Socio-dynamic discrete choice: Theory and application

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SOCIO-DYNAMIC BINARY LOGIT WITH CONSTANT BIAS: THEORY

In this appendix, we apply techniques from the mathematics of dynamical systems and bifurcation theory to continue our exploration of the binary logit model with social interactions originally studied by Aoki (1995), Brock and Durlauf (2001a), and Blume and Durlauf (2003), when adding a constant bias for one the choice alternatives. Doing so yields a bifurcation *cusp* separating two steady-state regimes. One regime has a single unique stable solution, and one regime has multiple equilibria of which two solutions are stable and one solution is unstable.

In section B.1, we describe the sociodynamic binary logit model with constant bias as a two parameter scalar autonomous differential equation in the utility parameter β for the generic aggregate social influence for both choice alternatives, and in the alternative specific constant bias h for one of the choice alternatives. Solving this equation for h, yields a three dimensional solution surface for h in terms of β and the mode share p_0 or p_1. The level sets of the surface provide classical bifurcation diagrams in the (β, p_0)-plane or in the (β, p_1)-plane for given values of the parameter h. In section B.2, we characterize the stability of solutions via the derivative of the original equation, and find values of the parameters (β, h) such that the solutions are bifurcation points. We show that the bifurcation points have quadratic degeneracy for $h \neq 0$, confirming the saddle-node bifurcation noticed visually from the level sets of the solution surface. Plotting this bifurcation curve on the solution surface and then projecting the curve onto the (β, h)-plane yields a bifurcation cusp. In section B.3, we see that a potential function can be derived, yielding an alternative way of determining the stability of the equilibria. In section B.4, we visualize the analytical results derived in this appendix in terms of the bifurcation cusp in the (β, h)-plane, showing the solution of the scalar autonomous differential equation and the corresponding potential function at various points in (β, h)-parameter space for the two solution regimes. We see in this visualization very naturally how symmetry is broken by the alternative specific constant bias.

B.1 SYMMETRY-BREAKING TWO PARAMETER BIFURCATION

Recall the formulation of the multinomial logit model in section 2.1. Under the assumption of independent and identically Gumbel distributed error terms, the probability $P_n(i|C)$ that the individual decision making entity n chooses alternative i within the binary choice set $C = 0, 1$ is then given by:

$$P_n(i|C) = \frac{e^{\mu V_{in}}}{\sum_{j=0}^{1} e^{\mu V_{jn}}} \tag{B.1}$$

where μ is a strictly positive scale parameter which we normalize to 1, following standard convention.

$$\mu \equiv 1 \tag{B.2}$$

If we now assume that the only contributions to the systematic utility of choices are a global field effect with utility parameter β real, finite on the proportion p_i of decision-making agents that have chosen alternative i, and a constant bias h real, finite for choice alternative 0, then in such a case, when the agents include their own choice with equal weight to others' choices in the calculation of the field effect for a given alternative, the agents' choice behavior is perfectly homogeneous across agents. The probabilities of choosing respectively alternatives 0, 1, among the possible alternatives in the choice set are:

$$P(i = 0|C) = \frac{e^{\beta p_0 + h}}{e^{\beta p_0 + h} + e^{\beta p_1}} \tag{B.3}$$

$$P(i = 1|C) = \frac{e^{\beta p_1}}{e^{\beta p_0 + h} + e^{\beta p_1}} \tag{B.4}$$

For a large sample population, the rate of change of the proportions p_0, p_1 of decision-making agents that have chosen each alternative is given by the probabilities $P(i|C)$ of choosing respectively alternatives 0, 1 among the two possible alternatives in the choice set, minus these proportions. This yields a system of two equations in two unknowns, with p_0, p_1 defined on [0, 1]. We refer to (B.5) as the *sociodynamic binary logit model with constant bias*. Given β and h real, finite, we will be interested to find the steady-state solutions p_0, p_1 of the system.

$$\begin{aligned} \dot{p}_0 &= \frac{e^{\beta p_0 + h}}{e^{\beta p_0 + h} + e^{\beta p_1}} - p_0 \\ \dot{p}_1 &= \frac{e^{\beta p_1}}{e^{\beta p_0 + h} + e^{\beta p_1}} - p_1 \end{aligned} \tag{B.5}$$

$$p_0, p_1 \in [0, 1]$$

At equilibrium:

$$\dot{p}_0 = 0 : \; p_0 = \frac{e^{\beta p_0 + h}}{e^{\beta p_0 + h} + e^{\beta p_1}} \tag{B.6}$$

$$\dot{p}_1 = 0: \ p_1 = \frac{e^{\beta p_1}}{e^{\beta p_0 + h} + e^{\beta p_1}} \tag{B.7}$$

Adding (B.6), (B.7):

$$p_0 + p_1 = \frac{e^{\beta p_0 + h} + e^{\beta p_1}}{e^{\beta p_0 + h} + e^{\beta p_1}} = 1 \tag{B.8}$$

Solving (B.8) for p_1:

$$p_1 = 1 - p_0 \tag{B.9}$$

Substituting (B.9) back into (B.5) at equilibrium:

$$\dot{p}_0 = \frac{e^{\beta p_0 + h}}{e^{\beta p_0 + h} + e^{\beta(1-p_0)}} - p_0 = 0 \tag{B.10}$$

Multiplying (B.10) through by the denominator of the first term which is always strictly positive for β and h real, finite with p_0, p_1 defined on $[0, 1]$, we have an alternative equation that is analytically easier to work with:

$$g \equiv e^{\beta p_0 + h} - p_0 e^{\beta p_0 + h} - p_0 e^{\beta(1-p_0)} = 0 \tag{B.11}$$

Or, combining terms to see the effect of h on the symmetry-breaking of p_0 and $p_1 = 1 - p_0$ more immediately:

$$g = (1 - p_0) e^{\beta p_0 + h} - p_0 e^{\beta(1-p_0)} = p_1 e^{\beta p_0 + h} - p_0 e^{\beta p_1} = 0 \tag{B.12}$$

This scalar equation can be solved conveniently graphically, for example, by plotting the curve g_0 and finding its intersection with the p_0-axis. Depending on the value of β, the equation may have more than one solution. If $h = 0$, we recover the symmetric equation (A.14). Note however, the symmetric solution (A.15) will no longer apply for (B.12) when $h \neq 0$ except in the limit $\beta \to \pm\infty$.

Re-writing (B.12), re-arranging terms and taking the natural logarithm to solve for h:

$$\begin{aligned} p_1 e^{\beta p_0} e^h - p_0 e^{\beta p_1} &= 0 \\ e^h &= \frac{p_0}{p_1} e^{\beta(p_1 - p_0)} \\ h &= \ln p_0 - \ln p_1 + \beta(p_1 - p_0) \end{aligned} \tag{B.13}$$

Or equivalently, substituting (B.9) into (B.13) to write h in terms of β and p_0 alone:

$$h = \ln p_0 - \ln(1 - p_0) + \beta(1 - 2p_0) \tag{B.14}$$

We can now plot a three dimensional surface in (β, p_0, h)-space for h in terms of β and p_0. The solutions of (B.12) are simply the level sets of (B.14).

B.2 STABILITY ANALYSIS

To find values of the parameters β and h which lead to bifurcations in behavior, that is, change in number or stability of stationary points, we are interested cases when the derivative dg/dp_0 evaluated at a stationary point is zero.

$$\begin{aligned} 0 &= \left.\frac{dg}{dp_0}\right|_{g(p_0)=0} = \frac{d}{dp_0}\left.\left((1-p_0)e^{\beta p_0+h} - p_0 e^{\beta(1-p_0)}\right)\right|_{g(p_0)=0} \\ &= \beta(1-p_0)e^{\beta p_0+h} - e^{\beta p_0+h} - (-\beta)p_0 e^{\beta(1-p_0)} - e^{\beta(1-p_0)}\Big|_{g(p_0)=0} \\ &= (\beta(1-p_0)-1)e^{\beta p_0+h} + (\beta p_0 - 1)e^{\beta(1-p_0)}\Big|_{g(p_0)=0} \end{aligned} \tag{B.15}$$

Dividing through by $\exp(\beta p_0 + h) + \exp(\beta(1-p_0))$ which is always positive for β and h real, finite, and then substituting (B.6), (B.7) and (B.9) at equilibrium:

$$\begin{aligned} 0 &= (\beta(1-p_0)-1)\left.\frac{e^{\beta p_0+h}}{e^{\beta p_0+h}+e^{\beta(1-p_0)}}\right|_{g(p_0)=0} \\ &\quad +(\beta p_0 - 1)\left.\frac{e^{\beta(1-p_0)}}{e^{\beta p_0+h}+e^{\beta(1-p_0)}}\right|_{g(p_0)=0} \\ &= (\beta(1-p_0)-1)p_0 + (\beta p_0 - 1)(1-p_0) \\ &= \beta p_0 - \beta {p_0}^2 - p_0 + \beta p_0 - 1 - \beta {p_0}^2 + p_0 \\ &= -2\beta {p_0}^2 + 2\beta p_0 - 1 \end{aligned} \tag{B.16}$$

Solving for β we recover the same result as in (A.22)

$$\beta = \frac{1}{2p_0(1-p_0)} = \frac{1}{2p_0 p_1} \tag{B.17}$$

Thus we find that the symmetry-breaking role of h has an effect on the value of p_0 and $p_1 = 1 - p_0$ at equilibrium, but it does not effect the relation between β and p_0, p_1. Furthermore, since p_0, p_1 are defined on $[0, 1]$ by (B.5), we see immediately that there can exist a bifurcation only for strictly positive values of the parameter β, independent of h. More precisely, since $p_0 = p_1 = \frac{1}{2}$ can never be a solution of (B.12) for $h \neq 0$ when β is real, finite we have:

$$\beta = \frac{1}{2p_0 p_1} > 2 \text{ if } h \neq 0 \tag{B.18}$$

By plotting the continuous string of values (β, p_0, h) in (β, p_0, h)-space with β given by (B.17) and h given by (B.14), we obtain curve of bifurcation points on the stationary solution surface that separates the behavior of the system into two different regimes in terms of the number and stability of stationary points for given β and h. See Figure B.1 on page 363.

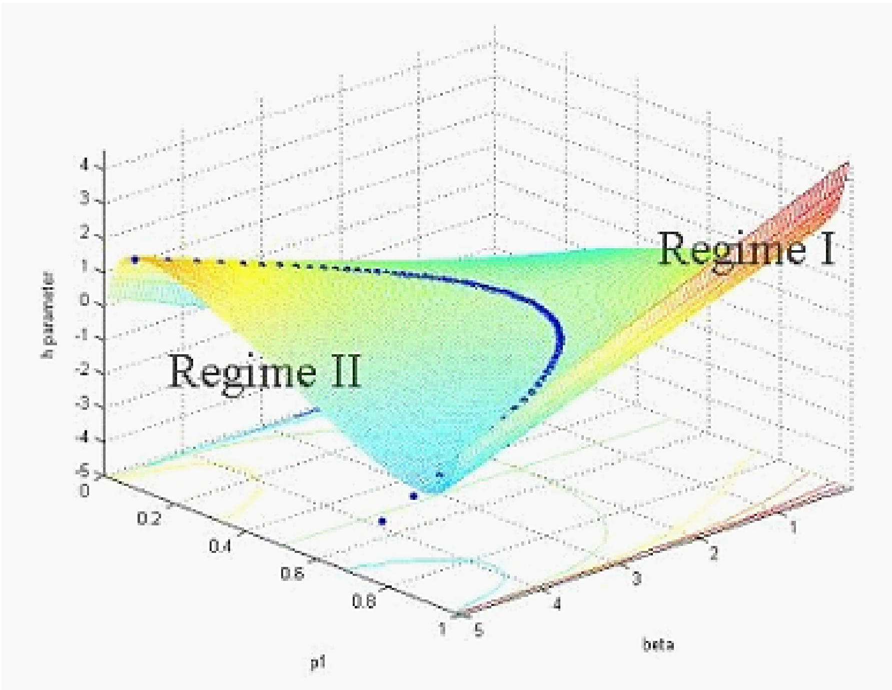


Figure B.1: Solution surface for the sociodynamic binary logit model with constant bias. The blue line depicts the continuous string of bifurcation points separating the two regimes. Level sets for various values of alternative specific constant are projected in color below the solution surface, retrieving the bifurcation diagrams in Figure A.1 on page 356 for $h = 0$ and Figure B.2 on page 365 for $h > 0$ and $h < 0$. The projection of the continuous string of bifurcation points onto the (β, h)-plane yields a cusp as shown in Figure B.3 on page 368.

Substituting (B.17) back into (B.14), we can also obtain an equation for h in terms of p_0.

$$h = \ln p_0 - \ln(1-p_0) + \frac{(1-2p_0)}{2p_0(1-p_0)} \tag{B.19}$$

To determine the behavior at a bifurcation point it is necessary to examine higher orders of the Taylor expansion of g. We compute the second derivative:

$$\begin{aligned} g''(p_0) &\equiv \frac{d^2 g}{dp_0{}^2} = \frac{d}{dp_0}\left((\beta(1-p_0)-1)e^{\beta p_0+h} + (\beta p_0 - 1)e^{\beta(1-p_0)}\right) \\ &= \beta(\beta(1-p_0)-1)e^{\beta p_0+h} - \beta e^{\beta p_0+h} \\ &\quad - \beta(\beta p_0 - 1)e^{\beta(1-p_0)} + \beta e^{\beta(1-p_0)} \\ &= (\beta^2(1-p_0)-2\beta)e^{\beta p_0+h} - (\beta^2 p_0 - 2\beta)e^{\beta(1-p_0)} \end{aligned} \tag{B.20}$$

For convenience let be Z a normalization factor which is by definition always positive for β and h real, finite

$$Z \equiv e^{\beta p_0+h} + e^{\beta(1-p_0)} \tag{B.21}$$

Substituting (B.6), (B.7) at equilibrium in terms of (B.21) and (B.9):

$$\begin{aligned} g''(p_0)\big|_{g(p_0)=0} &= (\beta^2(1-p_0)-2\beta)p_0 Z - (\beta^2 p_0 - 2\beta)(1-p_0)Z \\ &= Z(\beta^2 p_0 - \beta^2 p_0{}^2 - 2\beta p_0 - \beta^2 p_0 + 2\beta + \beta^2 p_0{}^2 - 2\beta p_0) \\ &= Z(-4\beta p_0 + 2\beta) = 2Z\beta(-2p_0+1) \end{aligned} \tag{B.22}$$

Let $p_0 = \underline{p_0}$ be a stationary point. Since Z is strictly positive as defined in (B.21), since β is strictly positive as defined in (B.17) with p_0 defined on [0, 1] by (B.5), since $p_0 = \frac{1}{2}$ can never be a solution of (B.12) for $h \neq 0$, then we have:

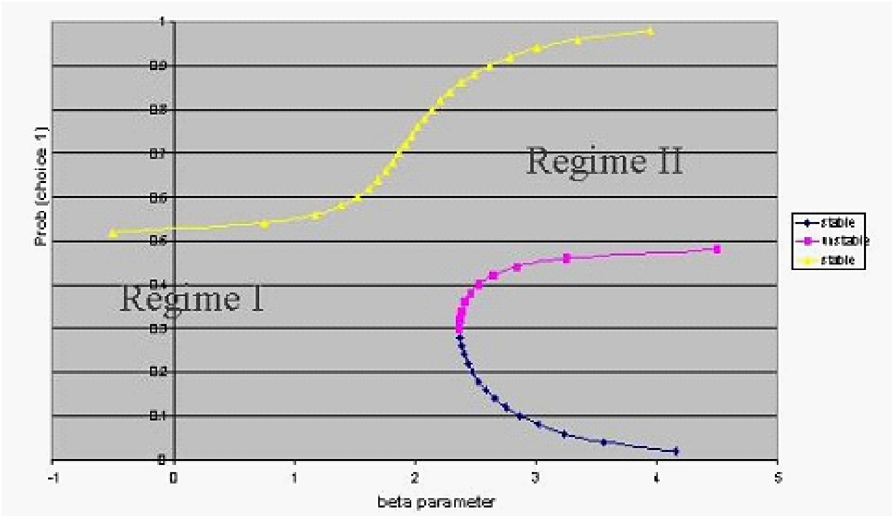
$$g''(p_0)\big|_{g(p_0)=0} = 2Z\beta(-2p_0+1) > 0 \text{ if } h \neq 0, \text{ and } \underline{p_0} \in [0, \frac{1}{2}) \tag{B.23}$$

$$g''(p_0)\big|_{g(p_0)=0} = 2Z\beta(-2p_0+1) < 0 \text{ if } h \neq 0, \text{ and } \underline{p_0} \in (\frac{1}{2}, 1] \tag{B.24}$$

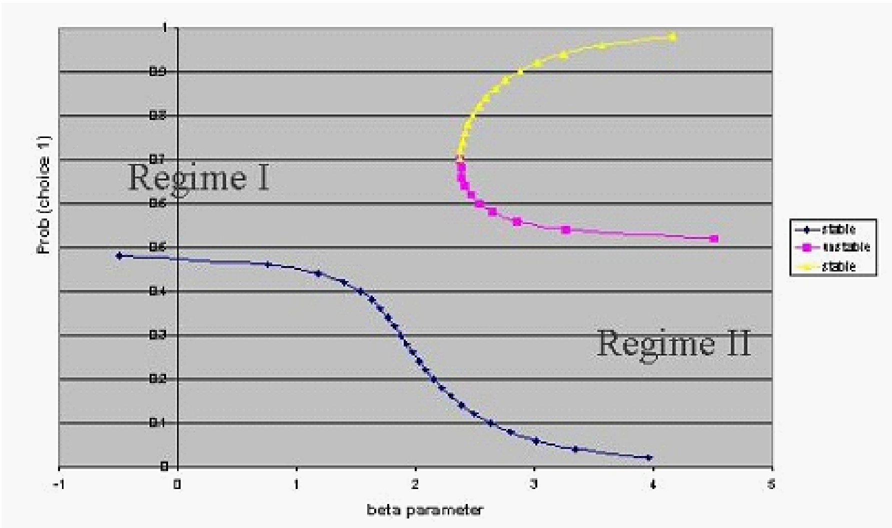
Since we have established behavior at second order for $h \neq 0$, there is no need to proceed to compute the third derivative.

Quadratic degeneracy: Saddle-node bifurcation

See Figure B.2 on page 365.



(a) $h > 0$



(b) $h < 0$

Figure B.2: Bifurcation diagrams for sociodynamic binary model with: (a) positive alternative specific constant for choice alternative 0 and (b) negative alternative specific constant for choice alternative 0 showing a bifurcation point at $\beta = \beta* > 2$, separating two regimes: there is one stable equilibrium for $\beta < \beta*$; for $\beta > \beta*$ we have the appearance of a pair of new equilibria, one stable and one unstable. Compare with Figure A.1 on page 356.

B.3 POTENTIAL FUNCTION

We try to find potential function G by integrating g with respect to p_0

$$\begin{aligned} G &= -\int g dp_0 = -\int \left((1-p_0)e^{\beta p_0 + h} - p_0 e^{\beta(1-p_0)}\right) dp_0 \\ &= -e^h \int (1-p_0)e^{\beta p_0} dp_0 + e^{\beta} \int p_0 e^{-\beta p_0} dp_0 \end{aligned} \tag{B.25}$$

We evaluate the integral in the first term in (B.25) by parts. Suppose

$$\begin{aligned} u &\equiv (1-p_0): \ du = -dp_0 \\ dv &\equiv e^{\beta p_0} dp_0: \ v = \int e^{\beta p_0} dp_0 = \frac{1}{\beta} e^{\beta p_0} \end{aligned} \tag{B.26}$$

then we have:

$$\begin{aligned} &\int (1-p_0)e^{\beta p_0} dp_0 = \int u dv = uv - \int v du \\ &= (1-p_0)\frac{1}{\beta}e^{\beta p_0} - \int -\frac{1}{\beta}e^{\beta p_0} dp_0 = (1-p_0)\frac{1}{\beta}e^{\beta p_0} + \frac{1}{\beta^2}e^{\beta p_0} \\ &= (\beta(1-p_0)+1)\frac{1}{\beta^2}e^{\beta p_0} \end{aligned} \tag{B.27}$$

Similarly, we evaluate the integral in the second term in (B.25). Suppose

$$\begin{aligned} u &\equiv p_0: \ du = dp_0 \\ dv &\equiv e^{-\beta p_0} dp_0: \ v = \int e^{-\beta p_0} dp_0 = -\frac{1}{\beta} e^{-\beta p_0} \end{aligned} \tag{B.28}$$

then integrating by parts:

$$\begin{aligned} &\int p_0 e^{-\beta p_0} dp_0 = \int u dv = uv - \int v du \\ &= -\frac{p_0}{\beta}e^{-\beta p_0} - \int (-\frac{1}{\beta}e^{-\beta p_0}) dp_0 = -p_0\frac{1}{\beta}e^{-\beta p_0} - \frac{1}{\beta^2}e^{-\beta p_0} \\ &= -(\beta p_0 + 1)\frac{1}{\beta^2}e^{-\beta p_0} \end{aligned} \tag{B.29}$$

Substituting (B.27) and (B.29) into (B.25) gives

$$\begin{aligned} G &= -\int g dp_0 = -e^h(\beta(1-p_0)+1)\frac{1}{\beta^2}e^{\beta p_0} - e^{\beta}(\beta p_0 + 1)\frac{1}{\beta^2}e^{-\beta p_0} \\ &= -(\beta(1-p_0)+1)\frac{1}{\beta^2}e^{\beta p_0 + h} - (\beta p_0 + 1)\frac{1}{\beta^2}e^{\beta(1-p_0)} \end{aligned} \tag{B.30}$$

Or, substituting (A.2.7) to re-write in terms of p_0, p_1

$$G = -(\beta p_1 + 1)\frac{1}{\beta^2}e^{\beta p_0}e^h - (\beta p_0 + 1)\frac{1}{\beta^2}e^{\beta p_1} \tag{B.31}$$

The symmetry in the potential function is broken by the factor $\exp(h)$ in the first term.

Alternatively, divide (B.6) by (B.7) to obtain:

$$\frac{p_0}{p_1} = \frac{e^{\beta p_0 + h}/(e^{\beta p_0 + h} + e^{\beta p_1})}{e^{\beta p_1}/(e^{\beta p_0 + h} + e^{\beta p_1})} = e^{\beta(p_0 - p_1) + h} \tag{B.32}$$

Taking the natural logarithm of both sides:

$$\ln \frac{p_0}{p_1} = \beta(p_0 - p_1) + h \tag{B.33}$$

Substituting in (B.9) at equilibrium:

$$\ln \frac{p_0}{1 - p_0} = \beta(p_0 - (1 - p_0)) + h = \beta(2p_0 - 1) + h \tag{B.34}$$

Alternative equation, rearranging terms:

$$\breve{g} \equiv -\ln p_0 + \ln(1 - p_0) + \beta(2p_0 - 1) + h = 0 \tag{B.35}$$

Alternative potential function:

$$\breve{G} = -\int \breve{g} dp_0 = \int \ln p_0 dp_0 - \int \ln(1 - p_0) dp_0 + \int (\beta(1 - 2p_0) - h) dp_0 \tag{B.36}$$

Substituting (A.52) and (A.54) into (B.36) gives

$$\begin{aligned}
\breve{G} = -\int \breve{g} dp_0 &= (p_0 \ln p_0 - p_0) - (-(1 - p_0)\ln(1 - p_0) + 1 - p_0) \\
&\quad + \int (\beta(1 - 2p_0) - h) dp_0 \\
&= p_0 \ln p_0 + (1 - p_0)\ln(1 - p_0) - 1 + \beta(p_0 - {p_0}^2) - hp_0 \\
&= p_0 \ln p_0 + (1 - p_0)\ln(1 - p_0) \\
&\quad - \frac{\beta}{2}{p_0}^2 - \frac{\beta}{2}(1 - 2p_0 + {p_0}^2) - hp_0 + (\frac{\beta}{2} - 1) \\
&= p_0 \ln p_0 + (1 - p_0)\ln(1 - p_0) - \frac{\beta}{2}{p_0}^2 - \frac{\beta}{2}(1 - p_0)^2 - hp_0 + C
\end{aligned} \tag{B.37}$$

Or, substituting (B.9) to re-write in terms of p_0, p_1

$$\breve{G} = p_0 \ln p_0 + p_1 \ln p_1 - \frac{\beta}{2}({p_0}^2 + {p_1}^2) - hp_0 + C \tag{B.38}$$

The symmetry in the alternative potential function is broken by the term $-hp_0$.

B.4 BIFURCATION CUSP IN (β, h)-PARAMETER SPACE

See Figure B.3 on page 368

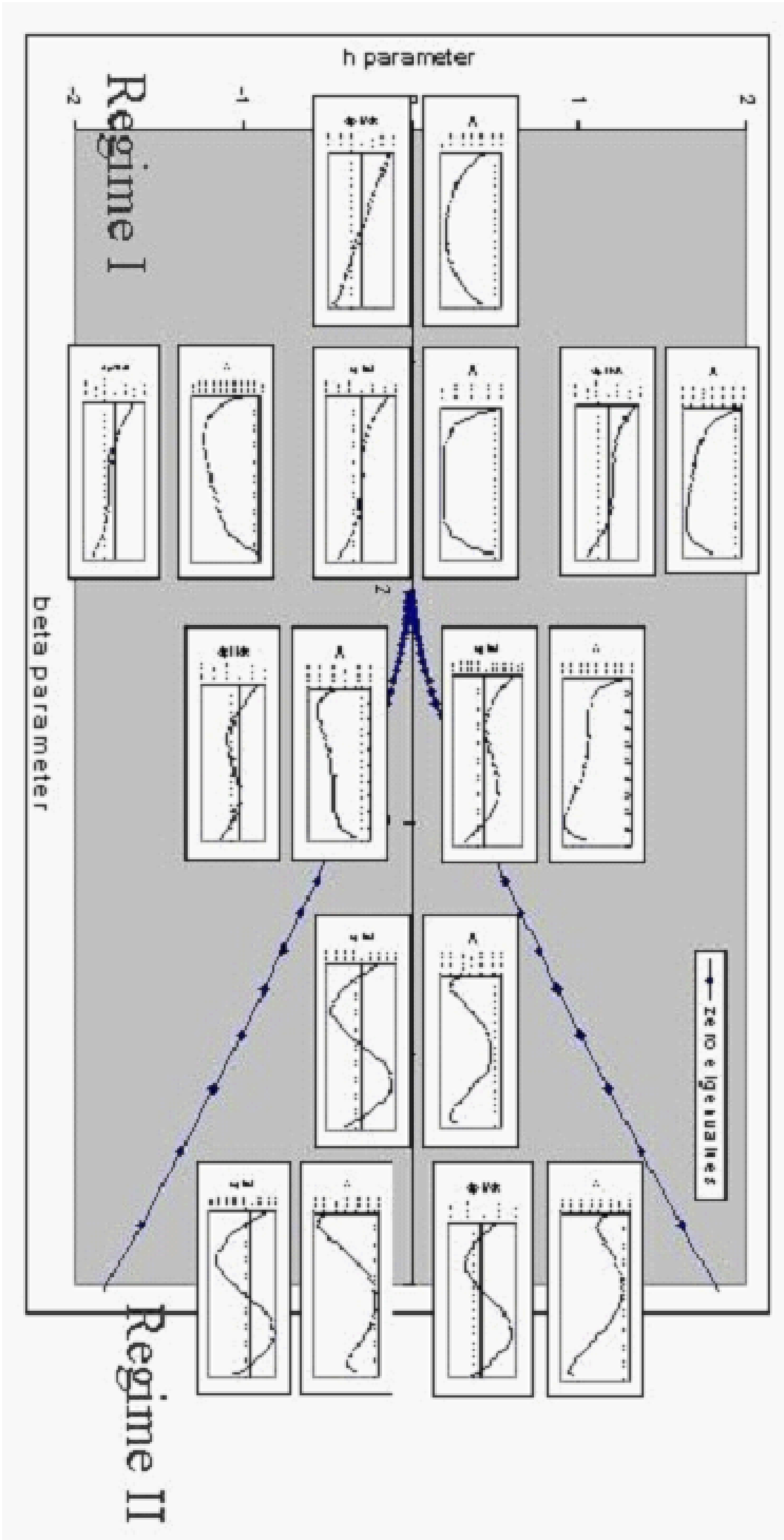


Figure B.3: Bifurcation curves in the (β, h)-plane showing the potential function and the graphical solution of the system at various points in parameter space. The blue cusp shows the bifurcation curve separating the two regimes.