UNIVERSITY OF AMSTERDAM

UvA-DARE (Digital Academic Repository)

History of constructivism in the 20th century

Troelstra, A.S.

DOI
10.1017/CBO9780511910616.009

Publication date
2011

Document Version
Submitted manuscript

Published in
Set Theory, Arithmetic, and Foundations of Mathematics: Theorems, Philosophies

Link to publication

Citation for published version (APA):
Troelstra, A. S. (2011). History of constructivism in the 20th century. In J. Kennedy, & R. Kossak (Eds.), *Set Theory, Arithmetic, and Foundations of Mathematics: Theorems, Philosophies* (pp. 150-179). (Lecture Notes in logic; Vol. 36). Cambridge University Press. https://doi.org/10.1017/CBO9780511910616.009

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

UvA-DARE is a service provided by the library of the University of Amsterdam (https://dare.uva.nl)

Download date:28 Jun 2022

History of constructivism in the 20th century

A.S.Troelstra

1 Introduction

In this survey of the history of constructivism, more space has been devoted to early developments (up till ca 1965) than to the work of the last few decades. Not only because most of the concepts and general insights have emerged before 1965, but also for practical reasons: much of the recent work is of a too technical and complicated nature to be described adequately within the limits of this article.

Constructivism is a point of view (or an attitude) concerning the methods and objects of mathematics which is normative: not only does it interpret existing mathematics according to certain principles, but it also rejects methods and results not conforming to such principles as unfounded or speculative (the rejection is not always absolute, but sometimes only a matter of degree: a decided preference for constructive concepts and methods). In this sense the various forms of constructivism are all 'ideological' in character.

Constructivism as a specific viewpoint emerges in the final quarter of the 19th century, and may be regarded as a reaction to the rapidly increasing use of highly abstract concepts and methods of proof in mathematics, a trend exemplified by the works of R. Dedekind[1] and G. Cantor[2].

The mathematics before the last quarter of the 19th century is, from the viewpoint of today, in the main constructive, with the notable exception of geometry, where proof by contradiction was commonly accepted and widely employed.

Characteristic for the constructivist trend is the insistence that mathematical objects are to be constructed (mental constructions) or computed; thus theorems asserting the existence of certain objects should by their proofs give us the means of constructing objects whose existence is being asserted.

L. Kronecker[3] may be described as the first conscious constructivist. For Kronecker, only the natural numbers were 'God-given'; all other mathematical objects ought to be explained in terms of natural numbers (at least in algebra). Assertions of existence should be backed up by constructions, and the properties of numbers

[1](Julius Wilhelm) Richard Dedekind 1831–1916
[2]Georg Ferdinand Ludwig Philipp Cantor 1845–1918
[3]Leopold Kronecker 1823–1891

defined in mathematics should be decidable in finitely many steps. This is in the spirit of finitism (see the next section). Kronecker strongly opposed Cantor's set theory, but in his publications he demonstrated his attitude more by example than by an explicit discussion of his tenets.

The principal constructivistic trends in this century are finitism already mentioned (section 2), semi-intuitionism and predicativism (section 3), intuitionism (section 4), Markov's constructivism (section 7) and Bishop's constructivism (section 8).

Constructivism, in particular intuitionism, has given rise to a considerable amount of metamathematical research (sections 5, 6).

Notations. We use **N**, **Q**, **R** for the set of natural, rational, and real numbers respectively.

For logical symbols we use $\neg$ (not), $\perp$ (falsehood), $\rightarrow$ (implication), $\wedge$ (conjunction, and), $\vee$ (disjunction, or), $\forall$ (for all), $\exists$ (there exists). A, B, C are arbitrary propositions or formulas of the language under discussion.

Usually n, m, k are elements of **N**, α, β number-theoretic functions (infinite sequences of natural numbers); $\bar{\alpha}n \equiv \langle \alpha 0, \alpha 1, \ldots, \alpha(n-1)\rangle$ is the initial segment of α of length n; $\langle\ \rangle$ is the empty sequence. $n * m$ is the concatenation of n and m.

IPC, **IQC**, **HA** are systems of intuitionistic propositional logic, predicate logic, and arithmetic respectively. In arithmetic, S is the sucessor function (i.e. $Sx = x + 1$). These intuitionistic systems differ from standard axiomatizations of the corresponding classical systems only by the absence of the *principle* PEM *of the excluded middle*

PEM $\qquad A \vee \neg A,$

or the principle of double negation $\neg\neg A \rightarrow A$.

2 Finitism

2.1. *Finitist mathematics*

Finitism may be characterized as based on the concept of natural number (or finite, concretely representable structure), which is taken to entail the acceptance of proof by induction and definition by recursion.

Abstract notions, such as 'constructive proof', 'arbitrary number-theoretic function' are rejected. Statements involving quantifiers are finitistically interpreted in terms of quantifier-free statements. Thus an existential statement $\exists x\, Ax$ is regarded as a partial communication, to be supplemented by providing an x which satisfies A. Establishing $\neg\forall x Ax$ finitistically means: providing a particular x such that Ax is false.

In this century, T. Skolem[4] was the first to contribute substantially to finitist

[4]Thoralf Skolem 1887–1963

mathematics; he showed that a fair part of arithmetic could be developed in a calculus without bound variables, and with induction over quantifier-free expressions only. Introduction of functions by primitive recursion is freely allowed (Skolem 1923). Skolem does not present his results in a formal context, nor does he try to delimit precisely the extent of finitist reasoning.

Since the idea of finitist reasoning was an essential ingredient in giving a precise formulation of Hilbert's programme (the consistency proof for mathematics should use intuitively justified means, namely finitist reasoning), Skolem's work is extensively reported by D.Hilbert[5] and P.Bernays[6]. Hilbert also attempted to circumscribe the extent of finitist methods in a global way; the final result is found in 'Die Grundlagen der Mathematik' (Hilbert & Bernays 1934, chapter 2).

In 1941 H. B. Curry[7] and R.L. Goodstein[8] independently formulated a purely equational calculus **PRA** for primitive recursive arithmetic in which Skolem's arguments could be formalized, and showed that the addition of classical propositional logic to **PRA** is conservative (i.e. no new equations become provable). **PRA** contains symbols for all primitive recursive functions, with their defining equations, and an induction rule of the form: if $t[0] = t'[0]$, and $t[Sx] = s[x, t[x]], t'[Sx] = s[x, t'[x]]$ has been derived, then we may conclude $t[x] = t'[x]$.

Goodstein carried the development of finitist arithmetic beyond Skolem's results, and also showed how to treat parts of analysis by finitist means (Goodstein 1957, 1959).

Recently W.W.Tait[9] made a new attempt to delimit the scope of finitist reasoning (Tait 1981). He defends the thesis that **PRA** is indeed the limit of finitist reasoning. Any finitely axiomatized part of **PRA** can be recognized as finitist, but never all of **PRA**, since this would require us to accept the *general* notion of a primitive recursive function, which is not finitist.

In recent years there has been a lot of metamathematical work showing that large parts of mathematics have an indirect finitist justification, namely by results of the form: a weak system **S** in a language with strong expressive power is shown to be consistent by methods formalizable in **PRA**, from which it may be concluded that **S** is conservative over **PRA**. A survey of such results is given in (Simpson 1988).

2.2. *Actualism*

A remark made in various forms by many authors, from G. Mannoury[10] in 1909 onwards, is the observation that already the natural number concept involves a strong idealization of the idea of 'concretely representable' or 'visualizable'. Such an idealization is implicit in the assumption that all natural numbers are constructions

[5]David Hilbert 1862–1943
[6]Paul Bernays
[7]Haskell Brooks Curry
[8]Reuben L. Goodstein
[9]William W. Tait
[10]Gerrit Mannoury 1857–1956

of the same kind, whether we talk about very small numbers such as 3 or 5, or extremely large ones such as 9^{9^9}. In reality, we cannot handle 9^{9^9} without some understanding of the general concept of exponentiation. The objection to finitism, that it is not restricted to objects which can be *actually* realized (physically, or in our imagination) one might call the 'actualist critique', and a programme taking the actualist critique into account, *actualism* (sometimes called 'ultra-intuitionism', or 'ultra-finitism').

The first author to defend an actualist programme, was A.S. Esenin-Vol'pin[11] in 1957. He intended to give a consistency proof for **ZF** using only 'ultra-intuitionist' means. Up till now the development of 'actualist' mathematics has not made much progress — there appear to be inherent difficulties associated with an actualist programme.

However, mention should be made of a paper by R. Parikh[12] (Parikh 1971), motivated by the actualist criticism of finitism, where is indicated, by technical results, the considerable difference in character between addition and multiplication on the one hand and exponentiation on the other hand. Together with work in complexity theory, this paper has stimulated the research on polynomially bounded arithmetic, as an example of which may be quoted the monograph 'Bounded Arithmetic' (Buss 1986).

3 Predicativism and semi-intuitionism

3.1. *Poincaré*

The French mathematician H. Poincaré[13] wrote many essays on the philosophy of mathematics and the sciences, collected in (Poincaré 1902, 1905, 1908, 1913); his ideas played an important role in the debate on the foundations of mathematics in the early part of this century. One cannot extract a unified and coherent point of view from Poincaré's writings. On the one hand he is a forerunner of the (semi-) intuitionists and predicativists, on the other hand he sometimes expresses formalist views, namely where he states that existence in mathematics can never mean anything but freedom from contradiction.

For the history of constructivism, Poincaré is important for two reasons:

(1) Explicit discussion and emphasis on the role of intuition in mathematics, more especially 'the intuition of pure number'. This intuition gives us the principle of induction for the natural numbers, characterized by Poincaré as a 'synthetic judgment a priori'. That is to say, the principle is neither tautological (i.e. justified by pure logic), nor is it derived from experience; instead, it is a consequence of our intuitive understanding of the notion of number. In this respect, Poincaré agrees with the semi-intuitionists and Brouwer.

[11] A.S. Esenin-Vol'pin ?
[12] Rohit J. Parikh
[13] Henri Poincaré 1854–1913

(2) According to Poincaré, the set-theoretic paradoxes are due to a vicious circle, namely the admission of *impredicative* definitions: defining an object N by referring to a totality E (in particular by quantifiying over E), while N itself belongs to E.

Poincaré's standard example is J.Richard's[14] paradox (Richard 1905): let E be the totality of all real numbers given by infinite decimal fractions, definable in finitely many words. E is clearly countable, so by a wellknown Cantor style diagonal argument we can define a real N not in E. But N has been defined in finitely many words! However, Poincaré, adopting Richard's conclusion, points out that the definition of N as element of E refers to E itself and is therefore impredicative. For a detailed discussion of Poincaré's philosophy of mathematics, see (Mooij 1966).

3.2. *The semi-intuitionists*

The term 'semi-intuitionists' or 'empirists' refers to a group of French mathematicians, in particular E. Borel[15] , H. Lebesgue[16] , R. Baire[17], and the Russian mathematician N.N.Luzin[18]. Their discussions of foundational problems are always in direct connection with specific mathematical developments, and thus have an 'ad hoc', local, character; also the views within the group differ.

What the semi-intuitionists have in common is the idea that, even if mathematical objects exist independently of the human mind, mathematics can only deal with such objects if we can also attain them by mentally constructing them, i.e. have access to them by our intuition; in practice, this means that they should be explicitly definable. In addition, pragmatical considerations occur: one is not interested in arbitrary objects of a certain kind, but only in the ones which play an important role in mathematics (which 'occur in nature' in a manner of speaking).

We shall illustrate semi-intuitionism by a summary of the views of Borel, the most explicit and outspoken of the semi-intuitionists. According to Borel, one can assert the existence of an object only if one can define it unambiguously in finitely many words. In this concern with definability there is a link with predicativism (as described below in 3.4); on the other hand, there is in Borel's writings no explicit concern with impredicativity and the vicious circle principle, as in the writings of Poincaré and H.Weyl[19] . (The Borel sets, introduced by Borel in the development of measure theory, are a 'continuous' analogue of the hyperarithmetic sets, which may be regarded as a first approximation to the general notion of a predicative subset of the natural numbers, but this was not known to Borel.)

With respect to the reality of the countably infinite, Borel takes a somewhat pragmatical attitude: while conceding the strength of the position of the strict

[14] Jules Richard
[15] (Félix Édouard) Émile Borel 1871–1970
[16] Henri (-Léon) Lebesgue 1875–1941
[17] (Louis) René Baire
[18] Nikolai Nikolaevich Luzin (Nicolas Lusin) 1893-1950
[19] Hermann Weyl 1885–1955

finitist, he observes that the countably infinite plays a very essential role in mathematics, and mathematicians have in practice always agreed on the correct use of the notion:

> ' The notion of the countably infinite appears therefore [...] as a limit possibility conceived by our imagination, just like the ideal line or the perfect circle.' (Borel 1914, p.179)

In other words, the natural numbers and the principle of complete induction are intuitively clear.

Borel explains the Richard paradox, not by referring to the impredicative character of the definition of N, but by the observation that N has not been unambiguously defined; the collection E is countable, but not effectively enumerable and hence the construction of N cannot be carried out (Borel 1914, p.165). This distinction would later obtain a precise formulation in recursion theory.

Borel explicitly introduced the notion of a calculable real number; a function f is calculable if $f(x)$ is calculable for each calculable x, and he observes that a calculable function must necessarily be continuous, at least for calculable arguments (Borel 1914, p.223). This foreshadows Brouwer's wellknown theorem on the continuity of all real functions (cf. section 6.1).

3.3. *Borel and the continuum*

Borel rejected the general notion of an uncountable set, as an essentially negative notion. According to Borel, Cantor's diagonal argument only shows that the continuum cannot be exhausted by a countable set. Obviously, on this view the continuum presents a problem. On the one hand it is the basic concept in analysis, on the other hand it cannot be described as 'the set of its (definable) elements' since it is uncountable. Thus Borel remarks in 1908:

> ' [...] the notion of the continuum, which is the only well-known example of a non-denumerable set, that is to say a set of which the mathematicians have a clear idea in common (or believe to have in common, which in practice amounts to the same thing). I regard that notion as acquired from the geometric intuition of the continuum; it is well-known that the complete arithmetical notion of the continuum requires that one admits the legitimacy of an infinity of countably many successive arbitrary choices'. (Borel 1914, p.162)

(The term 'arithmetic theory of the continuum', or 'arithmetization of the continuum', frequently appears in discussions of constructivism in the early part of this century. By this is meant the characterization of the continuum as a *set* of reals, where the real numbers are obtained as equivalence classes of fundamental sequences of rationals, or as Dedekind cuts. Since the rationals can be enumerated, this achieves a reduction of the theory of the continuum to numbertheoretical or 'arithmetical' functions.)

In 1912 Borel remarks that one can reason about certain classes of objects, such as the reals, since the class is defined in finitely many words, even if not all elements of the class are finitely definable. Therefore Borel had to accept the continuum as a primitive concept, not reducible to an arithmetical theory of the continuum.

Although the fact is not mentioned by Borel, the idea of a continuum consisting of only countably many definable reals suggests a 'measure-theoretic paradox', for if the reals in $[0,1]$ are countable, one can give a covering by a sequence of intervals I_0, I_1, I_2, ... with $\sum_{n=0}^{\infty} |I_n| < 1$, where $|I_n|$ is the length of I_n. (Such coverings are called *singular.*) The paradox is repeatedly mentioned in Brouwer's publications (e.g. Brouwer 1930), as a proof of the superiority of his theory of the continuum.

3.4. *Weyl*

Motivated by his rejection of the platonistic view of mathematics prevalent in Cantorian set theory and Dedekind's foundation of the natural number concept, Weyl, in his short monograph 'Das Kontinuum' (Weyl 1918) formulated a programme for predicative mathematics; it appears that Weyl had arrived at his position independently of Poincaré and B.Russell[20]. Predicativism may be characterized as 'constructivism' w.r.t. definitions of sets (but not w.r.t. the use of logic): sets are constructed from below, not characterized by singling them out among the members of a totality conceived as previously existing.

Weyl accepted classical logic and the set of natural numbers with induction and definition by recursion as unproblematic. Since the totality of natural numbers is accepted, all arithmetical predicates make sense as sets and we can quantify over them. The arithmetically definable sets are the *sets of rank 1*, the first level of a predicative hierarchy of ranked sets; sets of higher rank are obtained by permitting quantification over sets of lower rank in their definition. Weyl intended to keep the developments simple by restricting attention to sets of rank 1.

On the basis of these principles Weyl was able to show for example: Cauchy sequences of real numbers have a limit; every bounded monotone sequence of reals has a limit; every countable covering by open intervals of a bounded closed interval has a finite subcovering; the intermediate value theorem holds (i.e. a function changing sign on an interval has a zero in the interval); a continuous function on a bounded closed interval has a maximum and a minimum.

After his monograph appeared, Weyl became for a short period converted to Brouwer's intuitionism. Later he took a more detached view, refusing the exclusive adoption of either a constructive or an abstract axiomatic approach. Although Weyl retained a lifelong interest in the foundations of mathematics, he did not influence the developments after 1918. An excellent summary of Weyl's development, as well

[20]Bertrand Russell 1872–1970

as a technical analysis of 'Das Kontinuum' has been given by S.Feferman[21] (1988, Weyl).

3.5. *Predicativism after 'das Kontinuum'*

After Weyl's monograph predicativism rested until the late fifties, when interest revived in the work of M. Kondô[22], A. Grzegorczyk[23] and G. Kreisel[24]. Kreisel showed that the so-called hyperarithmetic sets known from recursion theory constituted an upper bound for the notion 'predicatively definable set of natural numbers'. Feferman and K. Schütte[25] addressed the question of the precise extent of predicative analysis; they managed to give a characterization of its proof theoretic ordinal. Type free formalizations for predicative analysis with sets of all (predicative) ranks were developed. In recent years also many formalisms have been shown to be indirectly reducible to predicative systems, cf. (Feferman 1988, Hilbert).

The books of P. Lorenzen[26] (Lorenzen 1955, 1965) may be regarded as a direct continuation of Weyl's programme.

4 Brouwerian intuitionism

4.1. *Early period*

In his thesis 'Over de Grondslagen der Wiskunde' (Brouwer 1907) the Dutch mathematician L.E.J. Brouwer[27] defended, more radically and more consistently than the semi-intuitionists, an intuitionist conception of mathematics. Brouwer's philosophy of mathematics is embedded in a general philosophy, the essentials of which are found already in (Brouwer 1905). To these philosophical views Brouwer adhered all his life; a late statement may be found in (Brouwer 1949). With respect to mathematics, Brouwer's main ideas are

1. Mathematics is not formal; the objects of mathematics are mental constructions in the mind of the (ideal) mathematician. Only the thought constructions of the (idealized) mathematician are exact.

2. Mathematics is independent of experience in the outside world, and mathematics is in principle also independent of language. Communication by language may serve to suggest similar thought constructions to others, but there is no guarantee that these other constructions are the same. (This is a solipsistic element in Brouwer's philosophy.)

---

[21] Solomon Feferman
[22] Motokiti Kondô
[23] Andrzej Grzegorczyk
[24] Georg Kreisel, born 1926
[25] Kurt Schütte
[26] Paul Lorenzen
[27] Luitzen Egbertus Jan Brouwer 1881–1966

3. Mathematics does not depend on logic; on the contrary, logic is part of mathematics.

These principles led to a programme of reconstruction of mathematics on intuitionistic principles ('Brouwer's programme' or 'BP' for short). During the early period, from say 1907 until 1913, Brouwer did the major part of his work in (classical) topology and contributed little to BP. In these years his view of the continuum and of countable sets is quite similar to Borel's position on these matters. Thus he writes

> ' The continuum as a whole was intuitively given to us; a construction of the continuum, an act which would create "all" its parts as individualized by the mathematical intuition is unthinkable and impossible. The mathematical intuition is not capable of creating other than countable quantities in an individualized way.' (Brouwer 1907, p.62, cf. also p.10)

On the other hand, already in this early period there are also clear differences; thus Brouwer did not follow Borel in his pragmatic intersubjectivism, and tries to explain the natural numbers and the continuum as two aspects of a single intuition ('the primeval intuition').

Another important difference with Borel c.s. is, that Brouwer soon after finishing his thesis realized that classical logic did not apply to his mathematics (see the next section). Nevertheless, until circa 1913 Brouwer some of the views of the semi-intuitionists and did not publicly dissociate himself from them.

4.2. *Weak counterexamples and the creative subject*

Already in (Brouwer 1908) a typically intuitionistic kind of counterexample to certain statements A of classical mathematics was introduced, not counterexamples in the strict sense of deriving a contradiction from the statement A, but examples showing that, assuming that we can prove A intuitionistically, it would follow that we had a solution to a problem known to be as yet unsolved. Undue attention given to these examples often created for outsiders the erroneous impression that intuitionism was mainly a negative activity of criticizing classical mathematics.

Example. Consider the set $X \equiv \{x : x = 1 \vee (x = 2 \wedge F)\}$ where F is any as yet undecided mathematical statement, such as the Riemann hypothesis. X is a subset of the finite set $\{1, 2\}$, but we cannot prove X to be finite, since this would require us to decide whether X has only one, or two elements, and for this we would have to decide F. (Intuitionistically, a set is finite if it can be brought into a constructively specified 1-1 correspondence with an initial part of the natural numbers.) So we have found a weak counterexample to the statement 'a subset of a finite set is finite'.

By choosing our undecided problems suitably, it is also possible to give a weak counterexample to: 'for reals x, $x < 0$ or $x = 0$ or $x > 0$', or to 'for all reals x, $x \leq 0$ or $x \geq 0$'. Brouwer used these examples to show the need for an intuitionistic revision of certain parts of the classical theory, and to demonstrate how classically equivalent definitions corresponded to distinct intuitionistic notions.

G.F.C. Griss[28], in a number of publications between 1944 and 1955, advocated a form of intuitionistic mathematics without negation, since one cannot have a clear conception of what it means to give proof of a proposition which turns out to be contradictory (cf. the interpretation of intuitionistic implication and negation in subsection 5.2).

In reaction to Griss, Brouwer published in 1948 an example of a statement in the form of a negation which could not be replaced by a positive statement. Brouwer's example involved a real number defined by explicit reference to stages in the activity of the ideal mathematician trying to prove an as yet undecided statement A. An essential ingredient of Brouwer's argument is that if the ideal mathematician (*creative subject* in Brouwer's terminology) is certain that at no future stage of his mathematical activity he will find A to be true, it means that he knows that $\neg A$ (Brouwer seems to have used such examples in lectures since 1927).

The argument illustrates the extreme solipsistic consequences of Brouwer's intuitionism. Because of their philosophical impact, these examples generated a good deal of discussion and inspired some metamathematical research, but their impact on BP was very limited, in fact almost nihil.

4.3. *Brouwer's programme*

After 1912, the year in which he obtained a professorship at the University of Amsterdam, Brouwer started in earnest on his programme, and soon discovered that for a fruitful reconstruction of analysis his ideas on the continuum needed revision. Around 1913 he must have realized that the notion of choice sequence (appearing in a rather different setting in Borel's discussion of the axiom of choice) could be legitimized from his viewpoint and offered all the advantages of an 'arithmetical' theory of the continuum. The first paper where the notion is actually used is (Brouwer 1919).

In the period up to 1928 he reconstructed parts of the theory of pointsets, some function theory, developed a theory of countable well-orderings and, together with his student B. de Loor[29], gave an intuitionistic proof of the fundamental theorem of algebra.

Brouwer's ideas became more widely known only after 1920, when he lectured on them at many places, especially in Germany.

After 1928, Brouwer displayed very little mathematical activity, presumably as a result of a conflict in the board of editors of the 'Mathematische Annalen' (cf. Van Dalen 1990). From 1923 onwards, M.J. Belinfante[30] and A. Heyting[31], and

[28] George François Cornelis Griss
[29] Barend de Loor
[30] Maurits Joost Belinfante
[31] Arend Heyting 1898–1980

later also Heyting's students continued BP. Belinfante investigated intuitionistic complex function theory in the thirties. Heyting dealt with intuitionistic projective geometry and algebra (in particular linear algebra and elimination theory). In the period 1952-1967, six of Heyting's Ph.D students wrote theses on subjects such as intuitionistic topology, measure theory, theory of Hilbert spaces, the Radon integral, intuitionistic affine geometry. After 1974, interesting contributions have been made by W. Veldman[32], he studied a.o. the intuitionistic analytic hierarchy.

The discovery of a precise notion of algorithm in the thirties (the notion of a (general) recursive function) as a result of the work of A.Church[33], K. Gödel[34], J.Herbrand[35], A.M.Turing[36] and S.C.Kleene[37], did not affect intuitionism. This is not really surprising: most of these characterizations describe algorithms by specifying a narrow language in which they can be expressed, which is utterly alien to Brouwer's view of mathematics as the languageless activity of the ideal mathematician. Turing's analysis is not tied to a specific formalism. Nevertheless he bases his informal arguments for the generality of his notion of algorithm on the manipulation of symbols and appeals to physical limitations on computing; and such ideas do not fit into Brouwer's ideas on mathematics as a 'free creation'.

Without Heyting's sustained efforts to explain Brouwer's ideas and to make them more widely known, interest in intuitionism might well have died out in the thirties. However, most of the interest in intuitionism concerned its metamathematics, not BP, contrary to Heyting's intentions. Heyting's monograph (Heyting 1956) was instrumental in generating a wider interest in intuitionism.

The next two sections will be devoted to the codification of intuitionistic logic and the gradual emergence of the metamathematics of constructive theories.

5 Intuitionistic Logic and Arithmetic

5.1. *L.E.J. Brouwer and intuitionistic logic*

The fact that Brouwer's approach to mathematics also required a revision of the principles of classical logic was not yet clearly realized by him while writing his thesis, but in 1908 Brouwer explicitly noted that intuitionism required a different logic. In particular, he noted that the principle of the excluded middle $A \vee \neg A$ is not intuitionistically valid. Implicitly, of course, the meaning of the logical operators had been adapted to the intuitionistic context, that is the intuitionistic meaning of a statement $A \vee \neg A$ is different from the classical one. With a quote from (Brouwer 1908):

[32] Willem Henri Maria Veldman
[33] Alonzo Church
[34] Kurt Gödel 1906–1978
[35] Jacques Herbrand 1908–1931
[36] Alan Mathison Turing 1912–1954
[37] Stephen Cole Kleene, born 1909

> 'is it allowed, in purely mathematical constructions and transformations, to neglect for some time the idea of the mathematical system under construction and to operate in the corresponding linguistic structure, following the principles of [...] and can we have confidence that each part of the argument can be justified by recalling to the mind the corresponding mathematical construction?'

A first important technical contribution to intuitionistic logic is made in (Brouwer 1924, Zerlegung), namely the observation that $\neg\neg\neg A \leftrightarrow \neg A$ is an intuitionistic logical law.

5.2. *The Brouwer-Heyting-Kolmogorov interpretation*

The standard informal interpretation of logical operators in intuitionistic logic is the so-called *proof-interpretation* or *Brouwer-Heyting-Kolmogorov interpretation* (*BHK-interpretation* for short). The formalization of intuitionistic logic started before this interpretation was actually formulated, but it is preferable to discuss the BHK-interpretation first since it facilitates the understanding of the more technical results. On the BHK-interpretation, the meaning of a statement A is given by explaining what constitutes a proof of A, and *proof of* A for logically compound A is explained in terms of what it means to give a proof of its constituents. Thus, for propositional logic:

1. A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.
2. A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B.
3. A proof of $A \rightarrow B$ is a construction which transforms any proof of A into a proof of B.
4. Absurdity $\perp$ ('the contradiction') has no proof; a proof of $\neg A$ is a construction which transforms any supposed proof of A into a proof of $\perp$.

Such an interpretation is implicit in Brouwer's writings (e.g. Brouwer 1908, 1924, Zerlegung) and has been made explicit by Heyting for predicate logic (Heyting 1934), and by A.N.Kolmogorov[38] (Kolmogorov 1932) for propositional logic.

Kolmogorov formulated what is essentially the same interpretation in different terms: he regarded propositions as problems, and logically compound assertions as problems explained in terms of simpler problems, e.g. $A \rightarrow B$ represents the problem of reducing the solution of B to the solution of A. Initially Heyting and Kolmogorov regarded their respective interpretations as distinct; Kolmogorov stressed that his interpretation also makes sense in a classical setting. Later Heyting realized that, at least in an intuitionistic setting, both interpretations are practically the same.

[38] Andrei Nikolaevich Kolmogorov 1903–1987

5.3. *Formal intuitionistic logic and arithmetic till 1940*

Kolmogorov's paper from 1925, written in Russian, is the earliest published formalization of a fragment of intuitionistic logic, and represents a remarkable achievement, but had very little effect on the developments (in 1933 still unknown to Gödel, and not seen by Heyting in 1934). Kolmogorov does not assume the 'ex falso sequitur quodlibet' $P \to (\neg P \to Q)$ which is justifiable on the basis of the BHK-interpretation. The system of intuitionistic logic with the 'ex falso' deleted became known as *minimal logic* and is of some interest in connection with completeness problems.

V.I.Glivenko[39] presented in 1928 a (not complete) formalization of intuitionistic propositional logic and derives from this informally $\neg\neg(\neg P \vee P)$, $\neg\neg\neg P \to \neg P$, $(\neg P \vee P \to \neg Q) \to \neg Q$, and uses these theorems to show that the interpretation of (Barzin and Errera 1927) of Brouwer's logic, according to which a proposition intuitionistically is either true, or false, or 'tierce', is untenable; a nice example of the use of formalization to settle a philosophical debate.

Heyting wrote a prize essay on the formalization of intuitionistic mathematics which was crowned by the Dutch Mathematical Association in 1928; the essay appeared in revised form in 1930. The first of the three papers (Heyting 1930) contains a formalization of intuitionistic propositional logic in its present extent. The second paper deals with predicate logic and arithmetic. Predicate logic does not yet appear in its final form, due to a defective treatment of substitution, and the (not quite consistent) germs of a theory permitting non-denoting terms. Arithmetic as presented in the second paper is a fragment of Heyting arithmetic as it is understood today, since there are axioms for addition (in the guise of a definition) but not for multiplication. The third paper deals with analysis. The system is very weak due to a lack of existence axioms for sets and functions.

In 1929 Glivenko formulated and proved as a result of his correspondence with Heyting the 'Glivenko theorem': in propositional logic $\neg\neg A$ is intuitionistically provable if and only if A is classically provable.

Kolmogorov in his paper from 1925 describes an embedding of classical propositional logic into (his fragment of) intuitionistic propositional logic, thereby anticipating the work of Gödel (1933) and G.Gentzen[40](also dating from 1933, but published only in 1965), and argues that this embedding is capable of generalization to stronger systems. Gödel's embedding is formulated for arithmetic, but can be adapted in an obvious way to predicate logic. In Gentzen's version prime formulas P are first replaced by $\neg\neg P$, and the operators $\ldots \vee \ldots$, $\exists x \ldots$ by $\neg(\neg \ldots \wedge \neg \ldots), \neg\forall x \neg \ldots$ respectively. In Kolmogorov's version $\neg\neg$ is inserted simultaneously in front of every subformula. The various embeddings are logically equivalent. If * is one of these embeddings, then $A^* \leftrightarrow A$ classically, and A^* is provable intuitionistically if and only if A is provable classically.

[39] Valeriĭ Ivanovich Glivenko 1897–1940
[40] Gerhard Gentzen 1909–1945

Gödel's embedding made it clear that intuitionistic methods went beyond finitism, precisely because abstract notions were allowed. This is clear e.g. from the clause explaining intuitionistic implication in the BHK-interpretation, since there the abstract notion of constructive proof and construction are used as primitives. This fact had not been realized by the Hilbert school until then; Bernays was the first one to grasp the implications of Gödel's result.

Quite important for the proof theory of intuitionistic logic was the formulation in (Gentzen 1935) of the sequent calculi LK and LJ. Using his cut elimination theorem, Gentzen showed that for **IQC** the *disjunction property* DP holds: if $\vdash A \vee B$, then $\vdash A$ or $\vdash B$. Exactly the same method yields the *explicit definability* or *existence property* ED: if $\vdash \exists x\, A(x)$ then $\vdash A(t)$ for some term t. These properties present a striking contrast with classical logic, and have been extensively investigated and established for all the usual intuitionistic formal systems.

The earliest semantics for **IPC**, due to the work of S.Jaskowski[41], M.H. Stone[42], A.Tarski[43], G.Birkhoff[44], T. Ogasawara[45]in the years 1936-1940 was algebraic semantics, with topological semantics as an important special case. In algebraic semantics, the truth values for propositions are elements of a *Heyting algebra* (also known as Brouwerian lattice, pseudo-complemented lattice, pseudo-Boolean algebra, or residuated lattice with bottom). A Heyting algebra is a lattice with top and bottom, and an extra operation $\rightarrow$ such that $a \wedge b \leq c$ iff $a \leq b \rightarrow c$, for all elements a, b, c of the lattice. An important special case of a Heyting algebra is the collection of open sets of a topological space T ordered under inclusion, where $U \rightarrow V := \text{Interior}(V \cup (T \setminus U))$. The logical operations $\wedge, \vee, \rightarrow, \neg$ correspond to the lattice operations $\wedge, \vee, \rightarrow$ and the defined operation $\neg a := a \rightarrow 0$, where 0 is the bottom of the lattice. (A boolean algebra is a special cases of a Heyting algebra.)

5.4. *Metamathematics of intuitionistic logic and arithmetic after 1940*

In the early forties Kleene devised an interpretation which established a connection between the notion of computable (= recursive) function and intuitionistic logic, the *realizability* interpretation (Kleene 1945).

The essence of the interpretation is that it so to speak *hereditarily* codes information on the explicit realization of disjunctions and existential quantifiers, recursively in numerical parameters. The definition is by induction on the number of logical symbols of sentences (= formulas without free variables): with every formula A one associates a predicate 'x realizes A', where x is a natural number. Typical clauses are

1. n realizes $t = s$ iff $t = s$ is true.

[41] Stanislaw Jáskowski
[42] M.H.Stone?
[43] Alfred Tarski
[44] Garrett Birkhoff
[45] Tôzirô Ogasawara

2. n realizes $A \to B$ iff for all m realizing A, $n \bullet m$ realizes B;

3. n realizes $\exists m B(m)$ iff n is a pair (m, k), and k realizes $B(m)$.

Here $\bullet$ is the operation of application between a number and the code of a partial recursive function. Kleene established the correctness of this interpretation: if $\mathbf{HA} \vdash A$, then for some number n, n realizes A.

The interest of the interpretation is, that it makes more true than just what is coded in the formalism **HA**. In particular, the following version of *Church's thesis* may be shown to be realizable:

$$\mathrm{CT}_0 \qquad \forall n \exists m A(n,m) \to \exists k \forall n A(n, k \bullet m)$$

a principle which is easily seen to be incompatible with classical logic. Realizability and its many variants have become a powerful tool in the study of metamathematics of constructive systems.

We now turn to the further development of truth-value semantics. Algebraic semantics was extended to predicate logic, and A.Mostowski[46]in 1949 was the first one to apply topological models to obtain underivability results for **IQC**. This development culminated in the monograph (Rasiowa and Sikorski 1963). Although algebraic semantics has proved to be technically useful in metamathematical research, it is so to speak only the algebraic version of the syntax, as witnessed by the fact that **IPC** itself can be made into a Heyting algebra (the *Lindenbaum algebra* of **IPC**). More important from a conceptual point of view are two other semantics, *Beth models*, due to E.W.Beth[47](Beth 1956, 1959), and *Kripke models*, due to S.Kripke[48] (Kripke 1965).

Both these semantics are based on partially ordered sets. We call the elements $k, k', k'' \ldots$ of a partially ordered set $(K, \leq)$ *nodes*. In Kripke models the partial order is arbitrary, in Beth models as defined by Beth it is a finitely branching tree. The interest of these models resides in the intuitive interpretation of the partial order: for Beth models, each node represents a state of information in time, and a higher node represents a possible state of information at a later point in time. The branching of the tree reflects the fact that there are different possibilities for the extension of our knowledge in the future. In the Kripke models, it is more natural to think of the nodes as representing possible stages of knowledge; a higher node in the ordering corresponds to an extension of our knowledge. (That is to say, passing to a later period in time does not force us to move upwards in a Kripke model, only extension of our knowledge does.) In these models one has a notion of 'A is true at k', or 'k forces A'. Falsehood $\perp$ is nowhere forced. It is possible to interpret Beth and Kripke models as topological models for special spaces.

An important aspect of Beth models is the connection with intuitive intuitionistic validity; a formula $A(R_1, \ldots, R_n)$ of **IQC**, containing predicate letters $R_1, \ldots, R_n$

[46]Andrzej Mostowski 1913–1975
[47]Evert Willem Beth 1908–1964
[48]Saul Kripke

is *intuitionistically valid* if for all domains D and all relations R_i^* of the appropriate arity (i.e. the appropriate number of argument places), $A^D(R_1^*, \ldots, R_n^*)$ holds intuitionistically; here $A^D(R_1^*, \ldots R_n^*)$ is obtained from A by restricting quantifiers to D, and replacing R_i by R_i^*.

From observations by G.Kreisel in 1958 it follows that, for propositional logic, validity in a Beth model is equivalent to intuitive validity for a collection of propositions P_1^α, P_2^α, …, P_n^α depending on a lawless parameter α (for the explanation of 'lawless' see the next section).

Beth and Kripke proved completeness for their respective kinds of semantics by classical methods. (Beth originally believed to have also an intuitionistic completeness proof for his semantics.) Veldman was able to show that if one extends the notion of Beth model to *fallible* Beth models, where it is permitted that in certain nodes falsehood is forced, it is possible to obtain an intuitionistic completeness proof for Kripke semantics. The idea was transferred to Kripke semantics by H. de Swart[49]. For the fragment of intuitionistic logic without falsehood and negation, fallible models are just ordinary models. For minimal logic, where $\perp$ is regarded as an arbitrary unprovable proposition letter, one has intuitionistic completeness relative to ordinary Beth models. The best results in this direction can be obtained from work by H.Friedman[50]from ca. 1976 (cf. Troelstra & van Dalen 1988, 2, chapter 13).

C.A.Smoryński[51] used Kripke models with great virtuosity in the study of the metamathematics of intuitionistic arithmetic (see Smoryński 1973).

5.5. *Formulas-as-types*

In essence, the 'formulas-as-types' idea (may be 'propositions-as-types' would have been better) consists in the identification of a proposition with the set of its (intuitionistic) proofs. Or stated in another form: in a calculus of typed objects, the types play the role of propositions, and the objects of a type A correspond to the proofs of the proposition A.

Thus, since on the BHK-interpretation a proof of an implication $A \to B$ is an operation transforming proofs of A into proofs of B, the proofs of $A \to B$ are a set of functions from (the proofs of) A to (the proofs of) B. Similarly, (the set of proofs of) $A \wedge B$ is the set of pairs of proofs, with first component a proof of A, and second component a proof of B. So $A \wedge B$ corresponds to a cartesian product.

A clear expression to this idea was given in the late sixties (circa 1968–1969), by W.A.Howard[52], and by N.G.de Bruijn[53]; H. Läuchli[54]around the same time used the idea for a completeness proof for **IQC** for a kind of realizability semantics.

[49] Henricus Cornelis Maria de Swart, born 1944

[50] Harvey Friedman

[51] Craig A. Smoryński

[52] William A. Howard

[53] Nicolaas Govert de Bruijn

[54] H. Läuchli ?

The analogy goes deeper: one can use terms of a typed lambda calculus to denote natural deduction proofs, and then normalization of proofs corresponds to normalization in the lambda calculus. So pure typed lambda calculus is in a sense the same as **IPC** in natural deduction formulation; similarly, second-order lambda calculus (polymorphic lambda calculus) is intuitionistic logic with propositional quantifiers.

The formulas-as-types idea is a guiding principle in much recent research in type theory on the borderline of logic and theoretical computer science. It is used in the type theories developed by P.Martin-Löf[55] (Martin-Löf 1975, 1984) to reduce logic to type theory; thus proof by induction and definition by recursion are subsumed under a single rule in these theories. Formulas-as-types plays a key role in the proof-checking language AUTOMATH devised by de Bruijn and collaborators since the late sixties.

The formulas-as-types idea has a parallel in category theory, where propositions correspond to objects, and arrows to (equivalence classes of) proofs. J. Lambek[56] investigated this parallel for **IPC** in (Lambek 1972) and later work, culminating in the monograph (Lambek & Scott 1986).

6 Intuitionistic analysis and stronger theories

6.1. *Choice sequences in Brouwer's writings*

As already remarked, the continuum presented a problem to the semi-intuitionists; they were forced to introduce it as a primitive notion, while Brouwer in his thesis tried to explain the continuum and the natural numbers as emanating both from a single 'primeval intuition'.

However, when Brouwer started (circa 1913) with his intuitionistic reconstruction of the theory of the continuum and the theory of pointsets, he found that the notion of choice sequence, appearing in Borel's discussion of the axiom of choice (as the opposite, so to speak, of a sequence defined in finitely many words, and therefore in Borel's view of a dubious character) could be regarded as a legitimate intuitionistic notion, and as a means of retaining the advantages of an arithmetic theory of the continuum.

In Brouwer's intuitionistic set theory the dominating concept is that of a spread (German: 'Menge'). Slightly simplifying Brouwer's original definition, we say that a *spread* consists essentially of a tree of finite sequences of natural numbers, such that every sequence has at least one successor, plus a law L assigning objects of a previously constructed domain to the nodes of the tree. Choice sequences within a given spread correspond to the infinite branches of the tree. Brouwer calls a sequence $\mathrm{L}(\bar{\alpha}1)$, $\mathrm{L}(\bar{\alpha}2)$, $\mathrm{L}(\bar{\alpha}3)$, ..., ($\alpha$ an infinite branch) an element of the spread. Below we shall use 'spread' only for trees of finite sequences of natural numbers without

[55] Per Martin-Löf
[56] Joachim Lambek

finite branches, corresponding to the trivial L satisfying $\mathrm{L}(\bar{\alpha}(n+1)) = \alpha n$. Since it is not the definition of a spread, but the way the choice sequences are given to us, which determines the properties of the continuum, we shall henceforth concentrate on the choice sequences.

The notion of spread is supplemented by the notion of species, much closer to the classical concept of set; one may think of a species as a set of elements singled out from a previously constructed totality by a property (as in the separation axiom of classical set theory).

The admissibility of impredicative definitions is not explicitly discussed in Brouwer's writings, though it is unlikely that he would have accepted impredicative definitions without restrictions. On the other hand, his methods allow more than just predicative sets over $\mathbf{N}$: Brouwer's introduction of ordinals in intuitionistic mathematics is an example of a set introduced by a so-called generalized inductive definition, which cannot be obtained as a set defined predicatively relative to $\mathbf{N}$.

A *choice sequence* α of natural numbers may be viewed as an unfinished, ongoing process of choosing values $\alpha 0$, $\alpha 1$, $\alpha 2$, ... by the ideal mathematician (IM); at any stage of his activity the IM has determined only finitely many values plus, possibly, some restrictions on future choices (the restrictions may vary from 'no restrictions' to 'choices henceforth completely determined by a law'). For sequences completely determined by a law or recipe we shall use *lawlike*; other mathematical objects not depending on choice parameters are also called lawlike. An important principle concerning choice sequences is the

> *Continuity principle* or *continuity axiom.* If to every choice sequence α of a spread a number $n(\alpha)$ is assigned, $n(\alpha)$ depends on an initial segment $\bar{\alpha}m = \alpha 0, \alpha 1, \ldots, \alpha(m-1)$ only, that is to say for all choice sequences β starting with the same initial segment $\bar{\alpha}m$, $n(\beta) = n(\alpha)$.

This principle is not specially singled out by Brouwer, but used in proofs (for the first time in course notes from 1916-17), more particularly in proofs of what later became known as the bar theorem. From the bar theorem (explained below) Brouwer obtained an important corollary for the finitary spreads, the

> *Fan theorem.* If to every choice sequence α of a finitely branching spread (fan) a number $n(\alpha)$ is assigned, there is a number m, such that for all α, $n(\alpha)$ may be determined from the first m values of α (i.e. an initial segment of length m).

The fan theorem may be seen as a combination of the compactness of finite trees with the continuity axiom; Brouwer uses the fan theorem in particular to derive

> *The uniform continuity theorem.* Every function from a bounded closed interval into $\mathbf{R}$ is uniformly continuous.

The essence of the bar theorem is best expressed in a formulation which appears in a footnote in (Brouwer 1927), and was afterwards used by Kleene as an axiom in his formalization of intuitionistic analysis.

> *Bar theorem.* If the 'universal spread' (i.e. the tree of all sequences of natural numbers) contains a decidable set A of nodes such that each choice sequence α has an initial segment in A, then for the set of nodes generated by (i) $X \subset A$, (ii) if all successors of a node n are in X, then $n \in X$, it follows that the empty sequence is in X.

Originally, in (Brouwer 1919), the fan theorem is assumed without proof. In 1923 Brouwer presents an unsatisfactory proof of the uniform continuity theorem, in 1924 he proves this theorem via the fan theorem which in its turn is obtained from the bar theorem; his 1924 proof of the bar theorem is repeated in many later publications with slight variations.

Brouwer's proof of the bar theorem has often been regarded as obscure, but has also been acclaimed as containing an idea of considerable interest. For in this proof Brouwer analyzes the possible forms of a constructive proof of a statement of the form $\forall\alpha\exists n A(\bar{\alpha}n)$ for decidable A.

More precisely, the claim made by Brouwer, in his proof of the bar theorem, amounts to the following. Let $\mathrm{Sec}(n)$ ('n is secured') mean 'All α through the node n pass through A', Now Brouwer assumes that a 'fully analyzed proof' of the statement $\langle\,\rangle \in \mathrm{Sec}$ or $\mathrm{Sec}(\langle\,\rangle)$ (i.e. all α pass through A), can consist of three kinds of steps only:

(i) $n \in A$, hence $\mathrm{Sec}(n)$;
(ii) if for all i $\mathrm{Sec}(n * \langle i\rangle)$, then $\mathrm{Sec}(n)$;
(iii) if $\mathrm{Sec}(n)$, then $\mathrm{Sec}(n * \langle i\rangle)$.

Brouwer then shows that steps of the form (iii) may be eliminated, from which one readily obtains the bar theorem in the form given above.

Nowadays the proof is regarded not so much obscure as well as unsatisfactory. As Kleene (Kleene & Vesley 1965, p.51) rightly observes, Brouwer's assumption concerning the form of 'fully analyzed proofs' is not more evident than the bar theorem itself. On the other hand, the notion of 'fully analyzed' proof or 'canonical' proof was used later by M.A.E.Dummett[57] (Dummett 1977) in his attempts to give a more satisfactory version of the BHK-interpretation of intuitionistic logic.

6.2. *Axiomatization of intuitionistic analysis*

In Heyting's third formalization paper from 1930 we find for the first time a formal statement of the continuity principle. For a long time nothing happened till Kleene in 1950 started working on the axiomatization of intuitionistic analysis; his work culminated in the monograph (Kleene & Vesley 1965). Kleene based his system on a language with variables for numbers and choice sequences; to arithmetical axioms he added an axiom of countable choice, the axiom of bar induction (equivalent to the bar theorem as formulated above) and a continuity axiom, a strengthening of the continuity principle as stated above ('Brouwer's principle for functions').

The continuity axiom was the only non-classical principle, and Kleene established its consistency relative to the other axioms using a realizability interpretation

[57]Michael A.E. Dummett

(function-realizability). He also showed, by means of another realizability notion, that Markov's principle (see 7.2) was not derivable in his system.

In 1963 Kreisel developed an axiomatization based on a language with number variables and two kinds of function variables, for lawlike sequences and for choice sequences. He sketched a proof of the conservativity of the axioms for choice sequences relative to the lawlike part of the system, by means of a translation of an arbitrary sentence into the lawlike part of the theory (the 'elimination translation'). This work finally resulted in (Kreisel & Troelstra 1970).

J. Myhill[58] in 1967 introduced an axiom intended to express the consequences in the language of analysis of Brouwer's solipsistic theory of the creative subject (as reported above in 4.2), called by him Kripke's scheme (KS) since Kripke was the first to formulate this principle. KS is

$$\exists\alpha((\exists x(\alpha x \neq 0) \to A) \wedge (\forall x(\alpha x = 0) \to \neg A)$$

for arbitrary A. KS conflicts with Brouwer's principle for functions. (Brouwer's reasoning seems to justify in fact the even stronger $\exists\alpha(\exists x(\alpha x \neq 0) \leftrightarrow A)$). Myhill's conceptual analysis of the notion of choice sequence is considerably more refined earlier attempts.

There is an obvious connection between Kripke's scheme and the theory of the creative subject mentioned in subsection 4.2. For any proposition A, the α in KS may be interpreted as: $\alpha n \neq 0$ if and only if the creative subject has found evidence for the truth of A at stage n of his activity.

Brouwer appears to have vacillated w.r.t. the precise form which restrictions on choice sequences could take, but in his published writings he does not explicitly consider subdomains of the universe of choice sequences which are characterized by the class of restrictions allowed, except for the trivial example of the lawlike sequences.

In 1958 Kreisel considered 'absolutely free' (nowadays *lawless*) sequences ranging over a finitely branching tree, where at any stage in the construction of the sequence no restriction on future choices is allowed; later this was extended to an axiomatization **LS** of the theory of lawless sequences ranging over the universal tree (Kreisel 1968). Lawless sequences are of interest because of their conceptual simplicity (when compared to other concepts of choice sequence), as a tool for studying other notions of choice sequence (see e.g. (Troelstra 1983)), and because they provide a link between Beth-validity and intuitive intuitionistic validity (cf. 5.4).

As a result of work of Kreisel, Myhill and A.S.Troelstra[59], mainly over the period 1963-1980, it became clear that many different notions of choice sequence may be distinguished, with different properties. For a survey of this topic and its history see (Troelstra 1977 , Appendix C; 1983).

[58] John Myhill

[59] Anne Sjerp Troelstra, born 1939

The publication of (Bishop 1967) led to several proposals for an axiomatic framework for Bishop's constructive mathematics, in particular a type-free theory of operators and classes (Feferman 1975, 1979), and versions of intuitionistic set theory (Friedman 1977). Martin-Löf's type theories have also been considered in this connection (Martin-Löf 1975, 1984). As shown by Aczel in 1978, one can interpret a constructive set theory **CZF** in a suitable version of Martin-Löf's type theory. (**CZF** does not have a powerset axiom; instead there are suitable collection axioms which permit to derive the existence of the set of all functions from x to y for any two sets x and y; moreover, the foundation axiom has been replaced by an axiom of $\in$-induction). Much of this work is reported in the monograph (Beeson 1985); Beeson[60] himself made substantial contributions in this area.

6.3. *The model theory of intuitionistic analysis*

Topological models and Beth models turned out to be very fruitful for the metamathematical study of intuitionistic analysis, type theory and set theory. D.S. Scott[61] was the first to give a topological model for intuitionistic analysis, in two papers from 1968 and 1970. In this model the real numbers are represented by the continuous functions over the topological space T underlying the model. For suitable T, all real-valued functions are continuous in Scott's model.

This later developed into the so-called sheaf models for intuitionistic analysis, type theory and set theory, with a peak of activity in the period 1977-1984. The inspiration for this development not only came from Scott's models just mentioned, but also from category theory, where W.Lawvere[62](Lawvere 1971) developed the notion of elementary topos — a category with extra structure, in which set theory and type theory based on intuitionistic logic can be interpreted. The notion of elementary topos generalized the notion of Grothendieck topos known from algebraic geometry, and is in a sense 'equivalent' to the notion of an intuitionistic type theory.

Even Kleene's realizability interpretation can be extended to type theory and be recast as an interpretation of type theory in a special topos (Hyland 1982).

Some of the models studied are mathematically interesting in their own right, and draw attention to possibilities not envisaged in the constructivist tradition (e.g. analysis without an axiom of countable choice, where the reals defined by Dedekind cuts are not isomorphic to the reals defined via fundamental sequences of rationals).

[60] Michael J. Beeson

[61] Dana Stewart Scott, born 1932

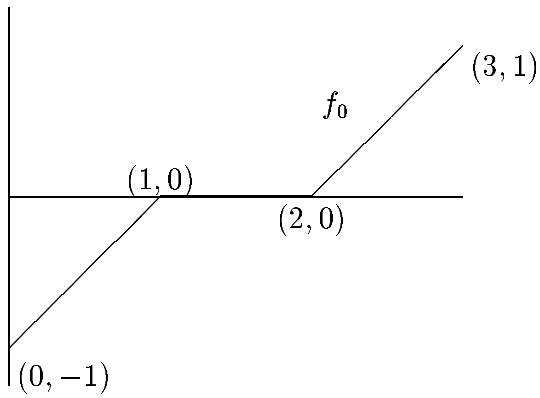
[62] William Lawvere

7 Constructive recursive mathematics

7.1. *Classical recursive mathematics*

Before we can discuss Markov's version of constructive mathematics, it is necessary to say a few words on classical recursive mathematics (RM for short).

In RM, recursive versions of classical notions are investigated, against a background of classical logic. The difference with a more strictly constructive approach is illustrated by the following example:



The function $f_a(x)$ given by $f_a(x) = f_0(x) + a$, where f_0 is as in the picture, cannot be constructively proved to have a zero as long as we do not know whether $a \leq 0$ or $a \geq 0$. But even classically we can show that f_a does not have a zero recursively in the parameter a - and this is typically a result of RM.

Where in constructive recursive mathematics the recursivity in parameters is built into the constructive reading of the logical operators, in RM the recursiveness has to be made explicit.

RM is almost as old as recursion theory itself, since already Turing introduced "computable numbers" (Turing 1937), and it is still expanding today. We mention a few examples.

E.Specker[63] was the first to construct an example of a recursive bounded monotone sequence of rationals without a recursive limit (Specker 1949). Such sequences are now known as *Specker sequences.*

Kreisel and D.Lacombe[64] constructed singular coverings of the interval in 1957; Kreisel, Lacombe and J.Shoenfield[65] showed, also in 1957, that every effective operation of type 2 is continuous (an effective operation of type 2 is a partial recursive operation with code u say, acting on codes x, y of total recursive functions such that $\forall z(x \bullet z = y \bullet z) \to u \bullet x = u \bullet y$; continuity of u means that $u \bullet x$ depends on finitely many values only of the function coded by x).

---

[63] Ernst Specker
[64] Daniel Lacombe
[65] Joseph Shoenfield

A.I.Maltsev[66] and Y.L.Ershov[67] developed (mainly in the period 1961-1974) the 'theory of numerations' as a systematic method to lift the notion of recursiveness from **N** to arbitrary countable structures (Ershov 1972). In 1974 G.Metakides[68] and A.Nerode[69] gave the first applications of the powerful priority method from recursion theory to problems in algebra; see (Metakides & Nerode 1979). As an example of a recent striking result we mention the construction of a recursive ordinary differential equation without *recursive solutions*, obtained by M.B. Pour-El[70] and J.I. Richards[71] in 1979.

For a recent monograph, see (Pour-El & Richards 1989).

7.2. *Constructive recursive mathematics*

A.A. Markov[72] formulated in 1948-49 the basic ideas of constructive recursive mathematics (CRM for short). They may be summarized as follows.

1. objects of constructive mathematics are constructive objects, concretely: words in various alphabets.
2. the abstraction of potential existence is admissible but the abstraction of actual infinity is not allowed. Potential realizability means e.g. that we may regard plus as a well-defined operation for all natural numbers, since we know how to complete it for arbitrarily large numbers.
3. a precise notion of algorithm is taken as a basis (Markov chose for this his own notion of 'Markov-algorithm').
4. logically compound statements have to be interpreted so as to take the preceding points into account.

Not surprisingly, many results of RM can be bodily lifted to CRM and vice versa. Sometimes parallel results were discovered almost simultaneously and independently in RM and CRM respectively. Thus the theorem by Kreisel, Lacombe and Shoenfield mentioned above is in the setting of CRM a special case of a theorem proved by G.S.Tseĭtin[73] in 1959: every function from a complete separable metric space into a separable metric space is continuous.

N.A.Shanin[74] (Shanin 1958) formulated a "deciphering algorithm which makes the constructive content of mathematical statements explicit. By this reinterpretation an arbitrary statement in the language of arithmetic is reformulated as a

[66] A.I.Maltsev ?
[67] Yuriĭ Leonidovich Ershov, born 1940
[68] Georges Metakides
[69] Anil Nerode
[70] Marian Boykan Pour-El
[71] (Jonathan) Ian Richards
[72] Andrei Andreevich Markov 1903–1979
[73] Grigorij Samuilovich Tseĭtin, born 1936
[74] Nikolai Aleksandrovich Shanin, born 1919

formula $\exists x_1 \ldots x_i A$ where A is normal, i.e. does not contain $\vee, \exists$ and the string of existential quantifiers may be interpreted in the usual way. Shanin's method is essentially equivalent to Kleene's realizability, but has been formulated in such a way that normal formulas are unchanged by the interpretation (Kleene's realizability produces a different although intuitionistically equivalent formula when applied to a normal formula).

The deciphering method systematically produces a constructive reading for notions defined in the language of arithmetic; classically equivalent definitions may obtain distinct interpretations by this method. But not all notions considered in CRM are obtained by applying the method to a definition in arithmetic (example: the 'FR-numbers' are the reals corresponding to the intuitionistic reals and are given by a code α for a fundamental sequence of rationals, together with a code β for a modulus of convergence, i.e. relative to a standard enumeration of the rationals $\langle r_n \rangle_n$ the sequence is $\langle r_{\alpha n} \rangle_n$, and $\forall mm'(|r_{\beta n+m} - r_{\beta n+m'}| < 2^{-k})$. An F-sequence is an FR-sequence with the β omitted. The notion of an F-sequence does not arise as an application of the deciphering algorithm.)

Markov accepted one principle not accepted in either intuitionism or Bishop's constructivism: if it is impossible that computation by an algorithm does not terminate, then it does terminate. Logically this amounts to what is usually called Markov's principle:

MP $\qquad \neg\neg\exists x(fx = 0) \to \exists x(fx = 0)$ ($f : \mathbf{N} \to \mathbf{N}$ recursive.)

The theorem by Tsejtin, mentioned above, needs MP for its proof.

The measure-theoretic paradox (cf. 3.3) is resolved in CRM in a satisfactory way: singular coverings of $[0, 1]$ do exist, but the sequence of partial sums $\sum_{n=0}^{k} |I_n|$ does not converge, but is a Specker sequence; and if the sequence does converge, the limit is ≥ 1, as shown by Tsejtin and I.D.Zaslavskiĭ[75] in 1962.

In recent years (after ca. 1985) the number of contributions to CRM has considerably decreased. Many researchers in CRM have turned to more computer-science oriented topics.

8 Bishop's constructivism

8.1. *Bishop's constructive mathematics*

In his book 'Foundations of constructive mathematics' the American mathematician E. Bishop[76] launched his programme for constructive mathematics. Bishop's attitude is both ideological and pragmatical: ideological, inasmuch he insists that we should strive for a type of mathematics in which every statement has empirical content, and pragmatical in the actual road he takes towards his goal.

[75] Igor' Dmitrievich Zaslavskiĭ, born 1932

[76] Errett Bishop 1928-1983

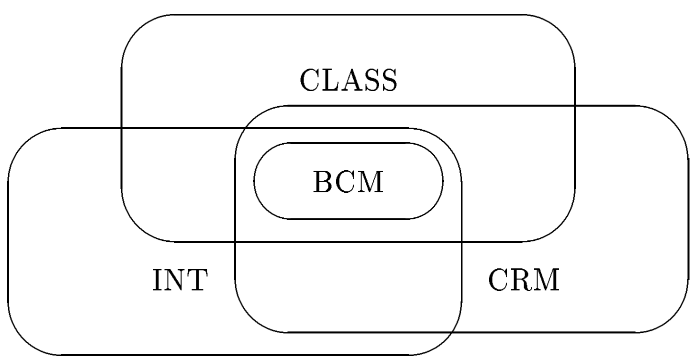
In Bishop's view, Brouwer successfully criticized classical mathematics, but had gone astray in carrying out his programme, by introducing dubious concepts such as choice sequences, and wasting much time over the splitting of classical concepts into many non-equivalent ones, instead of concentrating on the mathematically relevant versions of these concepts. In carrying out his programme, Bishop is guided by three principles:

1. avoid concepts defined in a negative way;
2. avoid defining irrelevant concepts — that is to say, among the many possible classically equivalent, but constructively distinct definitions of a concept, choose the one or two which are mathematically fruitful ones, and disregard the others;
3. avoid pseudo-generality, that is to say, do not hesitate to introduce an extra assumption if it facilitates the theory and the examples one is interested in satisfy the assumption.

Statements of Bishop's constructive mathematics (BCM for short) may be read intuitionistically without distortion; sequences are then to be regarded as given by a law, and accordingly, no continuity axioms nor bar induction are assumed.

Statements of BCM may also be read by a Markov-constructivist without essential distortion; the algorithms are left implicit, and no use is made of a precise definition of algorithm.

Thus BCM appears as a part of classical mathematics, and the situation may be illustrated graphically in the diagram below, where 'INT' stands for intuitionistic mathematics, and 'CLASS' for classical mathematics.



However, it should not be forgotten that this picture is not to be taken at face value, since the mathematical statements have different interpretations in the various forms of constructivism.

In the actual implementation of his programme, Bishop not only applied the three principles above, but also avoided negative results (only rarely did he present a weak counterexample), and concentrated almost exclusively on positive results.

There has been a steady stream of publications contributing to Bishop's programme since 1967 until now; two of the most prolific contributors are F. Richman[77] and D.E.Bridges[78]. The topics treated cover large parts of analysis, including the theory of Banach spaces and measure theory, parts of algebra (e.g. abelian groups, Noetherian groups) and topology (e.g. dimension theory, the Jordan curve theorem).

8.2. *The relation of BCM to INT and CRM*

The success of Bishop's programme has left little scope for traditional intuitionistic mathematics; to some extent this is also true of CRM. For all of BCM may at the same time be regarded as a contribution to INT; moreover, in many instances where in INT a routine appeal would be made to a typical intuitionistic result, such as the uniform continuity theorem for functions defined on a closed bounded interval, the corresponding treatment in in BCM would simply add an assumption of uniform continuity for the relevant function, without essential loss in mathematical content. Thus to find scope for specifically intuitionistic reasoning, one has to look for instances where the use of typically intuitionistic axioms such as the continuity axiom or the fan theorem results in a significantly better or more elegant result, and such cases appear to be comparatively rare.

To some extent the above, mutatis mutandis, also applies to constructive recursive mathematics. Thus, for example, the usefulness of the beautiful Kreisel-Lacombe-Shoenfield-Tsejtin theorem is limited by two factors: (1) an appeal to the theorem can often be replaced by an assumption of continuity in the statement of the result to be proved; (2) in many cases, continuity without uniform continuity is not enough, as witnessed e.g. by Kushner's example of a continuous function on [0,1] which is not integrable.

In the literature contributing to Brouwer's and Markov's programme, a comparatively large place is taken by counterexamples and splitting of classical notions. This may be compared with periods in the development of classical analysis and topology in which there was also considerable attention given to 'pathologies'. In this comparison, BCM exemplifies a later stage in constructivism.

Finally, let us note that in classical mathematics there arise questions of constructivity of a type which has not been considered in the constructivistic tradition For example, one may attempt to find a bound on the *number* of solutions to a number-theoretic problem, without having a bound for the *size* of the solutions; see (Luckhardt 1989).

[77] F. Richman ?

[78] Douglas E.Bridges

9 Concluding remarks

In the foundational debate in the early part of this century, constructivism played an important role. Nevertheless, at any time only a handful of mathematicians have been actively contributing to constructive mathematics (in the sense discussed here).

In the course of time, the focus of activity in constructive mathematics has shifted from intuitionism to Markov's constructivism, then to Bishop's constructivism. In addition there has been a steady flow of contributions to classical recursive mathematics, a subject which is still flourishing.

Much more research has been devoted to intuitionistic logic and the metamathematics of constructive systems. In this area the work has recently somewhat ebbed, but its concepts and techniques play a significant role elsewhere, e.g. in theoretical computer science and artificial intelligence, and its potential is by no means exhausted (example: the notion of 'formulas-as-types' and Martin-Löf-style type theories).

New areas of application lead to refinement and modification of concepts developed in another context. Thus a recent development such as 'linear logic' (Girard 1987) may be seen as a refinement of intuitionistic logic, obtained by pursuing the idea of a 'bookkeeping of resources' seriously.

References

A very extensive bibliography of constructivism is (Müller 1987), especially under F50–65.

BARZIN, M., ERRERA, A., Sur la logique de M. Brouwer, *Academie Royale des Sciences, des Lettres et des Beaux Arts de Belgique. Bulletin de la Classe des Sciences. Cinquième Série 13*, 1927, pp. 56–71.

BEESON, M.J., *Foundations of Constructive Mathematics*, Springer Verlag, Berlin 1985.

BETH, E.W., Semantic construction of intuitionistic logic, *Koninklijke Nederlandse Akademie van Wetenschappen. Afdeling Letterkunde: Mededelingen. Nieuwe Reeks 19*, 1956, pp. 357–388.

BETH, E.W., *The Foundations of Mathematics*, North-Holland Publ. Co., Amsterdam 1959, 1965^2.

BISHOP, E., *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.

BOREL, E., *Leçons sur la Théorie des Fonctions*, Gauthier-Villars, Paris, 1914^2, 1928^3, 1959^4.

BROUWER, L.E.J., *Leven, Kunst, Mystiek* (Dutch) (Life, Art and Mysticism). Waltman, Delft, 1905.

BROUWER,L.E.J., *Over de Grondslagen der Wiskunde* (Dutch) (On the foundations of mathematics). Maas en van Suchtelen, Amsterdam, 1907.

BROUWER,L.E.J., Over de onbetrouwbaarheid der logische principes (Dutch) (On the unreliability of the principles of logic), *Tijdschrift voor Wijsbegeerte 2*, 1908, pp.152–158. English translation: (Brouwer 1975, pp. 107–111).

BROUWER,L.E.J., Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten I: Allgemeine Mengenlehre, *Verhandelingen der Koninklijke Nederlandse Akademie van Wetenschappen. Tweede Reeks. Afdeling Natuurkunde 12/5*, 1918. Also (Brouwer 1975, pp. 150–190).

BROUWER,L.E.J., Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten II: Theorie der Punktmengen, *Verhandelingen der Koninklijke Nederlandse Akademie van Wetenschappen. Tweede Reeks. Afdeling Natuurkunde 12/7*, 1919. Also (Brouwer 1975, pp. 191–221).

BROUWER,L.E.J., Intuitionistische Zerlegung mathematischer Grundbegriffe, *Jahresbericht der Deutsche Mathematiker-Vereinigung 33*, 1924, pp.241–256. Also (Brouwer 1975, pp.429–440).

BROUWER,L.E.J. Beweis dass jede volle Funktion gleichmässig stetig ist, *Koninklijke Nederlandse Akademie van Wetenschappen. Proceedings of the Section of Sciences 27*, 1924, pp. 189–193. Also (Brouwer 1975, pp. 274–280).

BROUWER,L.E.J., Über Definitionsbereiche von Funktionen, *Mathematische Annalen 97*, 1927, pp. 60–75. Also (Brouwer 1975, pp. 390–405).

BROUWER, L.E.J., *Die Struktur des Kontinuums*, Gistel, Wien 1930. Also (Brouwer 1975, pp. 429–440).

BROUWER,L.E.J., Consciousness, Philosophy and mathematics, in *Library of the Tenth International Congress in Philosophy, August 1948, Vol. 1*, edited by E.W. Beth, H.J.Pos, H.J.A. Hollak, North-Holland Publ. Co., Amsterdam 1949, pp. 1235–1249. Also (Brouwer 1975, pp. 480–494).

BROUWER, L.E.J., *Collected Works, Vol. 1*, edited by A. Heyting, North-Holland Publ. Co., Amsterdam 1975.

BUSS, S., *Bounded Arithmetic*, Bibliopolis, Naples,1986.

DALEN, D. VAN, The war of the frogs and the mice, or the crisis of the Mathematische Annalen, *The Mathematical Intelligencer 12*, 1990, pp.17–31.

DUMMETT, M.A.E., *Elements of Intuitionism*, Clarendon Press, Oxford, 1977.

ERSHOV, Y.L., La théorie des énumérations, in *Actes du Congrès International des Mathématiciens 1970, Vol. 1.*, edited by M. Berger, J. Dieudonné, J. Leroy, J.-L. Lions, M.P. Malliavin, Gauthier-Villars, Paris 1972, pp. 223–227.

FEFERMAN, S., A language and axioms for explicit mathematics, in *Algebra and Logic*, edited by J. Crossley, Springer Verlag, Berlin 1975, pp. 87–139.

FEFERMAN, S., Constructive theory of functions and classes, in *Logic Colloquium '78*, edited by M. Boffa, D. van Dalen, K. McAloon, North-Holland Publ. Co., Amsterdam 1979, pp. 159–224

FEFERMAN, S., Weyl vindicated: 'Das Kontinuum' 70 years later, in: *Temi e Prospettive della Logica e della Filosofia Contemporanee I*, Cooperativa Libraria Universitaria Editrice Bologna, Bologna 1988, pp.59–93.

FEFERMAN, S., Hilbert's program relativized: proof-theoretical and foundational reductions, *The Journal of Symbolic Logic 53*, 1988, pp. 364–384.

FRIEDMAN, H., Set-theoretic foundations for constructive analysis, *Annals of Mathematics 105*, 1977, pp. 1-28.

GENTZEN, G., Untersuchungen über das logische Schliessen, *Mathematische Zeitschrift 39*, 1935, pp.176–210, 405–431.

GIRARD, J.-Y., Linear logic. *Theoretical Computer Science 50*, 1987, pp. 1–101.

GÖDEL, K., Zur intuitionistischen Arithmetik und Zahlentheorie, *Ergebnisse eines mathematischen Kolloquiums 4*, 1933, pp.34–38.

GOODSTEIN, R.L., *Recursive Number Theory*, North-Holland Publ. Co., Amsterdam, 1957.

GOODSTEIN, R.L., *Recursive Analysis*, North-Holland Publ. Co., Amsterdam, 1959.

HEIJENOORT, J. VAN, *From Frege to Gödel*, Harvard University Press, Cambridge (Mass.) 1967.

HEYTING, A., Die formalen Regeln der intuitionistischen Logik, *Sitzungsberichte der Preussischen Akademie von Wissenschaften, Physikalisch Mathematische Klasse*, 1930, pp.158–169; Die formalen Regeln der intuitionistischen Mathematik, *Ibidem* pp.57–71, 158–169.

HEYTING, A., *Mathematische Grundlagenforschung. Intuitionismus. Beweistheorie*, Springer Verlag, Berlin 1934.

HEYTING, A., *Intuitionism, an introduction*, North-Holland Publ. Co., Amsterdam 1956, 1966^2, 1971^3.

HILBERT, D., BERNAYS, P., *Grundlagen der Mathematik Vol. I*, Springer Verlag, Berlin, 1934, 1968^2.

HYLAND, J.M.E., The effective topos, in *The L.E.J. Brouwer Centenary Symposium*, edited by A.S. Troelstra, D. van Dalen, North-Holland Publ. Co., Amsterdam 1982, pp. 165–216.

KLEENE, S.C., On the interpretation of intuitionistic number theory, *The Journal of Symbolic Logic 10*, 1945, pp. 109–124.

KLEENE, S.C., VESLEY, R.E., *The Foundations of Intuitionistic Mathematics, especially in Relation to Recursive Functions*, North-Holland Publ. Co., Amsterdam 1965.

KOLMOGOROV, A.N., O printsipe tertium non datur (Russian) (On the principle of the excluded middle), *Matematicheskii Sbornik 32*, 1925, pp.646–667. Translation (van Heijenoort 1967, pp.414-437).

KOLMOGOROV, A.N., Zur Deutung der intuitionistischen Logik, *Mathematische Zeitschrift 35*, 1932, pp. 58–65.

KREISEL, G., A remark on free choice sequences and the topological completeness proofs, *The Journal of Symbolic Logic 23*, 1958, pp. 369–388.

KREISEL, G., Lawless sequences of natural numbers, *Compositio Mathematica 20*, 1968, pp. 222–248.

KREISEL, G., TROELSTRA, A.S., Formal systems for some branches of intuitionistic analysis, *Annals of Mathematical Logic 1*, 1970, pp. 229–387.

KRIPKE, S., Semantical analysis of intuitionistic logic, in *Formal Systems and Recursive Functions*, edited by J.Crossley, M.A.E. Dummett, North-Holland Publ. Co., Amsterdam 1965, pp. 92–130.

KUSHNER, B.A., *Lektsii po konstruktivnomu matematicheskomu analizu* (Russian), 1973. Translated as *Lectures on Constructive Mathematical Analysis*, American Mathematical Society, Providence, Rhode Island, U.S.A. 1984.

LAMBEK, J., Deductive systems and categories III, in *Toposes, Algebraic Geometry and Logic*, edited by F.W.Lawvere, Springer Verlag, Heidelberg 1972, pp.57–82.

LAMBEK, J., SCOTT, P.J., *Introduction to higher order categorical logic*, Cambridge University Press, Cambridge 1986.

LUCKHARDT, H., Herbrand-Analysen zweier Beweise des Satzes von Roth: polynomiale Anzahlschranken, *The Journal of Symbolic Logic 54*, 1989, pp. 234–263.

MARTIN-LÖF, P., An intuitionistic theory of types:predicative part, in *Logic Colloquium '73*, edited by H.E.Rose, J.C.Shepherdson, North-Holland Publ. Co., Amsterdam 1975, pp. 73–118.

MARTIN-LÖF, P., *Intuitionistic type theory*, Bibliopolis, Napoli 1984.

METAKIDES, G., NERODEE,A., Effective content of field theory, *Annals of Mathematical Logic 17*, 1979, pp. 289-320.

MOOIJ, J.J.A., *La Philosophie des Mathématiques de Henri Poincaré*, Gauthier-Villars, Paris 1966.

Müller, G.H., *Ω-Bibliography of Mathematical Logic* edited by G.H. Müller in collaboration with W. Lenski, *Volume VI. Proof theory, Constructive Mathematics*, edited by J.E. Kister, D. van Dalen and A.S. Troelstra, Springer Verlag, Berlin 1987.

Parikh, R.J., Existence and feasibility in arithmetic, *The Journal of Symbolic Logic 36*, 1971, pp.494–508.

Poincaré, H., *Science et Hypothèse*, Flammarion, Paris 1902.

Poincaré, H., *La Valeur de la Science*, Flammarion, Paris 1905.

Poincaré, H., *Science et Méthode*, Flammarion, Paris 1908.

Poincaré, H., *Dernières Pensées*, Flammarion, Paris 1913.

Pour-El, M.B., Richards, J. I., *Computability in analysis and physics*, Springer Verlag, Berlin 1989.

Rasiowa, H., Sikorski, R., *The Mathematics of Metamathematics*, Państwowe Wydawnictwo Naukowe, Warszawa 1963.

Richard, J., Les principes des mathématiques et le problème des ensembles, *Revue génénerale des sciences pures et appliquées* 16, 1905, p. 541. Also in *Acta Mathematica 30*, 1906, pp. 295–296.

Scott, D.S., Extending the topological interpretation to intuitionistic analysis, *Compositio Mathematica 20*, 1968, pp. 194–210.

Scott, D.S., Extending the topological interpretation to intuitionistic analysis II, in *Intuitionism and Proof Theory*, edited by J. Myhill, A. Kino, R.E. Vesley, North-Holland Publ. Co., 1970, pp.235–255.

Shanin, N.A., O konstruktiviom ponimanii matematicheskikh suzhdenij (Russian) (On the constructive interpretation of mathematical judgments), *Trudy Ordena Lenina Matematicheskogo Instituta imeni V.A. Steklova. Akademiya Nauk SSSR 52*, 1958, pp. 226–311. Translation: *American Mathematical Society Translations, Series 2, 23*, pp.108–189.

Simpson, S., Partial realizations of Hilbert's program, *The Journal of Symbolic Logic 53*, 1988, pp. 349–363.

Smoryński, C. A., Applications of Kripke models, in *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, edited by A.S.Troelstra, Springer Verlag, Berlin 1973, pp. 324–391.

Specker, E., Nicht konstruktiv beweisbare Sätze der Analysis, *The Journal of Symbolic Logic 14*, 1949, pp. 145–158.

Skolem, Th., Begründung der elementaren Arithmetik durch die rekurrirende Denkweise ohne Anwendung scheinbarer Veränderlichen mit unendlichen Ausdehnungsbereich, *Videnskapsselskapets Skrifter, I.Matematisk-naturvidenskabelig klasse 6*, 1923, pp. 1–38. Translation (van Heijenoort 1967, pp.302-333).

Stigt, W.P. van, *Brouwer's intuitionism*, North-Holland Publ. Co., Amsterdam 1990.

Tait, W.W., Finitism, *The Journal of Philosophy 78*, 1981, pp.524–546.

Troelstra, A.S., *Choice Sequences, a Chapter of Intuitionistic Mathematics.* Clarendon Press, Oxford 1977.

Troelstra, A.S., Analyzing choice sequences, *The Journal of Philosophical Logic 12*, 1983, pp.197–260.

Troelstra, A.S., Logic in the writings of Brouwer and Heyting, in: *Atti del Convegno Internazionale di Storia della Logica, San Gimignano, 4-8 dicembre 1982*, edited by V.M Abrusci, E. Casari, M. Mugnai, Cooperativa Libraria Universitaria Editrice Bologna, Bologna 1983, pp. 193–210.

Troelstra, A.S., On the origin and development of Brouwer's concept of choice sequence, in: *The L.E.J. Brouwer Centenary Symposium* edited by A.S. Troelstra and D. van Dalen, North-Holland Publ. Co., Amsterdam 1990, pp. 465–486.

Troelstra, A.S., On the early history of intuitionistic logic, in: *Mathematical Logic* edited by P.P.Petkov, Plenum Press, New York, 1990, pp. 3–17.

Troelstra, A.S., Dalen, D. van, Constructivism. An Introduction, 2 vols., North-Holland Publ. Co., Amsterdam 1988.

Turing, A.M., On computable numbers, with an application to the Entscheidungsproblem, *Proceedings of the London Mathematical Society 42*, 1937, pp. 230–265; Corrections, *Ibidem 43*, pp. 544–546.

Weyl, H., *Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis*, Veit, Leipzig 1918.