}{3} \tag{5.123}
\]

and

\[\beta > \frac{1}{3p_1 - 1} \ln \frac{2p_1}{1 - 2p_1} ; \ p_1 < \frac{1}{3} \tag{5.124}
\]

and where $\mu_L \geq 1$ requires:

\[-\ln(1 - 2p_1) + \ln p_1 + \beta - 3\beta p_1 \leq 0
\]

\[-\ln(1 - 2p_1) + \ln p_1 \leq \beta(3p_1 - 1) \tag{5.125}
\]

so that we have

\[\frac{1}{3p_1 - 1} \ln \frac{p_1}{1 - 2p_1} \leq \beta ; \ p_1 > \frac{1}{3} \tag{5.126}
\]

and

\[\frac{1}{3p_1 - 1} \ln \frac{p_1}{1 - 2p_1} \geq \beta ; \ p_1 < \frac{1}{3} \tag{5.127}
\]

See Figure 5.17 on page 114. ♦
Socio-dynamic trinary nested logit: Theory

Figure 5.17: Allowed steady-state values of $p_1 = p_2$, and the parameter $\beta$ shown in the range from 0 to 10, with $\mu_L \geq 1$ real, finite: (a) projection onto the $(p_1, \beta)$-plane of $1/\mu_L$ on the range from 0 to 1, with $1/\mu_L$ defined by (5.102); (b) limit curves given by (5.120).

**Theorem** (Solution surface of the sociodynamic trinary nested logit model for $p_0$ when $p_1 = p_2$) Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter $\beta$ real, finite on the proportion $p_i$ of decision-making agents that have chosen alternative $i$, and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter $\mu_L \geq 1$ real, finite. The solution surface for $p_0$ defined on $(0, 1)$ for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, is characterized by:

\[
\frac{1}{\mu_L} = \ln \left(1 - p_0\right) + \frac{(\beta/2)(3p_0 - 1) - \ln p_0}{\ln 2} \quad (5.128)
\]

**Proof.** Substituting (5.105) into (5.106) to write in terms of $p_0$:

\[
0 = g_0|_{p_1=p_2} = e^{\beta p_0} - p_0 e^{\beta p_0} - 2^{\frac{1}{\mu_L}} p_0 e^{\frac{\beta(1-p_0)}{2}}
\]

Dividing through by $\exp(\beta(1-p_0)/2)$ which is always nonzero for $\beta$ real, finite:

\[
0 = e^{\beta(1-p_0)/2} (1-p_0) - 2^{\frac{1}{\mu_L}} p_0 e^{\frac{\beta p_0}{2}} e^{-\frac{\beta p_0}{2}} \quad (5.129)
\]

Re-arranging terms:

\[
\frac{(1-p_0)}{p_0} e^{\beta(1-p_0)/2} = 2^{\frac{1}{\mu_L}} \quad (5.130)
\]

Taking the natural logarithm of both sides, we have

\[
\ln(1-p_0) + \frac{\beta}{2} (3p_0 - 1) - \ln p_0 = \frac{1}{\mu_L} \ln 2 \quad (5.132)
\]
where the values of $p_0$ are restricted for $\mu_L$ real, finite:

$$p_0 \neq 0; p_0 \neq 1$$  \hfill (5.133)

Solving (5.132) for $\beta$ we have:

$$\beta = \frac{2}{3p_0 - 1} \left( \frac{1}{\mu_L} \ln 2 + \ln p_0 - \ln (1 - p_0) \right)$$  \hfill (5.134)

$$= \frac{2}{3p_0 - 1} \ln \frac{2\pi}{1 - p_0}$$

Similarly solving (5.132) for $1/\mu_L$ we have:

$$\frac{1}{\mu_L} = \frac{\ln (1 - p_0) + (\beta/2)(3p_0 - 1) - \ln p_0}{\ln 2}$$  \hfill (5.135)

And solving (5.132) for $\mu_L$ we have:

$$\mu_L = \frac{\ln 2}{\ln (1 - p_0) + (\beta/2)(3p_0 - 1) - \ln p_0}$$  \hfill (5.136)

\hfill ♦

**Lemma** (Allowed values of $\beta$ for $p_0$ for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model) The allowed values of utility parameter $\beta$ real, finite for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, is characterized for the non-nested mode share $p_0$ defined on $(0, 1)$ and values of the scale parameter $\mu_L \geq 1$ real, finite by:

$$\frac{2}{1 - 3p_0} \ln \frac{1 - p_0}{2p_0} \leq \beta < \frac{2}{1 - 3p_0} \ln \frac{1 - p_0}{p_0}; \ 0 < p_0 < \frac{1}{3}$$

$$-\infty < \beta < +\infty; \ p_0 = \frac{1}{3}$$

$$\frac{2}{1 - 3p_0} \ln \frac{1 - p_0}{p_0} \leq \beta \leq \frac{2}{1 - 3p_0} \ln \frac{1 - p_0}{2p_0}; \ \frac{1}{3} < p_0 < 1$$  \hfill (5.137)

**Proof.** From an earlier lemma in this subsection, we know that the 3-way symmetric solution $p_0 = p_1 = p_2 = 1/3$ is valid for all values of the utility parameter $\beta$ real, finite, hereby proving the middle line of (5.137).

Now, since we have defined $\beta$ and $\mu_L$ real, finite with $\mu_L \geq 1$, then it follows from (5.136)

$$0 < \ln (1 - p_0) + (\beta/2)(3p_0 - 1) - \ln p_0 \leq \ln 2$$  \hfill (5.138)

where $\beta$ and $\mu_L$ real, finite requires:

$$0 < \ln (1 - p_0) + (\beta/2)(3p_0 - 1) - \ln p_0$$

$$\frac{1}{\mu_L} \ln (1 - p_0) < \ln (1 - p_0) - \ln p_0$$  \hfill (5.139)
so that we have
\[ \beta < \frac{2}{1 - 3p_0} \ln \frac{1 - p_0}{p_0} \; ; \; p_0 < \frac{1}{3} \] (5.140)
and
\[ \beta > \frac{2}{1 - 3p_0} \ln \frac{1 - p_0}{p_0} \; ; \; p_0 > \frac{1}{3} \] (5.141)
and where \( \muL \geq 1 \) requires:
\begin{align*}
\ln (1 - p_0) + (\beta/2)(3p_0 - 1) & - \ln p_0 \leq \ln 2 \\
\ln(1 - p_0) - \ln p_0 - \ln 2 & \leq (\beta/2)(1 - 3p_0)
\end{align*}
(5.142)
so that we have
\[ \frac{2}{1 - 3p_0} \ln \frac{1 - p_0}{2p_0} \leq \beta ; \; p_0 < \frac{1}{3} \] (5.143)
and
\[ \frac{2}{1 - 3p_0} \ln \frac{1 - p_0}{2p_0} \geq \beta ; \; p_0 > \frac{1}{3} \] (5.144)

\[ \diamond \]

**Lemma** (Relationship between \( g_0 \) and \( g_1 \) when \( p_1 = p_2 \)) The two equations of the two-parameter planar autonomous system (5.26) and (5.27) satisfy the following relation for all values of the utility parameter \( \beta \) and scale parameter \( \muL \) real, finite with \( \muL \geq 1 \), for the solution of the sociodynamic trinary nested logit model (5.17) with \( p_1 = p_2 \) defined on \((0, 1/2)\), where the mode shares of the correlated elemental alternatives in the lower level nest are equal.

\[ g_1|_{p_1=p_2} = -\frac{1}{2}g_0|_{p_1=p_2} \] (5.145)

**Proof.** Substituting (5.103) into (5.27):
\[
0 = g_1|_{p_1=p_2} = e^{\muL \beta p_1}(e^{\muL \beta p_1} + e^{\muL \beta(1-p_0-p_1)})^{-1/\muL} \\
- p_1 e^{\beta p_0} - p_1 (e^{\muL \beta p_1} + e^{\muL \beta(1-p_0-p_1)})^{1/\muL} \\
= e^{\muL \beta p_1}(e^{\muL \beta p_1} + e^{\muL \beta p_1})^{-1/\muL} \\
- p_1 e^{\beta p_0} - p_1 (e^{\muL \beta p_1} + e^{\muL \beta p_1})^{1/\muL} \\
= 2^{1-\muL} e^{\muL \beta p_1}(e^{(1-\muL)\beta p_1}) - p_1 e^{\beta p_0} - p_1 2^{1/\muL} e^{\beta p_1} \\
= 2^{1-\muL} e^{\beta p_0} - p_1 e^{\beta p_0} - p_1 2^{1/\muL} e^{\beta p_1} \\
= -\frac{1}{2}\left(2p_1 e^{\beta p_0} - 2^{1/\muL} (1 - 2p_1)e^{\beta p_1}\right)
\]
(5.146)
Substituting in (5.104) to write (5.146) in terms of \( p_1 \):
\[
0 = g_1|_{p_1=p_2} = -\frac{1}{2}\left(2p_1 e^{\beta (1-2p_1)} - 2^{1/\muL} (1 - 2p_1)e^{\beta p_1}\right)
\] (5.147)
Substituting (5.107) into (5.147) we obtain a simple relationship between $g_0$ and $g_1$ for the special solution $p_1 = p_2$:

$$g_1|_{p_1=p_2} = -\frac{1}{2} g_0|_{p_1=p_2}$$  \hspace{1cm} (5.148)

Furthermore (5.147) will yield the same relationship for $\mu_L$ in terms of $p_1 = p_2$ in (5.114), and for $\beta$ in terms of $p_1 = p_2$ in (5.112), and thus also for $\mu_L$ in terms of $p_0$ and for $\beta$ in terms of $p_0$. ♦

Finally, note also that we can substitute (5.104) back into (5.107) to see the effect of $\mu_L$ on the symmetry-breaking of $p_0$ and $p_1 = p_2$ more immediately:

$$0 = g_0|_{p_1=p_2} = 2p_1 e^{\beta (1-2p_1)} - 2p_1 e^{\beta p_1}$$

$$= 2p_1 e^{\beta p_0} - 2p_1 e^{\beta p_1}$$  \hspace{1cm} (5.149)

**Transcritical bifurcation, hysteresis loop and saddle-node bifurcation**

The level sets of the solution surfaces (5.128) and (5.102) yield bifurcation diagrams for the special solution $p_1 = p_2$ of the sociodynamic trinary nested logit model where the mode shares of the correlated elemental alternatives in the lower level nest are equal, respectively in the $(\beta, p_0)$-plane and in the $(\beta, p_1)$-plane, for given values of scale parameter $\mu_L$. We have already seen examples of these level sets for various values of $(1/\mu_L)$ projected in color below the solution surface (5.102) in Figure 5.16 on page 111 in the $(\beta, p_1)$-plane. In Figure 5.18 on page 118 we depict these bifurcation diagrams for the trinary nested logit model for the special solution where $p_1 = p_2$ in both the $(\beta, p_0)$-plane and in the $(\beta, p_1)$-plane for selected values of scale parameter $\mu_L$ showing the essential qualitative behavior. Recall that the parameter range $\mu_L < 1$ for the scale of the lower level nest is not empirically allowed when the scale of the upper level nest is normalized to unity.

Hereby we see visually the transcritical bifurcation point at $\beta = 3$, $\mu_L = 1$ as discussed in section 5.5 in the top panels (a) and (b) of Figure 5.18 on page 118. The transcritical bifurcation point disappears for scale parameter of the lower nests $\mu_L > 1$. We also see visually that the hysteresis loop as discussed in section 5.5 exists for parameter range $(0 \leq \mu_L < \mu_L^*)$ and disappears for $(\mu_L > \mu_L^*)$. In the next subsection we will derive the analytically this value of $\mu_L = \mu_L^*$. Furthermore we see visually that the transcritical bifurcation point at $\beta = 3$, $\mu_L = 1$ forms the limit of the saddle node bifurcation which then subsequently persists for all $\mu_L > 1$ in Curve C as discussed in section 5.5.
Figure 5.18: Bifurcation diagrams for the solution \( p_1 = p_2 \) of the trinary nested logit model where the mode shares of the correlated elemental alternatives in the lower level nest are equal, showing bifurcations in the scale of the lower nests at \( \mu_L = 1 \) and at \( \mu_L = \mu_L^* \).
5.6.4 Stability Analysis: $p_1 = p_2$

In this subsection, our goal is to derive an expression for Curves A and C as shown in panel (a) of Figure 5.12.

The saddle-node bifurcations exhibiting hysteresis behavior for Curve A and the saddle-node bifurcation for Curve C are all cases where the characteristic pairs of saddle points and nodes that suddenly appear as we cross into Regime VII from Regime I and into Regimes II and III from Regime IV are all solutions with $p_1 = p_2$ where the mode shares of the correlated elemental alternatives in the lower level nest are equal. To emphasize, this is true for both directions of approach of the hysteresis region with increasing or decreasing utility parameter $\beta$ for Curve A, as well as for the classic saddle-node bifurcation in Curve C. These bifurcations thus occur entirely on the solution surface (5.114) of the sociodynamic trinary nested logit model derived in the previous subsection 5.6.3. We can use this fact to significantly simplify our analysis by studying the ordinary derivative of just one of the equations $g_0$ or $g_1$ in the system (5.26) and (5.27). Which equation we choose is irrelevant since we have seen in the previous subsection that the equations are directly linearly related as in (5.148) for solutions with $p_1 = p_2$. This leads us to the following theorem which we prove in this subsection.

**Theorem** (Bifurcation curve for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model) Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter $\beta$ real, finite on the proportion $p_1$ of decision-making agents that have chosen alternative $i$, and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter $\mu_L \geq 1$ real, finite. The equilibrium solutions with $p_1 = p_2$ defined on $(0, 1/2)$ on the solution surface (5.114) of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, will be bifurcation points on this surface if values of the parameter $\beta$ satisfy:

$$\beta = \frac{1}{3p_1(1 - 2p_1)} = \frac{1}{3p_0p_1} \quad (5.150)$$

**Proof.** For the special solution $p_1 = p_2$ to find values of the parameters $\beta$ and $\mu_L$ which lead to bifurcations in behavior, that is, change in
number or stability of stationary points, we are interested cases when the derivative \( \frac{dg_0}{dp_1} \) evaluated at a stationary point is zero.

\[
0 = \left. \frac{dg_0}{dp_1} \right|_{p_1=p_2 \atop g_0=0} = \left. \frac{d}{dp_1} \left( 2p_1e^{\beta(1-2p_1)} - 2\pi \left( 1 - 2p_1 \right)e^{\beta p_1} \right) \right|_{p_1=p_2 \atop g_0=0} \\
= \left( 2e^{\beta(1-2p_1)} + (2p_1)(-2\beta e^{\beta(1-2p_1)}) \right) \bigg|_{p_1=p_2} \\
- \left( 2\pi \left( -2e^{\beta p_1} + (1 - 2p_1)\beta e^{\beta p_1} \right) \right) \bigg|_{p_1=p_2} \\
= \left( 2(1 - 2\beta p_1)e^{\beta(1-2p_1)} - 2\pi \left( -2 + \beta(1 - 2p_1) \right)e^{\beta p_1} \right) \bigg|_{p_1=p_2 \atop g_0=0} \tag{5.151}
\]

Dividing through by \( e^{(\beta p_1)} \) which is always nonzero for \( \beta \) real, finite:

\[
0 = 2(1 - 2\beta p_1)e^{\beta(1-3p_1)} - 2\pi \left( -2 + \beta(1 - 2p_1) \right) \tag{5.152}
\]

Re-arranging terms:

\[
\frac{2(1 - 2\beta p_1)e^{\beta(1-3p_1)}}{-2 + \beta(1 - 2p_1)} = 2\pi \tag{5.153}
\]

Taking the natural logarithm of both sides:

\[
\ln 2 + \ln(1 - 2\beta p_1) + \beta(1 - 3p_1) - \ln(-2 + \beta(1 - 2p_1)) = \frac{1}{\mu_L} \ln 2 \tag{5.154}
\]

where the values of \( p_1 \) are restricted for \( \mu_L \) real, finite:

\[
p_1 \neq \frac{1}{2\beta} \quad \text{and} \quad p_1 \neq \frac{\beta - 2}{2\beta} = \frac{1}{2} - \frac{1}{\beta} \tag{5.155}
\]

Solving (5.154) for \( \mu_L \):

\[
\mu_L = \frac{\ln 2}{\ln 2 + \ln(1 - 2\beta p_1) + \beta(1 - 3p_1) - \ln(-2 + \beta(1 - 2p_1))} \tag{5.156}
\]

where we require the denominator is nonzero:

\[
\ln 2 + \ln(1 - 2\beta p_1) + \beta(1 - 3p_1) - \ln(-2 + \beta(1 - 2p_1)) \neq 0 \tag{5.157}
\]

Now, setting (5.153) equal to (5.109) we have the condition for which a bifurcation will occur for the special solution \( p_1 = p_2 \).

\[
2\pi = \frac{2(1 - 2\beta p_1)e^{\beta(1-3p_1)}}{-2 + \beta(1 - 2p_1)} = \frac{2p_1e^{\beta(1-3p_1)}}{-1 - 2p_1} \tag{5.158}
\]

Re-arranging terms and simplifying:

\[
(1 - 2\beta p_1)(1 - 2p_1) = p_1(-2 + \beta(1 - 2p_1)) \\
1 - 2\beta p_1 - 2p_1 + 4\beta p_1^2 = -2p_1 + \beta p_1 - 2\beta p_1^2 \\
6\beta p_1^2 - 3\beta p_1 + 1 = 0
\]
Solving for $\beta$:
\[
\beta = \frac{1}{3p_1(1-2p_1)} = \frac{1}{3p_2p_1}
\] (5.160)

Or alternatively solving for $p_1$:
\[
p_1 = \frac{3\beta \pm \sqrt{(-3\beta)^2 - 4(6\beta)}}{2(6\beta)} = \frac{3\beta \pm \sqrt{9\beta^2 - 24\beta}}{12\beta}
\] (5.161)

Note that the null clines plot for \( F \equiv (5.156) - (5.114) \), or alternatively for \( F^* \equiv (5.153) - (5.109) \), gives the same result as (5.160) and (5.161).

See Figure 5.19 on page 122. Figure 5.20 on page 123 and Figure 5.21 on page 123 show projections of this detail respectively onto the \((\beta, p_1)\)-plane and onto the \((\beta, \mu_L)\)-plane. The leftmost area in Figure 5.21 corresponds to Regime I in Figure 5.3 on page 74; the area under the blue curve near $\beta = 2.8$ corresponds to Regime II in Figure 5.3 (the hysteresis regime, including also the Regime VII in Figure 5.4 on page 75); the area under the blue curve with $\beta > 3$ corresponds to Regime III in Figure 5.3. Regime IV in Figure 5.3 does not appear in the projection in Figure 5.21 because the region of the solution surface with the relevant value of $\mu_L$ is outside the selected detail in terms of $p_1$; the full solution surface for $\mu_L$ namely curves around to yield multiple solutions in $p_1$ as can be seen in the rotational visualization of the surface given in terms of $1/\mu_L$ in Figure 5.16 on page 111. Regime V and VI are also out of scope in the selected detail in Figure 5.21.

**Hysteresis regime: maximum value of $\mu_L$**

**Lemma** (Maximum value of $\mu_L$ for hysteresis for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model) The maximum value of scale parameter $\mu_L$ for hysteresis for an equilibrium solution with $p_1 = p_2$ defined on \((0, 1/2)\) on the solution surface (5.114) of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, and the corresponding value of utility parameter $\beta$, are given by:
\[
\beta = \frac{8}{3} \approx 2.6667
\]
\[
\mu_L = \frac{3\ln 2}{2} \approx 1.0397
\]
\[
p_0 = \frac{1}{2}
\]
\[
p_1 = p_2 = \frac{1}{4}
\]

**Proof.** To find the value of $\mu_L^*$ separating the regime where the hysteresis loop exists as in the top panel of Figure 5.18 on page 118 from...
Figure 5.19: Detail of 3D solution surface (5.114) at hysteresis region in \((\beta, \mu_L)\)-parameter space for the special solution \(p_1 = p_2\) of the sociodynamic trinary nested logit model where the mode shares of the correlated elemental alternatives in the lower level nest are equal. The blue curve (5.150) on the solution surface delineates a continuous string of bifurcation points separating regimes.
Figure 5.20: Projection onto $(\beta, p_1)$-plane of the 3D solution surface detail in Figure 5.19 on page 122. The area to the right of the blue curve in the lower left part of the plane is the hysteresis region. Compare with Figure 5.17 on page 114 to see the location of this region in larger context in $(\beta, p_1)$-parameter space.

Figure 5.21: Projection onto $(\beta, \mu_L)$-plane of 3D solution surface detail in Figure 5.19 on page 122. The area under the blue curve near $\beta = 2.8$ is the hysteresis region. Compare with Figure 5.3 on page 74. (Note that the $\mu_L$-axis is reversed.)
the regime where it does not exist in the bottom panel of Figure 5.18, we are interested in the value of $\mu_L$ for which the lower curve in the lower panel will first have a point of infinite slope for decreasing $\mu_L$. It is evident from the projection of the 3D string of bifurcation points onto the $(\beta, p_1)$-plane in Figure 5.20 on page 123 and onto the $(\beta, \mu_L)$-plane in Figure 5.21 on page 123 that this point occurs along the string of bifurcation points at the lowest value of $\beta$. This is namely the point where the derivative $\frac{dg_0}{dp_1}$ and the derivative $\frac{d\beta}{dp_1}$ evaluated at a stationary point are both zero.

From (5.160) we have:

$$0 = \frac{d\beta}{dp_1} \bigg|_{p_1=p_2, \frac{dg_0}{dp_1}=0} = \frac{d}{dp_1} \left( \frac{1}{3p_1(1-2p_1)} \right) = -\frac{1}{3} \cdot \frac{1-4p_1}{p_1^2(1-2p_1)^2}$$

(5.163)

Thus solving (5.163) for $p_1$ and $p_0$:

$$p_1 = \frac{1}{4}$$

(5.164)

$$p_0 = 1 - 2p_1 = 1 - 2 \cdot \frac{1}{4} = \frac{1}{2}$$

(5.165)

Substituting (5.164) back into (5.160) to determine $\beta$:

$$\beta = \frac{1}{3p_1(1-2p_1)} = \frac{1}{3 \cdot (1/4) \cdot (1-2 \cdot (1/4))} = \frac{8}{3}$$

(5.166)

And finally substituting (5.164) and (5.166) back into (5.156) to determine $\mu_L$:

$$\mu_L = \frac{\ln 2}{\ln 2 + \ln(1-2\beta p_1) + \beta(1-3p_1) - \ln(-2 + \beta(1-2p_1))}$$

$$= \frac{\ln 2}{\ln 2 + \ln(1-2 \cdot \frac{8}{3} \cdot \frac{1}{4}) + \frac{8}{3}(1-3 \cdot \frac{1}{4}) - \ln(-2 + \frac{8}{3}(1-2 \cdot \frac{1}{4}))}$$

$$= \frac{\ln 2}{\ln 2 + \ln(-\frac{1}{3}) + \frac{2}{3} - \ln(-\frac{2}{3})} = \frac{\ln 2 + \ln(-\frac{1}{3}) + \frac{2}{3} - \ln(-\frac{1}{3}) - \ln 2}{1.03972}$$

(5.167)

Hysteresis regime: minimum and maximum value of $p_1$

Lemma (Minimum and maximum value of $p_1$ for hysteresis for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model) The minimum and maximum values of mode share $p_1$ for hysteresis for equilibrium solutions with $p_1 = p_2$ defined on $(0, 1/2)$ on the solution surface (5.114) of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal,
and the corresponding value of utility parameter $\beta$ and scale parameter $\mu_L$, are given by:

$$\beta \approx 2.7456$$
$$\mu_L = 1$$
$$p_0 \approx 0.5848$$
$$p_1 = p_2 \approx 0.2076$$
$$\beta = 3$$
$$\mu_L = 1$$
$$p_0 = p_1 = p_2 = \frac{1}{3}$$  \hspace{1cm} (5.169)

**Proof.** We can also use the relation (5.160) to derive the global lower and upper value of the special solution $p_1 = p_2$ for the bifurcation to and from the hysteresis regime. It is evident from the projection of the 3D string of bifurcation points onto the $(\beta, p_1)$-plane in Figure 5.20 on page 123 and onto the $(\beta, \mu_L)$-plane in Figure 5.21 on page 123 that this point occurs when the string of bifurcation points reaches the limit of the allowed solution values in the $(\beta, p_1)$-plane defined by the constraint $\mu_L = 1$.

Thus at this bifurcation point, from (5.125) and (5.160) we have:

$$\beta = \frac{1}{3p - 1} \ln \frac{p}{1 - 2p} = \frac{1}{3p_1(1 - 2p_1)}$$  \hspace{1cm} (5.170)

Re-arranging terms:

$$\ln \frac{p_1}{1 - 2p_1} = \frac{3p_1 - 1}{3p_1(1 - 2p_1)} = \frac{p_1 - (1 - 2p_1)}{3p_1(1 - 2p_1)}$$  \hspace{1cm} (5.171)

Or alternatively, exponentiating both sides:

$$\frac{p_1}{1 - 2p_1} = \exp \left( \frac{p_1 - (1 - 2p_1)}{3p_1(1 - 2p_1)} \right)$$  \hspace{1cm} (5.172)

Substituting (5.104) to see the symmetry between $p_0$ and $p_1 = p_2$ more immediately for the multinomial logit case where $\mu_L = 1$

$$\frac{p_1}{p_0} = \exp \left( \frac{p_1 - p_0}{3p_1p_0} \right)$$  \hspace{1cm} (5.173)

Equation (5.172) can be solved conveniently graphically by plotting the left hand side (LHS) and the right hand side (RHS) and finding their intersection, or alternatively taking the difference between the LHS and RHS and finding the intersection with the $p_1$-axis. See Figure 5.22 on page 126.

Substituting the lower limit $p_1 = 0.20760$ back into (5.160) to determine $\beta$:

$$\beta = \frac{1}{3p_1(1 - 2p_1)} = \frac{1}{3 \cdot (0.20760) \cdot (1 - 2 \cdot (0.20760))} \approx 2.74564$$  \hspace{1cm} (5.174)
Figure 5.22: Graphical solution of value of $p_1 = p_2$ at $\mu_L = 1$ on the cusp bifurcation, yielding the global lower limit and global upper limit of the special solution $p_1 = p_2$ for the bifurcation to the hysteresis regime.

The value of $p_0$ at the lower limit $p_1 = 0.20760$ is given by (5.104):

$$p_0 = 1 - 2p_1 = 1 - 2 \cdot (0.20760) \approx 0.58480 \quad (5.175)$$

Substituting the upper limit $p_1 = 1/3$ back into (5.160) to determine $\beta$:

$$\beta = \frac{1}{3p_1(1-2p_1)} = \frac{1}{3 \cdot (1/3) \cdot (1 - 2 \cdot (1/3))} = 3 \quad (5.176)$$

The value of $p_0$ at the upper limit $p_1 = 1/3$ is given by (5.104):

$$p_0 = 1 - 2p_1 = 1 - 2 \cdot (1/3) = \frac{1}{3} \quad (5.177)$$

\[ \diamond \]

**Saddle-node bifurcation: behavior of $\beta$ in the limit $\mu_L \to +\infty$**

**Lemma** (Limit behavior of saddle-node bifurcation as $\mu_L \to +\infty$ for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model) The limit behavior of utility parameter $\beta$ as scale parameter $\mu_L \to +\infty$ for the saddle-node bifurcation of an equilibrium solution with $p_1 = p_2$ defined on $(0, 1/2)$ on the solution surface (5.114) of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, is given by:

$$\beta \approx 6.2260$$

$$\mu_L \to +\infty$$

$$p_0 \approx 0.1220$$

$$p_1 = p_2 \approx 0.4390 \quad (5.178)$$

**Proof.** Finally, we can use the relation (5.160) to derive the upper value of parameter $\beta$ as $\mu_L \to \infty$ for the bifurcation generating an unstable
node. It is evident from the projection of the 3D string of bifurcation points onto the $(\beta, p_1)$-plane in Figure 5.20 on page 123 and onto the $(\beta, \mu_L)$-plane in Figure 5.21 on page 123 that this point occurs when the string of bifurcation points reaches the limit of the allowed solution values in the $(\beta, p_1)$-plane defined by the constraint $\mu_L$ finite shown in Figure 5.17 on page 114.

Thus at this bifurcation point, from (5.122) and (5.160) we have:

$$\beta = \frac{1}{3p - 1} \ln \frac{2p}{1 - 2p} = \frac{1}{3p_1(1 - 2p_1)} \quad (5.179)$$

Re-arranging terms:

$$\ln \frac{2p_1}{1 - 2p_1} = \frac{3p_1 - 1}{3p_1(1 - 2p_1)} = \frac{p_1 - (1 - 2p_1)}{3p_1(1 - 2p_1)} \quad (5.180)$$

Or alternatively, exponentiating both sides:

$$\frac{2p_1}{1 - 2p_1} = \exp \left( \frac{p_1 - (1 - 2p_1)}{3p_1(1 - 2p_1)} \right) \quad (5.181)$$

Substituting (5.104) to see the symmetry between $p_0$ and $p_1 = p_2$ more immediately at the bifurcation

$$\frac{2p_1}{p_0} = \exp \left( \frac{p_1 - p_0}{3p_1p_0} \right) \quad (5.182)$$

Equation (5.181) can be solved conveniently graphically by plotting the left hand side (LHS) and the right hand side (RHS) and finding their intersection, or alternatively taking the difference between the LHS and RHS and finding the intersection with the $p_1$ axis. See Figure 5.23 on page 128.

Substituting $p_1 = 0.439025$ back into (5.160) to determine $\beta$:

$$\beta = \frac{1}{3p_1(1 - 2p_1)} = \frac{1}{3 \cdot 0.439025 \cdot (1 - 2 \cdot 0.439025)} \approx 6.22598 \quad (5.183)$$

The value of $p_0$ at the limit $p_1 = 0.439025$ is given by (5.104):

$$p_0 = 1 - 2p_1 = 1 - 2 \cdot 0.439025 \approx 0.12195 \quad (5.184)$$

5.6.5 Stability Analysis: $p_1 = p_2$, Continued

In this subsection, our goal is to derive an expression for Curve D as shown in panel (b) of Figure 5.12.

The pitchfork bifurcation for Curve D is a case where one equilibrium forming the "handle" of the pitchfork on one side of the bifurcation satisfies $p_1 = p_2$ (i.e. in Regimes I and IV) but the two new equilibria in the new "prongs" of the pitchfork on the other side of
the bifurcation satisfy \( p_1 \neq p_2 \) (i.e. in Regimes V and VI). Thus in order to characterize these bifurcation points, we will need to return our study of the eigenvalues of the Jacobian matrix of both equations of the two parameter planar autonomous system (5.26) and (5.27), instead of merely studying the derivative of a single differential equation as in the previous subsection 5.6.4. However we can use the fact that the bifurcation point occurs at a solution with \( p_1 = p_2 \) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, to still simplify our analysis.

At the special solution \( p_1 = p_2 \), the four terms in the Jacobian matrix of \( g = (g_0, g_1) \) given by (5.92) - (5.95) are:

\[
\frac{\partial g_0}{\partial p_0} \bigg|_{p_1=p_2} = \beta e^{\beta p_0} - e^{\beta p_0} - \beta p_0 e^{\beta p_0} \\
- (2 \frac{1}{\mu_L} e^{\beta p_1}) + \beta p_0 \frac{e^{\mu_L \beta p_1}}{2 e^{\mu_L \beta p_1}} (2 \frac{1}{\mu_L} e^{\beta p_1}) \\
= \beta e^{\beta p_0} - e^{\beta p_0} - \beta p_0 e^{\beta p_0} - (2 \frac{1}{\mu_L} e^{\beta p_1}) + \frac{1}{2} \beta p_0 (2 \frac{1}{\mu_L} e^{\beta p_1}) \tag{5.185}
\]

\[
= (\beta - 1 - \beta p_0) e^{\beta p_0} + \frac{1}{2} \beta p_0 - 1)(2 \frac{1}{\mu_L} e^{\beta p_1}) \\
= (2 \beta p_1 - 1)e^{\beta p_0} + (\frac{1}{2} \beta p_0 - 1)(2 \frac{1}{\mu_L} e^{\beta p_1})
\]

\[
\frac{\partial g_0}{\partial p_1} = -\beta p_0 (e^{\mu_L \beta p_1} - e^{\mu_L \beta p_1}) (2 e^{\mu_L \beta p_1}) \frac{1 - \mu_L}{\mu_L} = 0 \tag{5.186}
\]
\[
\frac{\partial g_1}{\partial p_0} = -\beta(1 - \mu_L) \frac{e^{\mu_L \beta p_1} e^{\mu_L \beta p_1}}{(2e^{\mu_L \beta p_1})(2e^{\mu_L \beta p_1})} \frac{1}{\mu_L} \\
- \beta p_1 e^{\beta p_0} + \beta p_1 \frac{e^{\mu_L \beta p_1}}{(2e^{\mu_L \beta p_1})} \frac{1}{\mu_L} \\
= -\frac{\beta}{4} (1 - \mu_L)(2\pi e^{\beta p_1}) - \beta p_1 e^{\beta p_0} + \frac{\beta}{2} p_1 (2\pi e^{\beta p_1}) \\
= \frac{\beta}{4}(\mu_L - 1 - 2p_1)(2\pi e^{\beta p_1}) - \beta p_1 e^{\beta p_0}
\]

\[
\frac{\partial g_1}{\partial p_1} = \mu_L \beta e^{\mu_L \beta p_1} \frac{(2e^{\mu_L \beta p_1})^{1 - \frac{\mu_L}{\mu_L}}}{\mu_L} \\
+ (1 - \mu_L) \beta e^{\mu_L \beta p_1} \frac{(2e^{\mu_L \beta p_1})^{1 - \frac{\mu_L}{\mu_L}}}{\mu_L} (e^{\mu_L \beta p_1} - e^{\mu_L \beta p_1}) \\
- e^{\beta p_0} - (2e^{\mu_L \beta p_1})^{1 - \frac{\mu_L}{\mu_L}} \beta p_1 (e^{\mu_L \beta p_1} - e^{\mu_L \beta p_1}) \\
= \mu_L \beta \frac{e^{\mu_L \beta p_1}}{(2e^{\mu_L \beta p_1})}(2\pi e^{\beta p_1}) + 0 - e^{\beta p_0} - (2\pi e^{\beta p_1}) - 0 \\
= \left(\frac{1}{2} \mu_L \beta - 1\right)(2\pi e^{\beta p_1}) - e^{\beta p_0}
\]  

(5.187)

(5.188)

To determine bifurcations in behavior, that is, change in number or stability of stationary points, we are interested cases when at least one eigenvalue of the Jacobian has zero real part. Recall from the outset of section 5.6 that there will exist at least one zero eigenvalue if the determinant of the Jacobian is equal to zero. There will exist a zero real part of a complex eigenvalue if the determinant is positive and trace is equal to zero.

\[
\det J = 0: \quad \lambda_{1,2} = \frac{1}{2} \text{tr} J \pm \frac{1}{2} \sqrt{\left(\text{tr} J\right)^2 - 4 * 0} = \frac{1}{2} \text{tr} J \pm \frac{1}{2} \text{tr} J \\
\lambda_1 = 0, \quad \lambda_2 = \text{tr} J
\]  

(5.189)

\[
\det J > 0, \quad \text{tr} J = 0: \quad \lambda_{1,2} = \frac{1}{2} \times 0 \pm \frac{1}{2} \sqrt{0^2 - 4 \det J} = \pm i \sqrt{\det J}
\]  

(5.190)

Let us first consider (5.189) where the determinant is equal to zero for the special solution with \( p_1 = p_2 \) where the mode shares of the correlated elemental alternatives in the lower level nest are equal:

\[
0 = \det J = \frac{\partial g_0}{\partial p_1} \frac{\partial g_1}{\partial p_1} - \frac{\partial g_0}{\partial p_1} \frac{\partial g_1}{\partial p_0} = \\
\left(2p_1 - 1\right)e^{\beta p_0} + \frac{1}{2} \beta p_0 - 1\right)(2\pi e^{\beta p_1}) \\
\times \left(\frac{1}{2} \mu_L \beta - 1\right)(2\pi e^{\beta p_1}) - e^{\beta p_0}
\]  

(5.191)

Since \( \frac{\partial g_0}{\partial p_1} = 0 \) for the special solution \( p_1 = p_2 \), we thus must have either \( \frac{\partial g_0}{\partial p_0} = 0 \) or \( \frac{\partial g_1}{\partial p_1} = 0 \) for the determinant to be zero.
We will consider each of these cases in turn below. Hereby we will retrieve results from subsection 5.6.4, but we will now also be able to express Curve D as shown in panel (b) of Figure 5.12. Then later at the end of this subsection we will consider the case (5.190) where the trace is equal to zero and the determinant is strictly positive for the special solution with $p_1 = p_2$.

Case I. Suppose $\partial g_0/\partial p_0 = 0$

From (5.185) we have:

\[
(2\beta p_1 - 1)e^{\beta p_0} + \left(\frac{1}{2}\beta p_0 - 1\right)(2\pi e^{\beta p_1}) = 0
\]

\[
(2\beta p_1 - 1)e^{\beta(1-2p_1)} + \left(\frac{1}{2}\beta(1-2p_1) - 1\right)(2\pi e^{\beta p_1}) = 0
\]

\[
(2\beta p_1 - 1)e^{\beta e^{-2\beta p_1}} + \left(\frac{1}{2}\beta - \beta p_1 - 1\right)(2\pi e^{\beta p_1}) = 0
\]

\[
(2\beta p_1 - 1)e^{\beta} + \left(\frac{1}{2}\beta - \beta p_1 - 1\right)(2\pi e^{3\beta p_1}) = 0
\]

Solving (5.192) for $\mu_L$ we re-gain the same result as (5.156)

\[
\mu_L = \frac{\ln 2}{\ln 2 + \ln(1-2\beta p_1) + \beta(1-3p_1) - \ln(-2+\beta(1-2p_1))}
\]

(5.193)

Thus as derived previously in (5.160), solutions where a bifurcation occurs will satisfy

\[
\beta = \frac{1}{3p_1(1-2p_1)} = \frac{1}{3p_0p_1}
\]

(5.194)

and equivalently:

\[
p_1 = \frac{3\beta \pm \sqrt{(-3\beta)^2 - 4(6\beta)}}{2(6\beta)} = \frac{3\beta \pm \sqrt{9\beta^2 - 24\beta}}{12\beta}
\]

(5.195)

Substituting both +/ - branches of (5.195) back into (5.114) we obtain the bifurcation cusp for $\mu_L$ in terms of $\beta$. See Figure 5.24 on page 131.

The example null clines solutions in panel (e) in Figure 5.2 on page 68 and panels (b) and (c) in Figure 5.14 on page 107, and also in panel (a) in Figure 5.14 and in Figure 5.13 on page 105 show that transition between Regime I to Regime VII to Regime I, and transition between Regime I to Regime VII and Regime II to Regime IV, that is, crossing the cusp in $(\beta, \mu_L)$-parameter space inwards and then outwards, yields a hysteresis loop. The example null clines solutions in panels (c) and (b) in Figure 5.5 show that transition between Regime IV to Regime III, yields a classic saddle-node bifurcation.
Case II. Suppose $\frac{\partial g_1}{\partial p_1} = 0$

We saw at the outset of this subsection that the element of the Jacobian matrix $\frac{\partial g_0}{\partial p_1} = 0$ for the special solution $p_1 = p_2$ and thus we must have either $\frac{\partial g_0}{\partial p_0} = 0$ or $\frac{\partial g_1}{\partial p_1} = 0$ for the determinant of the Jacobian matrix to be zero. Having retrieved the results from subsection 5.6.4 by considering the case $\frac{\partial g_0}{\partial p_0} = 0$ for the special solution $p_1 = p_2$, we now proceed to consider the case $\frac{\partial g_1}{\partial p_1} = 0$ for the special solution $p_1 = p_2$. This leads to the following theorem which we will prove in this subsection. Hereby we will now be able to express Curve D as shown in panel (b) of Figure 5.12.

**Theorem** (Bifurcation curve for the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model, continued) Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter $\beta$ real, finite on the proportion $p_1$ of decision-making agents that have chosen alternative $i$, and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter $\mu_L \geq 1$ real, finite. The equilibrium solutions with $p_1 = p_2$ defined on $(0, 1/2)$ of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, will be bifurcation points for which at least one zero eigenvalue occurs and for which emerging solutions upon
bifurcation are asymmetric in the mode shares $p_1 \neq p_2$, if values of the parameters $\beta$ and $\mu_L$ satisfy:

$$\beta = \frac{1}{\mu_L p_1} \quad (5.196)$$

**Proof.** Suppose $\partial g_1 / \partial p_1 = 0$ at the special solution $p_1 = p_2$. From (5.188) we have:

$$\frac{1}{2} \mu_L \beta - 1 (\frac{1}{2} \pi e^{\beta p_1}) - e^{\beta p_0} = 0$$

$$\frac{1}{2} \mu_L \beta - 1 (\frac{1}{2} \pi e^{\beta p_1}) = e^{\beta p_0} = e^{\beta (1 - 2 p_1)} = e^{\beta e^{-2 \beta p_1}}$$

$$\frac{1}{2} \mu_L \beta - 1 (\frac{1}{2} \pi e^{3 \beta p_1}) = e^{\beta}$$

$$e^{3 \beta p_1} = \frac{e^{\beta}}{2 \pi (\frac{1}{2} \mu_L \beta - 1)}$$

Re-arranging terms:

$$2^{\frac{1}{\pi}} = \frac{e^{\beta (1 - 3 \beta p_1)}}{\left(\frac{1}{2} \mu_L \beta - 1\right)} \quad (5.198)$$

Comparing the condition (5.109) for solution $g = 0$ under the special solution $p_1 = p_2$, and condition (5.198) for $\det J = 0$, we have

$$2^{\frac{1}{\pi}} = \frac{e^{\beta (1 - 3 \beta p_1)}}{(\frac{1}{2} \mu_L \beta - 1)} = \frac{2 p_1 e^{\beta (1 - 3 p_1)}}{1 - 2 p_1} \quad (5.199)$$

Re-arranging terms and simplifying:

$$1 - 2 p_1 = 2 p_1 (\frac{1}{2} \mu_L \beta - 1) = \mu_L \beta p_1 - 2 p_1$$

$$1 = \mu_L \beta p_1$$

and solving for $\beta$

$$\beta = \frac{1}{\mu_L p_1} \quad (5.201)$$

or similarly, solving for $p_1$

$$p_1 = \frac{1}{\mu_L \beta} \quad (5.202)$$

This gives us a condition for which a bifurcation will occur for the special solution $p_1 = p_2$.

Substituting (5.201) back into (5.114) we obtain

$$\mu_L = \frac{\ln 2}{\ln 2 + \ln \frac{1}{\mu_L \beta} + \beta (1 - 3 \frac{1}{\mu_L \beta}) - \ln (1 - 2 \frac{1}{\mu_L \beta})}$$

$$= \frac{\ln 2}{\ln 2 + \left(\frac{2}{\mu_L \beta}\right) + \beta - \frac{3}{\mu_L} + \beta - \frac{3}{\mu_L}} \quad (5.203)$$

$$= \frac{\ln 2}{\ln 2 + \left(\frac{2}{\mu_L \beta}\right) + \beta - \frac{3}{\mu_L} + \beta - \frac{3}{\mu_L}}$$
Re-arranging terms and simplifying

\[ 0 = \mu_L \beta - \ln 2 - \mu_L \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) - 3 \tag{5.204} \]

we obtain a transcendental relation between \( \mu_L \) and \( \beta \) for the bifurcation curve. ♦

![Figure 5.25: Bifurcation curve in \((\beta, \mu_L)\)-parameter space satisfying \( \partial g_1 / \partial p_1 = 0 \). Compare with Figure 5.3 on page 74.](image)

The example null cines solutions in panel (e) in Figure 5.2 on page 68 and panel (d) in Figure 5.5 on page 76, and panels (c) and (e) in Figure 5.5 show that transition between Regime I to Regime V, and transition between Regime IV to Regime VI, that is, crossing the bifurcation curve in \((\beta, \mu_L)\)-parameter space inwards, yields a pitchfork bifurcation.

Alternatively, we can solve (5.197) directly for \( p_1 \). Taking the natural logarithm of both sides of (5.197):

\[ 3\beta p_1 = \beta - \frac{1}{\mu_L} \ln 2 - \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \tag{5.205} \]

Thus:

\[ p_1 = \frac{1}{3} - \frac{1}{3\mu_L \beta} \ln 2 - \frac{1}{3\beta} \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \tag{5.206} \]

Now recall from (5.107) and (5.147) that considering (5.26) or (5.27) for the case \( p_1 = p_2 \), we have:

\[ 0 = g_0 \big|_{p_1=p_2} = -2 \cdot g_1 \big|_{p_1=p_2} = 2p_1 e^{\beta (1-2p_1)} - 2 \frac{1}{2\pi} (1 - 2p_1) e^{\beta p_1} \tag{5.207} \]
Multiplying (5.207) through by $3\beta \exp(2\beta p_1)$

$$0 = 2 \cdot 3\beta p_1 e^\beta - (3\beta - 2 \cdot 3\beta p_1)(2^{\frac{1}{2\beta}} e^{3\beta p_1}) \tag{5.208}$$

Substituting (5.197) and (5.205) into (5.208)

$$0 = 2e^\beta \left( \beta - \frac{1}{\mu_L} \ln 2 - \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \right)$$

$$- \left( 3\beta - 2(\beta - \frac{1}{\mu_L} \ln 2 - \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \right) \cdot 2^{\frac{1}{2\beta}} \frac{e^\beta}{2 \pi (\frac{1}{2} \mu_L \beta - 1)}$$

$$= 2e^\beta \left( \beta - \frac{1}{\mu_L} \ln 2 - \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \right)$$

$$- \left( \beta + \frac{2}{\mu_L} \ln 2 + 2 \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \right) \frac{e^\beta}{(\frac{1}{2} \mu_L \beta - 1)} \tag{5.209}$$

Multiplying (5.209) through by the inverse of the second term in the difference, and simplifying:

$$0 = (\mu_L \beta - 2) \left( \beta - \frac{1}{\mu_L} \ln 2 - \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \right)$$

$$- \left( \beta + \frac{2}{\mu_L} \ln 2 + 2 \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \right)$$

$$= \mu_L \beta^2 - \frac{\mu_L \beta}{\mu_L} \ln 2 - \mu_L \beta \ln \left( \frac{1}{2} \mu_L \beta - 1 \right)$$

$$- 2\beta + \frac{2}{\mu_L} \ln 2 + 2 \ln \left( \frac{1}{2} \mu_L \beta - 1 \right)$$

$$- \beta - \frac{2}{\mu_L} \ln 2 - 2 \ln \left( \frac{1}{2} \mu_L \beta - 1 \right)$$

$$= \mu_L \beta^2 - \beta \ln 2 - \mu_L \beta \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) - 3\beta \tag{5.210}$$

Thus we re-gain the same result as above (5.204):

$$0 = \mu_L \beta - \ln 2 - \mu_L \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) - 3 \tag{5.211}$$

Dividing (5.211) through by 3 and re-arranging terms:

$$1 = \frac{1}{3} \mu_L \beta - \frac{1}{3} \ln 2 - \frac{1}{3} \mu_L \ln \left( \frac{1}{2} \mu_L \beta - 1 \right)$$

$$= \mu_L \beta \left( \frac{1}{3} - \frac{1}{3\mu_L \beta} \ln 2 - \frac{1}{3\beta} \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \right) \tag{5.212}$$

Substituting (5.206) into (5.212), we re-gain (5.200)

$$1 = \mu_L \beta p_1 \tag{5.213}$$
Pitchfork bifurcation: minimum value of $\mu_L$

Lemma (Minimum value of $\mu_L$ for pitchfork bifurcation of the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model with emerging solutions $p_1 \neq p_2$) The minimum value of scale parameter $\mu_L$, and the corresponding value of utility parameter $\beta$, for pitchfork bifurcation of the equilibrium solution with $p_1 = p_2$ defined on $(0, 1/2)$ of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, but for which the new emerging solutions upon bifurcation are asymmetric in these mode shares $p_1 \neq p_2$, occurs for:

$$\begin{align*}
\beta &\approx 1.6238 \\
\mu_L &\approx 3.2060 \\
p_0 &\approx 0.6158 \\
p_1 = p_2 &\approx 0.1921
\end{align*}$$ (5.214)

Proof. To find the lower value of $\mu_L^*$ at which the bifurcation point appears that gives rise to the pair of pitchfork bifurcations that then subsequently exist for all $\mu_L > \mu_L^*$, we are thus interested in where the partial derivative of (5.204) with respect to $\beta$ evaluated along the bifurcation curve in $(\beta, \mu_L)$-parameter space is equal to zero.

$$\begin{align*}
0 &= \frac{\partial}{\partial \beta} \left( \mu_L \beta - \ln 2 - \mu_L \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) - 3 \right) \\
&= \mu_L - \frac{1}{2} \mu_L^2 \cdot \frac{1}{2 \mu_L \beta - 1} = \mu_L \left( 1 - \frac{\mu_L}{\mu_L \beta - 2} \right)
\end{align*}$$ (5.215)

Since $\mu_L \geq 1$, we must have

$$
0 = 1 - \frac{\mu_L}{\mu_L \beta - 2} : \mu_L \beta - 2 = \mu_L
$$ (5.216)

Solving for $\mu_L$

$$
\mu_L = \frac{2}{\beta - 1}
$$ (5.217)

or solving for $\beta$

$$
\beta = 1 + \frac{2}{\mu_L}
$$ (5.218)

Substituting (5.218) back into (5.204), we have

$$\begin{align*}
0 &= \mu_L \left( 1 + \frac{2}{\mu_L} \right) - \ln 2 - \mu_L \ln \left( \frac{1}{2} \mu_L \left( 1 + \frac{2}{\mu_L} \right) - 1 \right) - 3 \\
&= \mu_L - \ln 2 - \mu_L \ln \left( \frac{1}{2} \mu_L \right) - 1
\end{align*}$$ (5.219)

Re-arranging terms:

$$
\mu_L - 1 = \ln 2 + \mu_L \ln \left( \frac{1}{2} \mu_L \right) = \ln \left( 2 \cdot \left( \frac{1}{2} \mu_L \right)^{\mu_L} \right)
$$ (5.220)
Exponentiating both sides:

$$\exp(\mu_L - 1) = 2\left(\frac{1}{\mu_L}\right)^{\mu_L} \tag{5.221}$$

Or:

$$\frac{1}{2e} = \left(\frac{\mu_L}{2e}\right)^{\mu_L} \tag{5.222}$$

This equation can be solved conveniently graphically by plotting the left hand side (LHS) and the right hand side (RHS) of (5.222) and finding their intersection, or alternatively by plotting (5.219) and finding the intersection with the $\mu_L$ axis. See Figure 5.26 on page 136.

![Graphical solution of global lower limit of $\mu_L$ for pitchfork bifurcation.](image)

Figure 5.26: Graphical solution of global lower limit of $\mu_L$ for pitchfork bifurcation.

Then substituting $\mu_L \approx 3.20603$ back into (5.218) we can determine the value of $\beta$ at the bifurcation point:

$$\beta = 1 + \frac{2}{\mu_L} = 1 + \frac{2}{3.20603} \approx 1.62382 \tag{5.223}$$

Using (5.202) to solve for $p_1$

$$p_1 = \frac{1}{(3.20603)(1.62382)} \approx 0.192085 \tag{5.224}$$

The value of $p_0$ at the limit $p_1 \approx 0.192085$ is given by (5.104):

$$p_0 = 1 - 2p_1 = 1 - 2 \cdot (0.192085) \approx 0.61583 \tag{5.225}$$

\[\diamond\]

**Pitchfork bifurcation: behavior of $\beta$ in the limit $\mu_L \to +\infty$**

**Lemma** (Limit behavior of pitchfork bifurcation as $\mu_L \to +\infty$ of the solution $p_1 = p_2$ of the sociodynamic trinary nested logit model with emerging solutions $p_1 \neq p_2$) The limit behavior of utility parameter $\beta$ as scale parameter $\mu_L \to +\infty$ for pitchfork bifurcation of the equilibrium solution with $p_1 = p_2$ defined
on \((0, 1/2)\) of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, but for which the new emerging solutions upon bifurcation are asymmetric in these mode shares \(p_1 \neq p_2\), is given by:

\[
\begin{align*}
\beta &= 0 \\
\mu_L &\to +\infty \\
p_0 &= \frac{1}{2} \\
p_1 &= p_2 = \frac{1}{4} \\
\beta &\to +\infty \\
\mu_L &\to +\infty \\
p_0 &= 1 \\
p_1 &= p_2 = 0
\end{align*}
\] (5.226)

**Proof.** The null cline for (5.204) [equivalently (5.211)] gives us the bifurcation curve for the case \(\partial g_1/\partial p_1 = 0\) for the special solution \(p_1 = p_2\). We saw previously via (5.183) that the bifurcation curve for the case \(\partial g_0/\partial p_0 = 0\) for the special solution \(p_1 = p_2\) reaches a limit of \(\beta \approx 6.22598\) in the limit \(\mu_L \to +\infty\). We are thus interested to find whether \(\beta\) also tapers off to some finite value for the case \(\partial g_1/\partial p_1 = 0\) as well, or whether the branches of \(\beta\) instead go to \(\pm \infty\).

To determine the behavior of \(\beta\) for large \(\mu_L\) it is convenient to write (5.204) as follows:

\[
0 = \mu_L \left( \beta - \frac{\ln 2}{\mu_L} - \frac{1}{2} \mu_L \beta - 1 - \frac{3}{\mu_L} \right) \equiv \mu_L \cdot f(\mu_L, \beta) \quad (5.228)
\]

Since we have defined \(\mu_L \geq 1\), we require that the term \(f(\beta, \mu_L)\) be equal to zero in order for the product with \(\mu_L\) to be zero. Then, taking the limit \(\mu_L \to +\infty\) we have

\[
0 = \lim_{\mu_L \to \infty} f(\mu_L, \beta) \approx \beta - \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \quad (5.229)
\]

Re-arranging terms

\[
\beta \approx \ln \left( \frac{1}{2} \mu_L \beta - 1 \right) \quad (5.230)
\]

Exponentiating both sides

\[
e^\beta \approx \frac{1}{2} \mu_L \beta - 1 \quad (5.231)
\]

Solving for \(\mu_L\)

\[
\mu_L \approx \frac{2(e^\beta + 1)}{\beta} \quad (5.232)
\]
For the limit \( \mu_L \to +\infty \), we must have that either the denominator on the right hand side is zero, or the numerator goes to infinity. Thus

\[
\beta = 0 \; ; \; \beta \to \infty
\]  

\( \diamondsuit \)

**Case III.** Suppose \( \text{tr} \, J = 0 \) and \( \text{det} \, J > 0 \)

We now consider the case (5.190) where the trace of the Jacobian is equal to zero and the determinant of the Jacobian is strictly positive for the special solution with \( p_1 = p_2 \). If values of the parameters \( \beta \) and \( \mu_L \) real, finite exist such that (5.190) is true, we will have zero real part of a pair of complex eigenvalues, i.e. purely imaginary eigenvalues. We will now show that (5.190) cannot be satisfied for solutions of sociodynamic trinary nested logit model with \( p_1 = p_2 \). This leads us to the following lemma.

**Lemma** (No bifurcation points of the sociodynamic trinary nested logit model with \( p_1 = p_2 \) having purely imaginary eigenvalues)

Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_i \) of decision-making agents that have chosen alternative \( i \), and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter \( \mu_L \geq 1 \) real, finite. The equilibrium solutions with \( p_1 = p_2 \) defined on \((0, 1/2)\) of the sociodynamic trinary nested logit model (5.17) where the mode shares of the correlated elemental alternatives in the lower level nest are equal, exhibit no bifurcation points with purely imaginary eigenvalues.

**Proof.** Let us consider (5.190) for the special solution \( p_1 = p_2 \),

\[
0 = \text{tr} \, J|_{p_1 = p_2} = \left( \frac{\partial g_0}{\partial p_0} + \frac{\partial g_1}{\partial p_1} \right)_{p_1 = p_2} \\
= (2\beta p_1 - 1)e^\beta p_0 + \frac{1}{2}\beta p_0 - 1)(2\mu_L e^\beta p_1) \\
+ (\frac{1}{2}\mu_L \beta - 1)(2\mu_L e^\beta p_1) - e^\beta p_0 \\
= 2(\beta p_1 - 1)e^\beta p_0 + (\frac{1}{2}\beta p_0 - 2 + \frac{1}{2}\mu_L \beta)(2\mu_L e^\beta p_1) \\
= 2(\beta p_1 - 1)e^{\beta(1-2p_1)} + (\frac{1}{2}\beta(1-2p_1) - 2 + \frac{1}{2}\mu_L \beta)(2\mu_L e^\beta p_1) \\
= 2(\beta p_1 - 1)e^{\beta - \beta p_1} + (\frac{1}{2}\beta - \beta p_1 - 2 + \frac{1}{2}\mu_L \beta)(2\mu_L e^\beta p_1) \\
= 2(\beta p_1 - 1)e^\beta p_1 + (\frac{1}{2}\beta - \beta p_1 - 2 + \frac{1}{2}\mu_L \beta)(2\mu_L e^\beta p_1) \\
\]  

(5.234)
For there to exist a zero real part of a complex eigenvalue, we also require that the determinant of the Jacobian is strictly positive. However since \( \partial g_0 / \partial p_1 = 0 \) for the special solution \( p_1 = p_2 \), we have the simplification:

\[
0 < \det J = \frac{\partial g_0}{\partial p_0} \frac{\partial g_1}{\partial p_1} - \frac{\partial g_0}{\partial p_1} \frac{\partial g_1}{\partial p_0} = \frac{\partial g_0}{\partial p_0} \frac{\partial g_1}{\partial p_1} - 0 \tag{5.235}
\]

Without further computation, we thus can conclude already immediately that there can exist no purely imaginary eigenvalues, since (5.234) and (5.235) cannot be simultaneously satisfied. The requirement (5.234) that the trace is zero implies that \( \partial g_0 / \partial p_0 \) and \( \partial g_1 / \partial p_1 \) must both be zero or have opposite signs, but the requirement (5.235) that the determinant is strictly positive implies that \( \partial g_0 / \partial p_0 \) and \( \partial g_1 / \partial p_1 \) must be non-zero and have the same signs.

5.6.6 Special Case: \( \beta = 0 \), Re-visited

In this subsection we re-visit the special case of a null trinary nested logit model with no social interactions studied earlier in subsection 5.1.3 and show that a bifurcation exists in the limit \( \mu_L \to +\infty \) when \( \beta = 0 \).

Recall from (5.47) that for the case \( \mu_L \to +\infty \) when \( \beta = 0 \), we have:

\[
p_1 = p_2 = \frac{1}{2} \left( \frac{2^0}{1 + 2^0} - \frac{1}{4} \right) \tag{5.236}
\]

At the limiting solution (5.236), the four terms in the Jacobian matrix of \( g = (g_0, g_1) \) given by (5.92) - (5.95) are:

\[
\left. \frac{\partial g_0}{\partial p_0} \right|_{p_0=1/2, p_1=p_2=1/4} = \beta e^{\beta/2} - e^{\beta/2} - \beta \frac{1}{2} e^{\beta/2} - \left( e^{\mu_L \beta/4} + e^{\mu_L \beta/4} \right) \frac{1}{\mu_L} - \frac{\beta}{2} e^{\mu_L \beta/4} \left( e^{\mu_L \beta/4} + e^{\mu_L \beta/4} \right) \frac{1}{\mu_L} = \frac{\beta}{2} e^{\beta/4} - e^{\beta/4} - (2e^{\mu_L \beta/4}) \frac{1}{\mu_L} + \beta \frac{1}{2} e^{\mu_L \beta/4} (2e^{\mu_L \beta/4}) \frac{1}{\mu_L} = \left( \frac{\beta}{2} - 1 \right) e^{\beta/4} + (\frac{\beta}{4} - 1) (2\left( \frac{1}{\mu_L} e^{\beta/4} \right) = \left( \frac{\beta}{2} e^{\beta/4} - e^{\beta/4} + \frac{\beta}{4} \frac{1}{\mu_L} - 2 \frac{1}{\mu_L} e^{\beta/4} \right) e^{\beta/4} \tag{5.237}
\]

\[
\left. \frac{\partial g_0}{\partial p_1} \right|_{p_0=1/2, p_1=p_2=1/4} = -\beta \frac{1}{2} \left( e^{\mu_L \beta/4} - e^{\mu_L \beta/4} \right) \left( e^{\mu_L \beta/4} + e^{\mu_L \beta/4} \right) \frac{1}{\mu_L} = 0 \tag{5.238}
\]
\[ \frac{\partial g_1}{\partial p_0} \bigg|_{p_0 = 1/2, p_1 = p_2 = 1/4} = -\beta (1 - \mu L) e^{\mu L \beta / 4} e^{\mu L \beta / 4} (e^{\mu L \beta / 4} + e^{\mu L \beta / 4}) \frac{1 - \mu L}{\mu L} \]

\[ - \frac{\beta}{4} e^{\beta / 2} + \frac{\beta}{4} e^{\mu L \beta / 4} (e^{\mu L \beta / 4} + e^{\mu L \beta / 4}) \frac{1 - \mu L}{\mu L} \]

\[ = -\beta (1 - \mu L) \frac{e^{\mu L \beta / 4} e^{\mu L \beta / 4}}{(2e^{\mu L \beta / 4})(2e^{\mu L \beta / 4})} (2e^{\mu L \beta / 4}) \frac{1}{\mu L} \]

\[ - \frac{\beta}{4} e^{\beta / 2} + \frac{\beta}{4} e^{\mu L \beta / 4} (2e^{\mu L \beta / 4}) \frac{1}{\mu L} \]

\[ = \frac{\beta}{4} (1 - \mu L) (2 \frac{1}{\mu L} e^{\beta / 4}) - \frac{\beta}{4} e^{\beta / 4} e^{\beta / 4} + \frac{\beta}{4} (2 \frac{1}{\mu L} e^{\beta / 4}) \]

\[ = \frac{\beta}{4} e^{\beta / 4} (2 \frac{1}{\mu L} - 1) \frac{1}{2} \cdot 2 \frac{1}{\mu L} - e^{\beta / 4} \]

\[ (5.239) \]

\[ \frac{\partial g_1}{\partial p_1} \bigg|_{p_0 = 1/2, p_1 = p_2 = 1/4} = \mu L \beta e^{\mu L \beta / 4} (e^{\mu L \beta / 4} + e^{\mu L \beta / 4}) \frac{1 - \mu L}{\mu L} \]

\[ + (1 - \mu L) \frac{\beta}{4} e^{\mu L \beta / 4} (e^{\mu L \beta / 4} + e^{\mu L \beta / 4}) \frac{1 - \mu L}{\mu L} (e^{\mu L \beta / 4} - e^{\mu L \beta / 4}) \]

\[ - e^{\beta / 2} - (e^{\mu L \beta / 4} + e^{\mu L \beta / 4}) \frac{1}{\mu L} \]

\[ - \beta p_1 (e^{\mu L \beta / 4} + e^{\mu L \beta / 4}) \frac{1 - \mu L}{\mu L} (e^{\mu L \beta / 4} - e^{\mu L \beta / 4}) \]

\[ = \mu L \beta \frac{e^{\mu L \beta / 4}}{(2e^{\mu L \beta / 4})(2e^{\mu L \beta / 4})} (2e^{\mu L \beta / 4}) \frac{1}{\mu L} + 0 - e^{\beta / 2} - (2e^{\mu L \beta / 4}) \frac{1}{\mu L} - 0 \]

\[ = \mu L \beta \frac{1}{2} \frac{1}{\mu L} e^{\beta / 4} - e^{\beta / 4} e^{\beta / 4} - 2 \frac{1}{\mu L} e^{\beta / 4} \]

\[ = (\mu L \beta) \frac{1}{2} \frac{1}{\mu L} e^{\beta / 4} - 2 \frac{1}{\mu L} e^{\beta / 4} \]

\[ (5.240) \]

To check for bifurcations in behavior, that is, change in number or stability of stationary points, we are interested cases when at least one eigenvalue of the Jacobian has zero real part. Recall from (5.89) that there will exist at least one zero eigenvalue if the determinant of the Jacobian is equal to zero. There would exist a zero real part of a complex eigenvalue if the determinant is positive and trace is equal to zero; however, we have seen already in (5.234) and (5.235) in Case III at the end of the previous subsection that the conditions tr J = 0 and det J > 0 cannot be simultaneously satisfied for the special solution p_1 = p_2, since we have \( \frac{\partial g_0}{\partial p_1} = 0 \). Thus we do not need to consider any possibility of purely imaginary eigenvalues.
Evaluating the determinant at the limiting solution (5.236), for the case $\mu_L \to +\infty$ when $\beta = 0$:

$$
\det J_{\beta=0, \mu_L \to \infty} = \left( \frac{\partial g_0 \partial g_1}{\partial p_0 \partial p_1} - \frac{\partial g_0 \partial g_1}{\partial p_1 \partial p_0} \right)_{\beta=0, \mu_L \to \infty}
$$

$$
= \left( \frac{\beta}{2} e^{\beta/4} - e^{\beta/4} + \frac{\beta}{4} 2^{1/4} \mu_L - 2^{1/4} \right)_{\beta=0, \mu_L \to \infty}
$$

$$
\times \left( \mu_L \frac{1}{2} e^{\beta/4} - e^{\beta/4} - 2^{1/4} \mu_L \right)_{\beta=0, \mu_L \to \infty} - 0
$$

$$
= (0 \cdot e^0 - e^0 + 0 \cdot 2^0 - 2^0) e^0 (\mu_L \frac{1}{2} \mu_L - e^0 - 2^0) e^0
$$

$$
= (-1 - 1) \cdot (\mu_L \frac{1}{2} - 1 - 1) = -\mu_L \beta + 4
$$

Suppose the bifurcation curve in ($\beta$, $\mu_L$)-parameter space defined by $\partial g_1/\partial p_1 = 0$ for the special solution $p_1 = p_2$ approaches $\beta = 0$ as $\mu_L \to +\infty$. Then substituting (5.236) into (5.200) we have:

$$
\mu_L \beta = \frac{1}{p_1} = \frac{1}{(1/4)} = 4
$$

(5.242)

Substituting (5.242) into (5.241) we have that the determinant is indeed equal to zero, re-confirming the bifurcation at $\beta = 0$ as $\mu_L \to +\infty$ in our earlier result (5.233).

5.6.7 General Stability Analysis

In this subsection, our goal is to derive an expression for Curve B as shown in panel (c) of Figure 5.12.

The double saddle-node bifurcation for Curve B is a case where the resulting pairs of saddle points and stable nodes when crossing from Regime VII to II, from Regime I to IV, and from Regime V to VI, are all characterized by $p_1 \neq p_2$. In order to express the Jacobian matrix at these bifurcation points we will need to relax the analytically convenient simplification which we applied in subsections 5.6.4 and 5.6.5 that the bifurcation occurs at a solution with $p_1 = p_2$ where the mode shares of the correlated elemental alternatives in the lower level nest are equal, and consider now the stability analysis in the general case for all equilibrium solutions $p_0, p_1, p_2$ defined on $[0, 1]$ of the sociodynamic trinary nested logit model (5.17).

Let us return to the original stability analysis at the outset of section 5.6. Since we are interested in evaluating the Jacobian matrix of $g = (g_0, g_1)$ at equilibrium solutions, we can use equilibrium equations (5.18) - (5.20) to simplify the elements of the Jacobian (5.92) - (5.95).

For analytical convenience, we make the following definition:

$$
Z \equiv e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{1/4}
$$

(5.243)
Substituting (5.22) and (5.243) into (5.18) - (5.20)

\[ p_0 = \frac{e^{\beta p_0}}{e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{\frac{1}{n_1}}} = \frac{e^{\beta p_0}}{Z} \]  
(5.244)

\[ p_1 = \frac{e^{\mu_L \beta p_1}}{e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2}} \cdot \frac{(e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2})^{\frac{1}{n_1}}}{e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2})^{\frac{1}{n_1}}} \]
\[ = \frac{e^{\mu_L \beta p_1}(e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{\frac{1}{n_1}}}{Z} \]  
(5.245)

\[ p_2 = \frac{e^{\mu_L \beta p_2}}{e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2}} \cdot \frac{(e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2})^{\frac{1}{n_1}}}{e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2})^{\frac{1}{n_1}}} \]
\[ = \frac{e^{\mu_L \beta p_2}(e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{\frac{1}{n_1}}}{Z} \]  
(5.246)

Adding (5.245) and (5.246)

\[ p_1 + p_2 = \frac{(e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2})}{(e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2})^{\frac{1}{n_1}}} \cdot \frac{(e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2})^{\frac{1}{n_1}}}{e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_L \beta p_2})^{\frac{1}{n_1}}} \]
\[ = \frac{(e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{\frac{1}{n_1}}}{Z} \]  
(5.247)

Re-writing (5.92) - (5.95) using (5.244) - (5.247)

\[ \frac{\partial g_0}{\partial p_0} \bigg|_{g=0} = \beta e^{\beta p_0} (1-p_0) - (e^{\beta p_0} + (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{\frac{1}{n_1}}) \]
\[ + \beta p_0 e^{\mu_L \beta (1-p_0-p_1)} (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{\frac{1}{n_1}} \]
\[ = \beta(p_0Z)(1-p_0) - Z + \beta p_0 (p_2 Z) = Z[\beta p_0(1-p_0) - 1 + \beta p_0 p_2] \]
\[ = Z[\beta p_0(p_1 + p_2) - 1 + \beta p_0 p_2] = Z[\beta p_0 p_1 + 2\beta p_0 p_2 - 1] \]  
(5.248)

\[ \frac{\partial g_0}{\partial p_1} \bigg|_{g=0} = -\beta p_0 (e^{\mu_L \beta p_1} - e^{\mu_L \beta (1-p_0-p_1)}) \]
\[ \times (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{\frac{1}{n_1}} \]
\[ = -\beta p_0 (p_1 - p_2) Z = -Z[\beta p_0 p_1 - \beta p_0 p_2] \]  
(5.249)

\[ \frac{\partial g_1}{\partial p_0} \bigg|_{g=0} = -\beta (1-\mu_L) e^{\mu_L \beta p_1} e^{\mu_L \beta (1-p_0-p_1)} \]
\[ \times (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{\frac{1}{n_1}} \]
\[ = -\beta p_0 e^{\beta p_0} + \beta p_1 e^{\mu_L \beta (1-p_0-p_1)} (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)})^{\frac{1}{n_1}} \]
\[ = -\beta (1-\mu_L)(p_1Z)(p_2Z) \frac{1}{(p_1 + p_2) Z} - \beta p_1(p_0Z) + \beta p_1(p_2Z) \]
\[ = -Z[\beta(1-\mu_L) \frac{p_1 p_2}{(p_1 + p_2)} + \beta p_0 p_1 - \beta p_1 p_2] \]  
(5.250)
\[
\frac{\partial g_1}{\partial p_1} \bigg|_{g=0} = \mu_L \beta e^{\mu_L \beta p_1} (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)}) \frac{1-\mu_L}{\mu_L} 
+ (1 - \mu_L) \beta e^{\mu_L \beta p_1} (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)}) \frac{1-\mu_L+1-\mu_L-1}{\mu_L} 
\times (e^{\mu_L \beta p_1} - e^{\mu_L \beta (1-p_0-p_1)}) 
- \beta p_1 (e^{\mu_L \beta p_1} + e^{\mu_L \beta (1-p_0-p_1)}) \frac{1-\mu_L}{\mu_L} (e^{\mu_L \beta p_1} - e^{\mu_L \beta (1-p_0-p_1)}) 
= \mu_L \beta (p_1 Z) + (1 - \mu_L) \beta (p_1 Z) (p_1 - p_2) Z \frac{1}{(p_1 + p_2) Z} 
- Z - \beta p_1 (p_1 - p_2) Z 
= Z [\mu_L \beta p_1 + (1 - \mu_L) \beta p_1 \frac{(p_1 - p_2)}{(p_1 + p_2)} - 1 - \beta p_1 (p_1 - p_2)] 
\]

(5.251)

Having computed convenient general relations for the elements of the Jacobian matrix at equilibrium, we can now write general expressions for the determinant and the trace of the Jacobian matrix at equilibrium solutions for use in (5.90) and in (5.91) to characterize when at least one eigenvalue of the Jacobian has zero real part. We will first consider the case (5.90) where the determinant is equal to zero. Hereby we will naturally retrieve results for this case from subsections 5.6.4 and 5.6.5, but the general form will now also allow us to derive Curve B as shown in panel (c) of Figure 5.12 for the case when the bifurcation occurs at a solution with \( p_1 \neq p_2 \) where the mode shares of the correlated elemental alternatives in the lower level nest are not equal. Then later at the end of this subsection we will consider the case (5.91) where the trace is equal to zero and the determinant is strictly positive in the general case.

**Case I. Suppose det \( J = 0 \)**

Our goal is to find possible values of the parameters \( \beta \) and \( \mu_L \) real, finite such that general solutions to the system of equations (5.26) and (5.27) are bifurcation points. From (5.90), we know that there will exist at least one zero eigenvalue if the determinant of the Jacobian is equal to zero. This leads us to the following theorem which we prove in this subsection.

**Theorem** (General characterization of \( \beta \) and \( \mu_L \) at bifurcation points of the sociodynamic trinary nested logit model for which at least one zero eigenvalue occurs) Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_1 \) of decision-
making agents that have chosen alternative \( i \), and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter \( \mu_L \geq 1 \) real, finite. The equilibrium solutions \( p_0, p_1, p_2 \) defined on \([0, 1]\) of the sociodynamic trinary nested logit model (5.17) will be candidates for bifurcation points for which at least one zero eigenvalue occurs, if values of the parameters \( \beta \) and \( \mu_L \) satisfy:

\[
\mu_L = \frac{2\beta p_0[p_1 p_1 + p_1 p_2 + p_2 p_2] - (p_1 + p_2)}{\beta p_1 p_2[3\beta p_0[p_1 + p_2] - 2]}
\]

(5.252)

Proof. Let us consider (5.90).

\[
0 = \det J = \frac{\partial g_0}{\partial p_0} \frac{\partial g_1}{\partial p_1} - \frac{\partial g_0}{\partial p_1} \frac{\partial g_1}{\partial p_0} = Z[\beta p_0 p_1 + 2\beta p_0 p_2 - 1]
\]

\[
\times Z[\mu_L \beta p_1 + (1 - \mu_L)\beta p_1 \frac{(p_1 - p_2)}{(p_1 + p_2)} - 1 - \beta p_1(p_1 - p_2)]
\]

\[- Z[\beta p_0 p_1 - \beta p_0 p_2] \times Z[\beta (1 - \mu_L) \frac{p_1 p_2}{(p_1 + p_2)} + \beta p_0 p_1 - \beta p_1 p_2]
\]

(5.253)

Dividing through by \( Z^2 \), which is always positive for \( \beta \) and \( \mu_L \) real, finite with \( \mu_L \geq 1 \) and collecting terms in \( \beta \)

\[
0 = (\beta p_0(p_1 + 2p_2) - 1)[\beta p_1(\mu_L + \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)} - p_1 + p_2) - 1]
\]

\[- \beta^2 p_0 p_1(p_1 - p_2) \frac{(1 - \mu_L)p_2}{(p_1 + p_2)} + p_0 - p_2]
\]

(5.254)

Multiplying out

\[
0 = \beta^2 p_0 p_1(p_1 + 2p_2)[\mu_L + \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)} - p_1 + p_2] + 1
\]

\[- \beta[p_0(p_1 + 2p_2) + p_1(\mu_L + \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)} - p_1 + p_2)]
\]

\[- \beta^2 p_0 p_1 p_2 \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)} - \beta^2 p_0 p_1(p_1 - p_2)(p_0 - p_2)
\]

(5.255)
Using (5.22) to write \(-p = -1 + p_0 + p_2\)

\[
0 = \beta^2 p_0 p_1 (p_1 + 2p_2) [(1 - \mu_L)(p_1 - p_2)] + (-1 + p_0 + p_2) + p_2
\]

\[
+ 1 - \beta p_0 (p_1 + 2p_2)
\]

\[
+ \beta p_1 [\mu_L + \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)}] + (-1 + p_0 + p_2) + p_2
\]

\[
- \beta^2 p_0 p_1 p_2 \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)} - \beta^2 p_0 p_1 (p_1 - p_2)(p_0 - p_2)
\]

\[
= \beta^2 p_0 p_1 (p_1 + 2p_2) [-(1 - \mu_L) + \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)} + p_0 + 2p_2]
\]

\[
+ 1 - \beta p_0 (p_1 + 2p_2)
\]

\[
- \beta [-p_1 (1 - \mu_L) + p_1 \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)} + p_1 (p_0 + 2p_2)]
\]

\[
- \beta^2 p_0 p_1 p_2 \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)} - \beta^2 p_0 p_1 (p_1 - p_2)(p_0 - p_2)
\]

(5.256)

Collecting terms in \((1 - \mu)\)

\[
0 = -\beta^2 p_0 p_1 (p_1 + 2p_2) [-(1 - \mu_L) - \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)}]
\]

\[
+ \beta^2 p_0 p_1 (p_1 + 2p_2)(p_0 + 2p_2) + 1
\]

\[
+ \beta p_1 [(1 - \mu_L) - \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)}]
\]

\[
- \beta [p_0 (p_1 + 2p_2) + p_1 (p_0 + 2p_2)]
\]

\[
- \beta^2 p_0 p_1 p_2 \frac{(1 - \mu_L)(p_1 - p_2)}{(p_1 + p_2)} - \beta^2 p_0 p_1 (p_1 - p_2)(p_0 - p_2)
\]

(5.257)
Expanding terms and simplifying

\[
0 = -\beta^2 p_{01} p_2 (1 - \mu_L) \left( \frac{2 p_2 (1 - \mu_L)}{p_1 + p_2} \right) + \beta^2 p_{01} (p_0 p_1 + 2 p_0 p_2 + 2 p_1 p_2 + 4 p_2^2) + 1 \\
+ \beta p_{11} (2 p_2 (1 - \mu_L) - \mu_L) \left( \frac{1}{p_1 + p_2} \right) - \beta (p_0 p_1 + 2 p_0 p_2 + p_1 p_0 + 2 p_1 p_2) \\
- \beta^2 p_{01} p_2 \left( \frac{1 - \mu_L}{p_1 + p_2} \right) (p_1 - p_2) \\
- \beta^2 p_{01} p_2 \left( \frac{1 - \mu_L}{p_1 + p_2} \right) \left( \frac{p_1 - p_2}{p_1 + p_2} \right) \\
- \beta^2 p_{01} p_2 (p_0 p_1 - p_0 p_2 - p_1 p_2 + p_2^2) \\
= -\beta^2 p_{01} p_2 \frac{2 (1 - \mu_L) (p_1 + 2 p_2)}{p_1 + p_2} - \beta^2 p_{01} p_2 \frac{(1 - \mu_L)(p_1 - p_2)}{p_1 + p_2} \\
+ \beta^2 p_{01} (3 p_0 p_2 + 3 p_1 p_2 + 3 p_2^2) + 1 \\
+ \beta p_{11} \frac{2 p_2 (1 - \mu_L)}{p_1 + p_2} - \beta (2 p_0 p_1 + 2 p_0 p_2 + 2 p_1 p_2) \\
= -\beta^2 p_{01} p_2 \left( \frac{1 - \mu_L}{p_1 + p_2} \right) (2 p_1 + 4 p_2 + p_1 - p_2) \\
+ 3 \beta^2 p_{01} p_2 (p_0 + p_1 + p_2) + 1 \\
+ 2 \beta p_{11} \frac{(1 - \mu_L)}{p_1 + p_2} - 2 \beta (p_0 p_1 + p_0 p_2 + p_1 p_2) \\
= -3 \beta^2 p_{01} p_2 (1 - \mu_L) + 3 \beta^2 p_{01} p_2 + 2 \beta p_{11} p_2 \frac{(1 - \mu_L)}{p_1 + p_2} \\
- 2 \beta (p_0 p_1 + p_0 p_2 + p_1 p_2) + 1
\]

Thus

\[
0 = 3 \mu_L \beta^2 p_{01} p_2 + 2 \beta p_{11} p_2 \frac{(1 - \mu_L)}{p_1 + p_2} - 2 \beta (p_0 p_1 + p_0 p_2) + 1
\]

Comparing (5.259) with results in the Appendix for the trinary multinomial logit model, we see that symmetry is broken additively by the second term in the sum. Alternatively, we can see the symmetry-breaking as a multiplicative factor as follows:

\[
0 = 3 \mu_L \beta^2 p_{01} p_2 + 2 \beta p_{11} p_2 \frac{(1 - \mu_L)}{p_1 + p_2} - 2 \beta (p_0 p_1 + p_0 p_2) \\
- 2 \beta p_{11} p_2 + 1 \\
= 3 \mu_L \beta^2 p_{01} p_2 - 2 \beta (p_0 p_1 + p_0 p_2) - 2 \beta p_{11} p_2 \left( \frac{1 - \mu_L}{p_1 + p_2} \right) + 1 \\
= 3 \mu_L \beta^2 p_{01} p_2 - 2 \beta (p_0 p_1 + p_0 p_2) - 2 \beta p_{11} p_2 \left( \frac{1 + \mu_L}{p_1 + p_2} \right) + 1 \\
= 3 \mu_L \beta^2 p_{01} p_2 - 2 \beta (p_0 p_1 + p_0 p_2) - 2 \beta p_{11} p_2 \left( \frac{\mu_L - p_0}{p_1 + p_2} \right) + 1 \\
= 3 \mu_L \beta^2 p_{01} p_2 - 2 \beta (p_0 p_1 + p_0 p_2) - 2 \beta p_{11} p_2 \left( \frac{\mu_L - p_0}{1 - p_0} \right) + 1
\]
Re-arranging terms:

$$3\mu_L\beta^2 p_0 p_1 p_2 = 2\beta[p_0 p_1 + p_0 p_2 + p_1 p_2 \left(\frac{\mu_L - p_0}{1 - p_0}\right)] - 1$$

and thus we naturally re-gain (5.150) from subsection 5.6.4 as well as (5.196) from subsection 5.6.5 for this solution with \( p_1 = p_2 \) where the mode shares of the correlated elemental alternatives in the lower level nest are equal:

$$3\beta p_0 p_1 - 1 = 0 : \beta = \frac{1}{3p_0p_1} \quad (5.264)$$

$$\mu_L = \frac{1}{\beta p_1} : \beta = \frac{1}{\mu_L p_1} \quad (5.265)$$
Alternatively, note that we can also solve (5.260) for \( \beta \) using the quadratic formula:

\[
\beta_{+, -} = \frac{\left(p_0p_1 + p_0p_2 + p_1p_2 \frac{(\mu - p_2)}{(1 - p_0)}\right)}{3\mu_L p_0p_1p_2} \pm \frac{\sqrt{L}}{3\mu_L p_0p_1p_2} \left(p_0p_1 + p_0p_2 + p_1p_2 \frac{(\mu - p_0)}{(1 - p_0)}\right) - 3\mu_L p_0p_1p_2
\]

This yields a general expression for the utility parameter \( \beta \) at bifurcation points of the sociodynamic trinary nested logit model (5.17) for which at least one zero eigenvalue occurs. To determine whether or not the special solution \( p_1 = p_2 \) is the only solution when the condition (5.90) holds, we can substitute the general expression (5.266) for \( \beta \) at bifurcation points back into the planar autonomous system (5.26) and (5.27) and solve graphically, plotting the null clines of the surfaces \( g_0 \) and \( g_1 \) on a graph and finding their intersection for a sweep of the scale parameter \( \mu_L \geq 1 \). See Figures 5.27 through 5.30 on pages 149 - 152.

We see visually in Figures 5.27 through 5.30 that we indeed recover the bifurcation points given by (5.150) from subsection 5.6.4 as well as (5.196) from subsection 5.6.5 for the solution with \( p_1 = p_2 \), but we also in addition find bifurcation points with \( p_1 \neq p_2 \) where the mode shares of the correlated elemental alternatives in the lower level nest are not equal. This occurs for all values of the scale parameter \( \mu_L \geq 1 \). These asymmetric bifurcation points in the mode shares \( p_1 \neq p_2 \) have the special property that they always occur in pairs as mirror solutions across the line \( p_1 = p_2 = 1 - p_0 - p_1 \), or otherwise said, across the line \( p_1 = (1 - p_0)/2 \), in the figures. The trajectory of the bifurcation points with \( p_1 \neq p_2 \) in the panels for \( \beta \) on the left side of Figures 5.27 through 5.30 over the sweep of the scale parameter \( \mu_L \geq 1 \) forms the green curve in Figure 5.33 on page 154.

Alternatively, as the behavior for the special solution \( p_1 = p_2 \) is now well-understood and we are primarily interested at this stage in determining any possible solutions with \( p_1 \neq p_2 \) we can substitute (5.34) into the general equation for the determinant (5.253) and into one of either (5.26) or (5.27) for sweep of the parameter \( \mu_L \). The intersection of the null cline condition for bifurcation over \( \beta \) with the solution trajectory of the system over \( \beta \) gives the bifurcation points. See Figure 5.31 on page 153 and Figure 5.32 on page 153. Additional detailed plots of solution trajectories over \( \beta \) with \( p_1 \neq p_2 \) and null clines for the determinant of the Jacobian with \( p_1 \neq p_2 \) for a sweep of values of \( \mu_L \) are given in the Supplemental Resources in the electronic companion to this dissertation.

We can now obtain the trajectory of the bifurcation points in the \((p_0, p_1)\)-plane over all \( \beta \) and all \( \mu_L \) satisfying \( \text{det} J = 0 \). It is evident from Figures 5.27 and 5.28 and Figure 5.22 on page 126 that the bi-
5.6 Stability Analysis

\[
\beta - \mu_L = 1 \text{ (Multinomial logit)}: \text{Regime I} \rightarrow \text{[Bifurcation point]} \rightarrow \text{Regime II} \rightarrow \text{[Bifurcation point]} \rightarrow \text{Regime III}
\]

\[
\mu_L = 1.005: \text{Regime I} \rightarrow \text{Regime VII} \rightarrow \text{Regime II} \rightarrow \text{Regime IV} \rightarrow \text{Regime III}
\]

\[
\mu_L = 1.03: \text{Regime I} \rightarrow \text{Regime VII} \rightarrow \text{Regime I} \rightarrow \text{Regime IV} \rightarrow \text{Regime III}
\]

Figure 5.27: Null clines solution of system at values of $\beta$ satisfying $\text{det} J = 0$, for a sweep of selected values of $\mu_L$ having different bifurcation regime sequences over $\beta$. 
\( \mu_L \approx 1.0397 \): Regime I \( \rightarrow \) [Bifurcation point] \( \rightarrow \) Regime I \( \rightarrow \) Regime IV \( \rightarrow \) Regime III

\( \mu_L = 2 \): Regime I \( \rightarrow \) Regime IV \( \rightarrow \) Regime III

\( \mu_L \approx 3.20603 \): Regime I \( \rightarrow \) [Bifurcation point] \( \rightarrow \) Regime I \( \rightarrow \) Regime IV \( \rightarrow \) Regime III

Figure 5.28: Null clines solution of system at values of \( \beta \) satisfying \( \det J = 0 \), for a sweep of selected values of \( \mu_L \) having different bifurcation regime sequences over \( \beta \), continued from Figure 5.27.
\[ \beta^-_L = 3.5 : \text{Regime I} \rightarrow \text{Regime V} \rightarrow \text{Regime I} \rightarrow \text{Regime IV} \rightarrow \text{Regime III} \]

\[ \beta^+_L \approx 4 : \text{Regime I} \rightarrow \text{Regime V} \rightarrow \text{Regime IV} \rightarrow \text{Regime III} \]

\[ \beta^-_L = 4.5 : \text{Regime I} \rightarrow \text{Regime V} \rightarrow \text{Regime VI} \rightarrow \text{Regime IV} \rightarrow \text{Regime III} \]

Figure 5.29: Null clines solution of system at values of \( \beta \) satisfying \( \det J = 0 \), for a sweep of selected values of \( \mu_L \) having different bifurcation regime sequences over \( \beta \), continued from Figures 5.27 and 5.28.
Large $\mu_L$

(a) $\beta-$
$\mu_L = 10 : \text{Regime I} \rightarrow \text{Regime V} \rightarrow \text{Regime VI} \rightarrow \text{Regime IV} \rightarrow \text{Regime III}$

(b) $\beta+$

(c) $\beta-$
$\mu_L = 100 : \text{Regime I} \rightarrow \text{Regime V} \rightarrow \text{Regime VI} \rightarrow \text{Regime IV} \rightarrow \text{Regime III}$

(d) $\beta+$

Figure 5.30: Null clines solution of system at values of $\beta$ satisfying $\det J = 0$, for large values of $\mu_L$. 
Figure 5.31: Example solution trajectory over $\beta$ with $p_1 \neq p_2$ and null cline for $\det J$ with $p_1 \neq p_2$, yielding bifurcation in parameter $\beta$, for a selected value of $\mu_L = 1.1$. The blue curves are the solution trajectories over $\beta$ with $p_1 \neq p_2$. The red curve is the null cline for the determinant of the Jacobian with $p_1 \neq p_2$. The intersection shows two bifurcation points with the same value of $p_0$ and mirror values of $p_1$ and $p_2 = 1 - p_0 - p_1$.

(a) Bifurcation point in Figure 5.31 with $p_1 > p_2$. (b) Bifurcation point in Figure 5.31 with $p_1 < p_2$.

Figure 5.32: Detail of solution trajectory over $\beta$ with $p_1 \neq p_2$ and null cline for $\det J$ with $p_1 \neq p_2$, yielding bifurcation in parameter $\beta$, for a selected value of $\mu_L = 1.1$. 
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Bifurcation points satisfying det J = 0 with β- given by (5.266) traverse all valid values of the line \( p_1 = p_2 \), that is, \( p_1 = (1 - p_0)/2 \) on the segment from \( (p_0, p_1) = (1/3, 1/3) \) to \( (p_0, p_1) \approx (0.58480, 0.20760) \) over \( \mu_L = 1 \) to \( \mu_L = 3 \ln(2)/2 \approx 1.03972 \) as given by (5.167). Likewise, it is evident from Figures 5.28 through 5.30 and Figure 5.26 on page 136 that the bifurcation points satisfying det J = 0 with β- given by (5.266) traverse all valid values of the line \( p_1 = p_2 \), that is, \( p_1 = (1 - p_0)/2 \) on the segment from \( (p_0, p_1) = (1/5, 1/4) \) to \( (p_0, p_1) \approx (1/3, 1/3) \) to \( (p_0, p_1) \approx (0.12195, 0.439025) \) over \( \mu_L \approx 3.20603 \) to \( \mu_L \to \infty \). Finally, it is evident from Figures 5.27 through 5.30 and Figure 5.23 on page 128 that the bifurcation points satisfying det J = 0 with β+ given by (5.266) traverse all valid values of the line \( p_1 = p_2 \) on the segment from \( (p_0, p_1) = (1/5, 1/4) \) to \( (p_0, p_1) \approx (1/3, 1/3) \) to \( (p_0, p_1) \approx (0.12195, 0.439025) \) over \( \mu_L = 1 \) to \( \mu_L \to \infty \). These three segments together cover the blue line in Figure 5.33 on page 154 entirely from \( (p_0, p_1) \approx (0.12195, 0.439025) \) to \( (1, 0) \). We obtain the trajectory of the bifurcation points in the \( (p_0, p_1) \)-plane over all \( \beta \) and all \( \mu_L \) satisfying det J = 0 with \( p_1 \neq p_2 \) from the series of detailed graphical solutions in the Supplemental Resources in the electronic companion to the dissertation. These bifurcation points with \( p_1 \neq p_2 \) form the green curve in Figure 5.33 on page 154.

Figure 5.33: Bifurcation trajectory in the \( (p_0, p_1) \)-plane over all \( \beta \) and all \( \mu_L \).

Substituting the values of \( p_1 \) and \( p_2 \) in the green trajectory in Figure 5.33 (obtained in the Supplemental Resources from the detailed plots of solution trajectories over \( \beta \) with \( p_1 \neq p_2 \) and null clines for the determinant of the Jacobian with \( p_1 \neq p_2 \) for a sweep of values of \( \mu_L \)) back into (5.35) we obtain the bifurcation curve for \( \mu_L \) in terms of \( \beta \). See Figure 5.34 on page 155. ♦
5.6 Stability Analysis

Figure 5.34: Bifurcation curve in the $(\beta, \mu_L)$-plane satisfying $\det J = 0$ for $p_1 \neq p_2$. Compare with Figure 5.3 on page 74 and Figure 5.4 on page 75.

The example null clines solutions in panels (b) and (a) in Figure 5.14 on page 107, in panel (c) in Figure 5.13 on page 105 (or similarly panel (e) in Figure 5.2 on page 68 and panel (c) in Figure 5.5 on page 76), and in panels (d) and (e) in Figure 5.5 show that transition between Regime VII to Regime II, transition between Regime I to Regime IV, and transition between Regime V to Regime VI, that is, crossing the bifurcation curve in $(\beta, \mu_L)$-parameter space, yields a pair of saddle-node bifurcations with mirror solutions in $p_1$ and $p_2$. ♦

Twin saddle-node bifurcations: behavior of $\beta$ for $\mu_L = 1$

**Lemma** (Maximum value of $\beta$ for twin saddle-node bifurcations at $\mu_L = 1$ for the pair of mirror solutions $p_1 \neq p_2$ of the sociodynamic trinary nested logit model.) The maximum value of utility parameter $\beta$ at scale parameter $\mu_L = 1$ for the twin saddle-node bifurcation of the pair of mirror equilibrium solutions of the sociodynamic trinary nested logit model (5.17) across the line $p_1 = p_2 = 1 - p_0 - p_1$, where the mode shares of the correlated elemental alternatives in the lower level nest are asymmetric $p_1 \neq p_2$, occurs for:

$$\beta \approx 2.7456$$
$$\mu_L = 1$$
$$p_0 = p_1 \approx 0.2076$$
$$p_2 \approx 0.5848$$  (5.267)
\[ \beta \approx 2.7456 \]
\[ \mu_L = 1 \]
\[ p_0 = p_2 \approx 0.2076 \]
\[ p_1 \approx 0.5848 \]  

**Proof.** At \( \mu_L = 1 \), the sociodynamic trinary nested logit model reduces to the sociodynamic trinary multinomial logit model. By symmetry, the saddle node bifurcation of the solution \( p_1 = p_2 \) when crossing Curve A at \( \mu_L = 1 \) given by (5.168) must be accompanied by two other saddle-node bifurcations with \( p_1 \neq p_2 \) at \( \mu_L = 1 \) for mirror solutions across from each other on both sides of the line \( p_1 = p_2 = 1 - p_0 - p_1 \). Hereby we retrieve the result from the second lemma in subsection 4.2.2 as given by (4.45), (4.46) and (4.48). Indeed we have already seen visually in the computationally derived bifurcation curves in Figure 5.4 on page 75 that Curve A and Curve B come together at \( \mu_L = 1 \).

**Twin saddle-node bifurcation: behavior of \( \beta \) in the limit \( \mu_L \to \infty \)**

**Lemma** (Limit behavior of twin saddle-node bifurcation as \( \mu_L \to \infty \) for the pair of mirror solutions of the sociodynamic trinary nested logit model with \( p_1 \neq p_2 \)) The limit behavior of utility parameter \( \beta \) as scale parameter \( \mu_L \to +\infty \) for the twin saddle-node bifurcation of the pair of mirror equilibrium solutions of the sociodynamic trinary nested logit model (5.17) across the line \( p_1 = p_2 = 1 - p_0 - p_1 \), where the mode shares of the correlated elemental alternatives in the lower level nest are asymmetric \( p_1 \neq p_2 \), is given by:

\[ \beta = 2 \]
\[ \mu_L \to \infty \]
\[ p_0 = p_1 = \frac{1}{2} \]
\[ p_2 = 0 \]
\[ \beta = 2 \]
\[ \mu_L \to \infty \]
\[ p_0 = p_2 = \frac{1}{2} \]
\[ p_1 = 0 \]

**Proof.** Consider the following limiting case:

\[ \mu \to \infty \]  

We have already seen visually in panel (c) of Figure 5.11 on page 94 in subsection 5.4.1 that twin saddle-node bifurcations occur midway
along the boundaries at \( p_1 = 0 \) and \( p_2 = 1 - p_0 - p_1 = 0 \) of the potential function surface for \( \beta = 2 \). We will now prove this rigorously. Consider the following solution:

\[
p_0 = p_1 = \frac{1}{2} ; \ p_2 = 0 \tag{5.272}
\]

or alternatively:

\[
p_0 = p_2 = \frac{1}{2} ; \ p_1 = 0 \tag{5.273}
\]

Substituting (5.272) into (5.26) and (5.27) for \( \mu_L \) very large, we have:

\[
\lim_{\mu_L \to \infty} g_0 |_{p_0 = p_1 = \frac{1}{2}} = \lim_{\mu_L \to \infty} \left( e^{\beta/2} - \frac{1}{2} e^{\beta/2} - \frac{1}{2} (e^{\mu_L \beta/2} + e^0) \frac{1}{\mu_L} \right)
\]

\[
\approx \frac{1}{2} e^{\beta/2} - \frac{1}{2} (e^{\mu_L \beta/2} + e^0) \frac{1}{\mu_L} = 0 ; \ \beta > \frac{1}{\mu_L} \tag{5.274}
\]

\[
\lim_{\mu_L \to \infty} g_1 |_{p_0 = p_1 = \frac{1}{2}} = \lim_{\mu_L \to \infty} \left( e^{\mu_L \beta/2} (e^{\mu_L \beta/2} + e^0) \frac{1 - \mu_L}{\mu_L} \right)
\]

\[
+ \lim_{\mu_L \to \infty} \left( - \frac{1}{2} e^{\beta/2} - \frac{1}{2} (e^{\mu_L \beta/2} + e^0) \frac{1}{\mu_L} \right)
\]

\[
\approx e^{\mu_L \beta/2} (e^{\mu_L \beta/2} + e^0) \frac{1 - \mu_L}{\mu_L} - \frac{1}{2} e^{\beta/2} - \frac{1}{2} (e^{\mu_L \beta/2} + e^0) \frac{1}{\mu_L}
\]

\[
= e^{\mu_L \beta/2} e^{\beta/2} - \frac{1}{2} e^{\beta/2} - \frac{1}{2} e^{\beta/2} = 0 ; \ \beta > \frac{1}{\mu_L} \tag{5.275}
\]

At the solution (5.272) for \( \mu_L \) very large, the four terms in the Jacobian matrix of \( g = (g_0, g_1) \) given by (5.92) - (5.95) are:

\[
\lim_{\mu_L \to \infty} \frac{\partial g_0}{\partial p_0} |_{p_0 = p_1 = \frac{1}{2}} = \lim_{\mu_L \to \infty} \left( \beta e^{\beta/2} - e^{\beta/2} - \beta \frac{1}{2} e^{\beta/2} \right)
\]

\[
+ \lim_{\mu_L \to \infty} \left( -(e^{\mu_L \beta/2} + e^0) \frac{1}{\mu_L} + \beta \frac{1}{2} e^0 (e^{\mu_L \beta/2} + e^0) \frac{1 - \mu_L}{\mu_L} \right)
\]

\[
\approx \frac{\beta}{2} e^{\beta/2} - e^{\beta/2} - (e^{\mu_L \beta/2} + e^0) \frac{1}{\mu_L} + \beta \frac{1}{2} (e^{\mu_L \beta/2} + e^0) \frac{1 - \mu_L}{\mu_L}
\]

\[
= \frac{\beta}{2} e^{\beta/2} - 2e^{\beta/2} + \beta \frac{1}{2} e^{\mu_L \beta/2} \tag{5.276}
\]

\[
\lim_{\mu_L \to \infty} \frac{\partial g_0}{\partial p_1} |_{p_0 = p_1 = \frac{1}{2}} = \lim_{\mu_L \to \infty} \left( -\beta \frac{1}{2} (e^{\mu_L \beta/2} - e^0) (e^{\mu_L \beta/2} + e^0) \frac{1 - \mu_L}{\mu_L} \right)
\]

\[
\approx -\frac{\beta}{2} (e^{\mu_L \beta/2} e^{\beta/2}) = -\frac{\beta}{2} (e^{\beta/2}) \tag{5.277}
\]
\[
\begin{align*}
\lim_{\mu_L \to \infty} \frac{\partial g_1}{\partial p_0} \bigg|_{p_0 = p_1 = \frac{1}{2}} &= \lim_{\mu_L \to \infty} \left( -\beta (1 - \mu_L) e^{\mu_L \beta / 2} e^0 (e^{\mu_L \beta / 2} + e^0)^{1 - \mu_L / \mu_L} \right) \\
&\quad + \lim_{\mu_L \to \infty} \left( -\beta p_1 e^{\beta / 2} + \beta p_1 e^0 (e^{\mu_L \beta / 2} + e^0)^{1 - \mu_L / \mu_L} \right) \\
&\approx -\beta (1 - \mu_L) \left( e^{\mu_L \beta / 2} / (e^{\mu_L \beta / 2})^2 (e^{\mu_L \beta / 2} + e^0)^{1 / \mu_L} - \frac{\beta}{2} e^{\beta / 2} + \frac{\beta}{2} (e^{\beta / 2}) \right) \\
&\quad - e^{\beta / 2} - (e^{\mu_L \beta / 2} + e^0)^{1 / \mu_L} - \frac{\beta}{2} (e^{\mu_L \beta / 2} + e^0)^{1 - \mu_L / \mu_L} (e^{\mu_L \beta / 2} - e^0) \\
&\approx \mu_L \beta e^{\mu_L \beta / 2} (e^{\mu_L \beta / 2})^{1 - \mu_L / \mu_L} + (1 - \mu_L) \beta e^{\mu_L \beta / 2} (e^{\mu_L \beta / 2})^{1 - \mu_L / \mu_L} (e^{\mu_L \beta / 2} - e^0) \\
&\quad - 2e^{\beta / 2} - \frac{\beta}{2} (e^{\mu_L \beta / 2})^{1 - \mu_L / \mu_L} (e^{\mu_L \beta / 2}) \\
&\approx \mu_L \beta e^{\mu_L \beta / 2} (e^{\mu_L \beta / 2})^{1 - \mu_L / \mu_L} + (1 - \mu_L) \beta e^{\mu_L \beta / 2} (e^{\mu_L \beta / 2})^{2} (e^{\beta / 2}) \\
&\quad - 2e^{\beta / 2} - \frac{\beta}{2} (e^{\mu_L \beta / 2})^{2} (e^{\mu_L \beta / 2}) \\
&= \mu_L \beta (e^{\beta / 2}) + (1 - \mu_L) \beta (e^{\beta / 2}) - 2e^{\beta / 2} - \frac{\beta}{2} (e^{\beta / 2}) \\
&= \frac{\beta}{2} (e^{\beta / 2}) - 2e^{\beta / 2} = \left( \frac{\beta}{2} - 2 \right) (e^{\beta / 2})
\end{align*}
\]

(5.278)
Let us consider case (5.189)

\[ 0 = \det \mathbf{J} = \frac{\partial g_0}{\partial p_0} \frac{\partial g_1}{\partial p_1} - \frac{\partial g_0}{\partial p_1} \frac{\partial g_1}{\partial p_0} = \]

\[ = \left( \frac{\beta}{2} e^{\beta/2} - 2e^{\beta/2} + \frac{\beta}{2} \frac{e^{\beta/2}}{e^{\mu_L \beta/2}} \right) \left( \frac{\beta}{2} - 2 \right) (e^{\beta/2}) \]

\[ - \left( -\frac{\beta}{2} (e^{\beta/2}) \right) \left( -\beta \left( \frac{1}{2} - \mu_L \right) \frac{e^{\beta/2}}{e^{\mu_L \beta/2}} - \frac{\beta}{2} e^{\beta/2} \right) \]

\[ = \left( \frac{\beta}{2} e^{\beta/2} - 2e^{\beta/2} \right) \left( \frac{\beta}{2} - 2 \right) (e^{\beta/2}) - \left( -\frac{\beta}{2} (e^{\beta/2}) \right) \left( -\frac{\beta}{2} e^{\beta/2} \right) \]

\[ = (\frac{\beta}{2} - 2)^2 (e^{\beta}) - (\frac{\beta}{2})^2 (e^{\beta}) = (4 - 2\beta)(e^{\beta}) \]

(5.28o)

Solving for \( \beta \) real, finite we have:

\[ \beta = 2 \]

(5.281)

\[ \diamond \]

Case II. Suppose tr \( \mathbf{J} = 0 \) and \( \det \mathbf{J} > 0 \)

We continue our search for possible values of the parameters \( \beta \) and \( \mu_L \) real, finite such that the solutions to the system of equations (5.26) and (5.27) are bifurcation points. From (5.91), we know there will exist zero real part of a pair of complex eigenvalues, i.e. purely imaginary eigenvalues, if the determinant is positive and trace is equal to zero. However, since we have been able to express the qualitative behavior of the sociodynamic trinary nested logit model as a gradient system in section 5.3 we expect that there are no purely imaginary eigenvalues. Indeed in subsection 5.6.5 we have already shown that there are no solutions of the sociodynamic trinary nested logit model with purely imaginary eigenvalues satisfying \( p_1 = p_2 \) where the mode shares of the correlated elemental alternatives in the lower level nest are equal. In this subsection we will prove that there are no such purely imaginary eigenvalues for the sociodynamic trinary nested logit model in general. This leads us to the following theorem.

**Theorem** (No bifurcation points of the sociodynamic trinary nested logit model with purely imaginary eigenvalues) Suppose that individual choices in a large sample population are characterized by the probabilities (5.14), (5.15) and (5.16) where the only contribution to the systematic utility of choices is a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_i \) of decision-making agents that have chosen alternative \( i \), and where the elemental choice alternatives 1 and 2 are assumed to be correlated with scale parameter \( \mu_L \geq 1 \) real, finite. The equilibrium solutions \( p_0, p_1, p_2 \) defined on \([0, 1]\) of the sociodynamic trinary nested logit model (5.17) exhibit no bifurcation points with purely imaginary eigenvalues.
Proof. Let us consider (5.91).

\[0 = \text{tr} J = \frac{\partial g_0}{\partial p_0} + \frac{\partial g_1}{\partial p_1} = Z[\beta p_0 p_1 + 2\beta p_0 p_2 - 1] + Z[\mu L \beta p_1 + (1 - \mu L)\beta p_1 \frac{(p_1 - p_2)}{(p_1 + p_2)} - 1 - \beta p_1 (p_1 - p_2)]\]

(5.282)

Dividing through by \(Z\), which is always positive for \(\beta\) and \(\mu_L\) real, finite with \(\mu_L \geq 1\), and re-arranging terms

\[0 = \beta p_0 p_1 + 2\beta p_0 p_2 - 1 + \mu L \beta p_1 + (1 - \mu L)\beta p_1 \frac{(p_1 - p_2)}{(p_1 + p_2)} - 1 - \beta p_1 (p_1 - p_2)\]

(5.283)

Using (5.22) to write \(-p = -1 + p_0 + p_2\)

\[0 = \mu L \beta p_1 + (1 - \mu L)\beta p_1 \frac{(p_1 - p_2)}{(p_1 + p_2)} + \beta p_1 (-1 + p_0 + p_2) + \beta p_1 p_2 + \beta p_0 p_1 + 2\beta p_0 p_2 - 2\]

(5.284)

Collecting terms in \((1 - \mu)\)

\[0 = -(1 - \mu L)\beta p_1 \frac{(p_1 - p_2)}{(p_1 + p_2)} + (1 - \mu L)\beta p_1 \frac{(p_1 - p_2)}{(p_1 + p_2)} + 2\beta (p_0 p_1 + p_1 p_2 + p_0 p_2) - 2\]

(5.285)

Thus

\[0 = -\beta p_1 p_2 \frac{(1 - \mu L)}{(p_1 + p_2)} + \beta (p_0 p_1 + p_1 p_2 + p_0 p_2) - 1\]

(5.286)
Further re-arranging terms

\[
0 = -\beta p_1 p_2 \frac{(1 - \mu_L)}{(p_1 + p_2)} + \beta (p_0 p_1 + p_0 p_2) + \beta p_1 p_2 - 1
\]

\[
= \beta (p_0 p_1 + p_0 p_2) + \beta p_1 p_2 \left[1 - \frac{(1 - \mu_L)}{(p_1 + p_2)}\right] - 1
\]

\[
= \beta (p_0 p_1 + p_0 p_2) + \beta p_1 p_2 \frac{(p_1 + p_2 - 1 + \mu_L)}{(p_1 + p_2)} - 1
\]

\[
= \beta (p_0 p_1 + p_0 p_2) + \beta p_1 p_2 \frac{(\mu_L - p_0)}{(p_1 + p_2)} - 1
\]

\[
= \beta [p_0 p_1 + p_0 p_2 + p_1 p_2 \frac{(\mu_L - p_0)}{(1 - p_0)}] - 1
\]

Solving (5.287) for \(\beta\)

\[
\beta = \frac{1}{p_0 p_1 + p_0 p_2 + p_1 p_2 \frac{(\mu_L - p_0)}{(1 - p_0)}}
\] (5.288)

Alternatively, solving (5.287) for \(\mu_L\)

\[
\mu_L \beta p_1 p_2 \frac{1}{(1 - p_0)} = \beta [p_0 p_1 + p_0 p_2 + p_1 p_2 \frac{-p_0}{(1 - p_0)}] - 1
\]

\[
= \beta [p_0 p_1 (1 - p_0) + p_0 p_2 (1 - p_0) - p_0 p_1 p_2] - (1 - p_0)
\]

\[
= \beta [p_0 p_1 (1 - p_0) + p_0 p_2 (1 - p_0) - p_0 p_1 p_2 - (1 - p_0)]
\]

\[
= \beta [p_0 p_1 (1 - p_0) + p_1 p_2 + p_2 p_2] - (1 - p_0)
\]

(5.290)

\[
\mu_L = \frac{\beta p_0 [p_1 p_1 + p_1 p_2 + p_2 p_2] - (1 - p_0)}{\beta p_1 p_2}
\]

\[
= \frac{\beta p_0 [p_1 p_1 + p_1 p_2 + p_2 p_2] - (p_1 + p_2)}{\beta p_1 p_2}
\] (5.291)

For the special solution \(p_1 = p_2\), we have

\[
\mu_L = \frac{\beta p_0 [3p_1^2] - (2p_1)}{\beta p_1^2} = \frac{3\beta p_0 p_1 - 2}{\beta p_1}
\] (5.292)

Furthermore, the condition that the trace of the Jacobian equals zero requires from (5.286)

\[
\beta p_1 p_2 \frac{(1 - \mu_L)}{(p_1 + p_2)} - \beta (p_0 p_1 + p_1 p_2 + p_0 p_2) = -1
\] (5.293)

The condition that the determinant of the Jacobian is strictly positive requires from (5.259):

\[
3\mu_L \beta^2 p_0 p_1 p_2 + 2\beta p_1 p_2 \frac{(1 - \mu_L)}{(p_1 + p_2)} - 2\beta (p_0 p_1 + p_0 p_2 + p_1 p_2) + 1 > 0
\] (5.294)
Substituting (5.293) into (5.294)

\[3\mu L\beta^2 \rho_0 \rho_1 \rho_2 - 2 + 1 = 3\mu L\beta^2 \rho_0 \rho_1 \rho_2 - 1 > 0\]  \hspace{1cm} (5.295)

Finally, substituting (5.288) into (5.295)

\[0 < 3\mu L\beta^2 \rho_0 \rho_1 \rho_2 - 1 = \frac{3\mu L\rho_0 \rho_1 \rho_2}{\rho_0 \rho_1 + \rho_0 \rho_2 + \rho_1 \rho_2} - 1 \]  \hspace{1cm} (5.296)

To determine whether or not there exist any solutions with purely imaginary eigenvalues, we substitute (5.288) back into (5.26) and (5.27) and solve graphically for a sweep of the scale parameter \(\mu_L\), plotting the null clines of the surfaces \(g_0\) and \(g_1\) on a graph and finding their intersection, subject to the condition (5.296).

Alternatively, we can obtain the same result by invoking symmetry. Instead of using equations (5.18) and (5.19) as the basis for our study, use instead (5.19) and (5.20). Our system of converted equations is then:

\[0 = g_1 \equiv e^{\mu L} \beta_1 \left( e^{\mu L} \beta_1 p_1 + e^{\mu L} \beta_2 p_2 \right)^{1-\mu_L} p_1 e^{1-p_1-p_2} - p_1 \left( e^{\mu L} \beta_1 p_1 + e^{\mu L} \beta_2 p_2 \right)^{\frac{1}{\mu_L}} \]  \hspace{1cm} (5.297)

\[0 = g_2 \equiv e^{\mu L} \beta_2 \left( e^{\mu L} \beta_1 p_1 + e^{\mu L} \beta_2 p_2 \right)^{1-\mu_L} p_2 e^{1-p_1-p_2} - p_2 \left( e^{\mu L} \beta_1 p_1 + e^{\mu L} \beta_2 p_2 \right)^{\frac{1}{\mu_L}} \]  \hspace{1cm} (5.298)

Here we have the convenient simplification

\[g_2(p_1, p_2) = g_1(p_2, p_1)\]  \hspace{1cm} (5.299)

Thus

\[
\frac{\partial g_2}{\partial p_1} \bigg|_{g=0} = \frac{\partial g_1}{\partial p_2} \bigg|_{g=0} \]  \hspace{1cm} (5.300)

so that the Jacobian matrix of the vector field \(g = (g_1, g_2)\) is symmetric when evaluated at an equilibrium point \(p\) of the system (5.297) and (5.298).

\[
J \equiv \begin{bmatrix}
\frac{\partial g_1}{\partial p_1}(p) & \frac{\partial g_1}{\partial p_2}(p) \\
\frac{\partial g_2}{\partial p_1}(p) & \frac{\partial g_2}{\partial p_2}(p)
\end{bmatrix} \]  \hspace{1cm} (5.301)

Since the off-diagonal terms of the Jacobian are real for the system (5.297) and (5.298) for \(p_0, p_1, p_2\) defined on \([0, 1]\) and the parameters \(\beta\) and \(\mu_L\) real, finite with \(\mu_L > 1\), by (5.300) the product of the off-diagonal terms must be greater than or equal to zero.

\[
\frac{\partial g_2}{\partial p_1} \cdot \frac{\partial g_1}{\partial p_2} \geq 0 \]  \hspace{1cm} (5.302)
Recall from (5.91) at the outset of section 5.6 that there will exist zero real part of a pair of complex eigenvalues, i.e., purely imaginary eigenvalues, if the determinant is strictly positive and trace is equal to zero. For the trace to be equal to zero, we must have that the diagonal terms of the Jacobian are equal and opposite sign, or both equal to zero:

$$\text{tr } J = \frac{\partial g_1}{\partial p_1} + \frac{\partial g_2}{\partial p_2} = 0$$

As a consequence, the product of the diagonal terms of the Jacobian must thus be less than or equal to zero

$$\frac{\partial g_1}{\partial p_1} \cdot \frac{\partial g_2}{\partial p_2} \leq 0$$

Now, for the determinant to be strictly positive we must have that the product of the diagonal terms is strictly greater than the product of the off-diagonal terms:

$$\text{det } J = \frac{\partial g_1}{\partial p_1} \cdot \frac{\partial g_2}{\partial p_2} - \frac{\partial g_1}{\partial p_2} \cdot \frac{\partial g_2}{\partial p_1} > 0$$

$$\frac{\partial g_1}{\partial p_1} \cdot \frac{\partial g_2}{\partial p_2} > \frac{\partial g_1}{\partial p_2} \cdot \frac{\partial g_2}{\partial p_1}$$

The condition (5.304) then implies that the product of the off-diagonal terms must be strictly less than zero

$$0 \geq \frac{\partial g_1}{\partial p_1} \cdot \frac{\partial g_2}{\partial p_2} > \frac{\partial g_1}{\partial p_2} \cdot \frac{\partial g_2}{\partial p_1}$$

However this is a contradiction with the requirement (5.302) that the product of the off-diagonal terms must be greater than or equal to zero since the elements of the Jacobian are real in this case. Thus (5.303) and (5.305) cannot be simultaneously satisfied for the system (5.297) and (5.298) with \(p_0, p_1, p_2\) defined on \([0, 1]\) and the parameters \(\beta\) and \(\mu_L\) real, finite with \(\mu_L \geq 1\), and we have that there can be no purely imaginary eigenvalues. ☐
5.7 BRINGING IT ALL TOGETHER

In this section, we visualize the analytical results derived in this chapter for the sociodynamic trinary nested logit model in three convenient graphical ways. First, in subsection 5.7.1 we plot an overview of all analytically derived bifurcation curves together in \((\beta, \mu_L)\)-plane from subsections 5.6.4, 5.6.5 and 5.6.7. Hereby we retrieve the seven major regimes in parameter space which we determined computationally at the outset of this chapter in Figure 5.3 on page 74 and Figure 5.4 on page 75 in section 5.2. Next, in subsection 5.7.2 we depict the solution trajectories in the \((p_0, p_1)\)-plane over utility parameter \(\beta\) for a sweep of selected values of scale parameter \(\mu_L\). Finally, in subsection 5.7.3 we depict the general bifurcation diagrams for the sociodynamic trinary nested logit model, in the \((\beta, p_0)\)-plane and in the \((\beta, p_1)\)-plane, for a sweep of selected values of scale parameter \(\mu_L\), whereby we see visually the five different types of bifurcations separating the various regimes: transcritical, hysteresis, double saddle-node, saddle-node, pitchfork.

5.7.1 Analytical Bifurcation Curves

See Figure 5.35 on page 165.

In the limit \(\mu_L \to \infty\), we have \(\beta = 0\) and \(\beta \to \infty\) on the curve satisfying \(\partial g_1 / \partial p_1 = 0\). The orange curve eventually crosses the blue curve at \(\mu_L \approx 156\), yielding one additional regime with 9 solutions. However for large \(\mu_L\), the saddle point and the 2 stable node solutions near \(p_0 \approx 1\), \(p_1 \approx 0\) are so close together that these are hardly distinguishable from each other for \(\beta > 6\). See for example panel (f) in Figure 5.11 on page 94.

5.7.2 Solution Trajectories

See Figure 5.36 on page 166.

5.7.3 General Bifurcation Diagrams

See Figures 5.37 through 5.39 on pages 167-169.

See Figure 5.40 on page 170.

5.8 CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

This chapter has studied the theoretical equilibrium behavior of the trinary nested logit model with social interactions. First we have observed that the trinary sociodynamic choice problem can be written
5.8 Conclusions and Recommendations for Further Research

Figure 5.35: Analytically-derived bifurcation curves in the $(\beta, \mu_L)$-plane showing major regimes. Compare with Figure 5.3 on page 74 and Figure 5.4 on page 75.
Figure 5.36: Solution trajectories in the \((p_0, p_1)\)-plane over \(\beta\) for selected values of \(\mu_L\).
5.8 Conclusions and Recommendations for Further Research

(a) $p_0$ vs. $\beta$ 
$\mu_L = 1$ (Multinomial logit): Regime I $\rightarrow$ [Bifurcation point] $\rightarrow$ Regime II $\rightarrow$ [Bifurcation point] $\rightarrow$ Regime III

(b) $p_1$ vs. $\beta$ 
$\mu_L = 1.005$: Regime I $\rightarrow$ Regime VII $\rightarrow$ Regime II $\rightarrow$ Regime IV $\rightarrow$ Regime III

(c) $p_0$ vs. $\beta$ 
(d) $p_1$ vs. $\beta$ 
$\mu_L = 1.03$: Regime I $\rightarrow$ Regime VII $\rightarrow$ Regime I $\rightarrow$ Regime IV $\rightarrow$ Regime III

Figure 5.37: General bifurcation diagrams for the sociodynamic trinary nested logit model. Compare with panels with corresponding $\mu_L$ in Figure 5.18 on page 118. Compare with null clines solutions of system at values of $\beta$ satisfying $\text{det} J = 0$ showing bifurcation points in Figure 5.27 on page 149.
Figure 5.38: General bifurcation diagrams for the sociodynamic trinary nested logit model, continued from Figure 5.37. Compare with panels with corresponding $\mu_L$ in Figure 5.18 on page 118. Compare with null clines solutions of system at values of $\beta$ satisfying $\det J = 0$ showing bifurcation points in Figure 5.28 on page 150.
Figure 5.39: General bifurcation diagrams for the sociodynamic trinary nested logit model, continued from Figures 5.37 and 5.38. Compare with null clines solutions of system at values of $\beta$ satisfying $\det J = 0$ showing bifurcation points in Figure 5.29 on page 151.
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Table 5.5: Characterization of the number of solutions of the sociodynamic trinary nested logit model in different regimes (ASN: Asymptotically stable node; SP: Saddle point; UN: Unstable node)

Figure 5.40: Detail of bifurcation diagrams for p1 versus β near the hysteresis region: (a) Regime I → [Bifurcation Point] → Regime II → [Bifurcation Point] → Regime III; (b) Regime I → Regime VII → Regime II → Regime IV → Regime III; (c) Regime I → Regime VII → Regime I → Regime IV → Regime III; (d) Regime I → [Bifurcation Point] → Regime I → Regime IV → Regime III. Compare with the lower panel of Figure 5.35 on page 165 showing detail of bifurcation curves in the (β, μL)-plane with major regimes.
as a planar autonomous system. By applying a graphical null clines approach, the number of solutions is counted and charted across a sweep of the utility parameter for the aggregate social feedback and a sweep of the scale parameter for the lower nest. This reveals seven qualitatively distinct solution regimes. Next a gradient system is derived which allows us to easily visually characterize the stability of the solutions. Five types of bifurcations are observed which describe the transitions between the regimes. Finally, the bifurcation curves which were initially observed computationally are rigorously derived analytically. We will use these theoretical results in this chapter as a benchmark for our further exploration of the socio-dynamic multinomial logit model later in Chapter 7.

This work also marks a starting point for a number of elaborations and variations. Here are just a few important directions.

*Higher number of choice alternatives.* The consideration of a trinary choice model in this chapter is a direct extension of the binary choice work carried out by economists Aoki, Blume, Brock, Durlauf and Ioannides, guided by the general multinomial results by Brock and Durlauf (2002, 2006). By applying techniques from the mathematics of dynamical systems, it is possible to observe a previously unnoticed hysteresis regime in midrange parameter space for the utility of the social feedback in the trinary sociodynamic multinomial logit model. Consideration of the scale parameter for the nesting further introduces additional solution regimes. Presumably a similar study with a higher number of choice alternatives would yield even more complex regimes. It would also allow the estimation of additional parameters, such as alternative specific constants, scale parameters for different nests, and/or alternative specific feedback effects. In the Appendix, in a parallel exploration to that in this chapter, we see that even something seemingly as simple as the addition of an alternative specific constant in sociodynamic trinary multinomial logit model yields regimes qualitatively similar to that in this chapter, but also another regime not observed here.

*Alternative form of choice kernel.* This chapter considered the nested logit model in response to Brock and Durlauf’s suggestion (2002, 2006) in their multinomial logit work. The convenient closed form of the probabilities in the nested logit model also allows a rigorous analytical approach to derivation of bifurcation curves, bifurcation diagrams, trajectory of solutions, and trajectory of bifurcations. Via numerical techniques however, it would be possible to study a probit choice model or a mixed logit model. It could be very interesting to see if qualitatively similar regimes arise for a sociodynamic probit model and/or a sociodynamic mixed logit model as observed in this thesis for the sociodynamic nested logit model.

*Additional observed heterogeneity.* Consideration of the nested logit model accommodates the possibility of unobserved heterogeneity im-
Pacting the presumed correlation structure of the alternatives. A key aspect in the theoretical results in this chapter however is the assumption that the only observed explanatory variable in the systematic utility for the elemental choice alternatives within a nest is the feedback effect. While such a specification may be plausible for a fad, it is much less intuitive for transportation mode choice where other explanatory variables would be assumed to be significant, including both attributes of the alternatives such as travel time, as well as characteristics of the decision-making agents such as gender, age and income. In Part III of this thesis, a multi-agent based simulation model is therefore presented which gives straightforward possibility to test more realistic empirical cases. The multi-agent based model is docked against the analytical results in this chapter for the special case of homogeneous agents as a means to verify the implementation of the computational model, before proceeding to add additional heterogeneity.

Local social-spatial feedback. The theoretical results in this chapter are derived on the basis of global feedback. Another extension as indicated at the outset of Part II is the consideration of local social-spatial networks. In Part III of this thesis, we illustrate the multi-agent based simulation of a discrete choice model with local interactions using microdata on transportation mode choice of households in the Netherlands as a testbed, highlighting some hypothesized network interaction effects on the basis of socioeconomic peer group, spatial proximity of residential location and spatial proximity of work location in a trinary sociodynamic nested logit model. Much empirical work in understanding local social-spatial feedback is still needed as we will see the computational results prove to be highly dependent on the presumed structure of the local feedback in the econometric model estimated. A stream of research by Carrasco and Miller (2006), Carrasco et al (2008), Axhausen (2008), van den Berg et al (2009), Kowald and Axhausen (2012), Carrasco and Cid-Aguayo (2012) provide promising progress in the measurement of personal networks as related to travel behavior.

In conclusion, the multiplicity of equilibria and the bifurcations between regimes in parameter-space, is clearly accentuated and becomes obvious when the only systematic part of the utility function is the feedback effect. There is thus nothing else in the utility to dampen or counter the feedback effect. Nonetheless the theoretical model points at underlying qualitative patterns which plausibly may exist also in fully-specified empirical models with detailed systematic utility functions that account for heterogeneous choice based on individual characteristics of decision-makers and individual-specific attributes of choice alternatives, and that account for local social-spatial networks – even if this is less easy to expose and detect in empirical models. It is our hope that this thesis serves as a call to other re-
searchers, both to empiricists to try to quantify how possible bifurcations affect results in empirical models, as well as to methodologists in how to further tackle the problem.