Socio-dynamic discrete choice: Theory and application

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In this appendix, we apply techniques from the mathematics of dynamical systems and bifurcation theory to continue our exploration of the multinomial logit model with social interactions originally studied by Brock and Durlauf (2002, 2006), when adding a constant bias for one the choice alternatives. Doing so yields rich bifurcation diagrams revealing eight emergent steady-state regimes where symmetry is broken by the constant bias.

In section C.1, we describe the sociodynamic trinary multinomial logit model with constant bias as a two parameter planar autonomous system in the utility parameter $\beta$ for the level of aggregate social influence and in the constant bias $h$ for one of the choice alternatives. Then in section C.2, by applying a graphical nullclines approach, the number of solutions is counted and charted in the $(\beta, h)$-plane across a sweep of the utility parameter and a sweep of the bias parameter, paying particular attention to the inherent (broken) symmetries in the trinary choice model between the biased and non-biased alternatives. In section C.3, we derive expressions for the defining bifurcation curves in the $(\beta, h)$-plane rigorously analytically by characterizing the stability of solutions via the eigenvalues of the Jacobian matrix of the system. In order to do this, we draw heavily on qualitative observations regarding the number of 2-way symmetric steady-state solutions in each regime where the mode shares of the non-biased alternatives are equal. In section C.4, we see that a qualitatively similar alternative system of equations can be expressed as a gradient system, yielding a straightforward visual way of directly determining the stability of the equilibria. In section C.5, we visualize the analytical results derived in this appendix in terms of the bifurcation curves in the $(\beta, h)$-plane, in terms of the solution trajectories in the $(p_0, p_1)$-plane over utility parameter $\beta$ for a sweep of the bias parameter $h$, and in terms of classical bifurcation diagrams in the $(\beta, p_0)$-plane and in the $(\beta, p_1)$-plane for a sweep of the bias parameter $h$.

C.1 TWO PARAMETER PLANAR AUTONOMOUS SYSTEM

Recall the formulation of the multinomial logit model in section 2.1. Under the assumption of independent and identically Gumbel distributed error terms, the probability $P_n(i|C)$ that the individual deci-
sion making entity \( n \) chooses alternative \( i \) within the trinary choice set \( C = 0, 1, 2 \) is then given by:

\[
P_n(i|C) = \frac{e^{\mu V_{in}}}{\sum_{j=0}^{2} e^{\mu V_{jn}}}
\]  

where \( \mu \) is a strictly positive scale parameter which we normalize to 1, following standard convention.

\[
\mu = 1
\]

If we now assume that the only contributions to the systematic utility of choices are a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_i \) of decision-making agents that have chosen alternative \( i \), and a constant bias \( h \) real, finite for choice alternative 0, then in such a case, when the agents include their own choice with equal weight to others’ choices in the calculation of the field effect for a given alternative, the agents’ choice behavior is perfectly homogeneous across agents. The probabilities of choosing respectively alternatives 0, 1, 2 among all possible alternatives in the choice set are:

\[
P(i = 0|C) = \frac{e^{\beta p_0 + h}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}}
\]  

\[
P(i = 1|C) = \frac{e^{\beta p_1}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}}
\]  

\[
P(i = 2|C) = \frac{e^{\beta p_2}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}}
\]

For a large sample population, the rate of change of the proportions \( p_0, p_1, p_2 \) of decision-making agents that have chosen each alternative is given by the probabilities \( P(i|C) \) of choosing respectively alternatives 0, 1, 2 among the three possible alternatives in the choice set, minus these proportions. This yields a system of three equations in three unknowns, with \( p_0, p_1, p_2 \) defined on \([0, 1]\). We refer to (C.6) as the sociodynamic trinary multinomial logit model with constant bias.

Given \( \beta \) and \( h \) real, finite, we will be interested to find the steady-state solutions \( p_0, p_1, p_2 \) of the system.

\[
\dot{p}_0 = \frac{e^{\beta p_0 + h}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}} - p_0
\]

\[
\dot{p}_1 = \frac{e^{\beta p_1}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}} - p_1
\]

\[
\dot{p}_2 = \frac{e^{\beta p_2}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}} - p_2
\]

\( p_0, p_1, p_2 \in [0, 1] \)
At equilibrium:

\[ \dot{p}_0 = 0 : \quad p_0 = \frac{e^{\beta p_0 + h}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}} \]  \hspace{1cm} (C.7)

\[ \dot{p}_1 = 0 : \quad p_1 = \frac{e^{\beta p_1}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}} \]  \hspace{1cm} (C.8)

\[ \dot{p}_2 = 0 : \quad p_2 = \frac{e^{\beta p_2}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}} \]  \hspace{1cm} (C.9)

Adding (C.7), (C.8), (C.9):

\[ p_0 + p_1 + p_2 = \frac{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}} = 1 \]  \hspace{1cm} (C.10)

Solving (C.10) for \( p_2 \):

\[ p_2 = 1 - p_0 - p_1 \]  \hspace{1cm} (C.11)

Substituting (C.11) back into (C.6) at equilibrium:

\[ \dot{p}_0 = \frac{e^{\beta p_0 + h}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1-p_0-p_1)}} - p_0 = 0 \]  \hspace{1cm} (C.12)

\[ \dot{p}_1 = \frac{e^{\beta p_1}}{e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1-p_0-p_1)}} - p_1 = 0 \]  \hspace{1cm} (C.13)

Converted equations that are easier to work with, multiplying through by the denominator of the first term which is always strictly positive for \( \beta \) real, finite with \( p_0, p_1 \) defined on \([0, 1] \):

\[ g_0 \equiv e^{\beta p_0 + h} - p_0 e^{\beta p_0 + h} - p_0 e^{\beta p_1} - p_0 e^{\beta (1-p_0-p_1)} = 0 \]  \hspace{1cm} (C.14)

\[ g_1 \equiv e^{\beta p_1} - p_1 e^{\beta p_0 + h} - e^{\beta p_1} - p_1 e^{\beta (1-p_0-p_1)} = 0 \]  \hspace{1cm} (C.15)

This planar system of equations can be solved conveniently graphically, for example, by plotting the null clines of the surfaces \( g_0 \) and \( g_1 \) on a graph and finding their intersection. Depending on the value of \( \beta \), the system may have more than one solution.

Alternatively, grouping terms and substituting back (C.11), to see the symmetry breaking between \( p_0, p_1 \) and \( p_2 \) clearly:

\[ g_0 = (1 - p_0)e^{\beta p_0 + h} - p_0 (e^{\beta p_1} + e^{\beta p_2}) = 0 \]  \hspace{1cm} (C.16)

\[ g_1 = (1 - p_1)e^{\beta p_1} - p_1 (e^{\beta p_0 + h} + e^{\beta p_2}) = 0 \]  \hspace{1cm} (C.17)

Note that the system (C.16) and (C.17) can also be solved for \( h \) as follows:

\[ e^h = \frac{p_0 (e^{\beta p_1} + e^{\beta (1-p_0-p_1)})}{(1-p_0)e^{\beta p_0}} \]  \hspace{1cm} (C.18)
Taking the natural logarithm of both sides of the system (C.18) and (C.19):

\[ h = -\beta p_0 - \ln(1 - p_0) + \ln p_0 + \ln(e^{\beta p_1} + e^{\beta(1-p_0-p_1)}) \]  

(C.20)

\[ h = -\beta p_0 - \ln p_1 + \ln \left( (1 - p_1)e^{\beta p_1} - p_1 e^{\beta(1-p_0-p_1)} \right) \]  

(C.21)

**Lemma** (Characterization of solutions of the sociodynamic trinary multinomial logit model with constant bias) Suppose that individual choices in a large sample population are characterized by the probabilities (C.3), (C.4) and (C.5) where the only contributions to the systematic utility of choices are a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_i \) of decision-making agents that have chosen alternative \( i \), and a constant bias \( h \) real, finite for choice alternative 0. Then there will exist at least one equilibrium solution of the sociodynamic trinary multinomial logit model with constant bias (C.6) with \( p_1 = p_2 \) defined on \([0, 1/2]\) for all values of \( \beta \) and \( h \) real, finite. Any other solutions with \( p_1 \neq p_2 \) will be characterized by a transcendental relation between \( p_1 \) and \( p_2 \) that is dependent on \( \beta \):

\[ \ln \frac{p_1}{p_2} = \beta(p_1 - p_2) \]  

(C.22)

**Proof.** Making use of the symmetry between the mode shares \( p_1 \) and \( p_2 \) for the non-biased choice alternatives, it is convenient to divide (C.8) by (C.9) to obtain:

\[ \frac{p_1}{p_2} = \frac{e^{\beta p_1}/e^{\beta p_0} + e^{\beta p_1} + e^{\beta p_2}}{e^{\beta p_2}/e^{\beta p_0} + e^{\mu_1 p_1} + e^{\mu_1 p_2}} = e^{\beta(p_1-p_2)} \]  

(C.23)

Taking the natural logarithm of both sides:

\[ \ln \frac{p_1}{p_2} = \beta(p_1 - p_2) \]  

(C.24)

From (C.24) we see that there will exist at least one solution with \( p_1 = p_2 \) for all values of \( \beta \). In such case, from the definition of \( p_i \) on \([0, 1]\), (C.10) implies \( 0 \leq p_0 = 1 - p_1 - p_2 = 1 - 2p_1 \), so that we have also

\[ p_1 \leq 1/2 \]  

(C.25)

Any other solutions with \( p_1 \neq p_2 \) will be characterized by a transcendental relation between \( p_1 \) and \( p_2 \) that is dependent on \( \beta \), thus proving the lemma.\( \square \)
We can also use (C.22) to write a convenient relation for $\beta$ in terms of the mode shares $p_0, p_1, p_2$ for the case $p_1 \neq p_2$:

$$\beta = \frac{1}{(p_1 - p_2)} \ln \frac{p_1}{p_2} = \frac{1}{(p_1 - (1 - p_0 - p_1))} \ln \frac{p_1}{(1 - p_0 - p_1)} \quad \text{(C.26)}$$

For the case $p_1 \neq p_2$, we can substitute (C.26) into (C.32) to obtain a relation for $h$ in terms of the mode shares $p_0, p_1, p_2$:

$$h = \ln \frac{p_0}{p_1} + \beta(p_1 - p_0) = \ln \frac{p_0}{p_1} + \frac{(p_1 - p_0)}{(p_1 - p_2)} \ln \frac{p_1}{p_2} \quad \text{(C.27)}$$

\[\hat{\diamond}\]

**Lemma** (Limiting solutions of the sociodynamic trinary multinomial logit model with constant bias) Suppose that individual choices in a large sample population are characterized by the probabilities (C.3), (C.4) and (C.5) where the only contributions to the systematic utility of choices are a global field effect with utility parameter $\beta$ real, finite on the proportion $p_1$ of decision-making agents that have chosen alternative $i$, and a constant bias $h$ real, finite for choice alternative $0$. Then any solutions of the sociodynamic trinary multinomial logit model with constant bias (C.6) with $p_0 \to 0$ implies $h \to -\infty$ for $\beta$ real, finite when $p_1 \neq p_0$ or $p_2 \neq p_0$. Likewise a solution with $p_1 \to 0$ or $p_2 \to 0$ implies $h \to +\infty$ for $\beta$ real, finite when $p_1 \neq p_0$ or $p_2 \neq p_0$.

**Proof.** It is also convenient making use of the symmetry between the mode shares $p_1$ and $p_2$ of the non-biased choice alternatives, to divide (C.8) and (C.9) respectively by (C.7), to obtain:

$$\frac{p_1}{p_0} = \frac{e^{\beta p_1} / e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}}{e^{\beta p_0 + h} / e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}} = e^{\beta (p_1 - p_0) - h} \quad \text{(C.28)}$$

$$\frac{p_2}{p_0} = \frac{e^{\beta p_2} / e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}}{e^{\beta p_0 + h} / e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta p_2}} = e^{\beta (p_2 - p_0) - h} \quad \text{(C.29)}$$

Taking the natural logarithm of both sides:

$$\ln \frac{p_1}{p_0} = \beta(p_1 - p_0) - h \quad \text{(C.30)}$$

$$\ln \frac{p_2}{p_0} = \beta(p_2 - p_0) - h \quad \text{(C.31)}$$

From (C.30) and (C.31), we can derive simple relations for $h$ in terms of the mode shares $p_0, p_1, p_2$ and the parameter $\beta$:

$$h = \ln \frac{p_0}{p_1} + \beta(p_1 - p_0) \quad \text{(C.32)}$$

$$h = \ln \frac{p_0}{p_2} + \beta(p_2 - p_0) \quad \text{(C.33)}$$
Here we see immediately that a solution with \( p_0 \to 0 \) implies \( h \to -\infty \) for \( \beta \) real, finite when \( p_1 \neq p_0 \) or \( p_2 \neq p_0 \). Likewise a solution with \( p_1 \to 0 \) or \( p_2 \to 0 \) implies \( h \to +\infty \) for \( \beta \) real, finite when \( p_1 \neq p_0 \) or \( p_2 \neq p_0 \), thus proving the lemma.♦

Also noteworthy from (C.32) and (C.33) is the fact that for a solution with \( p_1 = p_0 \) or \( p_2 = p_0 \), the value of \( h \) is always null, independent of \( \beta \) for \( \beta \) real, finite. We will characterize these unusual symmetric solutions in the next lemma.

**Lemma** (Symmetric solutions of the sociodynamic trinary multinomial logit model with constant bias) Suppose that individual choices in a large sample population are characterized by the probabilities (C.3), (C.4) and (C.5) where the only contributions to the systematic utility of choices are a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_i \) of decision-making agents that have chosen alternative \( i \), and a constant bias \( h \) real, finite for choice alternative \( 0 \). Then any symmetric solutions of the sociodynamic trinary multinomial logit model with constant bias (C.6) with \( p_1 = p_0 \) or \( p_2 = p_0 \) imply \( \beta \to +\infty \) for all real, finite \( h > 0 \), and \( \beta \to -\infty \) for all real, finite \( h < 0 \). Otherwise, if \( h = 0 \), then there will exist a solution with \( p_1 = p_0 \) and \( p_2 = p_0 \) for all \( \beta \).

**Proof.** We can similarly solve (C.30) and (C.31) for \( \beta \) in terms of the mode shares \( p_0 \), \( p_1 \), \( p_2 \) and the parameter \( h \), respectively when \( p_1 \neq p_0 \) or \( p_2 \neq p_0 \):

\[
\beta = \frac{h + \ln(p_1/p_0)}{(p_1 - p_0)} \quad (C.34)
\]

\[
\beta = \frac{h + \ln(p_2/p_0)}{(p_2 - p_0)} \quad (C.35)
\]

Here we see that a solution with \( p_1 = p_0 \) or \( p_2 = p_0 \) implies \( \beta \to +\infty \) for all real, finite \( h > 0 \), and \( \beta \to -\infty \) for all real, finite \( h < 0 \). Otherwise, if \( h = 0 \), then there will exist a solution with \( p_1 = p_0 \) and \( p_2 = p_0 \) for all \( \beta \), thus proving the lemma.♦

**C.2 Equilibrium Regimes in \((\beta, h)\)-Parameter Space**

See Figure C.1 on page 375.

See Figure C.2 on page 376 and Figure C.3 on page 377.

**C.3 Stability Analysis**

In subsection C.2 we have seen computationally-derived bifurcation curves in the \((\beta, h)\)-plane showing major regimes for the planar autonomous two-parameter system given by (C.12) and (C.13). Figure C.1 on page 375 can be visually decomposed into three distinct curves.
### C.3 Stability Analysis

#### Figure C.1: Computationally-derived bifurcation curves in the \((\beta, h)\)-plane showing major regimes indicated with number of solutions of the system at various points in parameter space. In Regime I there is one solution; in Regimes II and III there are seven solutions; in Regimes V and VII there are five solutions; in Regime VIII there are 3 solutions. Two additional regimes exist for small \(h\) at finer resolution, namely Regime IV with 3 solutions and Regime VI with 5 solutions. The row \(h = 0\) corresponds to the sociodynamic multinomial logit case in Appendix section A.3, with the familiar bifurcation points at \(\beta \approx 2.74\) and at \(\beta = 3\).

See Figure C.4 on page 378. In this section we will derive the analytical relations for these curves one by one.

Since the stability type of an equilibrium point is a local property, the stability type of equilibrium points of planar autonomous systems can be determined under certain conditions from the approximation of the vector field \(g = (g_0, g_1)\) with its derivative, which is a linear vector field.

Suppose that \(g = (g_0, g_1)\) is a \(C^1\) function and let Jacobian of \(g\) at the point \(p\) be the matrix:

\[
J \equiv Dg(p) = \begin{bmatrix}
\frac{\partial g_0}{\partial p_0}(p) & \frac{\partial g_0}{\partial p_1}(p) \\
\frac{\partial g_1}{\partial p_0}(p) & \frac{\partial g_1}{\partial p_1}(p)
\end{bmatrix}
\]

To find possible values of the parameter \(\beta\) and \(h\) which lead to bifurcations in behavior, that is, change in number or stability of sta-
Figure C.2: Example null clines solution in each of the regimes in parameter space shown in Figure C.1 on page 375, indicated with the number of 2-way symmetric steady-state solutions where the mode shares of the nonbiased choice alternatives are equal, $p_1 = p_2 = 1 - p_0 - p_1$ and thus, $p_1 = (1 - p_0)/2$: (a) one equilibrium, always satisfies $p_1 = p_2$, compare with Regime I nested logit; (b) one equilibrium, always satisfies $p_1 = p_2$, compare with Regime I nested logit; (c) seven equilibria, 3 with $p_1 = p_2$, compare with Regime II nested logit; (d) seven equilibria, 3 with $p_1 = p_2$, compare with Regime II nested logit; (e) seven equilibria, 3 with $p_1 = p_2$, compare with Regime III nested logit; (f) seven equilibria, 3 with $p_1 = p_2$, compare with Regime III nested logit.
(a) $h = -0.01; \beta = 2.75$: Regime VI

(b) $h = 0.01; \beta = 2.75$: Regime IV

(c) $h = -0.5; \beta = 4$: Regime VII

(d) $h = 0.5; \beta = 4$: Regime V

(e) $h = -0.5; \beta = 3$: Regime VIII

(f) $h = 0.5; \beta = 3$: Regime I+

Figure C.3: Example null clines solution in each of the regimes in parameter space shown in Figure C.1 on page 375, indicated with the number of 2-way symmetric steady-state solutions where the mode shares of the nonbiased choice alternatives are equal, $p_1 = p_2 = 1 - p_0 - p_1$ and thus, $p_1 = (1 - p_0)/2$, continued from Figure C.2 on page 376: (a) five equilibria, 1 with $p_1 = p_2$, compare with Regime IV nested logit; (b) three equilibria, 3 with $p_1 = p_2$, compare with Regime VII nested logit; (c) five equilibria, 3 with $p_1 = p_2$; (d) five equilibria, 1 with $p_1 = p_2$, compare with Regime IV nested logit; (e) three equilibria, 1 with $p_1 = p_2$, compare with Regime V nested logit; (f) one equilibrium, always satisfies $p_1 = p_2$, compare with Regime I nested logit.
tionary points, we are interested cases when at least one eigenvalue of the Jacobian has zero real part. Recall that there will exist one zero eigenvalue if the determinant of the Jacobian is equal to zero. There will exist a zero real part of a complex eigenvalue if the determinant is positive and trace is equal to zero.

\[
\text{det } J = 0 : \lambda_{1,2} = \frac{1}{2} \text{tr } J \pm \frac{1}{2} \sqrt{\text{tr } J^2 - 4 \ast 0} = \frac{1}{2} \text{tr } J \pm \frac{1}{2} \text{tr } J
\]  
(C.37)

\[
\lambda_1 = 0, \lambda_2 = \text{tr } J
\]

\[
\text{det } J > 0, \text{tr } J = 0 : \lambda_{1,2} = \frac{1}{2} \ast 0 \pm \frac{1}{2} \sqrt{0^2 - 4 \ast \text{det } J} = \pm i \sqrt{\text{det } J}
\]  
(C.38)

\section{Elements of the Jacobian matrix}

For the system given by (C.14) and (C.15), the four terms in the Jacobian matrix (C.36) are computed directly using the sum rule, the product rule and the chain rule:

\[
\frac{\partial g_0}{\partial p_0} = \beta e^{\beta p_0} + h - e^{\beta p_1} - e^{\beta (1 - p_0 - p_1)} - \beta p_0 e^{\beta p_0} + \beta p_0 e^{\beta (1 - p_0 - p_1)}
\]  
(C.39)

\[
\frac{\partial g_0}{\partial p_1} = -\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta (1 - p_0 - p_1)}
\]  
(C.40)

\[
\frac{\partial g_1}{\partial p_0} = -\beta p_1 e^{\beta p_0} + \beta p_1 e^{\beta (1 - p_0 - p_1)}
\]  
(C.41)

\[
\frac{\partial g_1}{\partial p_1} = \beta e^{\beta p_1} - e^{\beta p_0} - e^{\beta (1 - p_0 - p_1)} - \beta p_1 e^{\beta p_1} + \beta p_1 e^{\beta (1 - p_0 - p_1)}
\]  
(C.42)
C.3.2 Special Solution: \( p_1 = p_2 \)

**Theorem** (Solution surface for the solution \( p_1 = p_2 \) of the sociodynamic trinary multinomial logit model with constant bias) Suppose that individual choices in a large sample population are characterized by the probabilities (C.3), (C.4) and (C.5) where the only contributions to the systematic utility of choices are a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_i \) of decision-making agents that have chosen alternative \( i \), and a constant bias \( h \) real, finite for choice alternative \( 0 \). The solution surface for the solution of the sociodynamic trinary multinomial logit model with constant bias (C.6) with \( p_1 = p_2 \) defined on \( (0, \frac{1}{2}) \) where the mode shares of the non-biased choice alternatives are equal, is characterized by:

\[
\begin{align*}
  h|_{p_1=p_2} &= \ln \frac{p_0}{p_1} + \beta (p_1 - p_0) \tag{C.43}
\end{align*}
\]

**Proof.** Given the special 2-way symmetry at equilibrium between \( p_1 \) and \( p_2 \) in (C.8) and (C.9), suppose the following symmetric solution:

\[
  p_1 = p_2 = 1 - p_0 - p_1 \tag{C.44}
\]

so that:

\[
  1 - p_0 = 2p_1 \quad ; \quad p_0 = 1 - 2p_1 \tag{C.45}
\]

Considering (C.18) for the case \( p_1 = p_2 \), we have:

\[
\begin{align*}
  e^h|_{p_1=p_2} &= \frac{p_0(e^{\beta p_1} + e^{\beta p_1})}{(1 - p_0)e^{\beta p_0}} = \frac{2p_0e^{\beta p_1}}{(1 - p_0)e^{\beta p_0}} = \frac{2p_0e^{\beta p_1}}{2p_1 e^{\beta p_0}} = \frac{p_0 e^{\beta p_1}}{p_1 e^{\beta (1-2p_1)}} \tag{C.46}
  \\
  &= \frac{(1 - 2p_1) e^{\beta (3p_1 - 1)}}{p_1}
\end{align*}
\]

Taking the logarithm of both sides [or equivalently, considering (C.20) for the case \( p_1 = p_2 \)], we re-gain the same general result as (C.32):

\[
\begin{align*}
  h|_{p_1=p_2} &= \ln p_0 + \ln(e^{\beta p_1} + e^{\beta p_1}) - \ln(1 - p_0) - \beta p_0 \\
  &= \ln p_0 + \ln 2 + \beta p_1 - \ln(1 - p_0) - \beta p_0 \\
  &= \ln(1 - 2p_1) + \ln 2 + \beta p_1 - \ln(2p_1) - \beta (1 - 2p_1) \\
  &= \ln(1 - 2p_1) + \ln 2 - \ln(2p_1) + \beta (3p_1 - 1) \\
  &= \ln \left( \frac{1 - 2p_1}{p_1} \right) + \beta (3p_1 - 1) = \ln \frac{p_0}{p_1} + \beta (p_1 - p_0) \tag{C.47}
\end{align*}
\]

Likewise, solving for \( \beta \) we re-gain the same general result as (C.34):

\[
  \beta|_{p_1=p_2} = \frac{h - \ln(1 - 2p_1) - \ln 2 + \ln(2p_1)}{(3p_1 - 1)} = \frac{h + \ln(p_1/p_0)}{(p_1 - p_0)} \tag{C.48}
\]
Note that considering (C.19) for the case \( p_1 = p_2 \), gives the same result as (C.46) [and thus also (C.47) and (C.48)], since:

\[
e^h = \frac{(1 - p_1)e^{\beta p_1} - p_1e^{\beta p_1}}{p_1 e^{\beta p_0}} = \frac{(1 - 2p_1)e^{\beta p_1}}{p_1 e^{\beta p_0}} \quad (C.49)
\]

\[
e^h = \frac{p_0 e^{\beta p_1}}{1 - p_0 e^{\beta p_0}} = \frac{2p_0 e^{\beta p_1}}{(1 - p_0) e^{\beta p_0}}
\]

\[\Diamond\]

See Figure C.5 on page 381 and Figure C.6 on page 382.

### c.3.3 Stability Analysis: \( p_1 = p_2 \)

In this subsection, our goal is to derive an expression for Curve A’ as shown in panel (a) of Figure C.4 on page 378.

**Theorem** (Bifurcation curve for the solution \( p_1 = p_2 \) of the sociodynamic trinary multinomial logit model with constant bias) Suppose that individual choices in a large sample population are characterized by the probabilities (C.3), (C.4) and (C.5) where the only contributions to the systematic utility of choices are a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_i \) of decision-making agents that have chosen alternative \( i \), and a constant bias \( h \) real, finite for choice alternative 0. The equilibrium solutions with \( p_1 = p_2 \) defined on \((0, 1/2)\) on the solution surface (C.43) of the sociodynamic trinary multinomial logit model with constant bias (C.6) where the mode shares of the non-biased choice alternatives are equal, will be bifurcation points on this surface if values of the parameter \( \beta \) satisfy:

\[
\beta = \frac{1}{3p_1(1 - 2p_1)} = \frac{1}{3p_0 p_1} \quad (C.50)
\]

**Proof.** For the special solution \( p_1 = p_2 \) to find values of the parameters \( \beta \) and \( h \) which lead to bifurcations in behavior, that is, change in number or stability of stationary points, we are interested cases when the derivative of \( g_0 \) (or alternatively \( g_1 \)) evaluated at a stationary point is zero.

\[
0 = \frac{dg_0}{dp_1} \bigg|_{p_1=p_2} = \frac{d}{dp_1} \left( 2p_1 e^{\beta (1-2p_1)+h} - 2(1 - 2p_1)e^{\beta p_1} \right) \bigg|_{p_1=p_2, g_0=0} = \left( 2e^{\beta (1-2p_1)+h} + (2p_1)(-2\beta e^{\beta (1-2p_1)+h}) \right) \bigg|_{p_1=p_2, g_0=0} = \left( 2(-2e^\beta p_1 + (1 - 2p_1)\beta e^{\beta p_1}) \right) \bigg|_{p_1=p_2, g_0=0} = \left( 2(1 - 2\beta p_1)e^{\beta (1-2p_1)+h} - 2(-2 + \beta (1 - 2p_1)e^{\beta p_1}) \right) \bigg|_{p_1=p_2, g_0=0} = \left( 2(1 - 2\beta p_1)e^{\beta (1-2p_1)+h} - 2(-2 + \beta (1 - 2p_1)e^{\beta p_1}) \right) \bigg|_{p_1=p_2, g_0=0} (C.51)
\]
Figure C.5: Bifurcation diagrams for the special solution of the sociodynamic trinary multinomial logit model where the mode shares of the unbiased alternatives are equal ($p_1 = p_2$), showing bifurcations at $h = h^*$ and $h = 0$ separating regimes without a hysteresis loop ($h \geq h^*$), with a hysteresis loop ($h^* < h \leq 0$) and without a hysteresis loop ($h < 0$).
Figure C.6: Bifurcation diagrams for the special solution of the sociodynamic trinary multinomial logit model where the mode shares of the unbiased alternatives are equal ($p_1 = p_2$), showing bifurcations at $h = h^*$ and $h = 0$ separating regimes without a hysteresis loop ($h \geq h^*$), with a hysteresis loop ($h^* < h \leq 0$) and without a hysteresis loop ($h < 0$), continued from Figure C.5 on page 381.
Re-arranging terms:
\[
e^h|_{p_1=p_2} = \frac{(-2 + \beta(1 - 2p_1))}{(1 - 2\beta p_1)}e^{\beta(3p_1 - 1)} \tag{C.52}
\]

Taking the natural logarithm of both sides:
\[
h|_{p_1=p_2} = \beta(3p_1 - 1) + \ln(-2 + \beta(1 - 2p_1)) - \ln(1 - 2\beta p_1) \tag{C.53}
\]

where the values of \(p_1\) are restricted for \(h\) real, finite:
\[
p_1 \neq \frac{1}{2\beta}; \quad p_1 \neq \frac{\beta - 2}{2\beta} = \frac{1}{2} - \frac{1}{\beta} \tag{C.54}
\]

Now, setting (C.52) equal to (C.46) we have the condition for which a bifurcation will occur for the special solution \(p_1 = p_2\).
\[
e^h = \frac{(-2 + \beta(1 - 2p_1))}{(1 - 2\beta p_1)}e^{\beta(3p_1 - 1)} = \frac{(1-2p_1)}{p_1}e^{\beta(3p_1 - 1)} \tag{C.55}
\]

Re-arranging terms and simplifying:
\[
(1 - 2\beta p_1)(1 - 2p_1) = p_1(-2 + \beta(1 - 2p_1)) \\
1 - 2\beta p_1 - 2p_1 + 4\beta p_1^2 = -2p_1 + \beta p_1 - 2\beta p_1^2 \\
6\beta p_1^2 - 3\beta p_1 + 1 = 0 \tag{C.56}
\]

Solving for \(\beta\):
\[
\beta = \frac{1}{3p_1(1 - 2p_1)} = \frac{1}{3p_0 p_1} \tag{C.57}
\]

Or alternatively solving for \(p_1\):
\[
p_1 = \frac{3\beta \pm \sqrt{(-3\beta)^2 - 4(6\beta)}}{2(6\beta)} = \frac{3\beta \pm \sqrt{9\beta^2 - 24\beta}}{12\beta} \tag{C.58}
\]

Substituting both +/ - branches of (C.58) back into (C.47) we obtain the bifurcation curve for \(h\) in terms of \(\beta\). See Figure C.7 on page 384. Note that the null clines plot for \(F \equiv (C.53) - (C.47)\) [or alternatively for \(F^* \equiv (C.52) - (C.46)\)] gives the same graphical result as (C.57) and (C.58). ♦

**Hysteresis regime: maximum value of \(h^*\)**

**Lemma** (Maximum value of \(h^*\)for hysteresis for the solution \(p_1 = p_2\) of the sociodynamic trinary multinomial logit model with constant bias) The maximum value of bias parameter \(h\) for hysteresis for an equilibrium solution with \(p_1 = p_2\) defined on \((0, 1/2)\) on the solution surface (C.43) of the sociodynamic trinary multinomial logit model with constant bias (C.6) where the mode shares of the non-biased choice alternatives are equal, and the corresponding value of utility parameter \(\beta\), are given by:
Figure C.7: Bifurcation cusp in $(\beta, h)$-parameter space. Compare with Figure C.1 on page 375.

\[ \beta = \frac{8}{3} \approx 2.6667 \]
\[ h = \ln 2 - \frac{2}{3} \approx 0.02648 \]  
\[ p_0 = \frac{1}{2} \]
\[ p_1 = p_2 = \frac{1}{4} \]  

Proof. To find the value of $h^*$ separating the regime where the hysteresis loop does not exist as in the top panel of Figure C.5 on page 381 from the regime where the hysteresis loop exists as in the bottom panel of Figure C.5, we are interested in the value of $h$ for which the upper curve of $p_0$ versus $\beta$ and the lower curve of $p_1 = p_2$ versus $\beta$ will first have a point of infinite slope for decreasing $h$ as shown in the mid panel of Figure C.5. This is namely the point where the derivatives $dg_0/dp_1$ (or alternatively $dg_1/dp_1$) and the derivative $d\beta/dp_1$ evaluated at a stationary point are both zero.

From (C.57) we have:

\[ 0 = \frac{d\beta}{dp_1} \bigg|_{p_1=p_2} \left. \frac{dg_0}{dp_0} \right|_{p_1=p_2} = \frac{d}{dp_1} \left( \frac{1}{3p_1(1-2p_1)} \right) = -\frac{1}{3} \frac{1-4p_1}{p_1^2(1-2p_1)^2} \]  

Thus solving (C.60) for $p_1$:

\[ p_1 = \frac{1}{4} = 0.25 \]  

(C.61)
Substituting (C.61) back into (C.57) to determine $\beta$:

$$\beta = \frac{1}{3p_1(1 - 2p_1)} = \frac{1}{3 \cdot (1/4) \cdot (1 - 2(1/4))} = \frac{8}{3} \approx 2.6667 \quad \text{(C.62)}$$

And finally substituting (C.61) and (C.62) back into (C.47) to determine $h$:

$$h|_{p_1=p_2} = \ln \left( \frac{1 - 2p_1}{p_1} \right) + \beta(3p_1 - 1) = \ln \left( \frac{1 - 2(1/4)}{1/4} \right) + \frac{8}{3}(3(1/4) - 1)$$

$$= \ln 2 - \frac{8}{3}(1/4) = \ln 2 - \frac{2}{3} \approx 0.026481 \quad \text{(C.63)}$$

\[ \diamond \]

\section*{3.4 Stability Analysis: $p_1 = p_2$, Continued}

In this subsection, our goal is to derive an expression for Curve B’ as shown in panel (b) of Figure C.4 on page 378.

At the solution (C.44), the four terms in the Jacobian matrix of $g = (g_0, g_1)$ simplify to:

$$\frac{\partial g_0}{\partial p_0} \bigg|_{p_1=p_2} = \beta e^{\beta p_0 + h} - e^{\beta p_0 + h} - 2e^{\beta p_1} - \beta p_0 e^{\beta p_0 + h} + \beta p_0 e^{\beta p_1}$$

$$= \beta e^{\beta p_0 + h} - \beta p_0 e^{\beta p_0 + h} - e^{\beta p_0 + h} + \beta p_0 e^{\beta p_1} - 2e^{\beta p_1}$$

$$= (\beta - \beta p_0 - 1)e^{\beta p_0 + h} + \beta p_0 e^{\beta p_1} - 2e^{\beta p_1}$$

$$= (2\beta p_1 - 1)e^{\beta p_0 + h} + (\beta p_0 - 2)e^{\beta p_1} \quad \text{(C.64)}$$

$$\frac{\partial g_0}{\partial p_1} \bigg|_{p_1=p_2} = -\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta p_1} = 0 \quad \text{(C.65)}$$

$$\frac{\partial g_1}{\partial p_0} \bigg|_{p_1=p_2} = -\beta p_1 e^{\beta p_0 + h} + \beta p_1 e^{\beta p_1} = \beta p_1(e^{\beta p_1} - e^{\beta p_0 + h}) \quad \text{(C.66)}$$

$$\frac{\partial g_1}{\partial p_1} \bigg|_{p_1=p_2} = \beta e^{\beta p_1} - e^{\beta p_0 + h} - e^{\beta p_1} - e^{\beta p_1} - \beta p_1 e^{\beta p_1} + \beta p_1 e^{\beta p_1}$$

$$= \beta e^{\beta p_1} - 2e^{\beta p_1} - e^{\beta p_0 + h} = (\beta - 2)e^{\beta p_1} - e^{\beta p_0 + h} \quad \text{(C.67)}$$

To determine bifurcations in behavior, let us first consider (C.37) for the special solution $p_1 = p_2$.

$$0 = \det J = \frac{\partial g_0}{\partial p_0} \frac{\partial g_1}{\partial p_1} - \frac{\partial g_0}{\partial p_1} \frac{\partial g_1}{\partial p_0}$$

$$= (2\beta p_1 - 1)e^{\beta p_0 + h} + (\beta p_0 - 2)e^{\beta p_1}((\beta - 2)e^{\beta p_1} - e^{\beta p_0 + h}) - 0 \quad \text{(C.68)}$$

Since $\partial g_0/\partial p_1 = 0$ for the special solution $p_1 = p_2$, we thus must have either $\partial g_0/\partial p_0 = 0$ or $\partial g_1/\partial p_1 = 0$ for the determinant to be zero.
Case I. Suppose $\partial g_0 / \partial p_0 = 0$

$$0 = (2\beta p_1 - 1)e^{\beta p_0 + h} + (\beta p_0 - 2)e^{\beta p_1} \quad (C.69)$$

Solving (C.69) for $h$:

$$e^h = -\frac{(\beta p_0 - 2)e^{\beta p_1}}{(2\beta p_1 - 1)e^{\beta p_0}} = \frac{(\beta(1 - 2p_1) - 2)e^{\beta p_1}}{(2\beta p_1 - 1)e^{\beta(1 - 2p_1)}} \quad (C.70)$$

Taking the natural logarithm of both sides:

$$h = \beta (3p_1 - 1) + \ln(\beta - 2\beta p_1 - 2) - \ln(1 - 2\beta p_1) \quad (C.71)$$

Or alternatively, to see the symmetry-breaking in terms of $p_0$ and $p_1$:

$$h = \beta (p_1 - p_0) + \ln(\beta p_0 - 2) - \ln(1 - 2\beta p_1) \quad (C.72)$$

Comparing the condition (C.46) for solution $g = 0$ for the case $p_1 = p_2$, and the condition (C.70) for $\det J = 0$ for the case $p_1 = p_2$, we have:

$$e^h = \frac{2p_0 e^{\beta p_1}}{(1 - p_0)e^{\beta p_0}} = -\frac{(\beta p_0 - 2)e^{\beta p_1}}{(2\beta p_1 - 1)e^{\beta p_0}} \quad (C.73)$$

Re-arranging terms:

$$2p_0(1 - 2\beta p_1) = (\beta p_0 - 2)(1 - p_0)$$

$$2p_0 - 4\beta p_0 p_1 = \beta p_0 - 2 - \beta p_0^2 + 2p_0$$

$$-4\beta p_0 p_1 = \beta p_0(1 - p_0) - 2$$

$$1 = 3\beta p_0 p_1 \quad (C.74)$$

Solving for $\beta$:

$$\beta = \frac{1}{3p_0 p_1} = \frac{1}{3p_1(1 - 2p_1)} \quad (C.75)$$

Or alternatively, re-writing (C.74)

$$0 = 1 - 3\beta p_0 p_1 = 1 - 3\beta p_1(1 - 2p_1) = 6\beta p_1^2 - 3\beta p_1 + 1 \quad (C.76)$$

Solving for $p_1$:

$$p_{1+,-} = \frac{3\beta \pm \sqrt{(-3\beta)^2 - 4(6\beta)}}{2(6\beta)} = \frac{3\beta \pm \sqrt{9\beta^2 - 24\beta}}{12\beta} \quad (C.77)$$

Note that the null clines plot for $F \equiv (C.46) - (C.70)$ [or alternatively for $F^* \equiv (C.47) - (C.71)$] gives the same result as (C.75) and (C.77).
Case II. Suppose $\partial g_1/\partial p_1 = 0$

**Theorem** (Bifurcation curve for the solution $p_1 = p_2$ of the sociodynamic trinary multinomial logit model with constant, continued) Suppose that individual choices in a large sample population are characterized by the probabilities (C.3), (C.4) and (C.5) where the only contributions to the systematic utility of choices are a global field effect with utility parameter $\beta$ real, finite on the proportion $p_1$ of decision-making agents that have chosen alternative $i$, and a constant bias $h$ real, finite for choice alternative 0. The equilibrium solutions with $p_1 = p_2$ defined on $(0, 1/2)$ of the sociodynamic trinary multinomial logit model with constant bias (C.6) where the mode shares of the non-biased choice alternatives are equal, will be bifurcation points for which at least one zero eigenvalue occurs and for which emerging solutions upon bifurcation are asymmetric in the mode shares $p_1 \neq p_2$, if values of the parameter $\beta$ satisfy:

$$p_1 = \frac{1}{\beta}$$  \hfill (C.78)

**Proof.** Suppose $\partial g_1/\partial p_1 = 0$ at the special solution $p_1 = p_2$:

$$(\beta - 2)e^{\beta p_1} - e^{\beta p_0 + h} = 0$$  \hfill (C.79)

Solving (C.79) for $h$:

$$h = (\beta - 2)e^{\beta p_1}e^{\beta p_0} = (\beta - 2)e^{\beta(p_1 - p_0)}(\beta - 2) = e^{\beta(p_1 - 1)(\beta - 2)}$$  \hfill (C.80)

Taking the natural logarithm of both sides:

$$h = \beta(3p_1 - 1) + \ln(\beta - 2)$$  \hfill (C.81)

Or alternatively, to see the symmetry-breaking in terms of $p_0$ and $p_1$:

$$h = \beta(p_1 - p_0) + \ln(\beta - 2)$$  \hfill (C.82)

Comparing the condition (C.46) for solution $g = 0$ for the case $p_1 = p_2$, and the condition (C.80) for $\det J = 0$ for the case $p_1 = p_2$, we have:

$$e^h = p_0 e^{\beta p_1} = (\beta - 2)e^{\beta p_1}$$  \hfill (C.83)

Re-arranging terms:

$$\frac{p_0}{p_1} = (\beta - 2)$$  \hfill (C.84)

Solving for $\beta$:

$$\beta = \frac{p_0}{p_1} + 2 = \frac{1 - 2p_1}{p_1} + 2 = \frac{1 - 2p_1 + 2p_1}{p_1} = \frac{1}{p_1}$$  \hfill (C.85)
Solving for $p_1$:

$$p_1 = \frac{1}{\hat{\beta}} \quad \text{(C.86)}$$

Substituting (C.86) back into (C.47), we obtain $h$ in terms of $\beta$:

$$h = \ln \left( \frac{1 - 2p_1}{p_1} \right) + \beta(3p_1 - 1) = \ln \left( \frac{1}{p_1} - 2 \right) + 3\beta p_1 - \beta \quad \text{(C.87)}$$

Alternatively, we can also solve (C.81) directly for $p_1$:

$$p_1 = \frac{h + \beta - \ln(\beta - 2)}{3\hat{\beta}} \quad \text{(C.88)}$$

Considering (C.16) for the case $p_1 = p_2$, we have:

$$0 = g_0|_{p_1=p_2} = (1 - p_0)e^{\beta p_0 + h} - p_0(e^{\beta p_1} + e^{\beta p_2})$$

$$= 2p_1 e^{\beta p_0 + h} - 2p_0 e^{\beta p_1}$$

$$= 2 \left( p_1 e^{\beta p_0 + h} - (1 - 2p_1) e^{\beta p_1} \right) \quad \text{(C.89)}$$

$$= 2 \left( p_1 e^{\beta (1-2p_1) + h} - e^{\beta p_1} + 2p_1 e^{\beta p_1} \right)$$

$$= 2 \left( p_1 (e^{\beta + h} e^{-2\beta p_1} + 2 e^{\beta p_1}) - e^{\beta p_1} \right)$$

Multiplying (C.89) through by $\exp(-\beta p_1/2)$:

$$0 = p_1 (e^{\beta + h} e^{-3\beta p_1} + 2) - 1 \quad \text{(C.90)}$$

Substituting (C.88) into (C.90) for the case $p_1 = p_2$, we have:

$$0 = \left( \frac{h + \beta - \ln(\beta - 2)}{3\hat{\beta}} \right) \left( \frac{e^{\beta + h}}{e^{h + \beta - \ln(\beta - 2)} + 2} \right) - 1$$

$$= \left( \frac{h + \beta - \ln(\beta - 2)}{3\hat{\beta}} \right) \left( \frac{e^{\beta + h}}{e^{h + \beta} \cdot e^{\ln(\beta - 2)} + 2} \right) - 1 \quad \text{(C.91)}$$

$$= \frac{1}{3} (h + \beta - \ln(\beta - 2)) - 1$$

Solving for $h$ in terms of $\beta$ yields the same result as (C.87):

$$h = 3 - \beta + \ln(\beta - 2) \quad \text{(C.92)}$$

We thus obtain the bifurcation curve for $h$ in terms of $\beta$. See Figure C.8 on page 389.

Note that considering (C.17) for the case $p_1 = p_2$, we regain the same result as (C.92) since:

$$0 = g_1|_{p_1=p_2} = (1 - p_1)e^{\beta p_1} - p_1(e^{\beta p_0 + h} + e^{\beta p_2})$$

$$= (1 - 2p_1)e^{\beta p_1} - p_1 e^{\beta p_0 + h} \quad \text{(C.93)}$$

$$= \frac{1}{2} \cdot 2 \left( p_1 e^{\beta p_0 + h} - (1 - 2p_1) e^{\beta p_1} \right) = -\frac{1}{2} g_0|_{p_1=p_2}$$

$\Diamond$
Case III. Suppose $\text{tr} \ J = 0$ and $\det \ J > 0$

**Lemma** (No bifurcation points of the sociodynamic trinary multinomial logit model with constant bias and with $p_1 = p_2$ having purely imaginary eigenvalues) Suppose that individual choices in a large sample population are characterized by the probabilities (C.3), (C.4) and (C.5) where the only contributions to the systematic utility of choices are a global field effect with utility parameter $\beta$ real, finite on the proportion $p_1$ of decision-making agents that have chosen alternative $i$, and and a constant bias $h$ real, finite for choice alternative 0. The equilibrium solutions with $p_1 = p_2$ defined on $(0, \frac{1}{2})$ of the sociodynamic trinary multinomial logit model with constant bias (C.6) where the mode shares of the non-biased choice alternatives are equal, exhibit no bifurcation points with purely imaginary eigenvalues.

**Proof.** Let us consider (C.38) for the special solution $p_1 = p_2,$

$$0 = \text{tr}J = \frac{\partial g_0}{\partial p_0} + \frac{\partial g_1}{\partial p_1}$$

$$= (2\beta p_1 - 1)e^{\beta p_0} + (\beta p_0 - 2)e^{\beta p_1} + ((\beta - 2)e^{\beta p_1} - e^{\beta p_0} + h)$$

$$= 2(\beta p_1 - 1)e^{\beta p_0} + (\beta p_0 - \beta - 4)e^{\beta p_1}$$

$$= 2(\beta p_1 - 1)e^{\beta p_0} + (\beta p_0 - 1 - 4)e^{\beta p_1}$$

$$= 2(\beta p_1 - 1)e^{\beta (1 - 2p_1) + h} + 2(\beta p_1 - 2)e^{\beta p_1}$$

(C.94)
In order for there to exist a zero real part of a complex eigenvalue, we also require that the determinant of the Jacobian is strictly positive. However since \( \frac{\partial g_0}{\partial p_1} = 0 \) for the case \( p_1 = p_2 \), we have the simplification:

\[
0 < \det J = \frac{\partial g_0}{\partial p_0} \frac{\partial g_1}{\partial p_0} - \frac{\partial g_0}{\partial p_1} \frac{\partial g_1}{\partial p_0} = \left( (2\beta p_1 - 1) e^{\beta p_0 + h} + (\beta p_0 - 2) e^{\beta p_1} \right) \left( (\beta - 2) e^{\beta p_1} - e^{\beta p_0 + h} \right)
\]

(C.95)

Without further computation, we thus can conclude already immediately that there can exist no purely imaginary eigenvalues, since (C.94) and (C.95) cannot be simultaneously satisfied. The requirement (C.94) that the trace is zero implies that \( \frac{\partial g_0}{\partial p_0} \) and \( \frac{\partial g_1}{\partial p_1} \) must be zero or have opposite signs, but the requirement (C.95) that the determinant is strictly positive implies that \( \frac{\partial g_0}{\partial p_0} \) and \( \frac{\partial g_1}{\partial p_1} \) must be non-zero and have the same signs.

C.3.5 General Stability Analysis

In this subsection, our goal is to derive an expression for Curve C’ as shown in panel (c) of Figure C.4 on page 378.

To do this, we want to find possible values of the parameters \( \beta \) and \( h \) real, finite such that general solutions to the system of equations (C.14) and (C.15) are bifurcation points. From (C.37), we know that there will exist at least one zero eigenvalue if the determinant of the Jacobian is equal to zero. This leads us to the following theorem which we prove in this subsection.

**Theorem** (General characterization of \( \beta \) and \( h \) at bifurcation points of the sociodynamic trinary multinomial logit model with constant bias for which at least one zero eigenvalue occurs) Suppose that individual choices in a large sample population are characterized by the probabilities (C.3), (C.4) and (C.5) where the only contributions to the systematic utility of choices are a global field effect with utility parameter \( \beta \) real, finite on the proportion \( p_i \) of decision-making agents that have chosen alternative i, and a constant bias \( h \) real, finite for choice alternative 0. The equilibrium solutions \( p_0, p_1, p_2 \) defined on \([0, 1]\) of the sociodynamic trinary multinomial logit model with constant bias (C.6) will be candidates for bifurcation points for which at least one zero eigenvalue occurs, if values of the parameter \( \beta \) satisfy:

\[
0 = 3p_0p_1p_2\beta^2 - 2(p_0p_1 + p_0p_2 + p_1p_2)\beta + 1 \quad (C.96)
\]
Proof. Let us consider case (C.37)

\[ 0 = \det J = \frac{\partial g_0}{\partial p_0} \frac{\partial g_1}{\partial p_1} - \frac{\partial g_0}{\partial p_1} \frac{\partial g_1}{\partial p_0} = \]
\[ (\beta e^{\beta p_0 + h} - e^{\beta p_0 + h} - e^{\beta p_1} - e^{\beta (1 - p_0 - p_1)} - \beta p_0 e^{\beta p_0 + h} + \beta p_0 e^{\beta (1 - p_0 - p_1)}) \]
\[ \bullet (\beta e^{\beta p_1} - e^{\beta p_0 + h} - e^{\beta p_1} - e^{\beta (1 - p_0 - p_1)} - \beta p_1 e^{\beta p_1} + \beta p_1 e^{\beta (1 - p_0 - p_1)}) \]
\[ - (-\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta (1 - p_0 - p_1)} \bullet (-\beta p_1 e^{\beta p_0 + h} + \beta p_1 e^{\beta (1 - p_0 - p_1)}) \]
\[ = [-e^{\beta p_0 + h} - e^{\beta p_1} - e^{\beta (1 - p_0 - p_1)} + \beta (e^{\beta p_0 + h} - p_0 e^{\beta p_0 + h}) + \beta p_0 e^{\beta (1 - p_0 - p_1)}] \]
\[ \bullet [- e^{\beta p_0 + h} - e^{\beta p_1} - e^{\beta (1 - p_0 - p_1)} + \beta (e^{\beta p_1} - p_1 e^{\beta p_1}) + \beta p_1 e^{\beta (1 - p_0 - p_1)}] \]
\[ - (-\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta (1 - p_0 - p_1)} \bullet (-\beta p_1 e^{\beta p_0 + h} + \beta p_1 e^{\beta (1 - p_0 - p_1)}) \]

(C.97)

Substituting (C.14) and (C.15) into (C.97)

\[ 0 = [-e^{\beta p_0 + h} - e^{\beta p_1} - e^{\beta (1 - p_0 - p_1)} + \beta p_0 e^{\beta p_1} + 2 \beta p_0 e^{\beta (1 - p_0 - p_1)}) \]
\[ \bullet [- e^{\beta p_0 + h} - e^{\beta p_1} - e^{\beta (1 - p_0 - p_1)} + \beta p_1 e^{\beta p_0 + h} + 2 \beta p_1 e^{\beta (1 - p_0 - p_1)}} \]
\[ - (-\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta (1 - p_0 - p_1)} \bullet (-\beta p_1 e^{\beta p_0 + h} + \beta p_1 e^{\beta (1 - p_0 - p_1)}) \]

(C.98)

Re-grouping terms:

\[ 0 = (e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1 - p_0 - p_1)}) \bullet (e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1 - p_0 - p_1)}) \]
\[ - (e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1 - p_0 - p_1)}) \bullet (\beta p_0 e^{\beta p_1} + 2 \beta p_0 e^{\beta (1 - p_0 - p_1)}) \]
\[ - (e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1 - p_0 - p_1)}) \bullet (\beta p_1 e^{\beta p_0 + h} + 2 \beta p_1 e^{\beta (1 - p_0 - p_1)}) \]
\[ + (\beta p_0 e^{\beta p_1} + 2 \beta p_0 e^{\beta (1 - p_0 - p_1)}) \bullet (\beta p_1 e^{\beta p_0 + h} + 2 \beta p_1 e^{\beta (1 - p_0 - p_1)}) \]
\[ - (-\beta p_0 e^{\beta p_1} + \beta p_0 e^{\beta (1 - p_0 - p_1}) \bullet (-\beta p_1 e^{\beta p_0 + h} + \beta p_1 e^{\beta (1 - p_0 - p_1)}) \]

(C.99)

Dividing through by the first term, which is always positive for \( \beta \) real, finite:

\[ 0 = 1 - \frac{\beta (p_0 e^{\beta p_1} + 2 p_0 e^{\beta (1 - p_0 - p_1)} + p_1 e^{\beta p_0 + h} + 2 p_1 e^{\beta (1 - p_0 - p_1)})}{(e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1 - p_0 - p_1)})} \]
\[ + \frac{\beta (p_0 e^{\beta p_1} + 2 p_0 e^{\beta (1 - p_0 - p_1)})}{(e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1 - p_0 - p_1)})} \bullet \frac{\beta (p_1 e^{\beta p_0 + h} + 2 p_1 e^{\beta (1 - p_0 - p_1)})}{(e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1 - p_0 - p_1)})} \]
\[ - \frac{\beta (p_0 e^{\beta p_1} - p_0 e^{\beta (1 - p_0 - p_1)})}{(e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1 - p_0 - p_1)})} \bullet \frac{\beta (p_1 e^{\beta p_0 + h} - p_1 e^{\beta (1 - p_0 - p_1)})}{(e^{\beta p_0 + h} + e^{\beta p_1} + e^{\beta (1 - p_0 - p_1)})} \]

(C.100)

Substituting (C.7), (C.8), (C.9) and (C.11) into (C.100), we re-gain the same result as the multinomial logit case:

\[ 0 = 1 - \beta[p_0 p_1 + 2 p_0 (1 - p_0 - p_1) + p_1 p_0 + 2 p_1 (1 - p_0 - p_1)] \]
\[ + \beta[p_0 p_1 + 2 p_0 (1 - p_0 - p_1)] \bullet \beta[p_1 p_0 + 2 p_1 (1 - p_0 - p_1)] \]
\[ - \beta[p_0 p_1 - p_0 (1 - p_0 - p_1)] \bullet \beta[p_1 p_0 - p_1 (1 - p_0 - p_1)] \]

(C.101)
Multiplying out and re-grouping terms, we similarly re-gain the same result as the multinomial logit case:

\[
0 = 1 - 2\beta[p_0p_1 + p_0(1 - p_0 - p_1) + p_1(1 - p_0 - p_1)] + 3\beta^2p_0p_1(1 - p_0 - p_1) \tag{C.102}
\]

Thus, substituting (C.11) and re-arranging terms:

\[
0 = 3p_0p_1p_2\beta^2 - 2(p_0p_1 + p_0p_2 + p_1p_2)\beta + 1 \tag{C.103}
\]

Solving for \( \beta \) using the quadratic formula:

\[
\beta = \frac{(p_0p_1 + p_0p_2 + p_1p_2) \pm \sqrt{(p_0p_1 + p_0p_2 + p_1p_2)^2 - 3p_0p_1p_2}}{3p_0p_1p_2} \tag{C.104}
\]

\( \diamond \)

For all thoroughness, given the special 2-way symmetry in the system (C.7), (C.8), (C.9), let us first re-consider the symmetric solution:

\[
p_1 = p_2 \tag{C.105}
\]

Thus, by (C.10)

\[
p_0 = 1 - 2p_1 \tag{C.106}
\]

Substituting (C.105) and (C.106) into (C.104):

\[
\beta = \frac{(2p_1(1 - 2p_1) + p_1^2) \pm \sqrt{(2p_1(1 - 2p_1) + p_1^2)^2 - 3p_1^2(1 - 2p_1)}}{3p_1^2(1 - 2p_1)}
\]

\[
= \frac{2p_1 - 3p_1^2 \pm \sqrt{(2p_1 - 3p_1^2)^2 - 3p_1^2 + 6p_1^3}}{3p_1^2(1 - 2p_1)}
= \frac{2p_1 - 3p_1^2 \pm \sqrt{4p_1^2 - 12p_1^3 + 9p_1^4 - 3p_1^2 + 6p_1^3}}{3p_1^2(1 - 2p_1)}
= \frac{2 - 3p_1 \pm \sqrt{1 - 6p_1 + 9p_1^2}}{3p_1(1 - 2p_1)}
= \frac{2 - 3p_1 \pm \sqrt{(1 - 3p_1)^2}}{3p_1(1 - 2p_1)}
= \frac{2 - 3p_1 \pm (1 - 3p_1)}{3p_1(1 - 2p_1}) \tag{C.107}
\]

For the case \( \beta_- \) with a minus sign in (C.104), we re-gain the same result as (C.75)

\[
\beta_- = \frac{2 - 3p_1 - (1 - 3p_1)}{3p_1(1 - 2p_1)} = \frac{1}{3p_1(1 - 2p_1)} = \frac{1}{3p_0p_1} \tag{C.108}
\]

Substituting (C.108) into (C.89) for the case \( p_1 = p_2 \), we have:

\[
0 = g_0|_{p_1=p_2} = 2p_1e^{\beta p_0 + h} - 2p_0e^{\beta p_1}
= 2p_1e^{\frac{1}{3p_0p_1}p_0 + h} - 2p_0e^{\frac{1}{3p_0p_1}p_1}
= 2p_1e^{\frac{1}{3p_0} + h} - 2p_0e^{\frac{1}{3p_0}} \tag{C.109}
\]
Re-arranging terms:
\[ p_1 e^{\frac{1}{p_1} h} e^{h} = p_0 e^{\frac{1}{p_0} h} \]
\[ e^{h} = \frac{p_0}{p_1} e^{\frac{1}{p_0} h} - \frac{1}{p_1} = \frac{p_0}{p_1} e^{\frac{p_1 - p_0}{p_0 p_1}} \] (C.110)

Taking the logarithm of both sides, we re-gain (C.47) with \( \beta_- \) given by (C.108):
\[ h = \ln \frac{p_0}{p_1} + \frac{1}{3p_0p_1}(p_1 - p_0) \] (C.111)

For the case \( \beta_+ \) with a plus sign in (C.104), we re-gain the same result as (C.86)
\[ \beta_+ = \frac{2 - 3p_1 + (1 - 3p_1)}{3p_1(1 - 2p_1)} = \frac{3 - 6p_1}{3p_1(1 - 2p_1)} = \frac{1 - 2p_1}{p_1(1 - 2p_1)} = \frac{1}{p_1} \] (C.112)

Substituting (C.112) into (C.89) for the case \( p_1 = p_2 \), we have:
\[ 0 = g_0|_{p_1=p_2} = 2p_1 e^{\beta_+ p_0 + h} - 2p_0 e^{\beta_+ p_1} \]
\[ = 2p_1 e^{\frac{1}{p_1} p_0 + h} - 2p_0 e^{\frac{1}{p_1} p_1} = 2p_1 e^{\frac{p_1 - p_0}{p_1}} + h - 2p_0 e \] (C.113)

Re-arranging terms:
\[ p_1 e^{\frac{p_0}{p_1} h} e^{h} = p_0 e \]
\[ e^{h} = \frac{p_0}{p_1} e^{\frac{1}{p_0} h} = \frac{p_0}{p_1} e^{\frac{p_1 - p_0}{p_0 p_1}} \] (C.114)

Taking the logarithm of both sides, we re-gain (C.47) for \( h \) with \( \beta_+ \) given by (C.112):
\[ h = \ln \frac{p_0}{p_1} + \frac{1}{p_1}(p_1 - p_0) \] (C.115)

We can also use (C.102) to write a general expression for \( p_0 \) in terms of \( p_1 \) and \( \beta \). Multiplying out and re-grouping terms in orders of \( p_0 \), we have:
\[ 0 = 1 - 2\beta[p_0 p_1 + p_0 (1 - p_0 - p_1) + p_1 (1 - p_0 - p_1)] \]
\[ + 3\beta^2 p_0 p_1 (1 - p_0 - p_1) \]
\[ = 1 - 2\beta(p_0 p_1 + p_0 - p_0^2 - p_0 p_1 + p_1 - p_0 p_1 - p_1^2) \]
\[ + 3\beta^2 (p_0 p_1 - p_0^2 p_1 - p_0 p_1^2) \]
\[ = 1 - 2\beta(-p_0^2 + p_0 - p_0 p_1 + p_1 - p_1^2) \]
\[ + 3\beta^2 (-p_0^2 p_1 + p_0 p_1 - p_0 p_1^2) \]
\[ = 1 + 2\beta p_0^2 - 2\beta(p_0 - p_0 p_1) - 2\beta(p_1 - p_1^2) \]
\[ - 3\beta^2 p_0^2 p_1 + 3\beta^2 (p_0 p_1 - p_0 p_1^2) \]
\[ = (2 - 3\beta p_1)\beta p_0^2 - 2\beta(1 - p_1)p_0 + 3\beta^2 p_1 (1 - p_1)p_0 \]
\[ + 1 - 2\beta(p_1 - p_1^2) \]
\[ = (2 - 3\beta p_1)\beta p_0^2 - (2 - 3\beta p_1)(1 - p_1)\beta p_0 + 1 - 2\beta p_1 (1 - p_1) \] (C.116)
For computational convenience define:

\[
\begin{align*}
    a &\equiv (2 - 3\beta p_1)\beta \\
    b &\equiv -(2 - 3\beta p_1)(1 - p_1)\beta \\
    c &\equiv 1 - 2\beta p_1(1 - p_1)
\end{align*}
\]

(C.117)

Solving for \(p_0\) using the quadratic formula:

\[
p_{0+}, 0- = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

(C.118)

Now, to verify whether or not (C.105) is the only solution when (C.37) holds, we substitute (C.104) back into (C.14) and (C.15) and solve graphically, plotting the null clines of the surfaces \(g_0\) and \(g_1\) on a graph and finding their intersection for a sweep of the parameter \(h\). See Figures C.9 through C.12 on pages 395-398.

Alternatively, we can substitute (C.34) and (C.35) into the general equation for the determinant (C.23) and into one of either (C.14) or (C.15) for sweep of the parameter \(h\). The intersection of the null cline solutions of the system with the conditions for bifurcation give the bifurcation points. See Figure C.13 on page 399. See Figure C.14 on page 400 and Figure C.15 on page 401.

It is evident from Figures C.9 through C.12 on pages 395-398, Figure C.13 on page 399 and Figures C.14 and C.15 on pages 400-401 that the bifurcation points traverse all valid values of the line \(p_1 = p_2\), that is, \(p_1 = (1 - p_0)/2\) over all \(\beta\) and all \(h\). We can obtain the trajectory of the bifurcation points in the \((p_0,p_1)\)-plane over all \(\beta\) and all \(h\) with \(p_1 \neq p_2\) from the null clines plot of (C.26) - (C.104).

Substituting the values of \(p_1\) and \(p_2\) in the green trajectory in Figure C.16 back into (C.27) we obtain the bifurcation curve for \(h\) in terms of \(\beta\) for \(p_1 \neq p_2\). See Figure C.17 on page 402.

\section*{C.4 Gradient System}

\textbf{Theorem} (The sociodynamic trinary multinomial logit model with constant bias as a gradient system) Suppose that individual choices in a large sample population are characterized by the probabilities (C.3), (C.4) and (C.5) where the only contributions to the systematic utility of choices are a global field effect with utility parameter \(\beta\) real, finite on the proportion \(p_1\) of decision-making agents that have chosen alternative \(i\), and a constant bias \(h\) real, finite for choice alternative 0. The \textit{qualitative} stability of hyperbolic equilibrium solutions \(p_0, p_1, p_2\) defined on \([0, 1]\) of the sociodynamic trinary multinomial logit model with constant bias (C.6) can be determined from the isolated maxima, isolated minima and saddles of the function \(G\) given by:
Figure C.9: Null clines solution of system for parameter values of $\beta_-$ and $\beta_+$ showing bifurcations, for selected values of $h > 0$ with different bifurcation regime sequences over $\beta$. Compare with Figure 4.1 on page 45.
Figure C.10: Null clines solution of system for parameter values of $\beta_-$ and $\beta_+$ showing bifurcations, for selected values of $h > 0$ with different bifurcation regime sequences over $\beta$. Compare with Figure 4.1 on page 45.
Figure C.11: Null clines solution of system for parameter values of $\beta_-$ and $\beta_+$ showing bifurcations, for selected values of $h > 0$ with different bifurcation regime sequences over $\beta$. Compare with Figure 4.1 on page 45.
(a) $\beta_-$

$h = -0.5 \text{ Regime I } \rightarrow \text{ Regime VIII } \rightarrow \text{ Regime VII } \rightarrow \text{ Regime III}$

(b) $\beta_+$

$h = -1.5 \text{ Regime I } \rightarrow \text{ Regime VIII } \rightarrow \text{ Regime VII } \rightarrow \text{ Regime III}$

(c) $\beta_-$

$h = -10 \text{ Regime I } \rightarrow \text{ Regime VIII } \rightarrow \text{ Regime VII } \rightarrow \text{ Regime III}$

(d) $\beta_+$

Figure C.12: Null clines solution of system for parameter values of $\beta_-$ and $\beta_+$ showing bifurcations, for selected values of $h > 0$ with different bifurcation regime sequences over $\beta$. Compare with Figure 4.4 on page 45.
Figure C.13: Solution trajectory over $\beta$ and null clines for $\det J$ for the socio-dynamic trinary multinomial logit model with $h = 0$, yielding bifurcation in parameter $\beta$. The blue curves are the solution trajectories over $\beta$. The red and pink curves are the null clines for the determinant of the Jacobian. The intersection shows three 2-way symmetrical bifurcation points for $\beta \approx 2.7456$ (ie. $p_0 = 0.2076, p_1 = 0.2076$; $p_0 = 0.2076, p_1 = 0.5848$; $p_0 = 0.5848, p_1 = 0.2076$), one 3-way symmetrical bifurcation point for $\beta = 3$ (ie. $p_0 = p_1 = p_2 = \frac{1}{3}$), and three 2-way symmetrical bifurcation points in the limit $\beta \to \infty$ (ie. $p_0 = 0, p_1 = 0$; $p_0 = 0, p_1 = 1$; $p_0 = 1, p_1 = 0$). Compare with Figure 4.1 on page 45: Regime I $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime II $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime III.
Figure C.14: Example solution trajectory $\beta$ and nullclines for $\text{det } J$, yielding bifurcation in parameter $\beta$, over a sweep of parameter $h > 0$. The blue curves are the solution trajectories over $\beta$. The red and pink curves are the nullclines for the determinant of the Jacobian. Compare with panels with corresponding $h$ in Figure C.9 on page 395 and Figure C.10 on page 396: (a) Regime I $\rightarrow$ Regime V $\rightarrow$ Regime III; (b) Regime I $\rightarrow$ Regime V $\rightarrow$ Regime III; (c) Regime I $\rightarrow$ Regime V $\rightarrow$ Regime III; (d) Regime I $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime I $\rightarrow$ Regime V $\rightarrow$ Regime III; (e) Regime I $\rightarrow$ Regime IV $\rightarrow$ Regime I $\rightarrow$ Regime V $\rightarrow$ Regime III; (f) Regime I $\rightarrow$ Regime IV $\rightarrow$ Regime II $\rightarrow$ Regime V $\rightarrow$ Regime III
Figure C.15: Example solution trajectory $\beta$ and null clines for $\text{det} J$, yielding bifurcation in parameter $\beta$, over a sweep of parameter $h < 0$. The blue curves are the solution trajectories over $\beta$. The red and pink curves are the null clines for the determinant of the Jacobian. Compare with panels with corresponding $h$ in Figure C.11 on page 397 and Figure C.12 on page 398:

(a) Regime I $\rightarrow$ Regime VI $\rightarrow$ Regime II $\rightarrow$ Regime VII $\rightarrow$ Regime III;
(b) Regime I $\rightarrow$ Regime VI $\rightarrow$ Regime VIII $\rightarrow$ Regime VII $\rightarrow$ Regime III;
(c) Regime I $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime VIII $\rightarrow$ Regime VII $\rightarrow$ Regime III;
(d) Regime I $\rightarrow$ Regime VIII $\rightarrow$ Regime VII $\rightarrow$ Regime III;
(e) Regime I $\rightarrow$ Regime VIII $\rightarrow$ Regime VII $\rightarrow$ Regime III;
(f) Regime I $\rightarrow$ Regime VIII $\rightarrow$ Regime VII $\rightarrow$ Regime III.
402 Socio-Dynamic Trinary Logit with Constant Bias: Theory

Figure C.16: Bifurcation trajectory in the \( (p_0, p_1) \)-plane over all \( \beta \) and all \( h \)

Figure C.17: Bifurcation curve in the \( (\beta, h) \)-plane satisfying \( \text{det} J = 0 \) for \( p_1 \neq p_2 \). Compare with Figure C.1 on page 375.
\[ G = p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 - \frac{\beta}{2} (p_0^2 + p_1^2 + p_2^2) - h p_0 + C_0 \]  
(C.119)

**Proof.** Since we are primarily interested in qualitative behavior, for computational convenience we derive an alternative system of equations using (C.30) and (C.31). Re-arranging terms and substituting in (C.11) at equilibrium:

\[
0 = \dot{g}_1 \equiv -\ln p_1 + \ln(1 - p_1 - p_2) + \beta(p_1 - (1 - p_1 - p_2)) - h
\]

\[
0 = \dot{g}_2 \equiv -\ln p_2 + \ln(1 - p_1 - p_2) + \beta(p_2 - (1 - p_1 - p_2)) - h
\]

Try to find potential function \( G \) by integrating \( \dot{g}_1 \) with respect to \( p_1 \) and integrating \( \dot{g}_2 \) with respect to \( p_2 \), then comparing terms.

\[
\int \dot{g}_1 \, dp_1 = \int (-\ln p_1 + \ln(1 - p_1 - p_2) + \beta(p_1 - (1 - p_1 - p_2)) - h) \, dp_1
\]

\[
= -\int \ln p_1 \, dp_1 + \int \ln(1 - p_1 - p_2) \, dp_1 + \beta \int p_1 - (1 - p_1 - p_2) \, dp_1 - h \int dp_1
\]

\[
= -(p_1 \ln p_1 - p_1) - (1 - p_1 - p_2) \ln(1 - p_1 - p_2) + (1 - p_1 - p_2)
\]

\[
+ \beta (\frac{p_1^2}{2} - p_1 + \frac{p_1^2}{2} + p_1 p_2) - h p_1
\]

\[
= -p_1 \ln p_1 - (1 - p_1 - p_2) \ln(1 - p_1 - p_2)
\]

\[
+ \frac{\beta}{2} p_1^2 + \frac{\beta}{2} (1 - 2p_1 + p_1^2 + 2p_1 p_2) - h p_1 + (1 - p_2 - \frac{\beta}{2})
\]

\[
= -p_1 \ln p_1 - (1 - p_1 - p_2) \ln(1 - p_1 - p_2)
\]

\[
+ \frac{\beta}{2} p_1^2 + \frac{\beta}{2} (1 - 2p_1 + p_1^2 + 2p_1 p_2) - h p_1 + C_1
\]  
(C.122)

Likewise

\[
\int \dot{g}_2 \, dp_2 = \int (-\ln p_2 + \ln(1 - p_1 - p_2) + \beta(p_2 - (1 - p_1 - p_2)) - h) \, dp_1
\]

\[
= -p_2 \ln p_2 - (1 - p_1 - p_2) \ln(1 - p_1 - p_2) + \frac{\beta}{2} p_2^2
\]

\[
+ \frac{\beta}{2} (1 - 2p_2 + p_2^2 + 2p_1 p_2) - h p_2 + C_2
\]  
(C.123)

In order to satisfy

\[
\dot{g}_1 = -\frac{\partial G}{\partial p_1}, \quad \dot{g}_2 = -\frac{\partial G}{\partial p_2}
\]

we must have

\[
G = -\int \dot{g}_1 \, dp_1 = -\int \dot{g}_2 \, dp_2
\]  
(C.124)

Without loss of generality, the terms in (C.123) that appear exclusively in terms of \( p_2 \) can be absorbed into the additive constant term.
in (C.122). Likewise, the terms in (C.122) that appear exclusively in terms of \( p_1 \) can be absorbed into the additive constant term in (C.123). Furthermore, the only mixed terms that contain both \( p_1 \) and \( p_2 \) appear in both in (C.122) and (C.123). Thus we can write:

\[
G = -( -p_1 \ln p_1 - p_2 \ln p_2 - (1 - p_1 - p_2) \ln (1 - p_1 - p_2) \\
+ \frac{\beta}{2} p_1^2 + \frac{\beta}{2} p_2^2 + \frac{\beta}{2} (1 - 2p_1 + p_1^2 - 2p_2 + p_2^2 + 2p_1 p_2) \\
- hp_1 - hp_2) + \tilde{C}
\]

The terms in (C.122) that appear exclusively in terms of \( p_1 \) can be absorbed into the additive constant term in (C.123).

\[
= p_1 \ln p_1 + p_2 \ln p_2 + (1 - p_1 - p_2) \ln (1 - p_1 - p_2) \\
- \frac{\beta}{2} (p_1^2 + p_2^2 + (1 - p_1 - p_2)^2) + h(p_1 + p_2) + \tilde{C}
\]

Or, substituting (C.11) to see the symmetry-breaking in \( p_0 \) versus \( p_1 \) and \( p_2 \) more immediately:

\[
G = p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 - \frac{\beta}{2} (p_0^2 + p_1^2 + p_2^2) \\
+ h(p_1 + p_2) + \tilde{C}
\]

\[
= p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 - \frac{\beta}{2} (p_0^2 + p_1^2 + p_2^2) + h(1 - p_0) + \tilde{C}
\]

\[
= p_0 \ln p_0 + p_1 \ln p_1 + p_2 \ln p_2 - \frac{\beta}{2} (p_0^2 + p_1^2 + p_2^2) - hp_0 + C_0
\]

See Figure C.18 on page 405 and Figure C.19 on page 406.

C.5 Bringing it All Together

C.5.1 Analytical Bifurcation Curves

See Figure C.20 on page 408.

C.5.2 Solution Trajectories

See Figure C.21 on page 409.

C.5.3 General Bifurcation Diagrams

See Figure C.23 on page 411.

See Figure C.26 on page 414.
Figure C.18: Example potential functions for the sociodynamic trinary multinomial logit model with constant bias in each of the regimes in parameter space shown in Figure C.1 on page 375. Compare with the null clines solution of the system in Figure C.2 on page 376: (a) 1 stable node \((p_1 = p_2)\); (b) 1 stable node \((p_1 = p_2)\); (c) 4 stable nodes \((2 \text{ with } p_1 = p_2, 2 \text{ with } p_1 \neq p_2)\), 3 saddle points \((1 \text{ with } p_1 = p_2, 2 \text{ with } p_1 \neq p_2)\); (d) 4 stable nodes \((2 \text{ with } p_1 = p_2, 2 \text{ with } p_1 \neq p_2)\), 3 saddle points \((1 \text{ with } p_1 = p_2, 2 \text{ with } p_1 \neq p_2)\); (e) 3 stable nodes \((1 \text{ with } p_1 = p_2, 2 \text{ with } p_1 \neq p_2)\), 3 saddle points \((1 \text{ with } p_1 = p_2, 2 \text{ with } p_1 \neq p_2)\), 1 unstable node \((1 \text{ with } p_1 = p_2)\); (f) 3 stable nodes \((1 \text{ with } p_1 = p_2, 2 \text{ with } p_1 \neq p_2)\), 3 saddle points \((1 \text{ with } p_1 = p_2, 2 \text{ with } p_1 \neq p_2)\), 1 unstable node \((1 \text{ with } p_1 = p_2)\).
Figure C.19: Example potential functions for the sociodynamic trinary multinomial logit model with constant bias in each of the regimes in parameter space shown in Figure C.1 on page 375. Compare with the null clines solution of the system in Figure C.3 on page 377: (a) 3 stable nodes (1 with $p_1 = p_2$, 2 with $p_1 \neq p_2$), 2 saddle points ($p_1 = p_2$), 1 saddle point ($p_1 = p_2$); (b) 2 stable nodes ($p_1 = p_2$), 1 saddle point ($p_1 = p_2$); (c) 3 stable nodes (1 with $p_1 = p_2$, 2 with $p_1 \neq p_2$), 2 saddle points ($p_1 = p_2$); (d) 3 stable nodes (1 with $p_1 = p_2$, 2 with $p_1 \neq p_2$), 2 saddle points ($p_1 \neq p_2$); (e) 2 stable nodes ($p_1 \neq p_2$), 1 saddle point ($p_1 = p_2$); (f) 1 stable node ($p_1 = p_2$).
Table C.1: Characterization of the number of solutions of the sociodynamic trinary multinomial logit model with constant bias in different regimes (ASN: Asymptotically stable node; SP: Saddle point; UN: Unstable node)

<table>
<thead>
<tr>
<th></th>
<th>ASN</th>
<th>ASN</th>
<th>SP</th>
<th>SP</th>
<th>UN</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
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Figure C.20: Analytically-derived bifurcation curves in the \((\beta, h)\)-plane showing major regimes. Compare with Figure C.1 on page 375.
Figure C.21: Solution trajectories in the \((p_0, p_1)\)-plane over all \(\beta\) for selected values of the constant bias \(h\) for choice alternative 0: (a) Regime I \(\rightarrow\) Regime V \(\rightarrow\) Regime III; (b) Regime I \(\rightarrow\) Regime V \(\rightarrow\) Regime III; (c) Regime I \(\rightarrow\) Regime V \(\rightarrow\) Regime III; (d) Regime I \(\rightarrow\) [Bifurcation Point] \(\rightarrow\) Regime I \(\rightarrow\) Regime V \(\rightarrow\) Regime III; (e) Regime I \(\rightarrow\) Regime IV \(\rightarrow\) Regime I \(\rightarrow\) Regime V \(\rightarrow\) Regime III; (f) Regime I \(\rightarrow\) Regime IV \(\rightarrow\) Regime II \(\rightarrow\) Regime V \(\rightarrow\) Regime III
Socio-dynamic trinary logit with constant bias: theory

Figure C.22: Solution trajectories in the \((p_0, p_1)\)-plane over all \(\beta\) for selected values of the constant bias \(h\) for choice alternative 0, continued from Figure C.21 on page 409: (a) Regime I \(\rightarrow\) Regime II \(\rightarrow\) [Bifurcation Point] \(\rightarrow\) Regime III; (b) Regime I \(\rightarrow\) Regime VI \(\rightarrow\) Regime II \(\rightarrow\) Regime VII \(\rightarrow\) Regime III; (c) Regime I \(\rightarrow\) Regime VI \(\rightarrow\) Regime VIII \(\rightarrow\) Regime VII \(\rightarrow\) Regime III; (d) Regime I \(\rightarrow\) [Bifurcation Point] \(\rightarrow\) Regime VIII \(\rightarrow\) Regime VII \(\rightarrow\) Regime III; (e) Regime I \(\rightarrow\) Regime VIII \(\rightarrow\) Regime VII \(\rightarrow\) Regime III; (f) Regime I \(\rightarrow\) Regime VIII \(\rightarrow\) Regime VII \(\rightarrow\) Regime III
Figure C.23: General bifurcation diagrams for the sociodynamic multinomial logit model with constant bias. Compare with panels with corresponding h in Figure C.5 on page 381. Compare with null clines solutions of system at values of β satisfying det\(J = 0\) showing bifurcation points in Figure C.10 on page 396.
Socio-dynamic trinary logit with constant bias: theory

Figure C.24: General bifurcation diagrams for the sociodynamic multinomial logit model with constant bias. Compare with panels with corresponding $h$ in Figure C.5 on page 381 and Figure C.6 on page 382. Compare with null clines solutions of system at values of $\beta$ satisfying $\det J = 0$ showing bifurcation points in Figure C.10 on page 396 and Figure C.11 on page 397.
Figure C.25: General bifurcation diagrams for the sociodynamic multinomial logit model with constant bias. Compare with panels with corresponding $h$ in Figure C.6 on page 382. Compare with null clines solutions of system at values of $\beta$ satisfying $\text{det} J = 0$ showing bifurcation points in Figure C.11 on page 397 and Figure C.12 on page 398.
Figure C.26: Detail of bifurcation diagrams for $p_1$ versus $\beta$ for the socio-dynamic trinary multinomial logit model with constant bias near the hysteresis region. Compare with the lower panel of Figure C.20 on page 408 showing detail of bifurcation curves in the $(\beta, h)$-plane: (a) Regime I $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime II $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime III; (b) Regime I $\rightarrow$ Regime IV $\rightarrow$ Regime II $\rightarrow$ Regime V $\rightarrow$ Regime III; (c) Regime I $\rightarrow$ Regime IV $\rightarrow$ Regime I $\rightarrow$ Regime V $\rightarrow$ Regime III; (d) Regime I $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime I $\rightarrow$ Regime V $\rightarrow$ Regime III.
Figure C.27: Detail of bifurcation diagrams for $p_1$ versus $\beta$ for the sociodynamic trinary multinomial logit model with constant bias near the hysteresis region, continued from Figure C.26 on page 414. Compare with the lower panel of Figure C.20 on page 408 showing detail of bifurcation curves in the $(\beta, h)$-plane: (a) Regime I $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime II $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime III; (b) Regime I $\rightarrow$ Regime VI $\rightarrow$ Regime II $\rightarrow$ Regime VII $\rightarrow$ Regime III; (c) Regime I $\rightarrow$ Regime VI $\rightarrow$ Regime VIII $\rightarrow$ Regime VII $\rightarrow$ Regime III; (d) Regime I $\rightarrow$ [Bifurcation Point] $\rightarrow$ Regime VIII $\rightarrow$ Regime VII $\rightarrow$ Regime III.