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Socio-dynamic discrete choice: Theory and application

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SOCIO-DYNAMIC BINARY LOGIT: THEORY, REVISITED

In this appendix we re-visit the binary logit model with social interactions reviewed in Chapter 3, applying techniques from the mathematics of dynamical systems and bifurcation theory.

In section A.1, we describe the sociodynamic binary logit model for choice between two alternatives as a scalar autonomous differential equation in the utility parameter β for the level of aggregate social influence. In section A.2, we characterize the stability of solutions via the derivative of the equation, and search for values of the parameter β such that a solution is a bifurcation point. We show that there is one bifurcation point for this equation, which has cubic degeneracy. This yields a pitchfork bifurcation. In section A.3, we see that a potential function can be derived, yielding an alternative method for determining the stability of the equilibria. .

A.1 SCALAR AUTONOMOUS EQUATION

Recall the formulation of the multinomial logit model in section 2.1. Under the assumption of independent and identically Gumbel distributed error terms, the probability $P_n(i|C)$ that the individual decision making entity n chooses alternative i within the binary choice set $C = 0, 1$ is given by:

$$P_n(i|C) = \frac{e^{\mu V_{in}}}{\sum\limits_{j=0}^{1} e^{\mu V_{jn}}} \tag{A.1}$$

where μ is a strictly positive scale parameter which we normalize to 1, following standard convention.

$$\mu \equiv 1 \tag{A.2}$$

If we assume that the only contribution to the systematic utility of choices is a global field effect with utility parameter β real, finite on the proportion p_i of decision-making agents that have chosen alternative i, then in such a case, when the agents include their own choice with equal weight to others' choices in the calculation of the field effect for a given alternative, the agents' choice behavior is perfectly homogeneous across agents. The probabilities of choosing respectively alternatives $0, 1$, among the two possible alternatives in the choice set are:

$$P(i = 0|C) = \frac{e^{\beta p_0}}{e^{\beta p_0} + e^{\beta p_1}} \tag{A.3}$$

$$P(i = 1|C) = \frac{e^{\beta p_1}}{e^{\beta p_0} + e^{\beta p_1}} \tag{A.4}$$

For a large sample population, the rate of change of the proportions p_0, p_1 of decision-making agents that have chosen each alternative is given by the probabilities $P(i|C)$ of choosing respectively alternatives 0, 1 among the two possible alternatives in the choice set, minus these proportions. This yields a system of two equations in two unknowns, with p_0, p_1 defined on $[0, 1]$. Given β real, finite, we will be interested to find the steady-state solutions p_0, p_1 of the system.

$$\dot{p}_0 = \frac{e^{\beta p_0}}{e^{\beta p_0} + e^{\beta p_1}} - p_0 \tag{A.5}$$

$$\dot{p}_1 = \frac{e^{\beta p_1}}{e^{\beta p_0} + e^{\beta p_1}} - p_1 \tag{A.6}$$

$$p_0, p_1 \in [0, 1] \tag{A.7}$$

At equilibrium:

$$\dot{p}_0 = 0: \ p_0 = \frac{e^{\beta p_0}}{e^{\beta p_0} + e^{\beta p_1}} \tag{A.8}$$

$$\dot{p}_1 = 0: \ p_1 = \frac{e^{\beta p_1}}{e^{\beta p_0} + e^{\beta p_1}} \tag{A.9}$$

Adding (A.8), (A.9):

$$p_0 + p_1 = \frac{e^{\beta p_0} + e^{\beta p_1}}{e^{\beta p_0} + e^{\beta p_1}} = 1 \tag{A.10}$$

Solving (A.10) for p_1:

$$p_1 = 1 - p_0 \tag{A.11}$$

Substituting (A.11) back into (A.5) at equilibrium:

$$\dot{p}_0 = \frac{e^{\beta p_0}}{e^{\beta p_0} + e^{\beta(1-p_0)}} - p_0 = 0 \tag{A.12}$$

Multiplying (A.12) through by the denominator of the first term which is always strictly positive for β real, finite with p_0, p_1 defined on $[0, 1]$, we have an alternative equation that is analytically easier to work with:

$$g \equiv e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 e^{\beta(1-p_0)} = 0 \tag{A.13}$$

Or, combining terms to see the symmetry in p_0 and $p_1 = 1 - p_0$ more immediately:

$$g = (1 - p_0)e^{\beta p_0} - p_0 e^{\beta(1-p_0)} = p_1 e^{\beta p_0} - p_0 e^{\beta p_1} = 0 \tag{A.14}$$

This scalar equation can be solved conveniently graphically, for example, by plotting the curve g and finding its intersection with the p_0-axis. Depending on the value of β, the equation may have more than

one solution. Note however, that by the symmetry of (A.14), there is always at least one solution, regardless of the value of β.

$$p_0 = 1 - p_0 : \ p_0 = \frac{1}{2} \tag{A.15}$$

A.2 STABILITY ANALYSIS

A stationary point of g is locally stable if the derivative dg/dp_0 evaluated at the point is negative:

$$
\begin{aligned}
\left. \frac{dg}{dp_0} \right|_{g(p_0)=0} &= \left. \frac{d}{dp_0} \left(e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 e^{\beta(1-p_0)} \right) \right|_{g(p_0)=0} \\
&= \beta e^{\beta p_0} - \beta p_0 e^{\beta p_0} + \beta p_0 e^{\beta(1-p_0)} - e^{\beta p_0} - e^{\beta(1-p_0)} \Big|_{g(p_0)=0} \\
&= \beta (e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 e^{\beta(1-p_0)}) \Big|_{g(p_0)=0} \\
&\quad + 2\beta p_0 e^{\beta(1-p_0)} - e^{\beta p_0} - e^{\beta(1-p_0)} \Big|_{g(p_0)=0} \\
&= (2\beta p_0 - 1) e^{\beta(1-p_0)} - e^{\beta p_0}
\end{aligned}
\tag{A.16}
$$

Thus we have the condition for local stability:

$$
\begin{aligned}
\left. \frac{dg}{dp_0} \right|_{g(p_0)=0} &= (2\beta p_0 - 1)e^{\beta(1-p_0)} - e^{\beta p_0} < 0 \\
&(2\beta p_0 - 1)e^{\beta(1-2p_0)} < 1
\end{aligned}
\tag{A.17}
$$

Note that if β is any real, non-positive value, then this local stability condition will always be satisfied. Since we have defined p_0 on the interval $[0, 1]$, then for β real, non-positive, we see $\exp(\beta(1-2p_0)$ will always take a non-negative value and $(2\beta p_0 - 1)$ will always take a strictly negative value, so that the left hand side is always non-positive. Thus, we can see already that any bifurcation in behavior must take place when β is a strictly positive value.

A stationary point of g is locally unstable if the derivative dg/dp_0 is positive. Thus we have the condition for local instability:

$$
\begin{aligned}
\left. \frac{dg}{dp_0} \right|_{g(p_0)=0} &= (2\beta p_0 - 1)e^{\beta(1-p_0)} - e^{\beta p_0} > 0 \\
&(2\beta p_0 - 1)e^{\beta(1-2p_0)} > 1
\end{aligned}
\tag{A.18}
$$

To find values of the parameter β which lead to bifurcations in behavior, that is, change in number or stability of stationary points, we are interested cases when the derivative dg/dp_0 is zero:

$$\left. \frac{dg}{dp_0} \right|_{g(p_0)=0} = (2\beta p_0 - 1)e^{\beta(1-p_0)} - e^{\beta p_0} = 0 \tag{A.19}$$

Dividing through by $\exp(\beta p_0) + \exp(\beta(1-p_0))$ which is always positive for β real, finite:

$$(2\beta p_0 - 1)\frac{e^{\beta(1-p_0)}}{e^{\beta p_0} + e^{\beta(1-p_0)}} - \frac{e^{\beta p_0}}{e^{\beta p_0} + e^{\beta(1-p_0)}} = 0 \qquad (A.20)$$

Substituting (A.8), (A.9) and (A.11) into (A.20):

$$\begin{aligned}
0 &= (2\beta p_0 - 1)(1 - p_0) - p_0 \\
&= -2\beta p_0{}^2 + 2\beta p_0 - 1
\end{aligned} \qquad (A.21)$$

Solving for β

$$\beta = \frac{1}{2p_0(1 - p_0)} \qquad (A.22)$$

Substituting (A.22) into (A.13)

$$\begin{aligned}
0 &= e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 e^{\beta(1-p_0)} \\
&= e^{\frac{1}{2p_0(1-p_0)}p_0} - p_0 e^{\frac{1}{2p_0(1-p_0)}p_0} - p_0 e^{\frac{1}{2p_0(1-p_0)}(1-p_0)} \\
&= (1 - p_0)e^{\frac{1}{2(1-p_0)}} - p_0 e^{\frac{1}{2p_0}}
\end{aligned} \qquad (A.23)$$

Re-arranging terms

$$(1 - p_0)e^{\frac{1}{2(1-p_0)}} = p_0 e^{\frac{1}{2p_0}} \qquad (A.24)$$

Note by symmetry that the left-hand side and the right-hand side are equal when the terms in the products individually are equal:

$$(1 - p_0) = p_0 \qquad (A.25)$$

$$e^{\frac{1}{2(1-p_0)}} = e^{\frac{1}{2p_0}} \qquad (A.26)$$

Furthermore, assuming (A.25) implies (A.26). Therefore, solving (A.25) for p_0

$$p_0 = \frac{1}{2} \qquad (A.27)$$

Substituting (A.1.23) into (A.1.18)

$$\beta = \frac{1}{2p_0(1 - p_0)} = \frac{1}{2\frac{1}{2}(1 - \frac{1}{2})} = 2 \qquad (A.28)$$

Verification that (A.27) is the only solution when (A.19) holds, can be seen conveniently graphically by plotting the left-hand-side and the right-hand-side of (A.24) on a graph and finding their intersection.

To determine the behavior at the bifurcation point it is necessary to examine higher orders of the Taylor expansion of g. For computational convenience and clarity of the symmetry we repeat the computation of the first derivative using the form of g in (A.14)

$$\begin{aligned}
g'(p_0) \equiv \frac{dg}{dp_0} &= \frac{d}{dp_0}\left((1 - p_0)e^{\beta p_0} - p_0 e^{\beta(1-p_0)}\right) \\
&= \beta(1 - p_0)e^{\beta p_0} - e^{\beta p_0} - (-\beta)p_0 e^{\beta(1-p_0)} - e^{\beta(1-p_0)} \\
&= (\beta(1 - p_0) - 1)e^{\beta p_0} + (\beta p_0 - 1)e^{\beta(1-p_0)}
\end{aligned} \qquad (A.29)$$

We re-confirm for (A.14), (A.29) with the values p_0 and β given by (A.27), (A.28) we have:

$$g(p_0 = \tfrac{1}{2})\Big|_{\beta=2} = (1 - \tfrac{1}{2})e^{2 \cdot \frac{1}{2}} - \tfrac{1}{2}e^{2(1-\frac{1}{2})} = \tfrac{1}{2}e - \tfrac{1}{2}e = 0 \quad (A.30)$$

$$g'(p_0 = \tfrac{1}{2})\Big|_{\beta=2} = (2(1 - \tfrac{1}{2}) - 1)e^{2 \cdot \frac{1}{2}} + (2 \cdot \tfrac{1}{2} - 1)e^{2(1-\frac{1}{2})}$$
$$= 0 \cdot e + 0 \cdot e = 0 \quad (A.31)$$

We compute the second derivative:

$$g''(p_0) \equiv \frac{d^2g}{dp_0{}^2} = \frac{d}{dp_0}\left((\beta(1 - p_0) - 1)e^{\beta p_0} + (\beta p_0 - 1)e^{\beta(1-p_0)}\right)$$
$$= \beta(\beta(1 - p_0) - 1)e^{\beta p_0} - \beta e^{\beta p_0} - \beta(\beta p_0 - 1)e^{\beta(1-p_0)} + \beta e^{\beta(1-p_0)}$$
$$= (\beta^2(1 - p_0) - 2\beta)e^{\beta p_0} - (\beta^2 p_0 - 2\beta)e^{\beta(1-p_0)}$$
$$(A.32)$$

Evaluating at the bifurcation point:

$$g''(p_0 = \tfrac{1}{2})\Big|_{\beta=2} = (2^2(1 - \tfrac{1}{2}) - 2 \cdot 2)e^{2 \cdot \frac{1}{2}} - (2^2 \cdot \tfrac{1}{2} - 2 \cdot 2)e^{2(1-\frac{1}{2})}$$
$$= -2e + 2e = 0$$
$$(A.33)$$

Since the bifurcation point is degenerate at second order, we proceed to compute the third derivative:

$$g'''(p_0) \equiv \frac{d^3g}{dp_0{}^3} = \frac{d}{dp_0}\left((\beta^2(1 - p_0) - 2\beta)e^{\beta p_0} - (\beta^2 p_0 - 2\beta)e^{\beta(1-p_0)}\right)$$
$$= \beta(\beta^2(1 - p_0) - 2\beta)e^{\beta p_0} - \beta^2 e^{\beta p_0}$$
$$\quad + \beta(\beta^2 p_0 - 2\beta)e^{\beta(1-p_0)} - \beta^2 e^{\beta(1-p_0)}$$
$$= (\beta^3(1 - p_0) - 3\beta^2)e^{\beta p_0} + (\beta^3 p_0 - 3\beta^2)e^{\beta(1-p_0)}$$
$$(A.34)$$

Evaluating at the bifurcation point:

$$g'''(p_0 = \tfrac{1}{2})\Big|_{\beta=2} = (2^3(1 - \tfrac{1}{2}) - 3 \cdot 2^2)e^{2 \cdot \frac{1}{2}} + (2^3 \cdot \tfrac{1}{2} - 3 \cdot 2^2)e^{2(1-\frac{1}{2})}$$
$$= -8e + -8e = -16e$$
$$(A.35)$$

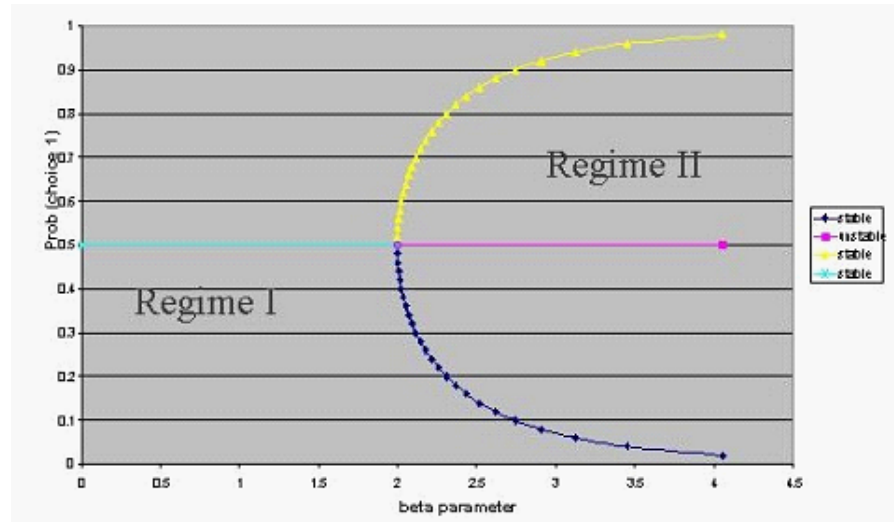*Cubic degeneracy: Pitchfork bifurcation*

See Figure A.1 on page .

Figure A.1: Bifurcation diagram for the sociodynamic binary logit model showing a bifurcation point at $\beta = 2$, separating two regimes: there is one stable equilibrium at $p = 0.5$ for $\beta < 2$; for $\beta > 2$ the equilibrium at $p = 0.5$ becomes unstable with the appearance of two new stable equilibria

A.3 POTENTIAL FUNCTION

The potential function is given by

$$
\begin{aligned}
G &= -\int g \, dp_0 = -\int \left(e^{\beta p_0} - p_0 e^{\beta p_0} - p_0 e^{\beta(1-p_0)} \right) dp_0 \\
&= -\int e^{\beta p_0} \, dp_0 + \int p_0 e^{\beta p_0} \, dp_0 + e^{\beta} \int p_0 e^{-\beta p_0} \, dp_0
\end{aligned}
\tag{A.36}
$$

We evaluate the first term by a simple change of variables. Let

$$
y \equiv \beta p_0 : \quad dy = \beta \, dp_0
\tag{A.37}
$$

Then

$$
\int e^{\beta p_0} \, dp_0 = \frac{1}{\beta} \int e^y \, dy = \frac{1}{\beta} e^y = \frac{1}{\beta} e^{\beta p_0}
\tag{A.38}
$$

We evaluate the second term in (A.36) by integration by parts. Let

$$
\begin{aligned}
u &\equiv p_0 : \quad du = dp_0 \\
dv &\equiv e^{\beta p_0} \, dp_0 : \quad v = \int e^{\beta p_0} \, dp_0
\end{aligned}
\tag{A.39}
$$

Then, applying the result in (A.38)

$$
\begin{aligned}
\int p_0 e^{\beta p_0} \, dp_0 &= \int u \, dv = uv - \int v \, du \\
&= p_0 \frac{1}{\beta} e^{\beta p_0} - \int \frac{1}{\beta} e^{\beta p_0} \, dp_0 = p_0 \frac{1}{\beta} e^{\beta p_0} - \frac{1}{\beta^2} e^{\beta p_0} \\
&= (\beta p_0 - 1) \frac{1}{\beta^2} e^{\beta p_0}
\end{aligned}
\tag{A.40}
$$

Similarly, we evaluate the third term in (A.36). By suitable change of variables

$$w \equiv -\beta p_0 : \ dw = -\beta \, dp_0 \tag{A.41}$$

we have:

$$\int e^{-\beta p_0} \, dp_0 = -\frac{1}{\beta} \int e^w \, dw = -\frac{1}{\beta} e^w = -\frac{1}{\beta} e^{-\beta p_0} \tag{A.42}$$

so that integrating by parts:

$$e^\beta \int p_0 e^{-\beta p_0} \, dp_0 = e^\beta \left(-\frac{p_0}{\beta} e^{-\beta p_0} - \int (-\frac{1}{\beta} e^{-\beta p_0}) \, dp_0 \right)$$

$$= e^\beta \left(-p_0 \frac{1}{\beta} e^{-\beta p_0} - \frac{1}{\beta^2} e^{-\beta p_0} \right) \tag{A.43}$$

$$= -(\beta p_0 + 1) \frac{1}{\beta^2} e^{\beta(1-p_0)}$$

Finally, substituting (A.38), (A.40), (A.43) into (A.36) gives

$$G = -\int g \, dp_0 = -\frac{1}{\beta} e^{\beta p_0} + (\beta p_0 - 1) \frac{1}{\beta^2} e^{\beta p_0} - (\beta p_0 + 1) \frac{1}{\beta^2} e^{\beta(1-p_0)}$$

$$= -(\beta(1-p_0) + 1) \frac{1}{\beta^2} e^{\beta p_0} - (\beta p_0 + 1) \frac{1}{\beta^2} e^{\beta(1-p_0)}$$

$$\tag{A.44}$$

Alternatively, since we are primarily interested in qualitative behavior, divide (A.8) by (A.9) to obtain:

$$\frac{p_0}{p_1} = \frac{e^{\beta p_0}/(e^{\beta p_0} + e^{\beta p_1})}{e^{\beta p_1}/(e^{\beta p_0} + e^{\beta p_1})} = e^{\beta(p_0 - p_1)} \tag{A.45}$$

Taking the natural logarithm of both sides:

$$\ln \frac{p_0}{p_1} = \beta(p_0 - p_1) \tag{A.46}$$

Substituting in (A.11) at equilibrium:

$$\ln \frac{p_0}{1 - p_0} = \beta(p_0 - (1 - p_0)) = \beta(2p_0 - 1) \tag{A.47}$$

Alternative equation, rearranging terms:

$$\breve{g} \equiv -\ln p_0 + \ln(1 - p_0) + \beta(2p_0 - 1) = 0 \tag{A.48}$$

This form can also be obtained by moving the second term in (A.14) to the right hand side, taking the natural logarithm and then moving the right hand side back to the left hand side as follows:

$$(1 - p_0) e^{\beta p_0} = p_0 e^{\beta(1-p_0)}$$

$$\ln(1 - p_0) + \beta p_0 = \ln p_0 + \beta(1 - p_0) \tag{A.49}$$

$$\breve{g} \equiv -\ln p_0 + \ln(1 - p_0) + \beta(2p_0 - 1) = 0$$

Alternative potential function is thus given by:

$$\check{G} = -\int \check{g}\,dp_0 = \int \ln p_0\,dp_0 - \int \ln(1-p_0)\,dp_0 + \int \beta(1-2p_0)\,dp_0$$

(A.50)

To evaluate the first term in (A.50), integrate by parts. Let

$$u \equiv \ln p_0 : \ du = \frac{1}{p_0}\,dp_0$$

$$dv \equiv dp_0 : \ v = \int dp_0 = p_0$$

(A.51)

Then

$$\int \ln p_0\,dp_0 = \int u\,dv = uv - \int v\,du$$

$$= p_0 \ln p_0 - \int p_0 \cdot \frac{1}{p_0}\,dp_0 = p_0 \ln p_0 - p_0$$

(A.52)

We evaluate the second term by a simple change of variables. Let

$$y \equiv (1-p_0) : \ dy = -dp_0 \tag{A.53}$$

Then, applying the result in (A.52)

$$\int \ln(1-p_0)\,dp_0 = -\int \ln y\,dy = -(y \ln y - y)$$

$$= -(1-p_0)\ln(1-p_0) + 1 - p_0$$

(A.54)

Finally, substituting (A.52) and (A.54) into (A.50) gives

$$\check{G} = -\int \check{g}\,dp_0 = (p_0 \ln p_0 - p_0) - (-(1-p_0)\ln(1-p_0) + 1 - p_0)$$

$$+ \beta \int (1-2p_0)\,dp_0$$

$$= p_0 \ln p_0 + (1-p_0)\ln(1-p_0) - 1 + \beta(p_0 - p_0{}^2)$$

$$= p_0 \ln p_0 + (1-p_0)\ln(1-p_0) - \frac{\beta}{2}p_0{}^2 - \frac{\beta}{2}(1 - 2p_0 + p_0{}^2) + (\frac{\beta}{2} - 1)$$

$$= p_0 \ln p_0 + (1-p_0)\ln(1-p_0) - \frac{\beta}{2}p_0{}^2 - \frac{\beta}{2}(1-p_0)^2 + C$$

(A.55)

Or, substituting (A.11) to see the symmetry in p_0 and $p_1 = 1 - p_0$ more immediately:

$$\check{G} = p_0 \ln p_0 + p_1 \ln p_1 - \frac{\beta}{2}(p_0{}^2 + p_1{}^2) + C \tag{A.56}$$